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Minimizing the Mean Majority Deficit: The Second Square-Root Rule

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Abstract

Let $\mathcal{W}$ be a composite (two-tier) simple voting game (SVG) consisting of a council, making yes/no decisions, whose members are delegates, each voting according to the majority view in his/her district. The council's decision rule is an arbitrary SVG $\mathcal{V}$. The *mean majority deficit* $\Delta[\mathcal{W}]$ is the mean difference between the size of the majority camp among all citizens and the number of citizens who agree with the council's decision. Minimizing $\Delta[\mathcal{W}]$ is equivalent to maximizing the sum of the voting powers of all the citizens, as measured by the (absolute) Banzhaf index β'. We determine the $\mathcal{V}$ which minimize $\Delta[\mathcal{W}]$. We discuss the difference between majoritarianism and equalization of the voting powers of all citizens.

Keywords: Banzhaf power index, equal suffrage, majoritarianism, mean majority deficit, simple voting game, square-root rules, two-tier decision-making.

Minimizing the Mean Majority Deficit: The Second Square-Root Rule*

1 Introduction

As far as we can tell, the first, better-known *square-root rule (SQRR)* was first stated by Lionel Penrose in his 1946 paper [9, p. 57]. Penrose considered a two-tier voting system, such as 'a federal assembly of nations' (an obvious reference to the newly established United Nations), in which a set of constituencies of various sizes elect one delegate each to a decision-making 'assembly of spokesmen'. He argued that in order to achieve an equitable distribution of voting power in the assembly, equalizing the (indirect) voting power of the people across all constituencies, the decision rule in the federal assembly should be such that 'the voting power of each nation in [the] assembly should be proportional to the square root of the number of people on each nation's voting list'. By 'voting power' Penrose meant, essentially, what later became known as the '(absolute) Banzhaf index', usually denoted by 'β'', a measure which he, in fact, was the first to propose.

Penrose does not give a rigorous proof of his SQRR, but justifies it by a semi-heuristic (and, in our view, inconclusive) argument. On the other hand, Banzhaf—who, unaware of Penrose's theory, re-invented it some twenty years later—provides in his paper [1] all the ingredients of a rigorous proof, but not a precise statement of the SQRR itself.

Although the Penrose–Banzhaf SQRR is quite well known, it is often misstated as though it prescribes a weighted voting system in which the *weights*, rather than voting *powers*, of the delegates ought to be proportional to the square roots of the sizes of their respective constituencies. This is a mistake, because of course voting weights are in general not proportional to voting powers. At one point ([9, p. 55]) Penrose himself seems to be guilty of a wrong, or at least ambiguous, statement of his rule. For more recent misstatements see, for example, [5, pp. 249, 254], [6, p. 171] and [7, p. 226]. This confusion is particularly unfortunate because, as we shall see below, there is in fact quite a different rule that does indeed require delegates' weights to be proportional to the square roots of the sizes of their respective constituencies.

This *second* SQRR is concerned with minimizing the mean deviation of the

*We are indebted to the editors for some helpul suggestions.

indirect two-tier decision-making rule from a 'direct democracy' simple majority rule.

Consider an arbitrary simple voting game (SVG) $\mathcal{W}$ with n voters. Suppose that in a given division of the voters a bill was passed by the 'yes' votes of a winning coalition of size k. If $k < n - k$, then the size of the majority camp, $n - k$, exceeds by $n - 2k$ the number k of voters who agree with the decision; in this case we say that in the given division there is a *majority deficit* of $n - 2k$. But if $k \geq n - k$ then the *majority deficit* is 0. Similarly, suppose that in a given division a bill was defeated by the 'no' votes of a blocking coalition of size k. If $k < n - k$, then again we say that in the given division there is a *majority deficit* of $n - 2k$. But if $k \geq n - k$ then the *majority deficit* is 0.

The majority deficit may be regarded as a random variable; we denote its mean (or expected) value—the *mean majority deficit*—by '$\Delta[\mathcal{W}]$'. This quantity is a measure of the deviation of $\mathcal{W}$ from a majority SVG with n voters.

In Section 2 we shall give a rigorous probabilistic definition of $\Delta[\mathcal{W}]$. We shall also derive a very simple relationship between this quantity and the sum of the (absolute) Banzhaf (Bz) powers of the voters of $\mathcal{W}$, which we denote by '$\Sigma[\mathcal{W}]$'. It transpires that minimizing $\Delta[\mathcal{W}]$ is equivalent to maximizing $\Sigma[\mathcal{W}]$.

