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Uncertainty Aversion and Equilibrium*

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Abstract. If rationality is not mutual knowledge in a game then standard expected utility theory requires a rational player to have a specific belief about the behaviour of a non-rational opponent. This paper argues that this problem does not arise under Choquet expected utility theory, which does not require a player's beliefs to be additive. Non-additive beliefs allow the formalization of the idea that a player who faces a non-rational opponent faces genuine uncertainty. Optimal strategies can then be derived from assumptions about the rational player's attitude towards uncertainty. This paper investigates the consequences of such a view of strategic interaction. We formulate equilibrium concepts, called Choquet-Nash equilibrium in normal forms and perfect Choquet equilibrium in extensive forms, that solves the infinite regress that arises in this situation and study existence and properties of these equilibria in normal and extensive form games.

Keywords: extensive form games, equilibrium, non-additive beliefs, uncertainty aversion.

1. Introduction

Game theory is about rational decision-making under strategic interaction. Players are assumed to be rational in a decision-theoretic sense: they act as if they possess a utility function over outcomes and beliefs given by a probability distribution over states, and maximise subjective expected utility (Savage, 1954). Beliefs in turn are derived from assumptions based on mutual knowledge of rationality.

In normal form games this leads to the game-theoretic solution concepts of dominance, rationalizability, equilibrium and correlated equilibrium (Tan and Werlang, 1988; Aumann and Brandenburger, 1995). In extensive form games this leads to equilibrium refinements (see, e.g., Selten, 1965; van Damme, 1992).

Yet this view leads to difficulties in extensive games (see, e.g., Selten, 1975, 1978; Binmore, 1987; Reny, 1993). Equilibrium refinements assume that players continue to regard their opponents as rational even after they have observed them deviating

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from rational play. More fundamentally, the natural explanation of experimental evidence is that players are not necessarily rational. But then rationality cannot even be mutual knowledge at the beginning of the game. Game theory suffers from the defect that its solution concepts are based on rationality alone.

What is a rational strategy if rationality is not mutual knowledge? This paper attempts to address this question by applying a weaker definition of decision-theoretic rationality to games. Schmeidler (1989) has shown that Savage's sure-thing Principle can be weakened so that players can still be modelled as maximising expected utility subject to their beliefs, but now beliefs do no longer have to be additive. Whereas Savage's approach reduces uncertainty to risk, Schmeidler's Choquet expected utility theory (henceforth CEU) gives rise to a qualitative difference between risk and uncertainty.

This difference is important in games if we distinguish between rational and non-rational players as in Kreps et al. (1982). A rational player is one who chooses his strategy as to maximise utility given his beliefs. A rational player who faces a rational opponent can anticipate her strategy if he knows her utility function and can anticipate her beliefs. Consequently, a rational player who faces a rational opponent faces risk, in the sense that his beliefs are given by objective probabilities determined by best-reply considerations. Thus his beliefs are necessarily additive.

On the other hand, a rational player who faces a non-rational opponent faces true uncertainty, if all he knows is that a non-rational player does not necessarily choose a utility-maximising strategy. Under CEU, a rational player's beliefs reflect his attitude towards uncertainty. If the player is not uncertainty-neutral, his beliefs will not be additive.

Schmeidler (1986, 1989) has also shown that maximisation of expected utility under uncertainty aversion is equivalent to minimax behaviour under uncertainty neutrality. We argue, therefore, that the rational strategy of an uncertainty-averse player when facing a non-rational opponent is maximin play, if there is neither a theory nor an empirical regularity that allows further restriction of the rational player's beliefs.

Overall, a player who is rational in the sense of CEU will therefore take both possibilities into account: that the opponent may be rational and that she may not be. He will therefore evaluate a strategy with the utility that results in case the opponent is rational and with the strategy's minimax utility in case the opponent is non-rational. The overall expected utility of the strategy is thus the weighted sum, where the weights come from a common prior about the opponent's rationality.

Clearly, a rational player will now choose the strategy that maximises expected utility. He will also anticipate that a rational opponent will do the same, and will anticipate that the rational opponent will anticipate that, ad infinitum. This gives, as usual, rise to an infinite hierarchy of beliefs.

This paper presents an equilibrium concept, called Choquet-Nash equilibrium, that cuts through this infinite regress that arises in games if rationality is not mutual knowledge. We formulate the equilibrium concept and discuss existence and properties in normal form games. An equilibrium for extensive form games also has to specify how players update their beliefs. Using the Dempster-Shafer rule, that generalizes Bayes' rule to non-additive probabilities, we define an equilibrium concept, called perfect Choquet equilibrium, for extensive form games and study its properties.

We show that in normal form games Choquet-Nash equilibria always exists, for every prior probability of mutual rationality. If the game is zero-sum, Choquet-Nash equilibria coincide with Nash equilibria. Thus the solution concept avoids the indeterminacy associated with rationalizability. If the game is nonzero-sum, every Nash equilibrium is a Choquet-Nash equilibrium, but not vice versa: A Choquet-Nash equilibrium may be non-rationalizable. Thus in normal form games Choquet-Nash equilibria are a proper generalization of Nash equilibria, thus allowing to model the dependence of the equilibrium outcome on the degree to which rationality is mutual knowledge.

In extensive form games, probability-zero histories only arise if rationality is mutual knowledge at the beginning of the game. For extensive games we show that, as a consequence, perfect Choquet equilibrium can differ fundamentally from subgame-perfect equilibrium and its refinements.

This paper joins a growing literature that applies CEU to games. The first of these were Dow and Werlang (1994) and Klibanoff (1993). Dow and Werlang (1994) consider normal form games in which players are CEU maximisers. Klibanoff (1993) similarly considers normal form games in which players follow maximin-expected utility theory (Gilboa and Schmeidler, 1989), which is closely related to CEU. In Hendon et al. (1995) players have belief functions, which amounts to a special case of CEU. Extensions and refinements have been proposed by Eichberger and Kelsey (1994), Lo (1995*a*) and Marinacci (1994). These authors consider normal form games and do not distinguish between rational and non-rational players. The paper closest to ours is Mukerji (1994), who considers normal form games but distinguishes between rational and non-rational players.[1] Extensive games have been studied by Lo (1995*b*) and Eichberger and Kelsey (1995). Lo (1995*b*) extends Klibanoff's approach to extensive games, Eichberger and Kelsey (1995) are the first to use the Dempster-Shafer rule in extensive games. They do not distinguish between rational and non-rational players.

