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Congestion-Dependent Pricing and Forward Contracts for Complementary Segments of a Communication Network

Miklós Reiter and Richard Steinberg, Member, IEEE

Abstract—Congestion-dependent pricing is a form of traffic management that ensures the efficient allocation of bandwidth between users and applications. As the unpredictability of congestion prices creates revenue uncertainty for network providers and cost uncertainty for users, it has been suggested that forward contracts could be used to manage these risks. We develop a novel game-theoretic model of a multi-provider communication network with two complementary segments, and investigate whether forward contracts would be adopted by service providers. Service on the upstream segment is provided by a single Internet Service Provider (ISP) and priced dynamically to maximize profit, while several smaller ISPs sell connectivity on the downstream network segment, with the advance possibility of entering into forward contracts with their users for some of their capacity. We show that the equilibrium forward contracting volumes are necessarily asymmetric, with one downstream provider entering into fewer forward contracts than the other competitors, thus ensuring a high subsequent downstream price level. In practice, network providers will choose the extent of forward contracting strategically based not only on their risk tolerance, but also on the market structure in the interprovider network and their peers’ actions.

Index Terms—Internet, contracts, traffic control (communication), communication systems, communication system traffic, game theory, economics.

I. INTRODUCTION

The pricing for Internet service is currently based on access bandwidth and usage. However, with the growing diversity of applications using the Internet, there is considerable interest in designing a future Internet architecture that would allow users to indicate the value they place on network service by purchasing end-to-end Quality of Service (QoS) from the service provider.

Congestion-dependent pricing for communication networks has been proposed [2]–[7] as a method of traffic management that can efficiently allocate bandwidth among users—e.g. households, small businesses, large service providers—who place different value on their applications. Congestion-dependent pricing ensures that users have an incentive to control congestion. The highly influential paper of Gibbens and Kelly [2] proposed a mechanism to implement usage-based charging. In that scheme, prices are set on the basis of aggregate traffic and communicated periodically to users, who can then decide for themselves how to best satisfy their requirements at the given price.

Financial contracts could be used to provide more predictable prices to both service providers and users in a network with congestion pricing. Semret and Lazar and their co-authors published a series of papers on bandwidth pricing and contracts. These include Semret and Lazar [8], which proposes a market for circuit switched calls, wherein calls are admitted or rejected at or soon after their arrival time and, if admitted, receive a fixed allocation of capacity and have the option of securing the resource at a guaranteed maximum price for a guaranteed minimum duration. The reservation fee is determined using the Black-Scholes option pricing approach. Semret, Liao, Campbell and Lazar [9] consider a game-theoretic model of capacity provisioning in a differentiated services Internet, where the players consist of one capacity seller per network, one broker per service per network, and a set of network users. The purchase of forward contracts by the network users is proposed by Anderson et al. as a “Contract and Balancing Mechanism” [10], which is shown to give users an incentive to control congestion, while avoiding the network provider’s perverse incentive to cause congestion. On the other hand, Yuksel et al. [11] propose a “contract-switched” Internet, featuring a dynamic inter-provider pricing system to provide end-to-end QoS, in conjunction with longer-term financial contracts used for risk management.

In this paper, we ask whether long-term forward contracts would be offered to users in a future Internet with a dynamic inter-provider pricing system. Our analysis differs from the above papers by considering the fraction of a provider’s capacity to be funded by long-term contracts as a strategic variable. While our analysis is motivated by contracts between Internet Service Providers and end-users, our model is sufficiently general to be applicable to contracting by large corporate customers, as considered in [10].

To study the dynamic interactions between multiple network providers in a tractable setting, we develop a two-stage model of bandwidth sold on two complementary segments of a multi-provider communication network by means of dynamic pricing (a spot market). Specifically, the upstream segment is provided by a single large Internet Service Provider, denoted UISP, and the downstream segment is provided by several smaller ISPs, denoted ISP1, ISP2, ..., ISPn. The upstream ISP connects the downstream ISPs to the Internet backbone. A schematic diagram of the business relationships is displayed in

1Two services are said to be complementary if they are used together because they have little or no value when used separately.
prove that an increase in this lowest volume has a negative marginal externality on other downstream ISPs’ utility, whereas an increase in any other contracting volume creates positive marginal externalities. In section VII, we present conclusions. In order to aid readability, we have relegated the more technical aspects of the proofs of the first two theorems to three lemmas, which are proved in the appendix.

II. Model Overview

We consider the following two-stage contracting and pricing game played by UISP and ISP₁, …, ISPₙ. In the first stage, the ISPᵢ simultaneously choose to sell capacities 0 ≤ fᵢ ≤ k by means of forward contracts, where k is each ISPᵢ’s total capacity. This bandwidth is sold at a price which is fixed in the first stage.

In the second stage, the providers UISP and ISP₁, …, ISPₙ simultaneously set prices p_U and p₁, …, pₙ, to maximize profits π_U and π₁, …, πₙ from their uncontracted capacity. The second-stage profits are functions of the prices and the forward contracts fᵢ chosen in the first stage.

The price sensitivity of bandwidth demand is not known at the time of contracting, but dynamic pricing allows the ISPs to choose their second-stage prices based on the realized price sensitivity. We therefore model the price sensitivity β as a random variable which is revealed between the two stages of the game. This means that a risk-averse ISPᵢ has an incentive to enter into forward contracts to hedge against demand uncertainty and maximize its total expected utility

$$\Pi_i(f₁, \ldots, fₙ) = E_β[U(I_i + E_pπ_i)],$$

where E_β denotes expectation over the random price sensitivity β, Uᵢ is ISPᵢ’s increasing and strictly concave utility function, Iᵢ is ISPᵢ’s income derived from forward contracting, E_p denotes expectation over the second-stage mixed strategy prices, and πᵢ is ISPᵢ’s second stage profit. Although the income Iᵢ from forward contracting is fixed during the first stage and so does not depend on the prices realized during the second stage, we will assume that it does depend on the expected second-stage prices E_pπ_i, as this is the fair-market level at which risk-neutral users are willing to enter into forward contracts.

Before we can fully define and analyze the first-stage contracting game in section VI, we first need to develop the second-stage pricing model.

III. Pricing Model

We model the second-stage behavior of the downstream ISPs as “Bertrand-Edgeworth” price competition with capacity constraints, first studied by Edgeworth who showed that the duopoly case might not have an equilibrium in prices [13]. The formulation of the problem with the “rationing rule” considered here is due to Levitan and Shubik [14]. They found that prices are competed down to the perfectly competitive level equal to marginal cost when demand is low;

$$\text{An externality of an economic transaction is an impact on a party that is not directly involved in the transaction. The marginal externality of an ISP’s contracting strategy is the impact of a unit increase in that ISP’s contracting volume.}$$
and there is a pure-strategy Nash equilibrium, a pair of prices such that neither firm can increase its profit by unilaterally changing its price when demand is high. For the intermediate region of demand, they derived a Nash equilibrium in mixed (random) strategies. Vives [15] established the mixed-strategy equilibrium for the case of symmetric oligopoly with more than two competitors and proved convergence to the perfectly competitive price as the number of firms increases. For any fixed choice of upstream price \( p_U \), our downstream pricing model differs by taking into account forward contracts previously sold by the ISP \( i \) for diverse fractions of their bandwidth. An important analytic contribution of this paper is the characterization of the mixed-strategy equilibrium for this more complicated asymmetric model. This result is used to find an equilibrium for the full second-stage pricing game where UISP and the ISP \( i \) choose prices simultaneously.

We assume that UISP is a large provider connecting the ISP \( i \) to the Internet backbone and has all the bargaining power. Thus, where the second-stage pricing game has multiple equilibria, the equilibrium with largest \( p_U \) arises. In the special case of \( n = 1 \), ISP \( i \) is another monopolist and our game describes a bilateral monopoly.

On the other hand, where the pricing game has no pure-strategy Nash equilibrium and prices fluctuate, a realistic analysis needs to take into account the timescales over which providers are likely to adjust their prices. This in turn depends on the technologies used for price updates. While the downstream providers can directly broadcast their prices to local users connected to their networks every few seconds, this approach does not scale to a large multi-provider network such as the Internet. The monopolistic transit provider is more likely to make use of a general pricing system. Proposals for implementing inter-provider pricing by extending the Border Gateway Protocol (BGP) [16] have been made by [17], [18]. Such a system would propagate price changes over the BGP convergence timescale of several minutes. For this reason, we assume the costs of building the firms’ infrastructure are sunk, and zero marginal costs are incurred during operation of the network. According to Odlyzko [19], “marginal costs are zero up to the point where congestion occurs or forces addition of new capacity.” Of course, ISPs also incur non-bandwidth marginal costs, such as the costs of billing and customer support. However, any constant marginal costs can be normalized to zero by redefining the prices, provided the marginal costs incurred by the competing downstream ISPs are equal. Let the upstream ISP’s payoff be

\[
\pi_i = p_i (D_i - f_i).
\]  

Suppose each ISP \( i \) has previously sold capacity \( f_i \) by means of forward contracts, so his (second-stage) payoff is

\[
\pi_i = p_i (D_i - f_i).
\]  

In order to obtain closed-form expressions for the equilibrium, we work with a linear demand function [14], which has been used in the network pricing literature, e.g., [20],

\[
d_{\text{market}}(p) = \alpha - \beta p,
\]  

where the total price \( p \equiv p_U + p_m \) and \( p_m \) is the price charged by the marginal ISP \( i \) with a positive market share, that is, the highest price charged by any ISP \( i \) with a positive market share. The downstream ISPs’ incentives for choosing their contracting volumes \( f_i \) under demand uncertainty are to be discussed in section VI. For the first-stage pricing model, we suppose simply that the market potential \( \alpha \) and the price sensitivity \( \beta \) are given non-negative constants, and the contracting volumes are given constants with \( 0 < f_i < k \) for some \( k \).