In Section 3 we set up a model of two-tier indirect decision-making, using a composite SVG $\mathcal{W} = \mathcal{V}[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_m]$, in which $\mathcal{V}$ provides the decision rule in a council of delegates, and the $\mathcal{W}_i$ are majority SVGs. We determine the $\mathcal{V}$ for which $\Sigma[\mathcal{W}]$ is maximal, hence $\Delta[\mathcal{W}]$ is minimal. We derive the second SQRR, which was stated, albeit imprecisely and without proper proof, by Morriss in [8, pp. 187–189].

In the concluding Section 4 we show by means of simple numerical examples that implementing the two SQRRs can lead to quite different results. In view of this we stress the difference between equal suffrage ('one person, one vote', briefly: 'OPOV') and majoritarianism, which have sometimes been conflated with each other.

2 Sensitivity and mean majority deficit

In this section we state some definitions and results from the general theory of a priori voting power.

Definition 1 (i) By an *SVG* we mean a set $\mathcal{W}$ of subsets of a finite set N such that $\emptyset \notin \mathcal{W}$; $N \in \mathcal{W}$; and whenever $X \subseteq Y \subseteq N$ and $X \in \mathcal{W}$ then also $Y \in \mathcal{W}$.

We refer to N as the *assembly*, to its members as *voters*, and to its subsets as *coalitions* [of $\mathcal{W}$]. The *winning* coalitions are just those belonging to $\mathcal{W}$; the rest are *losing* coalitions.

Voter a is *critical in* coalition S if $S \in \mathcal{W}$ but $S - \{a\} \notin \mathcal{W}$; a is *critical outside* S if $S \notin \mathcal{W}$ but $S \cup \{a\} \in \mathcal{W}$; a is *critical for* S if it is critical in or outside S.

We say that voter a *agrees with the outcome of* coalition S [in $\mathcal{W}$] if $a \in S \in \mathcal{W}$ or $a \notin S \notin \mathcal{W}$.[1]

We shall take $|N|$, the number of members of N, to be n. While generally N can be any finite set, we shall use the set $I_n = \{1, \ldots, n\}$ as a 'canonical' assembly of size n.

Definition 2 Let q be a positive real and let $w_1, \ldots, w_n$ be non-negative reals such that $0 < q \leq \sum_{i=1}^{n} w_i$. In this connection we refer to q as the *quota* and to the w_i as *weights*. Further, if $S \subseteq I_n$ we put $wS = \sum_{i \in S} w_i$ and refer to wS as the *weight of* S.

We denote by '$[q; w_1, \ldots, w_n]$' the SVG with assembly $I_n = \{1, \ldots, n\}$ whose winning coalitions are just those $S \subseteq I_n$ with $wS \geq q$.

Any SVG $\mathcal{W}$ isomorphic[2] to $[q; w_1, \ldots, w_n]$ is called a *weighted voting game (WVG)*. It 'inherits', via the isomorphism, a quota and weights.

The WVG

$$[\tfrac{n+1}{2}; \underbrace{1, 1, \ldots, 1}_{n \text{ times}}] \tag{1}$$

is denoted by '$\mathcal{M}_n$'; any WVG isomorphic to it is called a *majority* SVG.

Definition 3 For any finite set N we define the *Bernoulli space* of N, denoted by '$\mathbf{B}_N$', to be the probability space whose points are just the subsets of N, each of which is assigned the same probability, $1/2^n$.

Comment 1 (i) If each subset S of N is replaced by its characteristic function, then $\mathbf{B}_N$ becomes the well-known space of n Bernoulli trials, with equal probability, $1/2$, for success and failure.

(ii) In what follows, we shall denote by 'P' and 'E' the probability measure and expected value operator, respectively, on the space $\mathbf{B}_N$, where N is the

[1] This is short for saying that in a division in which S is the set of 'yes' voters, the decision goes according to the way a votes.

[2] We take the notion of isomorphism of SVGs to be self-explanatory.

assembly of some SVG. We shall usually not need to specify which SVG is being referred to, because this will be clear from the context.

Definition 4 If $\mathcal{W}$ is an SVG with assembly N and a is any of its voters, we put

$$\beta'_a[\mathcal{W}] =_{\text{def}} \mathrm{P}\{X \subseteq N : a \text{ is critical for } X \text{ in } \mathcal{W}\}. \tag{2}$$

We refer to $\beta'_a[\mathcal{W}]$ as the *Bz power* of a in $\mathcal{W}$, and to the function β' itself as the *Bz measure [of voting power]*.