This paper is organized as follows. Section 2 outlines the decision-theoretic approach to game theory, based on expected utility theory in subsection 2.1 and based on CEU in subsection 2.2. In section 3 the Choquet-Nash equilibrium is motivated and defined for normal form games. Section 4 contains a series of examples to illustrate the properties of Choquet-Nash equilibria. Section 5 defines the Dempster-Shafer updating rule in subsection 5.1 and the perfect Choquet equilibrium for extensive form games in subsection 5.2. Section 6 contains examples of extensive games to illustrate their properties. Section 7 concludes.

2. Decision-Theoretic Rationality and Nash Equilibrium

2.1. Expected Utility Theory

A game in normal form is defined by specifying the set of players N, for each player a set of strategies S_i and each player's von Neumann-Morgenstern utility

[1] Another difference is that Mukerji (1994) formulates the equilibrium in terms of beliefs, whereas we formulate it directly in mixed strategies, which is not equivalent under non-additive beliefs.

function u_i. Thus players are assumed to be rational: when faced with uncertainty they maximise subjective expected utility. This concept of rationality has been axiomatized by Savage (1954). The appendix contains details of his construction.

In a game rational beliefs must not only satisfy Savage's axioms, but must in addition be consistent with what players know about the structure of the game and about each other's rationality. In this way it is possible to derive game-theoretic solution concepts in Savage's framework from additional assumptions.

In particular, a rational player not only has a belief about the opponents' actions, but can also anticipate that rational opponents will hold such beliefs about himself. Consequently, a rational player will also form a belief about these opponents' beliefs. But again, he can anticipate that rational opponents who know that the player is rational do this as well, and this gives rise to an infinite regress. A Nash equilibrium solves this infinite regress, i.e. is consistent with such a hierarchy of beliefs.[2]

If rationality is not mutual knowledge the question thus arises how a rational player should act if if he *knew* that the opponent is not rational. In that case Savage's axioms imply that the rational player has a belief given by a unique probability measure over the opponent's actions.[3] If neither a theory of bounded rationality nor a stable empirical regularity of non-rational behaviour is available, there seems to be no way to derive this belief. On the other hand, if there is no restriction on this belief the folk theorem (see, e.g., Fudenberg and Maskin, 1986) applies and all feasible and individually rational payoffs can be achieved in some equilibrium.

The idea of this paper is that a weaker rationality concept allows further assumptions about the *rational* player from which rational actions can be derived. On this basis a solution concept can be defined that is consistent with the infinite regress that arises if rationality is not mutual knowledge.

Weakening the underlying rationality concept is also of independent interest. Savage's theory of rationality is normative (Savage, 1967), the question what a rational strategy is if rationality is not mutual knowledge is a descriptive one. A less restrictive concept of rationality will also be descriptively more adequate.

The next subsection describes a weaker version of decision-theoretic rationality.

2.2. Choquet Expected Utility Theory

Several axioms in Savage's framework are empirically very restrictive, in particular the assumptions that the preference relation over acts is complete, transitive and satisfies the Sure-Thing Principle.

Choquet expected utility theory weakens the sure-thing principle.[4] The descriptive validity of the sure-thing principle is questioned by the Allais paradox, the

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[2] We do not intend to give a formal or precise argument here, see e.g. Tan and Werlang (1988) or Aumann and Brandenburger (1995) for details.

[3] The celebrated Kreps et al. (1982) approach to bounded rationality follows this line. They show that it is possible to reconcile experimental evidence with game theoretic solutions by assuming a specific belief about the behaviour of non-rational players. This paper tries to extend their approach by relaxing the requirement of having a specific belief.

[4] Choquet expected utility has been developed by Schmeidler (1989) in the Anscombe and Aumann (1963) framework, in which acts ("horse lotteries") lead to additive probability measures over events ("roulette lotteries"). Gilboa (1987) has ex-

Ellsberg paradox and similar findings, and it places a high demand on a player's rationality. Under Choquet expected utility, the sure-thing principle is not assumed to hold for all acts, but only for acts that are comonotonic. Two acts[5] $f, f' \in \mathcal{F}$ are comonotonic iff $\neg\exists\omega, \omega' \in \Omega : f(\omega) \succ f(\omega')$ and $f'(\omega) \prec f'(\omega')$, i.e. both acts give rise to the same preference over states. In the following table, f, g and h are pairwise comonotonic, f (or g or h) and h' are not.

Table 1. Comonotonic and non-comonotonic acts.

	ω_1	ω_2
f	10	6
g	16	0
h	10	0
h'	0	4

Now consider objective mixtures between the above acts:

Table 2. Mixtures between comonotonic and non-comonotonic acts.

	ω_1	ω_2
$\frac{1}{2}f + \frac{1}{2}h$	10	3
$\frac{1}{2}g + \frac{1}{2}h$	13	0
$\frac{1}{2}f + \frac{1}{2}h'$	5	5
$\frac{1}{2}g + \frac{1}{2}h'$	8	2

Restricting the sure-thing principle to comonotonic acts means that if the player is indifferent between f and g then he must also be indifferent between $\frac{1}{2}f + \frac{1}{2}h$ and $\frac{1}{2}g + \frac{1}{2}h$, but he may, e.g., strictly prefer $\frac{1}{2}f + \frac{1}{2}h'$ to $\frac{1}{2}g + \frac{1}{2}h'$. The reason is that mixtures of non-comonotonic acts can be interpreted as "hedging", i.e. distributing utility across states. Uncertainty aversion means that the player prefers objective mixing (the hedging strategy) to his subjective weighting of pure acts.[6]

Schmeidler (1989) has shown that behaviour that is rational in this weaker sense can still be described by expected-utility maximisation. Players do still act as if they possess a von Neumann-Morgenstern utility function and beliefs, and take expected values.

These beliefs, however, are no longer given by a probability measure over events, but a capacity, i.e. non-additive measure over events. Formally: a capacity v maps Σ into $[0, 1]$ such that (i) $v(\emptyset) = 0$, (ii) $v(\Omega) = 1$ and (iii) $E \subseteq E' \Longrightarrow v(E) \leq v(E')$.

tended this approach to the Savage framework. Other important contributions are, e.g., Wakker (1989) and Sarin and Wakker (1992, 1994).