Assume the upstream ISP is not subject to any capacity constraint, other than the total capacity \( nk \) resulting from the capacity of the complementary network segment. To determine the market share of each ISP, we use the rationing rule maximizing consumer surplus chosen by [14], [21], which can intuitively be seen as a “water-filling” model: demand fills the downstream ISPs’ capacities in increasing order of price, up to the point where the total demand at the next ISP’s price would be insufficient to leave any market share to that ISP. Demand is split equally between several ISPs with the same price where there is not enough demand to fill their networks completely. An implicit economic assumption in the “water-filling” model is that there is no income effect\(^3\) on bandwidth consumption.

More formally, “water-filling” specifies the bandwidth \( D_i \) provided by ISP \( i \) and the total bandwidth provided by the upstream (and downstream) network \( D_U \) by the following four conditions. The bandwidth provided by ISP \( i \) must satisfy the capacity constraint

\[
0 \leq D_i \leq k,
\]  

the capacity of ISP \( i \) must be exhausted if market demand at price \( p_i \) exceeds the total bandwidth used in the network

\[
d_{\text{market}}(p_U + p_i) > D_U \Rightarrow D_i = k,
\]  

the bandwidth provided by ISP \( i \) must be zero if market demand at price \( p_i \) is less than the total bandwidth used in the network

\[
d_{\text{market}}(p_U + p_i) < D_U \Rightarrow D_i = 0,
\]  

and, finally, demand splits equally between ISPs choosing the same price

\[
p_i = p_j \Rightarrow D_i = D_j.
\]

\(^3\)A choice of price \( p_i \) such that \( D_i < f_i \) can be interpreted as ISP \( i \) purchasing bandwidth from the customers.

\(^4\)The market potential is the maximum achievable demand, which is given by the limit of the demand function as the price goes to zero.

\(^5\)The income effect occurs when a decrease in the price for a good, other things remaining the same, will leave the consumer with more income left over, some of which will be spent on buying more of the good.
In the rest of this paper, we shall assume without loss of generality that the ISP$_1$ are ordered by their contracting volumes as

$$0 < f_1 \leq f_2 \leq \cdots \leq f_n < k.$$  

IV. Pure-Strategy Equilibrium Analysis

The equilibrium outcome of the pricing game depends on the available bandwidth capacity compared to the market potential. More precisely, the following definition partitions the range of market potential $\alpha$ into three regions by comparing it with the number $n$ of ISP$_i$s, the capacity $k$ of each ISP$_1$, and the contracting volume $f_1$ of firm 1.

**Definition 1** (high, low, intermediate market potential). Let $0 \leq f_1 < k$. Consider the thresholds

$$\alpha_i(f_1) = 2(n-1)k + 2f_1,$$  
$$\alpha_h(f_1) = (2n+1)k - f_1.$$  

We say that market potential is $f_1$-high if

$$\alpha \geq \alpha_h(f_1);$$  
that market potential is $f_1$-low if

$$\alpha \leq \alpha_i(f_1);$$  
and that market potential is $f_1$-intermediate if

$$\alpha_i(f_1) < \alpha < \alpha_h(f_1).$$  

As we will now show, in the region of $f_1$-high market potential network capacity is exhausted. Thus, the total upstream and downstream price $p_1 + p_U$ is the congestion price, the lowest price at which demand can be satisfied. In the region of $f_1$-low market potential, competition forces the downstream market price $p_1$ down to marginal cost, which is normalized to zero. In the region of $f_1$-intermediate market potential, oscillatory price behavior follows, as will be explored in the next section. The following theorem characterizes the pure-strategy Nash equilibria in the three regions.

**Theorem 1.** Pure-strategy equilibria are characterized as follows:

(i) If market potential is $f_1$-high in the pricing game, then there is a range of pure-strategy equilibria given by

$$p_1 = p_2 = \cdots = p_n$$  
$$\beta(p_1 + p_U) = \alpha - kn$$  
$$k - \beta p_1 \leq f_i \forall i$$

moreover, any $f_1$-high pure-strategy equilibrium is of this form.

(ii) If market potential is $f_1$-low, then there is a unique pure-strategy equilibrium such that every ISP$_1$ sets a zero price ($p_1 = 0$) and UISP sets $p_U = \frac{\alpha}{2\beta}$.

(iii) If market potential is $f_1$-intermediate and $n = 1$, then there is a unique pure-strategy equilibrium given by

$$p_1 = \frac{\alpha - 2f_1}{3\beta}, \quad p_U = \frac{\alpha + f_1}{3\beta}.$$  

If market potential is $f_1$-intermediate and $n \geq 2$, then there is no pure-strategy equilibrium.

Some observations may be in order. To begin, note that the general form of the result only differs between the bilateral monopoly ($n = 1$) and the true downstream oligopoly case ($n \geq 2$) when market potential is $f_1$-intermediate and competition results in the non-existence of any pure-strategy equilibrium in the oligopoly case. However, the boundaries between the regions depend on the number $n$ of downstream firms. In the bilateral monopoly case, for example, the equilibrium with $p_1 = 0$ arises only if $f_1 \geq \frac{\alpha}{2}$. In the absence of competition to force the downstream price to zero, this will only happen when market potential is so low that, given the contracting volume $f_1$, provider ISP$_1$ cannot obtain a positive profit by setting $p_1 > 0$.

On the other hand, when $n \geq 2$, the theorem says that a pure-strategy equilibrium where the ISP$_i$s set positive prices $p_i > 0$ is necessarily of the form given by (15)-(18). It is easy to check that this system is inconsistent when market potential is not $f_1$-high, so an equilibrium of this form can only exist for $f_1$-high market potential.

Observe that none of the results stated in Theorem 1 depend on the contracting volumes $f_2, \ldots, f_n$, but only on the lowest contracting volume $f_1$. In general, any contracting weakens a downstream provider’s incentive to set a high price in the pricing game, and the provider with the lowest contracting volume, ISP$_1$, will have the strongest incentive to do so. When UISP holds all the bargaining power and market potential is $f_1$-high, the equilibrium with the highest $p_U$ arises, and the equilibrium price levels are determined by ISP$_1$ and UISP, the other downstream ISPs being able to follow ISP$_1$’s price $p_1$.

When the downstream ISPs have some of the bargaining power, the prices they set increase with market potential. The competition between the downstream ISPs is more significant in this case, and the game is closer to the classical Bertrand-Edgeworth price competition with capacity constraints.

**Proof of Theorem 1:** If market potential is $f_1$-high, this allows the choice of $p_1, p_U$ satisfying the outlined conditions. We verify that these choices of prices do indeed constitute a pure-strategy Nash equilibrium. Here, UISP serves a market of maximal size $nk$, and he can do no better by cutting his price. The effect on UISP’s profit of a rise in $p_U$ is

$$\frac{\partial \pi_U}{\partial p_U} = \alpha - \beta(p_U + p_1) - \beta p_U \leq nk - \beta p_U \leq 0,$$

at the chosen point as well as for any higher value of $p_U$. Therefore, UISP has no incentive to change his strategy.

Since firm ISP$_i$’s market share $\alpha - \beta(p_U + p_1) - (n-1)k$ is equal to $k$ at our chosen point, and $f_i \leq k$, it follows that ISP$_i$ cannot gain by cutting his price. Moreover ISP$_1$ cannot increase his profit by raising his price either, since

$$\frac{\partial \pi_1}{\partial p_1} = \alpha - \beta(p_U + p_1) - (n-1)k - \beta p_1 - f_1 \leq 0,$$

where the inequality follows from (17). We have shown that the chosen point is indeed a pure-strategy Nash equilibrium.

If market potential is $f_1$-low, consider the set of strategies $p_i = 0 \forall i, \beta p_U = \frac{\alpha}{2}$. The price $p_U$ is clearly UISP’s best
response to the zero strategy chosen by the ISP; it is the monopolistic price. Observe that the total market served is 
\[ D_U = \frac{a}{2} \leq (n-1)k + f_1. \] 
Therefore, if ISP \( i \) were to choose any other price \( p_i > 0 \), his profit would be negative. We have established that this set of strategies is indeed a Nash equilibrium.

Conversely, consider any pure-strategy equilibrium given by the tuple of prices \((p_U; p_1, p_2, \ldots, p_n)\). We will start by showing that the equilibrium satisfies (15)–(18) for \( n \geq 2 \) provided some \( p_iD_i > 0 \). Let \( i \) be such that \( p_iD_i > 0 \) with \( p_i \) maximal. Suppose there was some \( j \) such that \( p_j > p_i \). Then we would have \( D_j = 0 \) by the definition of \( i \), so ISP \( j \) would have an incentive to set \( p_j \) equal to \( p_i \). Suppose now that there was some \( j \) such that \( p_j < p_i \). It follows from assumptions (7)–(8) that \( D_j = k \) and ISP \( j \) would be able to increase his price to any \( p_j < p_i \) while retaining a market share of \( k \). Since \( f_j < k \), he would increase his profit by doing so. Therefore, we have shown that all prices are equal in our equilibrium (15).

Suppose we had \( D_i < k \). Then, if \( n \geq 2 \), ISP \( i \) would have an incentive to increase his market share by cutting his price by any small amount. Hence we must have \( D_i = k \) at equilibrium and the total market served is \( nk \) (16).

Our previous argument shows that (17) and (18) must hold at equilibrium, so the ISP \( i \) and ISP \( n \) respectively have no incentive to increase their price. We have therefore shown that every non-trivial pure-strategy equilibrium is of the given form.