Comment 2 (i) In the literature β' is often called the *absolute Banzhaf index* (as distinct from the ordinary, or *relative*, Banzhaf index β, which is obtained from β' by normalization so that $\sum_{x \in N} \beta_x[\mathcal{W}] = 1$.)

(ii) Our probabilistic definition of β' is easily seen to be equivalent to the definition more commonly given in the literature.

The measure proposed by Penrose in [9, p. 53] was in fact $\beta'/2$. In stating his definition, he asserts as an obvious fact the following result, whose proof is indeed quite simple.[3]

Lemma 1 *If $\mathcal{W}$ is an SVG and a is any of its voters, then*

$$\mathrm{P}\{X \subseteq N : a \text{ agrees with the outcome of } X \text{ in } \mathcal{W}\} = \frac{1 + \beta'_a[\mathcal{W}]}{2}, \tag{3}$$

where N is the assembly of $\mathcal{W}$. ■

Definition 5 For any SVG $\mathcal{W}$ we put

$$\Sigma[\mathcal{W}] =_{\text{def}} \sum_{x \in N} \beta'_x[\mathcal{W}], \tag{4}$$

where N is the assembly of $\mathcal{W}$. We refer to $\Sigma[\mathcal{W}]$ as the *sensitivity* of $\mathcal{W}$.

Further, we put $\Sigma_n =_{\text{def}} \Sigma[\mathcal{M}_n]$; so that Σ_n is the sensitivity of any majority SVG with exactly n voters.

Comment 3 By an easy combinatorial argument one obtains:

$$\Sigma_n = \frac{m}{2^{n-1}} \binom{n}{m}, \tag{5}$$

[3] For a proof see, for example, [3, pp. 124–125].

where $m = [n/2] + 1$ is the least integer greater than $n/2$. Hence, a routine calculation, using Stirling's well-known approximation formula for $n!$, yields

$$\Sigma_n \sim \sqrt{\frac{2n}{\pi}}, \tag{6}$$

where '$\sim$' means that the ratio of the two sides tends to 1 as n increases.

The sensitivity of an SVG has an interesting interpretation, as the expected value of an important random variable which we now proceed to define.

Definition 6 For any SVG $\mathcal{W}$ with assembly N, we define the random variable $\mathsf{Z}[\mathcal{W}]$ on the Bernoulli space $\mathbf{B}_N$ by stipulating that the value of $\mathsf{Z}[\mathcal{W}]$ at any coalition $S \subseteq N$ equals the number of voters who agree with the outcome of S in $\mathcal{W}$ minus the number of those who do not agree with the outcome of S in $\mathcal{W}$.

Further, if $\mathcal{W}$ is the majority SVG with assembly N, we denote $\mathsf{Z}[\mathcal{W}]$ by 'M_N', and refer to it as the *margin* [in N].

Comment 4 Note that the value of M_N at any coalition $S \subseteq N$ equals the absolute value of the difference between $|S|$ and $|N - S|$, that is, the excess of the size of the majority camp over that of the minority camp. If the voters are evenly divided, then the value of M_N is of course 0.

Theorem 1 *Let $\mathcal{W}$ be an SVG. Then*

$$\mathrm{E}(\mathsf{Z}[\mathcal{W}]) = \Sigma[\mathcal{W}]. \tag{7}$$

Proof. For each voter x and coalition S, let $\mathsf{Z}_x S$ equal 1 or -1 according as x agrees or does not agree with the outcome of S. From Lemma 1 it follows by an easy calculation that $\mathrm{E}(\mathsf{Z}_x) = \beta'_x[\mathcal{W}]$. But clearly $\mathsf{Z}[\mathcal{W}] = \sum_{x\in N} \mathsf{Z}_x$. Hence

$$\mathrm{E}(\mathsf{Z}[\mathcal{W}]) = \sum_{x\in N} \beta'_x[\mathcal{W}] = \Sigma[\mathcal{W}], \tag{8}$$

as claimed. ∎

Corollary 1 $\mathrm{E}(\mathsf{M}_N) = \Sigma_n$. ∎

Definition 7 For any SVG $\mathcal{W}$ with assembly N, we define the random variable $\mathsf{D}[\mathcal{W}]$, called the *majority deficit [of $\mathcal{W}$]*, on the Bernoulli space $\mathbf{B}_N$ by stipulating that the value of $\mathsf{D}[\mathcal{W}]$ at any coalition $S \subseteq N$ equals the size of the majority camp minus the number of voters who agree with the outcome of S in $\mathcal{W}$.