[5] Acts $f \in \mathcal{F}$ map states $\omega \in \Omega$ into consequences $z \in Z$. For a formal statement of the sure-thing Principle see the appendix.

[6] This forces these weights to add to less than 1. For further discussion of comonotonicity see, e.g., Chew and Wakker (1996).

Property (iii) weakens the finite-additivity requirement for finitely-additive measures: $E \cap E' = \emptyset \Longrightarrow v(E \cup E') = v(E) + v(E')$. Thus non-additive beliefs (which still may, but in general needn't be additive) have a qualitative flavour.

The expectation of a real-valued random variable X with respect to a non-additive measure v is defined in the sense of Choquet (1953). Formally:[7]

$$\int X dv := \int_0^\infty v(X \geq t)dt + \int_{-\infty}^0 [v(X \geq t) - 1]dt.$$

Schmeidler's theorem can now be stated:

Theorem 1 (Schmeidler, 1989). *If, in the Anscombe and Aumann (1963) version of the Savage framework, the sure-thing principle is restricted to comonotonic acts then there exist a utility function* $u : Z \to \mathbb{R}$, *bounded and cardinal,*[8] *and a capacity* $v : \Sigma \to [0,1]$, *unique and non-atomic, such that* $f \succeq f' \iff \int u(f)dv \geq \int u(f')dv$.

It seems that we did not gain much so far: in order to derive a rational player's action when facing a non-rational opponent we must now specify the capacity v instead of the measure p. However, Choquet expected utility allows, in contrast to Savage's subjective expected utility, the introduction of an additional assumption about rational preferences over acts that characterizes the player's attitude towards uncertainty.

Formally, uncertainty aversion can be characterized in terms of the capacity[9] v. Note that the domain Σ of v is a lattice with respect to set inclusion. v displays uncertainty aversion iff it is supermodular, i.e. $v(E)+v(E') \leq v(E \cap E')+v(E \cup E')$.

Maximisation of Choquet expected utility under uncertainty aversion can now be related to restrictions on *sets* of *additive* beliefs. Formally, the capacity v is a set function just like a cooperative game. The core of v can be defined analogously: $C(v) := \{p : \Sigma \to [0,1] |$ *p a finitely additive measure such that* $p(E) \geq v(E),\ \forall E \in \Sigma\}$. The core is nonempty if v is supermodular (Shapley, 1971).

Schmeidler (1986, 1989) has shown that maximisation of Choquet expected utility under uncertainty aversion is equivalent to maximinimization of subjective expected utility over the core.

Theorem 2 (Schmeidler, 1986, 1989).

$$\max_{f \in A} \int u(f)dv = \max_{f \in A} \min_{p \in C(v)} \int_\Omega u(f)dp.$$

Let us summarize the argument so far: Starting from Savage's subjective expected utility theory, the restriction of the sure-thing principle to comonotonic acts leads to the maximisation of Choquet expected utility, in which beliefs may, but

[7] As usual we write $v(X \geq t)$ for $v(\{\omega \in \Omega | X(\omega) \geq t\})$. The integrals on the right hand side are extended Riemann integrals. If v is additive this is the usual expectation.

[8] i.e. unique up to affine transformations

[9] Schmeidler (1989) expresses uncertainty aversion in the Anscombe-Aumann framework directly in terms of acts and proves that these definitions are equivalent.

needn't be additive. Under uncertainty aversion, maximisation of Choquet expected utility is equivalent to maximinimise over a set of additive beliefs given by the core of the non-additive belief.

As a result, we are led to the question which restrictions can or should be placed on the set of additive beliefs, over which the player maximinimizes in order to determine the rational action vis-á-vis a non-rational opponent. We suggest to make such a restriction part of the definition of an equilibrium.

3. Normal-Form Games

We consider finite[10] 2-player games in normal form in which rationality is not mutual knowledge.

Assumption 1: *(Choquet Rationality)*
Rational players are assumed to be rational in the sense of Choquet expected utility.

To avoid complications and to be able to concentrate on the issues that arise from lack of mutual knowledge of rationality alone, we make the usual assumption that players have a common prior ϵ.

Assumption 2: *(Common Prior)*
Rational players believe that the probability that the opponent is rational is $1 - \epsilon$. ϵ *is common knowledge among the rational players, i.e. it is a common prior.*[11]

Under the objective prior, a rational player will evaluate a strategy with the expected payoff, given that he faces either a rational or a non-rational opponent. The strategy of the rational opponent will be determined endogenously. A rational player must therefore evaluate the strategy in case he faces a non-rational opponent. For Choquet-rational players this will depend on their attitude towards uncertainty.

Assumption 3: *(Uncertainty Aversion)*
Rational players are uncertainty averse.[12]

There is both an empirical and a pragmatic justification for imposing this assumption. First, uncertainty aversion seems to be an empirical phenomenon. It is usually proposed as an explanation for the Ellsberg paradox. Dow and Werlang (1992) have proposed uncertainty aversion as an explanation of the phenomenon that there is a whole range of asset prices (as opposed to a unique price) for which an agent neither buys an asset nor sells the asset short. In general, just as risk aversion is usually assumed in decision making under risk, uncertainty aversion seems a reasonable assumption in decision making under uncertainty.[13] Second, there are pragmatic justifications for uncertainty aversion. Under uncertainty aversion the core of a non-additive belief is nonempty. Moreover, there is a long tradition of previous literature dealing with risk aversion, uncertainty aversion and the min-

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[10] The concept can easily be generalized to a finite number of players.
[11] In particular, it is treated as an objective probability.
[12] Note that we are in no way suggesting that it is rational to be uncertainty averse, just as Assumption 2 does not suggest that it is rational to have a common prior. Uncertainty aversion is an empirical assumption about rational players.
[13] Note, however, that the notions of risk aversion and uncertainty aversion are logically independent. Formally, risk aversion corresponds to a property of the utility function, uncertainty aversion corresponds to a property of beliefs. So it is theoretically possible, for example, to be both risk-loving and uncertainty-averse.

imax criterion.[14] More fundamentally, however, the alternative assumptions are uncertainty-appeal, which seems implausible, or uncertainty neutrality, in which case Choquet expected utility reduces to expected utility theory, which implies that a player has a *unique* belief about how non-rational opponents play.