To show the unique characterization for the equilibrium, consider any pure-strategy Nash equilibrium in prices \((p_U; p_1, p_2, \ldots, p_n)\). We use the following two results, which are direct consequences of the definitions of the downstream ISPs’ demand and payoff functions (4)–(9) and of the thresholds for high and low market potential given in Definition 1.

- Suppose market potential is not \( f_1 \)-low. Then there exists \( 1 \leq i \leq n \) such that ISP \( i \) has \( p_iD_i > 0 \) in equilibrium.
- Suppose market potential is not \( f_1 \)-high. If \( n \geq 2 \) then every ISP \( i \) has \( p_i = 0 \) in equilibrium.

If market potential is \( f_1 \)-high, the first result shows that some \( p_iD_i > 0 \) in equilibrium. For \( n \geq 2 \), we have shown that any such equilibrium must be of the form given by (15)–(18). For \( n = 1 \), the same argument shows (17)–(18), and it is easy to see that, if no provider has an incentive to cut his price, then we have (16).

If market potential is \( f_1 \)-low, \( n \geq 2 \), the second result shows that every ISP \( i \) has \( p_iD_i = 0 \). If market potential is \( f_1 \)-low and \( n = 1 \), it is easy to see that the unique pure-strategy equilibrium is given by \( p_i = 0 \), \( p_U = 2f_1 \).

If market potential is \( f_1 \)-intermediate and \( n \geq 2 \), the two results are contradictory, so there is no pure-strategy equilibrium. Finally, if market potential is \( f_1 \)-intermediate and \( n = 1 \), it is easy to see that the unique pure-strategy equilibrium is given by (19). This completes the proof of the theorem.

V. MIXED-STRATEGY EQUILIBRIUM ANALYSIS

From Theorem 1, we know that for \( f_1 \)-intermediate market potential there is no pure-strategy Nash equilibrium when the downstream market is a true oligopoly \((n \geq 2)\). Since the downstream ISPs set their prices on a shorter timescale than the upstream ISP, we assume they use mixed strategies, interpreted as distributions of fluctuating prices following (14). The pricing game can be shown to have an equilibrium point.

**Theorem 2.** Suppose \( n \geq 2 \) and market potential is \( f_1 \)-intermediate in the pricing game. Then there exists a unique equilibrium point \((p_U; p_1, \ldots, p_n)\) where the price \( p_U \) is a pure strategy for UISP and the prices \( p_i \) are mixed strategies for each ISP \( i \), respectively, such that \( p_U \) is locally optimal and each \( p_i \) is optimal given the other ISPs’ strategies.

Local optimality of the upstream equilibrium price \( p_U \) means that UISP has no incentive to make small-scale deviations. The question of global optimality of \( p_U \) is of little importance, since the other ISPs can in any case not be expected to maintain their strategies if UISP makes large-scale deviations. However, an interesting question that remains is whether allowing UISP to play a mixed strategy leads to a different equilibrium point. We will consider this in Theorem 3.

**Proof of Theorem 2:** The proof of this theorem makes use of a generalization of the solution of the Bertrand-Edgeworth oligopoly in (14), (15), taking forward contracting into account.

**Preliminaries: Reduced Pricing Game:** We start by considering the reduced pricing game arising between the ISP, if UISP has precommitted to a fixed price \( p_U \). In analogy with Definition 1, the following regions turn out to be useful.

**Definition 2.** Let \( 0 \leq f_1 \leq k \). We say that market potential is \((f_1, p_U)\)-high if

\[ \beta p_U \leq \alpha - k(n+1) + f_1; \]

that market potential is \((f_1, p_U)\)-low if

\[ \beta p_U \geq \alpha - k(n-1) - f_1; \]

and that market potential is \((f_1, p_U)\)-intermediate if

\[ \alpha - k(n+1) + f_1 < \beta p_U < \alpha - k(n-1) - f_1. \]

The form of the equilibrium depends on the level of market potential. The following lemma (proved in the appendix) shows that the reduced pricing game between the ISP, if UISP has a unique pure-strategy equilibrium if market potential is \((f_1, p_U)\)-high or low, and a unique mixed-strategy equilibrium if market potential is \((f_1, p_U)\)-intermediate. For high market potential, every ISP sets the same positive price, while for low market potential, every ISP sets price zero. For intermediate market potential, each ISP sets a random price chosen from an interval whose upper bound is a decreasing function in its contracting volume \( f_1 \). ISP’s strategy may include setting the price to the upper bound with a positive probability.

\[ \text{This argument for the stability of local equilibria is made in (22).} \]
Lemma 1. The reduced pricing game has the following Nash equilibria.

(i) If market potential is \((f_1, p_U)\)-high, then there is a unique pure-strategy equilibrium, in which each ISP \(i\) chooses almost surely (i.e., with probability one)

\[
p_i = \frac{\alpha - \beta p_U - kn}{\beta},
\]

(ii) If market potential is \((f_1, p_U)\)-low, then there is a unique pure-strategy equilibrium, in which each ISP \(i\) chooses almost surely

\[
p_i = 0.
\]

(iii) If market potential is \((f_1, p_U)\)-intermediate, then the reduced pricing game has the following unique mixed-strategy equilibrium. Let

\[
p^*_i = \frac{\alpha - \beta p_U - kn - f_i}{2\beta}.
\]

Define \(p^*_i \in [0, p^*_i]\) to be the unique value satisfying

\[
h(p^*_{i+1}) \equiv \frac{(k - f_i)^{i+1}}{\prod_{j=1}^{i} (k - f_j)}
\]

\[
\text{for } 2 \leq (i+1) \leq n
\]

\[
p^{n+1}_i \equiv p_0.
\]

For each \(1 \leq j \leq n\), define the function \(G_j(p)\) on \([p_0, p^*_j]\) piecewise for \(p \in [p^*_j, p^*_i]\), \(i \geq j, i \geq 2\)

\[
G_j(p) \equiv \left\{ \begin{array}{ll}
\left( \frac{\prod_{j \leq i < j, p} H_j(p)}{(H_j(p))^j} \right)^{\frac{1}{j-i}} & \text{if } p > p_0,
0 & \text{if } p = p_0.
\end{array} \right.
\]

Then the reduced pricing game has a unique mixed-strategy Nash equilibrium, in which each ISP \(j\) plays a random \(p_j \in [p_0, p^*_i]\) according to the cumulative density function \(G_j\), and ISP \(i\) chooses the value \(p_i = p^*_i\) with positive probability \(1 - \frac{k-f_i}{k-f_j}\). The mixed strategies \(p_i\) (as random variables) almost surely satisfy

\[
\max \left\{ 0, \frac{k - kn}{\beta} - p_U \right\} < p_i < \frac{k - kn - f_i}{\beta} - p_U,
\]

and ISP \(i\)'s expected payoff over every mixed strategy \(p_j\) is

\[
E_{p_i} = E_{p_U}(\alpha - k(n+1) - f_i).
\]

Moreover, \(E_{p_{\text{max}}} = E_{\max\{p_i\}}\) is everywhere a continuous function of \(p_U\). It is continuously differentiable in the region of \((f_1, p_U)\)-intermediate market potential, but it is not differentiable at the boundary points \(\beta p_U = \alpha - k(n+1) + f_1\) and \(\beta p_U = \alpha - k(n-1) + f_1\) towards \((f_1, p_U)\)-low and \((f_1, p_U)\)-high market potential.

Existence: This lemma allows us to complete the proof of Theorem 2 by showing the existence of the equilibrium point. Let \(p_U\) be such that

\[
\max \{k(n-1), \alpha - k(n+1) + f_1\} \leq \beta p_U \leq \min \left\{ kn, \frac{\alpha}{2} \right\}.
\]

It follows that

\[
\beta p_U \leq \frac{\alpha}{2} = \alpha - \frac{\alpha}{2} < \alpha - k(n-1) - f_1,
\]

since \(\alpha > 2(n-1)k + 2f_1\).

Let \(\{p_i\}\) be the mixed-strategy equilibrium of Lemma 1. Then the mixed strategy \(p_i\) maximizes ISP \(i\)'s profit. To prove our theorem, we just need to show that UISP's expected profit is at a local maximum at some \(p_U\) in this range.

First, suppose that \(\beta p_U > \alpha - k(n+1) + f_1\). Then UISP's expected profit is

\[
E_{\pi_U} = p_U (\alpha - \beta (p_U + \hat{E}_{p_{\text{max}}}))\]

which is locally maximized by \(p_U\) if and only if

\[
p_U = \frac{\alpha - \beta \hat{E}_{p_{\text{max}}}}{2\beta}.
\]

At the upper bound of the allowed range for \(p_U\), if \(p_U = \min \{ kn, \frac{\alpha}{2} \}\), then

\[
E_{p_{\text{max}}} \geq \frac{\alpha}{\beta} - 2p_U.
\]

At the lower bound of the allowed range, if \(\beta p_U = k(n-1) > \alpha - k(n+1) + f_1\), then

\[
E_{p_{\text{max}}} \leq \frac{\alpha - k(n-1)}{\beta} - p_U = \frac{\alpha}{\beta} - 2p_U.
\]

Since \(E_{p_{\text{max}}}\) is continuous in \(p_U\), the Intermediate Value Theorem shows that there exists a value \(p^*_U \in [k(n-1), \min \{ kn, \frac{\alpha}{2} \}]\) such that (35) holds.