Thus, if the voters who agree with the outcome of S in $\mathcal{W}$ *are* the majority, or if the voters are evenly divided, then the value of $\mathsf{D}[\mathcal{W}]$ at S is 0; but if the voters who agree with the outcome of S in $\mathcal{W}$ are the minority then the value of $\mathsf{D}[\mathcal{W}]$ at S equals that of the margin M_N.

Further, we put $\Delta[\mathcal{W}] =_{\text{def}} \mathrm{E}(\mathsf{D}[\mathcal{W}])$. We call $\Delta[\mathcal{W}]$ the *mean majority deficit (MMD) [of $\mathcal{W}$]*.

Clearly, the MMD is a measure of the average absolute deviation of an SVG from majority rule.

Theorem 2 *For any SVG $\mathcal{W}$ with exactly n voters,*

$$\Delta[\mathcal{W}] = \frac{\Sigma_n - \Sigma[\mathcal{W}]}{2}. \tag{9}$$

Proof. By Comment 4, the value of M_N always equals the margin by which the size of the majority camp exceeds that of the minority camp. Hence the size of the majority camp itself is given by the value of

$$\frac{n + \mathsf{M}_N}{2}. \tag{10}$$

Also, by Definition 6 the number of voters who agree with the outcome is always given by the value of

$$\frac{n + \mathsf{Z}[\mathcal{W}]}{2}. \tag{11}$$

Thus by Definition 7 we have:

$$\mathsf{D}[\mathcal{W}] = \frac{n + \mathsf{M}_N}{2} - \frac{n + \mathsf{Z}[\mathcal{W}]}{2} = \frac{\mathsf{M}_N - \mathsf{Z}[\mathcal{W}]}{2}. \tag{12}$$

We now apply the expected value operator E to both sides. By Definition 7 we obtain on the left-hand side $\Delta[\mathcal{W}]$; and by Corollary 1 and Theorem 1 we obtain on the right-hand side

$$\frac{\Sigma_n - \Sigma[\mathcal{W}]}{2}, \tag{13}$$

as claimed. ∎

Comment 5 From Theorem 2 it is clear that, for a given assembly N, an SVG $\mathcal{W}$ maximizes $\Sigma[\mathcal{W}]$ iff it minimizes $\Delta[\mathcal{W}]$.

Now, since the random variable $\mathsf{D}[\mathcal{W}]$ has no negative values, it follows that its expected value $\Delta[\mathcal{W}]$ is minimized—in fact, vanishes—iff $\mathsf{D}[\mathcal{W}]$ vanishes everywhere in the space $\mathbf{B}_N$. By Definition 7 this means that all coalitions of size $> n/2$ must win and all those of size $< n/2$ must lose. For odd n there is just one such SVG: the majority SVG with assembly N. For even n there are other such SVGs, all having sensitivity Σ_n, since the vanishing of $\mathsf{D}[\mathcal{W}]$ imposes no condition on any coalition of size $n/2$: it may be winning or losing.

This characterization of the SVGs that maximize $\Sigma[\mathcal{W}]$ is proved by Dubey and Shapley [3, pp. 106–107] using a very different argument. Below we shall use a somewhat modified form of their argument to prove Theorem 3.

3 The second square-root rule

We begin by recalling the definition of a composite SVG.[4]

Definition 8 Let m be a positive integer and let $\mathcal{V}$ be an SVG with canonical assembly $I_m = \{1, \ldots, m\}$. For each $i \in I_m$, let $\mathcal{W}_i$ be an arbitrary SVG, with assembly N_i. We put $N =_{\text{def}} \bigcup_{i=1}^m N_i$.

First, we define the *quotient map* q from the power set of N to that of I_m: for any $X \subseteq N$ we put

$$\mathrm{q}X =_{\text{def}} \{i \in I_m : X \cap N_i \in \mathcal{W}_i\}. \tag{14}$$

We now define the SVG $\mathcal{V}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$, called *the composite of* $\mathcal{W}_1, \ldots, \mathcal{W}_m$ (in this order!) *under* $\mathcal{V}$. We put

$$\mathcal{V}[\mathcal{W}_1, \ldots, \mathcal{W}_m] =_{\text{def}} \{X \subseteq N : \mathrm{q}X \in \mathcal{V}\}. \tag{15}$$

We refer to $\mathcal{V}$ as the *top* and to $\mathcal{W}_i$ as the *i-th component* of the composite SVG $\mathcal{V}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$.