Assumption 3 implies minimax-behaviour over a set of additive beliefs. It remains to determine the size of this set.

Any restriction should come from a theory of bounded rationality. However, no such theory seems available. It could also come from a stable empirical regularity that could, for instance, be observed in experiments. But such regularities are only available for specific games. Moreover, these regularities will not result from non-rational play alone.

Thus we are led to argue that no restriction at all can be placed on this set. As a consequence, it is rational for an uncertainty-averse player who maximises Choquet expected utility to play his maximin-strategy against a non-rational opponent.

Assumption 4: *(Unrestricted Non-Rationality)*
Rational players treat non-rational players as unpredictable, i.e. no a priori restriction is imposed on non-rational players.

A rational player will now anticipate that a rational opponent will maximise his utility given his beliefs, that the rational opponent has the same prior probability about being matched with a rational player and also reacts with minimax play to non-rational opponents. But, surely, the rational opponent will anticipate this as well, and the player should anticipate this as well, ad infinitum. This gives, as usual, rise to an infinite hierarchy of beliefs.

In a Choquet-Nash equilibrium this infinite regress does not lead to an inconsistency. Each rational player maximises utility given his beliefs. These beliefs take the common prior into account and that a rational opponent does the same. Under Assumptions 1 – 4 a Choquet-Nash equilibrium is a strategy profile such that a rational, uncertainty-averse player has no incentive to deviate.

Definition 1. Let G be a 2-player normal form game with finite pure strategy sets S_1, S_2 and utility functions u_1, u_2. Denote mixed strategies by $\sigma_i \in \varSigma_i$, $i = 1, 2$. A strategy profile $(\sigma_1^*, \sigma_2^*) \in \varSigma_1 \times \varSigma_2$ is a *Choquet-Nash equilibrium* iff

$$\sigma_1^* \in \arg\max_{\sigma_1 \in \varSigma_1} [(1-\epsilon) u_1(\sigma_1, \sigma_2^*) + \epsilon \min_{\sigma_2 \in \varSigma_2} u_1(\sigma_1, \sigma_2)]$$

and

$$\sigma_2^* \in \arg\max_{\sigma_2 \in \varSigma_2} [(1-\epsilon) u_2(\sigma_1^*, \sigma_2) + \epsilon \min_{\sigma_1 \in \varSigma_1} u_2(\sigma_1, \sigma_2)].$$

Note that if $\epsilon = 0$, i.e. if rationality is mutual knowledge, this definition reduces to the Nash equilibrium condition.

Theorem 3. *For every $\epsilon \in [0, 1]$ a Choquet-Nash equilibrium exists.*

Proof. The proof is a standard argument. We can define a maximin-correspondence[15] $\hat{\sigma_j}(\sigma_i) := \arg\min_{\sigma_j} u_i(\sigma_i, \sigma_j)$ Since S_j is finite, this correspondence is well

[14] "A minimax solution seems, in general, to be a reasonable solution of the decision problem when an a priori distribution in Ω does not exist or is unknown to the experimenter." (Wald, 1950)

[15] Recall that a minimax-strategy maximises a player's security level and that a maximin-strategy holds the opponent down to his security level.

defined on a compact, convex domain and upper-hemicontinuous. Thus we can define best-reply correspondences $\sigma_i^* = \sigma_i^*(\sigma_i, \sigma_j, \epsilon) = \arg\max_{\sigma_i}[(1-\epsilon)u_i(\sigma_i, \sigma_j) + \epsilon u_i(\sigma_i, \hat{\sigma_j}(\sigma_i))]$ which are again, for arbitrary ϵ, well-defined on a convex, compact domain and upper hemi-continuous. Kakutani's fixed point theorem then gives the existence of a fixed point, which is a Choquet-Nash equilibrium. □

Note the important difference between subjective expected utility and Choquet expected utility in justifying the minimax-strategy against non-rational opponents. Under subjective expected utility the maximin-strategy is rational only if the rational player believes that the non-rational opponent minimaxes him. This belief seems difficult to justify. Under Choquet expected utility the maximin-strategy is rational because the rational player cannot exclude the possibility that the non-rational opponent plays, perhaps by chance, a minimax-strategy and because he reacts aversely towards the uncertainty created by the lack of possibility to forecast a non-rational opponent's play.

4. Choquet-Nash Equilibria in Normal Form Games

In this section we go through a series of examples to illustrate the properties of Choquet-Nash equilibria (henceforth CNE) in normal form games. We will only consider two-player games.

4.1. Strict Dominance

Consider the following version of the prisoner's dilemma:

	L	R
T	2,2	0,3
B	3,0	1,1

Both players have a strictly dominant strategy. Consequently, it does not matter to them whether the opponent is rational or not, a strictly dominant strategy is the only rational one. (B,R) is also the unique CNE, because a strictly dominant strategy is of course also a minimax strategy.

4.2. Iterated Strict Dominance

Consider the following game, (similar to Fudenberg and Tirole (1991, p.6)):

	L	R
T	1,1	−99,0
B	0,1	0,0

In this game playing L is a strictly dominant strategy for player 2 (who chooses columns). Consequently, iterated strict dominance yields T as the unique rational strategy for player 1, if rationality is mutual knowledge. In particular, (T,L) is the unique equilibrium and the unique rationalizable strategy profile of the game.

Still, (T,L) is not a plausible profile. For T to make sense for player 1 he must be convinced that player 2 is rational. However, player 2 does not gain that much from playing L instead of R. In addition, player 1 does not gain much from playing

T instead of B if player 2 plays L, but he looses very much when he does so if player 2 plays R.

The CNE in this game depends on ϵ. In every CNE, player 2 will play L because this is his strictly dominant strategy. However, if ϵ is not sufficiently low, player 1 will play B. Note that this shows that non-rationalizable strategies may be CNE-strategies.

4.3. Weak Dominance

	L	R
T	2,2	0,2
B	2,0	1,1

This game has two Nash equilibria. (T,L) is the payoff-dominant Nash equilibrium, but it involves weakly dominated strategies. (B,R) is a Nash equilibrium in weakly dominant strategies but it is payoff-dominated.

The literature on cautious rationalizability maintains that it is not rational to play weakly dominated strategies. This is formalized through the concept of caution, according to which a player's belief should assign some positive probability to every possible strategy of the opponent. This solution concept thus excludes the Nash equilibrium (T,L).