On the other hand, at the lower bound \(\beta p_U = \alpha - k(n+1) + f_1 \geq k(n-1)\) the mixed-strategy equilibrium of \(p_i\) turns out to be the pure-strategy equilibrium given by

\[
p_i = E_{p_{\text{max}}} = \frac{\alpha - \beta p_U - kn}{\beta} - 2p_U\]

since \(\beta p_U < kn\).

Here, by the Intermediate Value Theorem, there exists a value \(p^*_U \in (\alpha - k(n+1) + f_1, \min \{ kn, \frac{\alpha}{2} \}]\) such that (35) holds. Since \(\beta p^*_U > \alpha - k(n+1) + f_1\), the total demand served by UISP retains its functional form in some neighborhood of \(p^*_U\), and \(p^*_U\) does indeed locally maximize UISP's profit.

Uniqueness: To prove that there is only one equilibrium point with the given properties, we first need a technical lemma, proved in the appendix, on the variation with the constant price \(p_U\) of the expected maximum price chosen by an ISP \(i\).

Lemma 2. Suppose market potential is \((f_1, p_U)\)-intermediate. Let the expected maximum downstream price be \(E_{p_{\text{max}}} = E_{p_{\text{max}}} (p_U, (f_1)_{i=1}^n)\) as specified in Lemma 1. Let \(p^*_U\) be the pure strategy followed by UISP at the equilibrium point constructed above. Then, at \(p^*_U\), the function \(E_{p_{\text{max}}}\) satisfies

\[
\frac{\partial E_{p_{\text{max}}}}{\partial p_U} \bigg|_{p_U=p^*_U} > -2.
\]


Consider any equilibrium point \( (p_U; p_1, \ldots, p_n) \), where \( p_U \) is a locally optimal pure strategy and each \( p_i \) is an optimal mixed strategy. It follows from the non-existence of a pure-strategy equilibrium, proved in Theorem 1, that market potential is \( (f_1, p_U) \)-intermediate.

Consider the function
\[
f(p_U) = \alpha - 2\beta p_U - \beta \mathbb{E}p_{max}(p_U).
\]
At any equilibrium point satisfying our assumptions, we have \( f(p_U) = 0 \). We have already shown the existence of such a point \( p_U = p_U^{(1)} \). It follows from Lemma 1 that \( f \) is continuously differentiable. By Lemma 2 \( f'(p_U^{(1)}) = -2\beta - \frac{\beta \mathbb{E}p_{max}}{\partial p_{U}} < 0 \).

Suppose, for a contradiction, that there exists \( p_U^{(2)} \neq p_U^{(1)} \) with the same properties. Without loss of generality \( p_U^{(1)} < p_U^{(2)} \). It follows from the sign of the derivative of \( f \) that we can find \( 0 < \epsilon_1, \epsilon_2 < \frac{1}{2}(p_U^{(2)} - p_U^{(1)}) \) such that \( f(p_U^{(1)} + \epsilon_1) < 0 \) and \( f(p_U^{(2)} - \epsilon_2) > 0 \). Since \( f \) is a continuous function, the Intermediate Value Theorem gives \( p_U^{(3)} \in (p_U^{(1)} + \epsilon_1, p_U^{(2)} - \epsilon_2) \) such that \( f(p_U^{(3)}) = 0 \).

Inductively, we obtain an infinite sequence \( p_U^{(1)}, p_U^{(2)}, \ldots \) of distinct points in \([p_U^{(1)}, p_U^{(2)}]\) such that \( f(p_U^{(1)}) = f(p_U^{(2)}) = \cdots = 0 \). By the Bolzano-Weierstrass Theorem, this sequence must have an accumulation point \( p_{U}^{*} \). Clearly then \( f(p_{U}^{*}) = 0 \) and \( f'(p_{U}^{*}) = 0 \), which contradicts Lemma 2. We have therefore established uniqueness of UISP’s equilibrium price \( p_U^{(1)} \). By Lemma 1, the equilibrium point is unique.

One remaining question is whether allowing the upstream ISP to play any mixed strategy gives rise to a different equilibrium. It turns out that this is not the case for mixed-strategy Nash equilibria where bandwidth demand can be served completely and is sufficient to fill all but one downstream ISPs’ networks almost surely.

**Theorem 3.** Let market potential be \( f_1 \)-intermediate. Suppose there exists a mixed-strategy Nash equilibrium in the pricing game such that almost surely
\[
k(n-1) \leq \alpha - \beta(p_U + p_i) \leq kn.
\]
Then \( p_U \) is a pure strategy and the equilibrium is the equilibrium point given in Theorem 2.

**Proof of Theorem 3:** Let
\[
p_U = \sup \{p : \mathbb{P}\{p_U < p\} = 0\}, \quad p_U = \inf \{p : \mathbb{P}\{p_U > p\} = 0\}.
\]
Inequality (37) must still hold almost surely if UISP plays any pure strategy \( p_U \in [p_U, p_U^*] \). For any such pure strategy, UISP’s expected profit is
\[
\mathbb{E} \pi_U(p_U) = p_U(\alpha - \beta p_U - \beta \mathbb{E}p_{max}).
\]
This is a quadratic function with a unique maximum on the domain \( p_U \in [p_U, p_U^*] \). Therefore, UISP plays a pure strategy.

Given the forward contracts entered into by the downstream providers, we have thus completely characterized the ISPs’ pricing behavior. In general, the size of the market potential relative to the available capacity determines whether the game has a pure or mixed-strategy equilibrium.

When market potential is low, there is a pure-strategy Nash equilibrium with downstream prices equal to zero or marginal cost. The downstream ISPs compete the price down in this case, or, for a single downstream firm operating as part of a bilateral monopoly, the capacity sold by forward contracts absorbs all demand.

When market potential is high, there is a range of pure-strategy Nash equilibria with different divisions of the same total network price between the upstream and downstream industries. Bandwidth demand attains the level of available capacity. At this point the total price is equal to the value of a marginal unit of capacity. This price is commonly referred to as the congestion price. The balance of bargaining power between the firms determines which equilibrium arises. When the upstream ISP has all the bargaining power, the fraction of the total income obtained by the downstream industry is a decreasing function of the lowest contracting volume \( f_1 \), but is independent of all other contracting volumes.

For intermediate market potential, there is a pure-strategy Nash equilibrium only in the case of a bilateral monopoly (and capacity is not exhausted in this case). For a downstream oligopoly \( (n \geq 2) \), there exists an equilibrium point consisting of optimal mixed strategies for each downstream ISP and a locally optimal pure strategy for the upstream ISP.

Despite the different pricing outcome in the two non-trivial cases of intermediate and high market potential, the next section shows that the incentives for forward contracting can be analyzed in a uniform way over both regions.

VI. **Forward Contracting**

Having analyzed the second-stage pricing subgame in sections III through V, by backward induction we can turn our attention to the first stage choice of forward contracting in the game described in section II. In particular, we will analyze the network providers’ choice of contracting under uncertain bandwidth demand. We will establish that the equilibrium contracting volumes are always asymmetric, with one provider choosing the unique lowest contracting volume, before deriving the form of the externalities within the oligopoly that are due to the choice of contracting volumes in equilibrium. The results of the previous sections show that the lowest contracting volume is an important factor in determining second-stage prices. In the case of the pure-strategy equilibrium outcome, the lowest contracting volume is the only contracting volume that determines the second-stage outcome. As the smallest contracting volume increases, downstream prices decline, hurting all downstream providers. However, the firm with the smallest contracting volume is clearly subject to more price risk than the other providers.

We now relax the assumption that \( 0 < f_i < k \) to allow capacities \( 0 \leq f_i \leq k \) sold by forward contracting. When some \( f_i = 0 \) or \( f_i = k \), we assume the outcome of the second-stage pricing game is the continuous extension of the pure-strategy
equilibrium of Theorem 1 or the equilibrium of Theorem 2, as appropriate.7

We formally define the first-stage income from forward contracts sold at the expected second-stage price as

\[ I_i = f_i E_\beta E_p p_i. \]  

(38)

Recall from Definition 1 that the market potential \( \alpha \) is 0-high if \( \alpha \geq (2n+1)k \) and 0-low if \( \alpha \leq 2(n-1)k \). In the case of 0-high market potential, it is easy to show that a pure-strategy Nash equilibrium of contracting volumes exists and all but one contracting volumes are maximal \( f_2 = f_3 = \cdots = f_n = k \) in equilibrium. In the more general case where we only know that market potential is not 0-low (so the downstream ISPs may not compete prices down to zero in the second stage), we do not know whether there is a pure-strategy Nash equilibrium in the first-stage choice of contracting volumes. However, any such equilibrium must satisfy the following result.

**Theorem 4.** Suppose market potential is not 0-low and the ISPs’ second-stage moves are the ones predicted by Theorems 1 and 2, assuming the greatest \( p_U \) when there are multiple equilibria. Suppose there is a pure-strategy equilibrium of positive first-stage contracting volumes, so without loss of generality

\[ 0 < f_1 \leq f_i \quad \text{for every } i. \]  

(39)

Then the lowest contracting volume is unique, i.e.,

\[ f_1 < f_i \quad \text{for every } i > 1. \]  

(40)

Thus when market potential is not 0-low, any contracting equilibrium where the downstream ISPs obtain positive payoffs must be asymmetric. A risk-averse provider would seek to set a high contracting volume as insurance against price risk, hoping that some other provider will choose a low contracting volume and thereby raise the downstream second-stage price level.

How would the lowest contracting \( ISP_i \) be chosen in practice? Although no provider would want to be the one choosing the lowest contracting volume, such a provider may arise naturally in practice, for example, due to asymmetries in information, risk aversion, or timing. Nevertheless, the lack of symmetric equilibrium may be a source of uncertainty for network providers considering investment into bandwidth.