Comment 6 A composite SVG $\mathcal{W} = \mathcal{V}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$ may be used to model two-tier decision-making. To this end, we regard $\mathcal{V}$ as a council of 'delegates': voter i of $\mathcal{V}$ is a delegate representing the 'citizens' belonging to the i-th 'constituency' N_i. When a bill is proposed, the entire citizenry N divides on it, and the decision within each N_i is made in accordance with the component $\mathcal{W}_i$, the decision rule of that constituency. Then the council divides, each delegate i voting 'yes' or 'no', depending on the decision reached in the i-th constituency. The decision of the council, reached in accordance with the decision rule $\mathcal{V}$, is regarded as the final decision of the entire composite $\mathcal{W}$.

[4] Our definition is essentially that given by Shapley in [10, p. 63].

To proceed, we set up a model consisting of a composite SVG, satisfying three additional conditions.

Model. Our model is a composite SVG $\mathcal{W} = \mathcal{V}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$. In this connection we use the terminology and notation introduced above, in Definition 8 and Comment 6. We impose the following three conditions.

(i) The N_i are pairwise disjoint: each citizen belongs to just one constituency.

(ii) Each of the $\mathcal{W}_i$ is a majority SVG.

(iii) The size n_i of each N_i is so large that the error in the approximations $\Sigma_{n_i} \sim \sqrt{2n_i/\pi}$ (Comment 3) is negligibly small.

Comment 7 If n_i is odd, the a priori probability that delegate i will vote 'yes' on a given bill is of course exactly 1/2. But by virtue of our Condition (iii) that the n_i are very large, we may also assume that where n_i is even the probability that delegate i will vote 'yes' is as close to 1/2 as makes no difference, because the probability that the citizens of N_i will be evenly split is negligibly small.

Also, the votes of the delegates are mutually independent, since the vote of delegate i depends only on the votes of citizens in the i-th constituency, and the constituencies are pairwise disjoint.

It follows from what was said in the preceding two paragraphs that the probability of an event $\mathbf{G}$ in the space $\mathbf{B}_{I_m}$ is as near as makes no difference to the probability of $\mathrm{q}^{-1}[\mathbf{G}]$ in the space $\mathbf{B}_N$. (Here q is the quotient map defined in Definition 8.)

Stated less formally: if $\mathbf{G}$ is any event defined in terms of division of the council and votes of the delegates, without direct reference to the votes of citizens, then the probability $\mathrm{P}(\mathbf{G})$ can be computed, with negligible error, as though $\mathcal{V}$ were a stand-alone SVG rather than the top of the composite SVG $\mathcal{W}$.

Let us now consider a citizen x belonging to the i-th constituency N_i. We wish to find $\beta'_x[\mathcal{W}]$, the (indirect) Bz power of x as a voter of $\mathcal{W}$. By Definition 4, $\beta'_x[\mathcal{W}]$ is equal to the probability of the event $\mathbf{E}$ that x is critical as a voter of $\mathcal{W}$. Stated more fully, $\mathbf{E}$ is the event that all other citizens, across all constituencies, vote in such a way that x's vote will decide the final fate of the bill.

Now, x's vote can have this effect iff the following two things happen: first, x's vote is able to tip the balance within x's own constituency, N_i; and, second, the vote of delegate i in the council is able to tip the balance there. Therefore **E** is the conjunction (intersection) of two events, **F** and **G**, where **F** is the event that x is critical as a voter of $\mathcal{W}_i$, and **G** is the event that delegate i is critical in the council.

Moreover, **F** and **G** are independent, because **F** is completely determined by the votes of the citizens of N_i, whereas **G** is determined by the votes in council of all delegates other than i, which in turn are determined by the votes of citizens of all constituencies other than N_i. Therefore,

$$\beta'_x[\mathcal{W}] = \mathrm{P}(\mathbf{F})\mathrm{P}(\mathbf{G}). \tag{16}$$

The P here is of course the probability measure on the space $\mathbf{B}_N$, which is the product space of the Bernoulli spaces of all the constituencies. Now, the event **F** involves only the i-th constituency; so its probability $\mathrm{P}(\mathbf{F})$ can be computed as though in $\mathbf{B}_{N_i}$. Thus $\mathrm{P}(\mathbf{F}) = \beta'_x[\mathcal{W}_i]$.