In contrast, (T,L) is a CNE if $\epsilon = 0$. Thus it is not a priori nonrational to play the payoff dominant equilibrium strategies in weakly dominated strategies if rationality is mutual knowledge. However, (B,R) is the only CNE if $\epsilon > 0$. In this sense CNE replaces the ad hoc concept of caution.

4.4. Payoff Dominance

	L	R
T	2,2	0,0
B	0,0	1,1

This game has three Nash equilibria, (T,L), (B,R) and a Nash equilibrium in mixed strategies. As can be checked easily, the mixed Nash equilibrium strategies are also the players' minimax strategies. It follows that the mixed Nash equilibrium is also a CNE independently of the degree ϵ of mutual knowledge of rationality, thus providing a rationale for mixed equilibria.[16] Further, for every ϵ for which (B,R) is a CNE (T,L) is a CNE as well. The opposite direction is not true, i.e. (T,L) is a CNE for a lower degree of mutual knowledge of rationality than (B,R). In this sense CNE provides some rationale for payoff dominance.

4.5. Risk Dominance

	L	R
T	9,9	0,7
B	7,0	8,8

[16] However, this only holds for mixed equilibria in minimax strategies, not for all mixed equilibria.

In this game the payoff dominant equilibrium (T,L) is risk dominated by the equilibrium (B,R). Here, for every ϵ for which (T,L) is a CNE, (B,R) is a CNE as well. The opposite direction is not true, i.e. (B,R) is a CNE for a lower degree of mutual knowledge of rationality than (T,L). In this sense CNE provides a rationale for risk dominance.

The last two examples show CNE allows a formal argument whether payoff or risk dominance should have precedence.

4.6. Battle of the Sexes

Consider the following Battle-of-the-Sexes game:

	L	R
T	3,1	0,0
B	0,0	1,3

This game has 3 Nash equilibria, (T,L), (B,R) and $p^* = \frac{3}{4}$, $q^* = \frac{1}{4}$. (T,L) and (B,R) are also Choquet-Nash equilibria for $\epsilon < \frac{1}{4}$. The mixed strategy Nash equilibrium, however, is only a Choquet equilibrium for $\epsilon = 0$. Thus, Choquet-Nash equilibria represent a formal approach to distinguish between the plausibility of mixed equilibria according to the specific game in question.

4.7. Zero-Sum Games

Consider the following zero-sum game ("Matching Pennies"):

	L	R
T	1,−1	−1,1
B	−1,1	1,−1

This game has a unique Nash equilibrium in mixed strategies $p^* = Prob(T) = \frac{1}{2}$, $q^* = Prob(L) = \frac{1}{2}$. Note that this is also the unique CNE. This is easy to see: Since it is already rational to play the minimax-strategy against rational opponents and since minimaxing is also rational against non-rational opponents it is overall rational. Note that this holds for all zero-sum games.

Remember, however, that *all* strategy profiles are rationalizable. Note that this shows that Choquet-Nash equilibria differ from rationalizability.

4.8. Duopoly

Consider the Cournot duopoly, i.e. the quantity-setting game between two identical profit-maximizing firms. It is clear that the minimax-strategies will in general depend on the specification of both the demand and the cost functions. As a consequence, the CNE will also depend on these specifications.

In contrast, consider the Bertrand duopoly, i.e. the price-setting game. Here the competitive prices are both equilibrium prices and minimax prices. As a result, the CNE coincides with the competitive equilibrium, independent of the specification of the demand and cost functions.

This finding reinforces the Bertrand paradox that these two models of duopolistic competition do not lead to the same outcome, given that the specification of the strategies is the modeller's choice. It lends support to the – counterintuitive – hypothesis that the Bertrand model is the more robust model of duopolistic competition.

5. Extensive-Form Games

In normal forms Choquet-Nash equilibrium generalizes Nash equilibrium, taking potential lack of mutual knowledge of rationality into account. Rationality may indeed be mutual knowledge however, in which case the analysis does not add anything new.

This changes dramatically in extensive form games. On the one hand, mutual knowledge of rationality may well be an assumption to start with, but it cannot be maintained after a player deviated from whatever action is defined as rational. Lack of mutual knowledge of rationality thus arises endogenously in extensive forms[17]. On the other hand, a rational strategy must certainly not give a rational player an incentive to pretend not to be rational. As a consequence, in extensive forms Choquet-Nash equilibria will not be just a generalization of equilibrium analysis, but differ substantially from equilibrium refinements.

In order to define the analogue of Choquet-Nash equilibrium for extensive forms we must first specify how rational players update their beliefs about the opponents after observing their actions. Since beliefs are not necessarily additive, Bayes' rule does not apply.

5.1. Updating Non-Additive Beliefs

Capacities can be updated according to the Dempster-Shafer rule **?** which generalizes Bayes' rule.

Definition 2. Let v be a capacity $v : \Sigma \to [0,1]$. Let $A, B \ \in \Sigma$. The posterior capacity according to the *Dempster-Shafer rule* is given by:

$$v(A|B) := \frac{v(A \cup \overline{B}) - v(\overline{B})}{1 - v(\overline{B})}.$$

There are no probability-zero-events unless rationality was mutual knowledge at the beginning of the game, i.e. $\epsilon_0 = 0$. In all other cases the Dempster-Shafer rule yields a well-defined posterior capacity.[18]. But even if $\epsilon_0 = 0$ a probability-zero-event arises only after a non-rational action, because, by definition, rational players should play rationally, and all players were assumed to be rational. Consequently, even in this case it can be argued that a player's non-rationality is revealed.

Applying the Dempster-Shafer rule to a prior ϵ that the opponent is non-rational and the rational opponent chooses a certain action A, with probability σ gives the posterior ϵ' in the following way:[19]

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[17] Selten (1975) argues explicitly that we must define the rational solution as a limiting form of bounded rationality. But of course trembles do not capture the essence of bounded rationality, nor can the assumption be justified that trembles are independent.

[18] See Gilboa and Schmeidler (1993) for more on updating ambiguous beliefs.

[19] For details see appendix 2.

$$1 - \epsilon' = v(R|A) = \frac{(1-\epsilon)\sigma}{1-(1-\epsilon)(1-\sigma)}.$$

5.2. Perfect Choquet Equilibrium

Consider now a finite two-player game in extensive form. In order to extend our equilibrium concept we add the Dempster-Shafer updating rule as a perfection requirement.