**Proof of Theorem 4:** Clearly market potential is not 0-low, since \( ISP_1 \) can achieve a positive profit by choosing a sufficiently low contracting volume \( f_1 > 0 \), subject to market potential not being 0-low.

Suppose first that market potential is \( f_1 \)-high. The second-stage subgame has a pure-strategy equilibrium, which is independent of \( f_j \), for \( j > 1 \). Since \( ISP_j, j > 1 \), is strictly risk-averse, he has an incentive to choose \( f_j > f_1 \).

Suppose that market potential is \( f_1 \)-intermediate instead. Suppose, for a contradiction, that \( f_2 = f_1 \). We will show that, if \( ISP_1 \) has no incentive to choose a lower contracting volume, then he must have an incentive to choose a higher one. For each \( \beta, ISP_1 \)’s profit varies with \( f_1 = f_2 \) according to

\[
\frac{d}{df_1} \left( E_p \pi_1 + I_1 \right) = \left( p_0(\beta)(k-f_1) + f_1 E_\beta E_p \right) \]  

\[ = -p_0 + (k-f_1) \left( \frac{\partial p_0}{\partial f_1} \pm \frac{\partial p_0}{\partial p_u} \frac{\partial E_p}{\partial f_1} \right) \]  

\[ + E_\beta E_p + f_1 E_\beta \left( \frac{\partial E_p}{\partial f_1} \pm \frac{\partial E_p}{\partial p_u} \frac{\partial E_p}{\partial f_1} \right), \]

where

\[
\frac{dE_p}{df_1} \pm = - \frac{\partial E_{p_{max}}}{\partial f_1} \left( 2 \mp \frac{\partial E_{p_{max}}}{\partial p_u} \frac{\partial E_p}{\partial f_1} \right)^{-1}. \]

It is easy to check that

\[
\frac{\partial E_p}{\partial f_1} \pm \leq \frac{\partial E_p}{\partial f_1} + \text{ and } \frac{\partial E_{p_{max}}}{\partial f_1} \pm \leq \frac{\partial E_{p_{max}}}{\partial f_1} + . \]

(41)

Trivially

\[
\frac{\partial p_0}{\partial f_1} < 0 = \frac{\partial p_0}{\partial f_1} + . \]

Since \( \frac{\partial E_p}{\partial p_u} < 0 \) and \( \frac{\partial p_0}{\partial p_u} < 0 \), clearly

\[
\frac{d}{df_1} \left( E_p \pi_1 + I_1 \right) < \frac{d}{df_1} \left( E_p \pi_1 + I_1 \right), \]

so

\[
\frac{\partial}{\partial f_1} \left( E_\beta E_p \left( E_p \pi_1 + I_1 \right) > \frac{\partial}{\partial f_1} \left( E_\beta E_p \pi_1 + I_1 \right). \right) \]

The right-hand side must be non-negative since \( ISP_1 \) has no incentive to decrease his contracting volume. Hence the left-hand side is positive, and \( ISP_1 \) can increase his expected utility by raising his contracting volume slightly. This is a contradiction, so \( f_2 \neq f_1 \) as required.

We now quantify the impact of one downstream provider’s choice of contracting volume on its competitors’ utility.

**Theorem 5.** Suppose

\[ 0 \leq f_1 < f_2 \leq \cdots \leq f_n < k, \quad (41) \]

and the ISPs’ second-stage moves are the ones predicted by Theorems 1 and 2, assuming the greatest \( p_U \) when there are multiple equilibria.

If market potential is \( f_1 \)-intermediate, an increase of \( f_1 \) by \( ISP_1 \) results in a negative marginal externality on the other downstream ISPs’ payoffs; and an increase of \( f_j \) by \( ISP_j \), for any \( j > 1 \), results in a positive marginal externality on the other downstream ISPs’ payoffs.
If market potential is \( f_1 \)-high, an increase of \( f_j \) by ISP \( i \) results in a negative marginal externality on the other down-stream ISPs’ payoffs; and an increase of \( f_j \) by ISP \( j \), for any \( j > 1 \), results in zero marginal externality on the other downstream ISPs’ payoffs.

Choosing a low contracting volume \( f_1 \) is like providing a “public good”\(^8\) to the oligopoly, by raising the general price level, but doing so is privately costly to ISP \( i \), as it implies a low level of insurance against demand uncertainty. In the case of \( f_1 \)-intermediate market potential, the choices of the contracting volumes \( f_2, \ldots, f_n \) result in externalities with the opposite sign, so greater contracting volumes benefit other ISPs. The presence of externalities means that downstream providers have an incentive to coordinate their actions by collusion. In this case, there is a particular incentive for a provider to make side-payments to a competitor in return for this provider agreeing to refrain from entering into forward contracts.

**Proof of Theorem 5:** If market potential is \( f_1 \)-high, every ISP \( i \) charges price \( p_1 = \frac{k-f_1}{\beta} \) in the second stage. The theorem is trivial in this case.

If market potential is \( f_1 \)-intermediate, let \( p_0 = p_0(p_T) \) and \( \mathbb{E}_p \pi^*_j = (\mathbb{E}_p p_j)(p_T) \). Then:

\[
\frac{dp_0}{df_1} = \frac{\partial p_0}{\partial f_1} - \frac{\partial p_0}{\partial p_T} \frac{\partial \mathbb{E} p_{\max}}{\partial f_1} \left( 2 + \frac{\partial \mathbb{E} p_{\max}}{\partial p_T} \right)^{-1},
\]

\[
\frac{d\mathbb{E} p_j}{df_1} = \frac{\partial \mathbb{E} p_j}{\partial f_1} - \frac{\partial \mathbb{E} p_j}{\partial p_T} \frac{\partial \mathbb{E} p_{\max}}{\partial f_1} \left( 2 + \frac{\partial \mathbb{E} p_{\max}}{\partial p_T} \right)^{-1}.
\]

When \( f_1 > f_1, \frac{\partial p_0}{\partial f_1} = 0, \frac{\partial p_0}{\partial p_T} < 0 \) and \( \frac{d\mathbb{E} p_j}{df_1} > 0 \). Hence \( \frac{dp_0}{df_1}, > 0 \).

On the other hand, \( \frac{\partial p_0}{\partial f_1} < 0 \) and \( \frac{d\mathbb{E} p_j}{df_1} > 0 \), so \( \frac{dp_0}{df_1} < 0 \).

Similarly, when \( 1 < i \neq j \), \( \frac{\partial p_0}{\partial f_1} \geq 0 \) and \( \frac{\partial p_0}{\partial p_T} < 0 \), so we have \( \frac{dp_0}{df_1} > 0 \). On the other hand, if \( j > 1, \frac{\partial \mathbb{E} p_j}{\partial f_1} \leq 0 \), so \( \frac{d\mathbb{E} p_j}{df_1} < 0 \).

Since ISP \( j \)'s profit is the stochastic quantity \( I_j + \mathbb{E}_p \pi^*_j \) where \( I_j = f_j \mathbb{E}_p \mathbb{E}_p p^*_j \) and \( \mathbb{E}_p \pi^*_j = p_0^*(k - f_j) \), the result follows immediately.

**VII. CONCLUSIONS**

This article started with the observation that a dynamic pricing system for the Internet would ensure a more efficient allocation of resources. However, without forward contracting, providers would be exposed to substantial price risk due to the uncertainty in market demand. Could forward contracting remove this price risk? In the absence of any strategic interaction, e.g. in a communication network operated by a single provider, the answer is yes. When strategic interaction is considered in a multi-provider network, the situation is more complex. Forward contracting weakens a provider’s strategic incentive to charge high prices. Thus, in the presence of an upstream monopoly, the optimal forward contracting strategy is a trade-off between reducing price risk and seeking to ensure high prices in the future. When the contracting provider is part of an oligopoly, the optimal contracting strategy will also be dependent on its competitors’ strategies.

In this paper, we have analyzed the incentives for forward contracting by ISPs competing to supply bandwidth on a downstream network segment, when a single ISP with significant market power supplies bandwidth on a complementary upstream network segment. In order to determine the incentives for contracting, we have first studied the subsequent pricing equilibrium which arises in different contracting scenarios. Depending on the level of market potential compared with the available bandwidth capacity, the pricing outcome can be characterized as an equilibrium in pure or mixed strategies.

We can draw some conclusions on the choice of forward contracts over two stages assuming the market’s price-sensitivity is random and the downstream firms are risk-averse. Note that in addition to the benefits, there are also risks associated with forward contracting. Provided that market potential is not so low that downstream prices are competed down to zero, we prove that any pure-strategy Nash equilibrium of positive contracting volumes must be asymmetric and have a unique lowest contracting volume. This gives rise to a version of the game of “Chicken”: as the provider who chooses this lowest contracting volume is exposed to the risk of more price uncertainty than the other competitors, no selfish risk-averse provider would want to be the one choosing the lowest equilibrium contracting volume. In practice, this instability may discourage investment into bandwidth. The reason is that forward contracts have a negative impact on a provider’s strategic incentives during the pricing stage. A natural low-contracting provider may arise in the presence of asymmetries, for example, in risk aversion or timing.

We further prove that the choice of contracting volumes causes externalities, both negative and positive. An increase in the lowest contracting volume has a negative marginal externality on other downstream ISPs. An increase in any other contracting volume has no externality for high market potential, but a positive marginal externality for intermediate market potential. In this sense, we can think of the downstream ISP with the least forward contracting as providing a public good to the oligopoly. A consequence is an incentive for providers to collude on contracting choice, as discussed below.