Also, by Comment 7 $\mathrm{P}(\mathbf{G})$ is equal, or as close as makes no difference, to $\beta'_i[\mathcal{V}]$. Thus

$$\beta'_x[\mathcal{W}] \sim \beta'_x[\mathcal{W}_i]\beta'_i[\mathcal{V}]. \tag{17}$$

From this it is very easy to derive the Penrose–Banzhaf SQRR. But we are headed in another direction. Summing (17) over N_i we obtain

$$\sum_{x \in N_i} \beta'_x[\mathcal{W}] \sim \Sigma_{n_i}\beta'_i[\mathcal{V}], \tag{18}$$

which, in view of Condition (iii) of our model can be written as

$$\sum_{x \in N_i} \beta'_x[\mathcal{W}] \sim \kappa\sqrt{n_i}\beta'_i[\mathcal{V}], \tag{19}$$

where $\kappa = \sqrt{2/\pi}$, a numerical constant. Summing (19) over all i we get

$$\Sigma[\mathcal{W}] \sim \kappa \sum_{i=1}^{m} \sqrt{n_i}\beta'_i[\mathcal{V}]. \tag{20}$$

Let us introduce a few abbreviations. First, we put

$$w_i =_{\mathrm{def}} \sqrt{n_i}, \tag{21}$$

and regard these w_i as weights on I_m. Thus, if $T \subseteq I_m$ we shall take wT to be $\sum_{i \in T} w_i$. Next, we put

$$w =_{\mathrm{def}} wI_m = \sum_{i=1}^{m} w_i. \tag{22}$$

Finally, we put

$$\Theta[\mathcal{V}] =_{\mathrm{def}} \sum_{i=1}^{m} w_i \beta'_i[\mathcal{V}]. \tag{23}$$

In terms of these abbreviations we can re-write (20) as

$$\Sigma[\mathcal{W}] \sim \kappa\Theta[\mathcal{V}]. \tag{24}$$

Taking m and the w_i as fixed, we shall now determine precisely the $\mathcal{V}$ that maximize $\Theta[\mathcal{V}]$.

Lemma 2 *For given m and w_i, $\Theta[\mathcal{V}]$ attains its maximal value iff the council $\mathcal{V}$ satisfies the following two conditions:*

(i) Every coalition T such that $w(T) < w/2$ is a losing coalition of $\mathcal{V}$;

(ii) Every coalition T such that $w(T) > w/2$ is a winning coalition of $\mathcal{V}$.

In particular, $\Theta[\mathcal{V}]$ attains its maximal value if

$$\mathcal{V} = [w/2 + \epsilon; w_1, \ldots, w_m], \tag{25}$$

where ϵ is a sufficiently small non-negative real.

Proof (outline). We proceed as in the proof of Theorem 2 in [3, p. 107]. Let T be a minimal winning coalition of $\mathcal{V}$ (that is, a member of $\mathcal{V}$ that does not include any other member) and let $\mathcal{V}'$ be the SVG obtained from $\mathcal{V}$ by removing T. In $\mathcal{V}'$ each $a \in T$ is no longer critical in T, whereas each $a \notin T$ becomes critical outside T. Threfore

$$\Theta[\mathcal{V}'] = \Theta[\mathcal{V}] - \frac{w(T) - w(I_m - T)}{2^{n-1}}. \tag{26}$$

On the other hand, if T is a maximal losing coalition of $\mathcal{V}$ and $\mathcal{V}' = \mathcal{V} \cup \{T\}$, then by the same token

$$\Theta[\mathcal{V}'] = \Theta[\mathcal{V}] + \frac{w(T) - w(I_m - T)}{2^{n-1}}. \tag{27}$$

From these facts the claim of our lemma follows easily. ∎

In view of (24), we now have:

Theorem 3 (Morriss) *If the n_i are sufficiently large, then—with vanishing or negligibly small error—the conditions of Lemma 2 provide a solution to the problem maximizing the sensitivity $\Sigma[\mathcal{W}]$ and thus minimizing the MMD $\Delta[\mathcal{W}]$.* ∎

Comment 8 We have attributed this result to Morriss because in [8, pp. 187–189] he states its most salient part: namely, that $\Sigma[\mathcal{W}]$ is maximized by making the council a WVG in which delegates' weights are proportional to the square roots of the sizes of their respective constituencies. (He omits to mention the crucial proviso that the quota should equal, or be just greater than, half the total weight. He also omits to mention that this solution need not be the only one. But, as shown in Example 2 below, there may be other solutions satisfying the conditions of Lemma 2.) He provides no rigorous proof, but some heuristic arguments that we do not find convincing.[5]

He himself attributes his result to Penrose, citing [9]; but this seems to us mistaken, as the issue of *maximizing* total voting power is never broached in [9]. The source of the error appears to be Penrose's misstatement of his SQRR, to which we have alluded in the Introduction.

4 Discussion

Both the Penrose–Banzhaf SQRR and the second SQRR, that of Theorem 3, depend on our model. Therefore the implementation of these rules as a way of achieving, or at least approximating, equal suffrage or majority rule, respectively, in a real-life case can be justified only to the extent that the assumptions of that model are satisfied.