Assumption 5: *(Dempster-Shafer Rule)*
Rational players use the Dempster-Shafer rule to update their belief about the opponents' rationality whenever this is possible, i.e. as long as an observed play did not have capacity zero.

Under Assumptions 1 − 5 a perfect Choquet equilibrium is again a strategy profile from which a rational, uncertainty-averse player will not deviate.

Definition 3. Let Γ be a 2-player game in extensive form[20] with perfect recall. Let the finite pure strategy set of player i at information set h_i be S_{i,h_i} and his utility functions u_i. Denote (local) mixed strategies at information set h_i by $\sigma_{i,h_i} \in \Sigma_{i,h_i}$. Player i's behaviour strategy in Γ is $\sigma_i = (\sigma_{i,h_i})_{h_i \in H_i}$, where H_i denotes the set of all information sets of player i. A strategy profile $(\sigma_1^*, \sigma_2^*) \in \Sigma_1 \times \Sigma_2$ is a *perfect Choquet equilibrium* iff

$$\sigma^*_{i,h_i} \in \arg \max_{\sigma_{i,h_i} \in \Sigma_{i,h}} [(1-\epsilon_{h_i}) u_i(\sigma_{i,h_i}, \sigma^*_{i,h'_i}, \sigma^*_j) + \epsilon_{h_i}) \min_{\sigma_j \in \Sigma_j} u_i(\sigma_{i,h_i}, \sigma^*_{i,h'_i}, \sigma_j)],$$

for both players i and j and for all information sets h_i and h'_i, $h'_i \neq h_i$. Here ν_{h_i} is player i's posterior belief that the opponent is rational, given that the information set h_i has been reached. This belief comes from a common prior, updated by the Dempster-Shafer rule:

$$1 - \epsilon_{h_i} = \frac{(1-\epsilon_{\tilde{h}_i})\sigma^*(h_i)}{1-(1-\epsilon_{\tilde{h}_i}) \cdot [1-\sigma^*(h_i)]}.$$

Here, $\tilde{h}_i$ is player i's information set that immediately precedes h_i and $\sigma^*(h)$ is the probability that the information set h_i is reached by the equilibrium strategies. The common prior assumption means that by default $\epsilon_\emptyset = \epsilon$.

The intended interpretation is this: In equilibrium, a rational player must not have an incentive to deviate, and this must hold at every information set. So, at each information set, an equilibrium strategy must maximise his expected utility given his beliefs. A rational player will have two kinds of beliefs. The first belief specifies at which node in the information set the player thinks he is if his opponent is rational, i.e. the belief specifies what a rational opponent does. In equilibrium, this belief must be consistent with the player's own play at different information sets σ_{i,h'_i}, and the rational opponent's equilibrium strategies σ^*_j. Since the updating rule pins down rational beliefs at every information set, and since in equilibrium

[20] For a formal definition see Selten (1975) or Kreps and Wilson (1982). The restriction to 2 players is for simplicity only.

beliefs are correct, this first belief needn't be made explicit. The second belief ν_{h_i} specifies whether the opponent is regarded as rational or not. This belief results from the belief whether the opponent is rational at the preceding information set of that player and the opponent's rational strategies, combined by the Dempster-Shafer rule. Players start with a common prior and are uncertainty averse, so that they evaluate a strategy with its security level when facing a non-rational opponent.

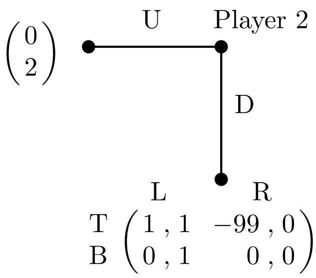
Note that in a perfect Choquet equilibrium the equilibrium path is supported by a different solution concept, i.e. minimax play, off the equilibrium path. Consequently, the solution concept does not suffer from the logical deficiency of subgame perfection, where the equilibrium path is supported by equilibrium reasoning off the equilibrium path.

6. Perfect Choquet Equilibria in Extensive Form Games

In this section we discuss examples of extensive games to clarify the concept of a perfect Choquet equilibrium (henceforth PCE).

6.1. Iterated Strict Dominance with Outside Option

Consider the following game:



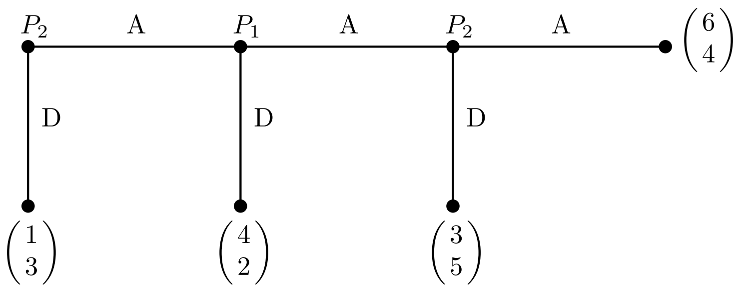
This game is similar to the iterated-strict-dominance normal form game discussed in section 5, except that player 2 now has a strictly dominant outside option. Clearly, player 1 can anticipate that his choice between T and B will only matter after a non-rational choice by player 2, in which case T is very risky.

Since U is a strictly dominant strategy for player 2, it is the only PCE-strategy for player 2. However, it follows that player 1, after observing D, can conclude that his opponent is non-rational.[21] If player 1 believes, however, that the opponent is non-rational he will play B, because player 1 is uncertainty-averse. It follows that the only PCE in this game, independent of the degree of mutual knowledge of rationality ϵ, is (U,B,L).

6.2. The Centipede Game

Consider the following version of the Centipede game:

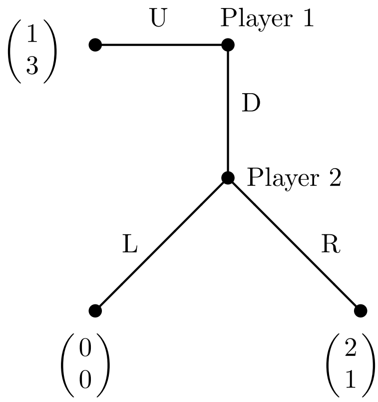
[21] If $\epsilon = 0$, D is a capacity zero event. As argued earlier, this event also reveals the player's non-rationality. This can be formalized by the Dempster-Shafer rule and a limit argument for $\epsilon \longrightarrow 0$.