In summary, for risk averse ISPs operating under this market structure employing forward contracts, this paper provides some initial practical guidelines. First, if an ISP believes that every competitor will choose a high volume of forward contracting, then he would be well-advised to choose a low contracting volume. Second, a provider with a high contracting volume might want to act in such a way that a low-contracting provider would choose a lower contracting volume than would be privately optimal. It could achieve this through side-payments or other strategic behavior. Third, given that forward contracts have a negative impact on a provider’s strategic incentives during the pricing stage, network providers might want to vertically integrate with the upstream provider in order to eliminate this effect. Of course, this paper is an initial investigation into this topic, and our model is somewhat restrictive. One interesting direction for future research would be to consider interactions between ISPs linked by other

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\(^8\)A public good is a good that is non-excludable and non-rivalrous, i.e., it is not possible to exclude someone from using the good, and one individual’s usage does not prevent another from using it.
network shapes.

Finally, our framework could have other networking applications; for example, similar risk-return trade-offs might exist in last-hop wireless spectrum markets, see [23].

APPENDIX

Proof of Lemma 1:

(i) Assume market potential is \((f_1, p_U)\)-high. At the given prices, assumptions (5)–(7) imply that capacity is exhausted, so no ISP has an incentive to lower his price. From (21) and (24), it follows that \(\beta p_i \geq k - f_i \geq k - f_1\), and it was shown in (20) that, together with the fact that capacity is exhausted, this implies that ISP \(i\) has no incentive to raise his price. Therefore this point is indeed a pure-strategy equilibrium.

To establish uniqueness, consider any pure-strategy equilibrium. Note that every ISP must have a positive profit and, in particular, a positive market share \(D_i > 0\) in equilibrium, since ISP \(i\) can achieve a positive profit by choosing the price given in (24) regardless of its competitors’ strategies. It follows that any two ISP and ISP \(j\) must choose the same price \(p_i = p_j\), since otherwise the ISP with the lower price would have an incentive to raise its price.

Next, assumptions (7)–(9) imply that, unless the price \(p_i\) equals the value given in (24), each downstream ISP’s market share is less than its capacity \(k\) or the total demand cannot be served by the downstream ISPs. In both cases, a downstream ISP would have an incentive to change its price, which shows that (24) must hold in equilibrium. This establishes uniqueness.

(ii) Assume market potential is \((f_1, p_U)\)-low. If every downstream network chooses a price of zero, then from (5), (22) the total demand satisfies \(d_{\text{market}} \leq k(n-1) + f_1\). Assumptions (7)–(8) imply that ISP \(i\)’s second-stage profit when choosing \(p_i > 0\) is negative. Therefore, this point is a pure-strategy equilibrium.

For uniqueness, consider any pure-strategy equilibrium. We will show that if \(p_i > 0\) then \(\pi_i < 0\) so ISP \(i\) has an incentive to set \(p_i = 0\). Let ISP \(i\) be the network choosing the highest price. First, if \(D_i = 0\) then \(\pi_i < 0\) is trivial.

Second, in the case where \(D_i > 0\), suppose there are \(m\) downstream ISPs choosing price \(p_i\). From assumptions (7)–(9), the market share obtained by each is \(D_i = (\alpha - \beta p_U - \beta p_i - k(n-m))/m < k\), where the inequality follows from (22) and the fact that \(f_i < k\). If \(m > 1\) then every provider choosing price \(p_i\) has an incentive to just undercut the other providers choosing price \(p_i\), contradicting the equilibrium assumption. On the other hand, if \(m = 1\), then assumptions (7)–(9), (22) imply that \(\pi_i < 0\). We have therefore shown uniqueness.

(iii) Assume market potential is \((f_1, p_U)\)-intermediate. The following results are direct consequences of the definitions stated in the lemma.

- We have
  \[
  0 < p_0 < p_1^* \leq \frac{k-f_1}{\beta}, \tag{42}
  \]
  - For \(p_0 \leq p \leq p_1^*\), we have
    \[k(n-1) < d_{\text{market}}(p + p_U) < kn. \tag{43}\]

- The functions \(h(p)\) and \(H_1(p)\) are continuous and strictly increasing on \([p_0, p_1]\), with \(H_1(p_0) = 0\), \(H_1(p_1^*) = 1\).

- We have
  \[
  p_0 = p_1^{n+1} \leq p_1^n \leq \cdots \leq p_1^2 = p_1^*, \tag{44}
  \]
  where, for \(1 < i < n\), \(p_i = p_i^{i+1}\) if and only if \(f_i = f_{i+1}\). Also \(p_1^1 > p_0\) since \(f_i < k\).

- For any \(j\), \(G_j\) is a continuous and strictly increasing function on \([p_0, p_1]\). For \(j > 1\), \(G_j(p_1^j) = 1\).

- Finally, for \(p_0 \leq p \leq p_1^*\), the cumulative density function of \(\max_{j \neq i} \{p_j\}\) satisfies
  \[
  G_{-i}(p) = \prod_{j \neq i, p_i > p} G_j(p) = H_i(p). \tag{45}
  \]

Given this, we can show that the strategies defined in the lemma form a Nash equilibrium. Note that by inequality (43) and assumptions (7)–(8), if \(\max_{j \neq i} \{p_j\} < p_i\), then ISP \(i\)’s market share \(D_i = k\), whereas if \(\max_{j \neq i} \{p_j\} > p_i\), then ISP \(i\)’s market share is the residual demand after the other \((n-1)\) downstream networks’ capacities are exhausted, \(D_i = d_{\text{market}}(p_i + p_U) - k(n-1)\). Thus ISP \(i\)’s market share depends only on \(p_i\) and \(\max_{j \neq i} \{p_j\}\).

Since the probability distributions have no point mass at any \(p_0 < p < p_1^*\), and at least one ISP \(j\) with \(j \neq i\) has \(p_j > p_0\) almost surely, the event that \(\max_{j \neq i} \{p_j\} = p\) has zero probability for any \(p < p_1^*\).

Thus ISP \(i\)’s profit, when choosing some \(p_0 \leq p < p_1^*\), is
\[
\pi_i(p) = (1 - G_{-i}(p))p(k-f_i) + G_{-i}(p)d_{\text{market}}(p + p_U) - k(n-1) - f_i = p(k-f_i) - \pi_i(p_0).
\]

Moreover, for ISP \(i\), \(\pi_i(p_1^j) = p_0(k-f_i) = \pi_i(p_0)\).

To establish the equilibrium, we just need to prove that, conditional on the other ISPs’ strategies, no ISP \(i\) can increase his profit by choosing a price \(p\) outside the support of \(G_i, [p_0, p_1^*]\).

First, since each ISP \(i\) can set price \(p_0\) for a market share of \(k\), setting a lower price \(p < p_0\) leads to lower profits:
\[
\pi_i(p) = p(k-f_i) - p_0(k-f_i) = \pi_i(p_0).
\]

Second, if ISP \(i\) sets price \(p_1^j < p < p_1^1\), then (44), \(p_1^{j+1} \leq p \leq p_1^j\) for some \(j\). Let \(p_{\text{max}}^{(j)} = \max_{j \neq i} \{p_j\}\) and let \(G_{\text{max}}^{(j)}\) be the cumulative density function of \(p_{\text{max}}^{(j)}\). Observe that, under the equilibrium strategies, for \(p_1^{j+1} \leq p \leq p_1^j\), we have
\[
G_{\text{max}}^{(j)}(p) = \prod_{i=1}^j G_i(p) = \left( h(p) \prod_{i=1}^j (k-f_i) \right)^{\frac{1}{j+1}} h(p)
\]
\[
\geq \left( h(p_1^{j+1}) \prod_{i=1}^j (k-f_i) \right)^{\frac{1}{j+1}} h(p) \geq H_i(p), \tag{46}
\]
where the first inequality follows from the monotonicity of \(h\) and the second inequality follows from \(i \leq j + 1\).
Thus ISP_i’s profit is $E \pi_i(p) = (1 - G_{\text{max}}^{(1)}(p))p(k-f_i) + G_{\text{max}}^{(1)}(p)p(d_{\text{market}}(p + p_U) - k(n-1) - f_i) \leq E \pi_i(p_0)$, where the inequality follows from (43) and (46). This shows that ISP_i has no incentive to set a price $p_1 < p \leq p_1^\ast$. It now only remains to consider a third case where ISP_i sets price $p > p_1^\ast$. His profit function takes the form $E \pi_i(p) = p(d_{\text{market}}(p + p_U) - k(n-1) - f_i)$, which is a quadratic function attaining its maximum in $(0, p_1^\ast]$. However, from (26)–(27), (42) and $f_i \geq f_1$, we have $E \pi_i(p_1^\ast) = (k-f_1)p_0 - (f_1-f_i)p_1^\ast \leq E \pi_i(p_0)$, showing that ISP_i has no incentive to deviate by setting $p > p_1^\ast$. This establishes that ISP_i has no incentive to deviate from his equilibrium strategy, and therefore the given mixed strategies form a Nash equilibrium.

Conversely, to prove uniqueness, consider any mixed-strategy Nash equilibrium given by cumulative density functions $G_j(p) = P\{p_j < p\}$. Consider the well-defined low- and high-price thresholds for each ISP_j

$p_0^\ast = \sup\{p: G_j(p) = 0\}$,

$p_1^\ast = \inf\{p: G_j(p) = 1\}$.

Note that

- Every ISP_j obtains a positive expected profit $E \pi_j$ in equilibrium. Indeed, we have already shown that ISP_i’s profit is positive when choosing $p_1^\ast$ regardless of his competitors’ strategies. But ISP_i must then have a positive low-price threshold $p_0^\ast$ and any competitor ISP_j can obtain a positive profit by slightly undercutting this price.