Apart from the three conditions (i)–(iii) of the model, there are implicit assumptions built into it. Thus, delegates are assumed to vote in the council according to the majority view in their respective constituencies. This need not mean that they conduct a referendum or opinion poll on each bill. Perhaps it is good enough if delegates, wishing to be re-elected, are reasonably sensitive to the wishes of their electorates. But if councillors tend to vote in disregard of the views of their constituents then the model is unrealistic.

A second in-built assumption is even more problematic. In using the probability space $\mathbf{B}_N$, it is tacitly assumed that all citizens, irrespective of constituency, vote independently of each other; in particular, this implies that in the long run the votes of citizens of the same constituency are not more highly correlated than votes of citizens of different constituencies. This is quite realistic if the division of the citizenry into constituencies is more or less random, say a matter of mere administrative convenience, unconnected in any systematic way with the attitudes of citizens to the issues that are to be decided by the council. But if citizens of the same constituency habitually vote as a single bloc, then again the model is unrealistic.

[5]See also op. cit., Note 9 on p. 249, and pp. 229–231.

However, let us put such misgivings on one side and confine ourselves to situations in which the model is realistic. The following simple numerical example illustrates the difference between the prescriptions of the two SQRRs.

Example 1 Consider a county made up of a constituency numbering 9 million citizens and two constituencies, each numbering 1 million.

First let us see how the Penrose–Banzhaf SQRR may be applied here. It prescribes that the Bz powers of delegates 1, 2 and 3 in $\mathcal{V}$ should be in the proportion 3:1:1. It is not difficult to verify by trial and error[6] that there is just one acceptable solution, namely

$$\mathcal{V} = [3; 2, 1, 1], \tag{28}$$

for which $\beta'_1[\mathcal{V}] = 3/4$ and $\beta'_i[\mathcal{V}] = 1/4$ for $i = 2, 3$. (The same Bz powers are also obtained for the dual of this SVG, $\mathcal{V}^* = [2; 2, 1, 1]$, but this is an improper SVG, having disjoint winning coalitions, which is normally unacceptable.)

With this $\mathcal{V}$ we obtain $\Sigma[\mathcal{W}] \approx 2194$ to the nearest unit. This should be compared with the sensitivity of the 'direct democracy' majority model: the majority SVG with the 11m citizens as voters. Using the standard approximation (Comment 3) we obtain $\Sigma_{11\cdot 10^6} \approx 2646$ to the nearest unit. Hence

$$\Delta[\mathcal{W}] \approx 226. \tag{29}$$

Now let us see what happens if we maximize $\Sigma[\mathcal{W}]$. Theorem 3 yields a unique solution for $\mathcal{V}$, namely

$$\mathcal{V} = [3; 3, 1, 1]. \tag{30}$$

Here delegate 1 is a dictator, and the other two delegates, and consequently also their constituents, are dummies! This is surely unacceptable, precisely because it is purely an artefact of the choice of $\mathcal{V}$: subject to our present assumptions, the citizens of the two smaller constituencies would not be dummies under the Penrose–Banzhaf SQRR or under direct democracy. It is no consolation that with this maximizing $\mathcal{V}$ we obtain a higher sensitivity, $\Sigma[\mathcal{W}] \approx 2394$ to the nearest unit; and for the MMD we now have the absolute minimum for the given set-up,

$$\Delta[\mathcal{W}] \approx 126, \tag{31}$$

which is very small indeed.

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[6] Or by consulting the table in [11, pp. 310–312], which lists all isomorphism types of SVGs with at most four voters.

Of course, the solutions provided by Theorem 3 need not always be unacceptable. This is illustrated by the following example.

Example 2 Consider a county that has one constituency with 9 million citizens, as in Example 1, but *three* (instead of two) constituencies with 1 million citizens. The Penrose–Banzhaf SQRR prescribes that the Bz powers of delegates $1, 2, 3$ and 4 in $\mathcal{V}$ should be in the proportion $3 : 1 : 1 : 1$. Here again there is a unique proper solution, namely

$$\mathcal{V} = [3; 2, 1, 1, 1], \tag{32}$$

for which $\beta'_1[\mathcal{V}] = 3/4$ and $\beta'_i[\mathcal{V}] = 1/4$ for $i = 2, 3, 4$.