Clearly, the only PCE-strategy for player 2 at his last information node is D, because this is strictly dominant and his beliefs about player 1 do not matter. Consider now player 1. If he plays D, he gets 4, if he plays A he will either meet a rational or a non-rational opponent. The rational opponent will play D. Player 1 does not know what a non-rational player 2 will do, but he cannot exclude that a nonrational player 2 will play D and player 1 is uncertainty averse. Consequently, the only PCE strategy for player 1 is D.

6.3. The Chain-Store stage game:

The following game is the stage game of a Chain-Store game. Here we are interested in the one-shot version only.



This game has two Nash equilibria, (U,L) and (D,R). Only (D,R) is a subgame-perfect equilibrium.

In every PCE, player 2 will play R, because R maximises his utility and it does not matter whether player 1 is rational or not. Player 2 is in a decision situation, not in a game situation. For player 1 the PCE will depend on his belief, i.e. the prior probability ϵ, whether the opponent is rational or not. If player 1 plays U with probability p, then his expected payoff will be $1 \cdot p$ from U and $(1-p) \cdot [(1-\epsilon)2 + \epsilon 0]$ from D, because with probability $1-\epsilon$ player 2 is rational and play his equilibrium strategy R, with probability ϵ player 2 will not be rational, and because player 1 is uncertainty averse he cannot exclude that the resulting payoff is 0. The overall payoff from p is thus $2 - p - 2\epsilon(1-p)$, so if $\epsilon > \frac{1}{2}$ the PCE strategy for player 1 is U, if $\epsilon < \frac{1}{2}$ the PCE strategy for player 1 is D.

7. Conclusion

Game Theory assumes that rationality is mutual knowledge. Yet this assumption is empirically questionable, and it cannot be maintained in extensive games after a deviation from rational play. This raises the normative question what a rational strategy is if rationality may, but need not be mutual knowledge.

We drew a distinction between rational and non-rational players and modelled rational players as rational in the sense of Choquet expected utility theory. This means that rational players maximise expected utility given their beliefs, but that these beliefs need not be additive. As a consequence rational play depends on players' attitude towards uncertainty.

The assumptions that players are uncertainty averse and that there is no theory of non-rationality that restricts beliefs about non-rational play gives rise to the concept of Choquet-Nash equilibrium that solves the infinite regress of beliefs in normal form games. A Choquet-Nash equilibrium exists for every common prior probability of mutual rationality. This solution concept generalizes Nash equilibrium, but differs both from rationalizability and from the Kreps et al. (1982) approach, and sheds light on equilibrium and dominance arguments in analyzing normal forms.

The assumption that players use the Dempster-Shafer rule to update their non-additive beliefs in extensive form games and that players conclude from probability zero events that they face a non-rational opponent gives rise to the concept of perfect Choquet equilibrium in extensive forms. This solution concepts avoids the logical difficulty associated with Nash equilibrium refinements in the sense that no equilibrium arguments are used off the equilibrium path. Perfect Choquet equilibria differ from equilibrium refinements.

This work can be extended in several ways. First, the foundations of this solution concept and its mathematical structure need further study. In particular, the solution concept can be refined if more restrictions are imposed on beliefs about non-rational play. Secondly, the solution concepts can be more systematically applied to classes of normal and extensive form games. This also raises the issue of economic applications, for instance in mechanism design. Finally, the solution concept can be taken as a reference point and compared with experimental evidence.

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Appendix

1. First Appendix

This appendix summarizes the concept of rationality as axiomatized by Savage (1954).

The basic elements of this theory are an infinite set Ω of states, $\omega \in \Omega$, an σ-algebra $\mathcal{A}$ of events, $E \in \mathcal{A} \subseteq 2^{\Omega}$, and a set Z of outcomes, $z \in Z$. An act f, $f \in \mathcal{F}$, is a measurable function that associates an outcome with each state, formally: $f : \Omega \to Z, f^{-1}(z) \in \mathcal{A}, \forall z \in Z$. The player has a preference ordering $\succeq$ over acts, i.e. $\succeq \subseteq \mathcal{F} \times \mathcal{F}$.

Savage's Theorem says that if this preference relation satisfies the following seven postulates then players act as if they possess a utility function over consequences and beliefs given by a probability measure over states, and players maximise subjective expected utility.

The first postulate (P1) says that the preference relation is complete and transitive, i.e. a complete preorder. Formally: $f \succeq f' \;\vee\; f \preceq f', \;\; \forall f, f' \in \mathcal{F}$ and $f \succeq f' \;\wedge\; f' \succeq f'' \Rightarrow f \succeq f'', \;\; \forall f, f', f'' \in \mathcal{F}$.

From the preference relation over acts three more preference relations can be derived. First, we can identify outcomes with those acts that lead with certainty to that outcome, and thus derive a preference relation over outcomes, also denoted by $\succeq$. Formally:[22] $\succeq \subseteq Z \times Z$, with $z \succeq z'$ iff (if and only if) $f =_{\Omega} z, f' =_{\Omega} z'$ and $f \succeq f'$. It is clear that (P1) implies that this is also a complete preorder.

Second, we can define a qualitative probability relation $\geq$ over events: Consider two acts f and f' and two outcomes z and z' with $z \succeq z'$, such that f leads to outcome z if event E obtains and to z' otherwise and f' leads to the outcome z if E' obtains and to z' otherwise. If the player prefers f to f' it must be because he believes that E is more probable than E'. Formally:[23] $\geq \subseteq \mathcal{A} \times \mathcal{A}$, with $E \geq E'$ iff $f = (z_E, z'_{\overline{E}}), f' = (z_{E'}, z'_{\overline{E'}})$ and $f \succeq f', z \succeq z'$.

The second postulate (P2) demands that this qualitative probability relation is well-defined, i.e. that the relation between events does not depend on the particular acts chosen to define it. Whenever two acts differ on some event, then the preference between every other two acts that differ only on that event and agree with the first two acts there, must be the same. This is the sure-thing principle. Formally: if $f =_{\overline{E}} g, f' =_{\overline{E}} g', f' =_E f$ and $g' =_E g$ then $f \succeq g \iff f' \succeq g', \;\; \forall E \in \mathcal{A}$ and $\forall f, f', g, g' \in \mathcal{F}$.