- In equilibrium, there is sufficient capacity for the total demand at each low-price threshold:

$\alpha - \beta(p_0^\ast + p_U) \leq kn$;

and each ISP_j’s market share is positive even at his high-price threshold:

$\alpha - \beta(p_1^\ast + p_U) > k(n-1)$.

It is easy to check that the first inequality is required for ISP_j to have no incentive to play a mixed strategy with a higher $p_0^\ast$, and that the second inequality is required for ISP_j with the highest price $p_1^\ast$ to have no incentive to play a mixed strategy with a lower $p_1^\ast$.

It follows that every ISP_j has the low-price threshold $p_0^\ast$ defined in (27), i.e. $p_0^\ast = p_0 \forall j$; every ISP_j has the expected profit given in (34), i.e. $E \pi_j(p) = p_0(k-f_j)$; and ISP_i has the high-price threshold $p_1^\ast$ defined in (26), and no high-price threshold exceeds it: $p_1^\ast = p_1^\ast \geq p_1^\ast \forall j$. Define cumulative density functions for max$_{i \neq j}\{p_i\}$ as before:

$G_{-j}(p) = \prod_{i \neq j} G_i(p)$.

From the equilibrium requirement that ISP_j should have no incentive to change his mixed strategy, it is straightforward to verify the following. There exists an open interval $U \supset [p_0, p_1^\ast]$ such that whenever $p \in U$, we have

$G_{-j}(p) \geq H_j(p)$; \hspace{1cm} (47)

and, whenever $G_{-j}(p) > H_j(p)$, we have

$\exists \epsilon > 0: G_j(p - \epsilon) = G_j(p + \epsilon)$.

By the definition of $p_1^\ast$, $G_j$ cannot be locally constant at $p_1^\ast$, so

$G_{-j}(p_1^\ast) = H_j(p_1^\ast).$ \hspace{1cm} (49)

Further, the following is easily shown:

- Each $G_j$ is continuous on $(p_0, p_1^\ast]$. (So the mixed strategies have no point mass, except possibly at $p_0$ and $p_1^\ast$.). Moreover, for $j \neq 1$, $G_j$ has no point mass at $p_1^\ast$, so

$G_j(p_1^\ast) = 1.$ \hspace{1cm} (50)

- If $f_i < f_j$, then $p_1^\ast \leq p_1^\ast$. Whenever $f_i = f_{i+1}$, we can re-order ISP_i, ISP_{i+1}, so that $p_1^{i+1} \leq p_1^i$. Letting $p_1^{i+1} \equiv p_0$, without loss of generality

$p_1^{i+1} = p_0 \leq p_1^i \leq p_1^{i-1} \leq \cdots \leq p_1^1 = p_1^\ast$. \hspace{1cm} (51)

Then (49)–(51) imply, for $2 \leq i \leq n$:

$G_{-j}(p_1^i) = \prod_{j=1}^{i-1} G_j(p_1^i) = H_i(p_1^i).$ \hspace{1cm} (52)

We have $p_1^i = p_1^{i+1} = p_1^\ast$.

We are ready to prove that the mixed strategies employed are indeed those of our constructed equilibrium. We now prove by induction that, for each $2 \leq i \leq n$:

(a) $p_1^i = p_1^\ast$;

(b) $G_j(p) = G_j^\ast(p)$ piecewise for $p \in [p_1^{i+1}, p_1^i], \; i \geq j$,

$G_j^\ast(p) = \begin{cases} (\prod_{j \neq i} G_j(p))^{1-t} & \text{if } p > p_0, \\ (H_j(p))^{1-t} & \text{if } p = p_0. \end{cases}$

For the case $i = 2$, we already know $p_1^2 = p_1^2 = p_1^\ast$, so (a) holds. For part (b), we have already shown that $G_j(p_1^2) = G_j^\ast(p_1^2)$ for $j \neq 1$. For the case $j = 1$, equation (52) implies that $G_{-2}(p_1^1) = G_1(p_1^1) = H_2(p_1^1) = \frac{(k-f_1)^{i-2}}{\prod_{j=1}^{i-1} (k-f_j)}$, so (b) holds.

Now assume the inductive hypothesis holds for some $i - 1 < n$. We first show (a). Using part (b) of the inductive hypothesis for $i - 1$ allows us to rearrange (52) as

$h(p_1^i) = \frac{(k-f_1)^{i-2}}{\prod_{j=1}^{i-1} (k-f_j)}$.

The unique solution of this equation is $p_1^i = p_1^\ast$, by the definition (30) of $p_1^\ast$, so (a) holds.

We now show (b). In the case $p_1^{i+1} \neq p_1^i$, we have

$G_j(p) = G_j(p_1^i) = G_j^\ast(p_1^i) = G_j^\ast(p_1^i)$ by the
inductive hypothesis and the definition (30) of \( p^i_1 \), so (b) holds.
Consider the case \( p^i_{1+t} < p^i_1 \). For every \( p^i_{1+t} \leq p \leq p^i_1 \), if \( p > p_0 \), then
\[
G_j(p) = \left( \prod_{l < i, l \neq j} G_{-i}(p) \right) \frac{1}{p^i_1} \left( G_{-j}(p) \right)^{\frac{1}{p^j_1}}.
\]
(53)
To establish (b), it is sufficient to show that \( G_{-i}(p) = H_i(p) \) for every \( l \leq i, p \in (p^i_{1+t}, p^i_1) \); then \( G_j(p) = G_j^i(p) \) for \( p \in [p^i_{1+t}, p^i_1) \) by (53) (using continuity at the interval bounds).
Suppose, for a contradiction, that there exists some \( l \leq i, p \in (p^i_{1+t}, p^i_1) \) such that \( G_{-i}(p) \neq H_i(p) \). Then \( G_{-i}(p) > H_i(p) \) by property (47). We start by showing that, for this value \( p \), we have \( G_j(p) > G_j^i(p) \) for every \( j \leq i \). We show this separately for \( j \) such that \( G_{-j}(p) = H_j(p) \) and \( j \) such that \( G_{-j}(p) > H_j(p) \). First, for every \( j \) such that \( G_{-j}(p) = H_j(p) \), we have
\[ G_j(p) > G_j^i(p) \] by (53). Second, for every \( l \) satisfying \( G_{-j}(p) > H_j(p) \), define
\[
\vec{p}_n = \sup \{ q : G_{-j}(q) > H_j(q), \forall p < q \}\. (54)
By the inductive hypothesis for \( i-1 \), property (48) does not hold at \( p^i_1 \), so the supremum exists and \( \vec{p}_n \leq p^i_1 \).
\[ G_{-i}(\vec{p}_n) = H_i(\vec{p}_n) \] follows by continuity if \( \vec{p}_n < p^i_1 \), and by the inductive hypothesis for \( i-1 \) if \( \vec{p}_n = p^i_1 \). Using expression (53) for \( G_i(\vec{p}_n) \), equation (55), and inequality (47) for \( G_{-k}, k \neq l \), we have
\[ G_{-i}(\vec{p}_n) \leq G_{-i}(p^i_1) \] by (56). Note that by the choice of \( l \) and (55), we must have \( p < \vec{p}_n \). From (54), for \( p \leq \vec{p}_n \), we have \( G_{-i}(q) > H_i(q) \), so property (48) implies that \( G_i \) is constant on \( (p, \vec{p}_n) \). Continuity at \( p \) and left-continuity at \( \vec{p}_n \) imply \( G_i(p) = G_i(\vec{p}_n) \). From (56) and the fact that \( G_i^j \) is strictly increasing: \( G_i(p) = G_i(\vec{p}_n) \geq G_i^j(\vec{p}_n) > G_i^j(p) \). Thus we have shown that \( G_i(p) > G_i^j(p) \) for every \( j \leq i \).
It follows directly that, for every \( j \leq i \),
\[ G_{-j}(p) = \prod_{l < i, l \neq j} G_i(p) > H_j(p). \]
Next, note that the set \( S = \{ p' \in [p^i_{1+t}, p^i_1] : G_i(p') > G_i^j(p') \forall p' \leq p'' \leq p, l \leq i \} \) is open in \([p^i_{1+t}, p^i_1]\), since each \( G_i^j \) is locally constant at every point inside it, and each \( G_i^j \) is increasing. Again, using the monotonicity of \( G_i^j \), it is easy to check that \( S = \{ p' \in [p^i_{1+t}, p^i_1] : G_i(p') = G_i(p) \ \forall l \leq i \} \), which is closed by continuity of \( G_i \). But since \( S \) is non-empty, open, and closed, it must be the entire interval \([p^i_{1+t}, p^i_1]\).
To obtain the desired contradiction, we consider the cases \( i = n \) and \( i \leq n \) separately. First, in the case \( i = n \), we have \( \vec{p}_n = p^i_{1+t} \in S \), so \( G_i(\vec{p}_n) > G_i^j(\vec{p}_n) \) for every \( l \leq i \). This implies that property (48) holds at \( \vec{p}_n \), which contradicts the definition of \( \vec{p}_n \). Second, in the case \( i < n \), each \( G_i \), \( l \leq i \), is constant on \( S \), so \( G_{-i+1}(p) \) is constant on \( S \). Thus \( G_{n-1}(p) = G_{-i+1}(p) \geq H_{i+1}(p) > H_{i+1}(p^i_{1+t}) \), where the inequalities follow from (47) and the fact that \( H_{i+1} \) is strictly increasing. This contradicts (52). In both cases, we have a contradiction, so we have shown part (b) of the inductive hypothesis. This completes the inductive argument.

Since \( G_j(p) = 0 \) for \( p \leq p_0 \), \( 1 \leq j \leq n \), we have proved that the cumulative density functions specifying the mixed strategies employed by the ISP \( j \) in any equilibrium coincide with those in the equilibrium we have explicitly constructed. Hence the mixed-strategy equilibrium of our game is unique.