For this $\mathcal{V}$ we obtain $\Sigma[\mathcal{W}] \approx 2394$ to the nearest unit. On the other hand, for the corresponding majority SVG (with 12m citizens) the standard approximation (Comment 3) yields $\Sigma_{12\cdot 10^6} \approx 2764$ to the nearest unit. Hence

$$\Delta[\mathcal{W}] \approx 185. \tag{33}$$

Happily, this $\mathcal{V}$ also satisfies conditions (i) and (ii) of Lemma 2. Indeed, from (21) we have here $w_1 = 3000$ and $w_i = 1000$ for $i = 2, 3, 4$; and note that Lemma 2 does not impose any condition on coalitions T for which $wT = 3000$, so it allows $\{1\}$ to be a losing coalition and $\{2, 3, 4\}$ to be a winning one.

But Lemma 2 admits also another proper solution, namely

$$\tilde{\mathcal{V}} = [4; 3, 1, 1, 1]. \tag{34}$$

Here $\beta'_1[\tilde{\mathcal{V}}] = 7/8$ and $\beta'_i[\tilde{\mathcal{V}}] = 1/8$ for $i = 2, 3, 4$. This solution yields of course the same (minimal) value for Δ but does not equalize the citizens' indirect voting powers (as measured by β'); indeed, it follows from (17) that the ratio between the power of a member of the large constituency and that of her fellow-citizen in a small constituency is $7 : 3$. So on egalitarian grounds $\mathcal{V}$ is to be preferred.

Most lay people, as well as many scholars, often confuse majority rule with equal suffrage. Such confusion is evident, for example, in some of the opinions of US Supreme Court judges in cases involving imposition of the OPOV principle on state legislatures and local government. The confusion was shared by liberals, such as the great radical reformer William O Douglas, who ardently supported Supreme Court activism in enforcing OPOV, and by their more conservative opponents, such as the formidable John M Harlan. Thus Douglas, delivering the Supreme Court's majority opinion in the famous case of *Gray v Sanders*,[7] supported the Court's pro-OPOV ruling by some arguments that are in fact majoritarian. Conversely, Harlan, in his separate opinion in the case of *Whitcomb v Chavis et al*,[8] attacked the imposition of OPOV using anti-majoritarian arguments.[9]

Also, some of the debate around the indirect two-tier system used in electing the president of the US, via an Electoral College, seems to be confused in a similar way.[10] The opponents of the present system sometimes seem to imply that it is *inherently* incapable of implementing the OPOV principle. This is of course false. True, the present allocation of Electoral votes (in effect, weights) to the various states results in inequality of suffrage as between citizens.[11] But this inequality can be eliminated or very greatly reduced by a proper re-allocation. On the other hand, the MMD of the system cannot possibly be eliminated or brought close to 0. Therefore a case against the two-tier system as such can be made on majoritarian grounds.

In cases where a two-tier system of decision-making is preferred or obligatory (as in federal or international bodies), it may sometimes be possible, as in Example 2, to satisfy the OPOV principle and at the same time minimize the MMD; but as Example 1 shows, the prescriptions of majoritarianism and equality of suffrage do not necessarily coincide, and may in fact sharply clash. The MMD cannot be made to vanish (except in some trivial cases, with degenerate constituencies), but it may be possible to reduce it considerably. However, this may result in an unequal—sometimes extremely unequal—distribution of voting power. How much inequality, or how high a value of the MMD, one is prepared to tolerate depends on the relative values one attaches to egalitarianism and majoritarianism.

Be that as it may, in our view the main value of Theorem 3 is descriptive rather than prescriptive. It provides a benchmark for the sensitivity of a composite SVG $\mathcal{W} = \mathcal{V}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$ where the number m of constituencies and their respective sizes n_i are given. It is surely always of interest to find

[7] 372 US Reports (1963), p. 368 ff.

[8] 403 US Reports (1971), p. 143 ff.

[9] For detailed analysis of these and other US judicial opinions in cases involving OPOV, see [4, Ch. 4].

[10] As is well known, the Electoral College operates in effect as a weighted voting council, with the Electors of each state as a single bloc voter. There are however two minor exceptions: Maine and Nebraska, whose Electors may split.

[11] See for example Banzhaf [2], where he shows, using the SQRR, that the present allocation over-represents the biggest states. Formerly, ignorance of the SQRR led to the general belief that the big states are *under*-represented.

out how close $\mathcal{W}$ is to majority rule. To this end, the sensitivity $\Sigma[\mathcal{W}]$ of any such $\mathcal{W}$ should be compared with the maximal value achievable with the given m and n_i, specified by Theorem 3, as well as with the sensitivity Σ_n of the corresponding direct majority rule.[12]

[12]Since the *minimal* sensitivity of an SVG with n voters, obtained for the *unanimity* rule, is $n/2^{n-1}$, a logarithmic scale is more appropriate for these comparisons than a linear one.

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