The third preference relation that can be derived from preferences between acts is a relation over acts conditional on events, denoted by $\succeq_E$. An act is preferred to another conditionally on event E iff every other two acts, that agree with the first two acts on that event and with each other on its complement, are preferred in the same way. Formally: $f \succeq_E g$ iff $\forall f', g' \in \mathcal{F}$: if $f' =_{\overline{E}} g'$ then $f' =_E f, g' =_E g \iff f' \succeq g'$.

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[22] We write $f =_E z$ for $f(\omega) = z, \; \forall \omega \in E$, for $f \in \mathcal{F}$ and $z \in Z$.

[23] $\overline{E}$ is the complement of E and we write $(z_E, z'_{\overline{E}})$ for $f(\omega) = \begin{cases} z \;, \text{if} & \omega \in E \\ z' \;, \text{if} & \omega \in \overline{E} \end{cases}$.

The third postulate (P3) says that conditional preference is consistent with the relation over outcomes. Formally: if E is non-null[24] and if $f =_E z, f' =_E z'$ then $f \succeq_E f' \iff x \succeq y,\ \forall f, f' \in \mathcal{F},\ \forall z, z' \in Z$ and $\forall E \in \mathcal{A}$.

The fourth postulate (P4) demands consistency between the qualitative probability relation and the relation between outcomes. Formally: if $x \succ y, x' \succ y'$ then $(x_E, y_{\overline{E}}) \succeq (x_{E'}, y_{\overline{E'}}) \iff (x'_E, y'_{\overline{E}}) \succeq (x'_{E'}, y'_{\overline{E'}}),\ \forall E, E' \in \mathcal{A}$ and $\forall x, x', y, y' \in Z$.

The fifth postulate (P5) demands that the relation between outcomes is non-trivial, i.e. that not all outcomes are equivalent. Formally: $\exists z, z' \in Z : z \succ z'$.

The sixth postulate (P6) demands, roughly, that no state is an atom, i.e. that there is always a partition of the state space such that it is possible for each cell of that partition to substitute an arbitrary outcome for the act on that cell without altering the preference between any two acts. Formally: whenever $f \succ g$ then there is a finite partition $(\Omega_i)_{i=1,\dots,n}$ of Ω such that $\forall \Omega_i$: if $f' =_{\Omega_i} x, f' =_{\overline{\Omega_i}} f$ then $f' \succ g$ and if $g' =_{\Omega_i} x, g' =_{\overline{\Omega_i}} g$ then $f \succ g',\ \forall f, g \in \mathcal{F}$ and $\forall x \in Z$.

The seventh and final postulate (P7) allows the extension of simple, i.e. finitely-valued, acts to general acts. Formally: if $\forall \omega \in E : f \succeq f'(\omega)$ then $f \succeq_E f'$, $\forall f, f' \in \mathcal{F},\ \forall E \in \mathcal{A}$, and analogously for $f \preceq f'(\omega)$.

Savage's Theorem can now be stated formally:

Theorem 4 (Savage, 1954). *If $\succeq \subseteq \mathcal{F} \times \mathcal{F}$ satisfies (P1) – (P7) then there exist a utility function $u : Z \to \mathbb{R}$, bounded and cardinal,*[25] *and a probability measure $p : \mathcal{A} \to [0,1]$, unique, non-atomic and finitely additive, such that $f \succeq f' \iff \int u(f)dp \geq \int u(f')dp$.*

2. Second Appendix

Applying the Dempster-Shafer rule to a prior ϵ that the opponent is non-rational and the rational opponent chooses a certain action A, with probability σ gives the posterior ϵ' in the following way.

Formally, all given data are additive capacities:

$$\begin{aligned} v(R = \text{ player is rational}) &= 1 - \epsilon, \\ v(\overline{R} = \text{ player is non-rational}) &= \epsilon, \\ v(A = \text{ rational action}|R) &= \sigma, \\ v(\overline{A} = \text{ non-rational action}|R) &= 1 - \sigma, \end{aligned}$$

If non-rational play is unrestricted, the associated capacity of a non-rational player is given by

$$\begin{aligned} v(A|\overline{R}) &= 0, \\ v(\overline{A}|\overline{R}) &= 0. \end{aligned}$$

According to the Dempster-Shafer rule the posterior is given by

$$v(R|A) = \frac{v(R \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})}$$

[24] An event E is null iff $f \succeq_E f', \forall f, f'$, i.e. all acts are indifferent conditional on E.

[25] i.e. unique up to affine transformations

and since the opponent is either rational or not we must have

$$v(R|A) + v(\overline{R}|A) = 1.$$

Since

$$v(\overline{A}|\overline{R}) = 0 = \frac{v(\overline{A} \cup R) - v(R)}{1 - v(R)} = \frac{v(\overline{A} \cup R) - (1-\epsilon)}{\epsilon}$$

we have for $\epsilon \neq 0$:

$$v(\overline{A} \cup R) = 1 - \epsilon.$$

Similarly:

$$v(\overline{A}|R) = 1 - \sigma = \frac{v(\overline{A} \cup \overline{R}) - v(\overline{R})}{1 - v(\overline{R})} = \frac{v(\overline{A} \cup \overline{R}) - (\epsilon)}{1 - \epsilon},$$

so that for $\epsilon \neq 1$:

$$v(\overline{A} \cup \overline{R}) = (1-\epsilon) \cdot (1-\sigma) + \epsilon = 1 - \sigma(1-\epsilon).$$

Now

$$\begin{aligned} & v(R|A) + v(\overline{R}|A) = 1 \\ \Longleftrightarrow\ & \frac{v(R \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})} + \frac{v(\overline{R} \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})} = 1 \\ \Longleftrightarrow\ & v(R \cup \overline{A}) + v(\overline{R} \cup \overline{A}) = 1 + v(\overline{A}), \end{aligned}$$

so that substituting gives

$$v(\overline{A}) = (1-\epsilon)(1-\sigma).$$

Consequently, the Dempster-Shafer rule becomes

$$v(R|A) = \frac{(1-\epsilon) - (1-\epsilon)(1-\sigma)}{1 - (1-\epsilon)(1-\sigma)} = \frac{(1-\epsilon)\sigma}{1 - (1-\epsilon)(1-\sigma)}.$$

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