Continuous differentiability of \( \mathbb{E}_{p_{max}} \) as a function of \( p_U \) is trivial inside the regions of \( (f_1, p_U) \)-high and \( (f_1, p_U) \)-low market potential. For \( (f_1, p_U) \)-intermediate market potential, it is obvious that \( p^i_1 \) and \( p_0 \) are continuously differentiable functions of \( p_U \). The existence of a continuous derivative of \( p^i_1 \) is a consequence of \( p^i_1 \) when \( f_i = f_1 \), and from \( h'(p^i_1) > 0 \) by the implicit function theorem when \( f_i \neq f_1 \). We can write
\[
\mathbb{E}_{p_{max}} = \int_0^\infty \left( 1 - \mathbb{P}\{p_{max} < p \} \right) \mathbb{P}(p_U) \] dp
\[
= p_0 + \sum_{i=2}^n \int_0^{p^i_1} \left( 1 - \mathbb{P}\{p_U < p \} \right) \mathbb{P}(p_U) \] dp. (57)

We now check that \( \mathbb{E}_{p_{max}} \) is continuously differentiable with respect to \( p_U \). The limits of each integral are continuously differentiable with respect to \( p_U \). Moreover, each integrand is continuously differentiable with respect to \( p_U \) and with respect to \( p \), where the derivative with respect to \( p_U \) can be bounded above by an integrable function independently of \( p_U \), for values of \( p_U \) in some sufficiently small interval. These conditions are sufficient for continuous differentiability of each integral with respect to \( p_U \). Therefore \( \mathbb{E}_{p_{max}} \) is a continuously differentiable function of \( p_U \) for \( (f_1, p_U) \)-intermediate market potential. Continuity and lack of differentiability are easy to verify at the boundary points, completing the proof of the lemma.

Proof of Lemma 2: As we have seen in the proof of Lemma 1, the function \( \mathbb{E}_{p_{max}} \) satisfies the assumptions required for the existence of a continuous derivative which can be found by differentiating the expression in (57) after substituting the definition of \( H_i \) given in (29):
\[
\frac{\partial \mathbb{E}_{p_{max}}}{\partial p_U} = \frac{1}{2} \left( 1 - \frac{k - f_2}{k - f_1} \right) - Q, \] (58)
where
\[
Q \equiv \sum_{j=2}^n \int_0^{p^i_1} \left( \mathbb{P}\{p_{max} < p \} \right) \frac{1}{p^j_1} \frac{\partial h(p)}{\partial p_U} \] dp. (59)

The proof that \( \frac{\partial \mathbb{E}_{p_{max}}}{\partial p_U} \) is always greater than \(-2\) is done in two parts: for \( \delta = \alpha - 2(n-1)k - 2f_1 \) smaller than \( \frac{1}{2}(k-f_1) \),
and for $\delta$ greater than $\frac{2\alpha}{\beta}(k-f_1)$. (Note that these regions overlap.)

Consider the first case, $\delta < \frac{12\delta}{5}(k-f_1)$. The function $h$ is increasing, and $p \leq p'_1$ in each integral in (59), so using (30):

$$
\left( h(p) \prod_{l=1}^{j} (k-f_l) \right)^{\frac{1}{j+1}} \leq k-f_j \leq k-f_1,
$$

(60)

Using $j \geq 2$ and (60) in (59):

$$
Q \leq 2(k-f_1) \sum_{j=2}^{n} \int_{p'_1}^{p_1} \frac{\partial h(p)}{\partial p_U} \, dp
$$

$$
= 2(k-f_1) \int_{p_0}^{p_1} \frac{\beta \left( 1 - \frac{p}{p_1} \right) (p_1 - p)}{p \left( \frac{p}{p_1} \beta (p_1 - p_0) - \beta (p_1 - p) \right)^2} \, dp \equiv \overline{Q}.
$$

Letting $\gamma \equiv \frac{p_0}{p_1}$, with the change of variable $t \equiv \frac{p - p_0}{p_1 - p_0}$, yields

$$
\overline{Q} = 2\gamma \left( \gamma - 1 \right) \log \gamma - \log(1 - \gamma) + (2\gamma - 1) \left( \frac{\gamma - 1}{2\gamma} \right)^2.
$$

(61)

Although the evaluated integral is undefined for $\gamma = \frac{1}{2}$, an application of L'Hôpital's Rule shows that it can be extended to this point, giving a continuous function of $\gamma$ on $(0, 1)$. Note that $\overline{Q}$ is an increasing function of $\gamma$.

By the definitions in Lemma 1, $\mathbb{E}_{p_{max}} \leq p_1$. From (35), $p_U = \frac{\alpha - \beta \mathbb{E}_{p_{max}}}{2\beta} \geq \frac{\alpha - \beta p_1}{2\beta}$. Substituting this inequality into the definition of $p'_1$ gives $p'_1 \leq \frac{\alpha - \beta p_1}{2\beta}$. This together with (27) gives $\gamma = \frac{p_0}{p_1} = \frac{\beta p_1}{k-f_1} \leq \frac{\delta}{k-f_1} \leq \frac{1}{2}$, since $\overline{Q}$ is increasing everywhere on $0 < \gamma < 1$ and $\overline{Q}(\frac{1}{2}) < \frac{3}{2}$. Therefore, it follows that $\overline{Q}(\gamma) < \frac{3}{2}$ for any $0 < \gamma \leq \frac{1}{2}$. Substituting $f_1 \leq f_2$ and $Q \leq \overline{Q} < \frac{3}{2}$ into (58) establishes the lemma for $\delta > \frac{2\alpha}{\beta}(k-f_1)$.

Consider now the second case, $\delta > \frac{2\alpha}{\beta}(k-f_1)$. The following bound is straightforward to verify:

$$
\frac{\partial h(p)}{\partial p_U} \leq \frac{\partial h(p)}{\partial p} + \frac{h(p)}{p}.
$$

(62)

Substituting this inequality into (59) gives

$$
Q \leq Q_1 + Q_2,
$$

(63)

where

$$
Q_1 = \sum_{j=2}^{n} \int_{p'_1}^{p_1} \frac{j}{j-1} \left( h(p) \prod_{l=1}^{j} (k-f_l) \right)^{\frac{1}{j+1}} \frac{\partial h(p)}{\partial p} \, dp,
$$

$$
Q_2 = \sum_{j=2}^{n} \int_{p'_1}^{p_1} \frac{j}{j-1} \left( h(p) \prod_{l=1}^{j} (k-f_l) \right)^{\frac{1}{j+1}} \frac{h(p)}{p} \, dp.
$$

Integrating:

$$
Q_1 = \sum_{j=2}^{n} \frac{(k-f_j)^j - (k-f_{j+1})^j}{\prod_{l=1}^{j} (k-f_l)} = \frac{k-f_2}{k-f_1},
$$

(64)

Using $j \geq 2$ with $p \geq p_0$ and $\mathbb{E}_{p_{max}} \geq p_0$:

$$
Q_2 \leq \frac{2}{p_0} (p'_1 - \mathbb{E}_{p_{max}}) \leq 2 \left( \frac{k-f_1}{\beta p'_1} - 1 \right).
$$

(65)

From $\mathbb{E}_{p_{max}} \geq p_0$ and (35) we have $p_U \leq \frac{\alpha - \beta p_0}{2\beta}$, whence

$$
p'_1 \geq \frac{\alpha + \beta p_0 - 2k(n-1) - 2f_1}{4\beta}.
$$

Using the definition of $p_0$, we can re-state this as

$$
(p'_1)^2 - \frac{4(k-f_1)}{\beta} p'_1 + \frac{\delta (k-f_1)}{\beta^2} \leq 0.
$$

Thus $p'_1$ is at least as large as the smaller root of the quadratic:

$$
p'_1 \geq \frac{2(k-f_1)}{\beta} \left( 1 - \sqrt{1 - \frac{\delta (\alpha)}{4(k-f_1)^2}} \right) \geq \frac{2(k-f_1)}{3\beta},
$$

(66)

where the second inequality follows from the assumption that $\delta (\alpha) > \frac{2\alpha}{\beta}(k-f_1)$.

We now substitute (66) into (65), obtaining $Q_2 < 1$. Combining (58), (63), (64) and $Q_2 < 1$, we get

$$
\frac{\partial \mathbb{E}_{p_{max}}}{\partial p_U} > -\frac{1}{2} \left( 1 - \frac{k-f_2}{k-f_1} \right) \frac{k-f_2}{k-f_1} - 1 \geq -2,
$$

which establishes the lemma for $\delta > \frac{2\alpha}{\beta}(k-f_1)$.

\[\blacksquare\]

\textbf{REFERENCES}


Richard Steinberg (M ’10) received the B.A. degree from Reed College, Portland, Oregon in 1976, the M.Math. and Ph.D. degrees in combinatorics and optimization from the University of Waterloo, Waterloo, Ontario, Canada, in 1976 and 1979, respectively, and the M.B.A. from the University of Chicago, Chicago, IL, in 1980.

He has worked at AT&T Bell Laboratories and has served on the faculties of the University of Chicago, Columbia University, and the University of Cambridge. He is currently Chair in Operations Research and Head of the Management Science Group at the London School of Economics. His current research interests include Internet economics and auctions.

Miklós Reiter received the B.A., M.Math. and the Ph.D. degrees in mathematics and operations research from the University of Cambridge in 2003, 2004 and 2007 respectively. He now works as Senior Quantitative Researcher at eValue FE Ltd., London. His research interests include game theory and financial risk modelling.