THE OPTIMAL CONSUMPTION FUNCTION IN A BROWNIAN MODEL OF ACCUMULATION PART B: EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEM* **

Lucien Foldes
London School of Economics and Political Science

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The Suntory Centre
Suntory and Toyota International Centres for
Economics and Related Disciplines
London School of Economics and Political Science
Houghton Street
London WC2A 2AE
Tel.: 020-7955 6679

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THE OPTIMAL CONSUMPTION FUNCTION IN A BROWNIAN MODEL OF ACCUMULATION

PART B: EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS * **

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Lucien Foldes

London School of Economics

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Theoretical Economics Workshop, The Suntory Centre, Suntory and Toyota International Centres for Economics and Related Disciplines,

London School of Economics and Political Science, Houghton Street,

London WC2A 2AE, Tel. 0171–405–7686

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Abstract

In Part A of the present study, subtitled '*The Consumption Function as Solution of a Boundary Value Problem*' Discussion Paper No. TE/96/297, STICERD, London School of Economics, we formulated a Brownian model of accumulation and derived sufficient conditions for optimality of a plan generated by a logarithmic consumption function, i.e. a relation expressing log-consumption as a time-invariant, deterministic function $H(z)$ of log–capital z (both variables being measured in 'intensive' units). Writing $h(z) = H'(z)$, $\theta(z) = \exp\{H(z)-z\}$, the conditions require that the pair (h,θ) satisfy a certain non-linear, non-autonomous (but asymptotically autonomous) system of o.d.e.s (F,G) of the form $h'(z) = F(h,\theta,z)$, $\theta' = G(h,\theta) = (h-1)\theta$ for $z \in \Re$, and that $h(z)$ and $\theta(z)$ converge to certain limiting values (depending on parameters) as $z \to \pm\infty$. The present paper, which is self-contained mathematically, analyses this system and shows that the resulting two-point boundary value problem has a (unique) solution for each range of parameter values considered. This solution may be characterised as the connection between saddle points of the autonomous systems $(F_{-\infty},G)$ and $(F_{+\infty},G)$, where $F_{\pm\infty}(h,\theta) = F(h,\theta,\pm\infty)$.

Key words: Consumption, capital accumulation, Brownian motion, optimisation, ordinary differential equations, boundary value problems

AMS(MOS) subject classifications: 34B10, 49A05, 90A16, 93E20.

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Abbreviated Title: Optimal Consumption as a Boundary Value Problem, Part B

INTRODUCTION TO PART B

As stated in the Abstract, this paper follows on from Foldes [1996] – hereinafter 'Part A' – comprising Sections 1 and 2 of our study The present Introduction restates the boundary value problems (b.v.p.s) and certain assumptions formulated in Part A. This is followed by Sections 3–4, which are devoted to analysis of these problems and proof of the existence of solutions, with little further reference to the economic and probabilistic background.

We consider the following non–linear, non–autonomous system $S = (F,G)$ of o.d.e.s

$$(0.1) \qquad h' = F(h,\theta,z) = bh^2 + (2/\sigma^2)h[\theta - n + m/b - \tfrac{1}{2}b\sigma^2 - A] - 2[m-M]/b\sigma^2$$
$$\theta' = G(h,\theta,z) = (h-1)\theta$$

defined for $z \in \Re$, where $h' = dh(z)/dz$, $\theta' = d\theta(z)/dz$. Here $b > 0$, $\sigma^2 > 0$, n, m are constants satisfying conditions stated below. Recall that z stands for log-capital and $h(z) = dH(z)/dz$, $\theta(z) = \exp\{H(z)-z\}$, where $H(z)$ is log-consumption (both capital and consumption being measured in 'intensive' units). The functions A and M are defined for $z \in \Re$ in terms of the 'intensive' production function ψ by

$$(0.2) \qquad A(z) = \psi(\kappa)/\kappa, \quad M(z) = \psi'(\kappa), \quad z = \ln \kappa, \ \kappa > 0,$$

and are (at least) $\mathbf{C}^1$ with one–sided limits

$$(0.3) \qquad A(-\infty) = M(-\infty) = \psi'(0) = \psi'_0, \quad A(\infty) = M(\infty) = \psi'(\infty) = 0.$$

Recall that ψ is defined for $\kappa \geq 0$, $\psi(0) = 0$, and is (at least) $\mathbf{C}^2$ with $\psi'(\kappa) > 0 > \psi''(\kappa)$ for $0 < \kappa < \infty$ and limits

$$(0.4) \qquad 0 < \psi'_0 < \infty, \quad \psi'(\infty) = 0.$$

We further introduce the following constants:

$$(0.5) \qquad N = n + (b-1)\psi'_0/b$$

$$(0.6) \qquad q = n + (b-1)(m + \tfrac{1}{2}b\sigma^2)/b$$

$$(0.7) \qquad Q = n - m/b + \tfrac{1}{2}b\sigma^2 = q - m + \tfrac{1}{2}\sigma^2,$$

cf (1.15–16). In line with the statements of Theorems 2 and 3 of Part A, we adopt throughout, without special mention, the following

STANDING ASSUMPTIONS

(0.8) If $b > 1$, then $N > 0$ and {either $n > 0$ or $q > 0$}.

(0.9) If $b < 1$, then $n > 0$ and {either $N > 0$ or $q > 0$}.

(0.10) If $b = 1$, then $N = n = q > 0$.

The main object of this Part is to show that each of the b.v.p.s defined by the statements of Theorems 2 and 3 has a (unique) solution. The following theorem is based on these statements, but a more precise and elegant version will be given later.

THEOREM 4A (Existence of Solutions of b.v.p.s)

In each of the following cases the system $S = (F,G)$ defined by (0.1) has a solution $(h^*,\theta^*) = (h^*(z),\theta^*(z))$ which is defined for all $z \in \Re$ and converges for $z \to \pm\infty$ to limits satisfying the following conditions:

'Type 1' b.v.p.s (cf. Theorem 2): If $b > 0$, $n > 0$ and $N > 0$, the limits are

(0.11) $h^*(+\infty) = 1, \quad \theta^*(+\infty) = n.$

(0.12) $h^*(-\infty) = 1, \quad \theta^*(-\infty) = N.$

'Type 0' b.v.p.s. (cf. Theorem 3):

(i) If $b > 1$, $N > 0$ and $q > 0 \geq n$, the limits are (0.12) and $(h^*(+\infty),0)$ for some $h^*(+\infty)$ satisfying

(0.13) $1/b < h^*(+\infty) \leq 1.$

(ii) If $b < 1$, $n > 0$ and $q > 0 \geq N$, the limits are (0.11) and $(h^*(-\infty),0)$ for some $h^*(-\infty)$ satisfying

(0.14) $1/b > h^*(-\infty) \geq 1.$

We call a solution of S which satisfies one of the sets of conditions of this theorem a solution of 'the' (appropriate) b.v.p. – of Type 1, 0(i) or 0(ii) – or simply a 'star' solution. Taking into account the results of Part A, *a proof that a star solution exists is a proof that the underlying model of accumulation admits an optimal log-consumption function* $H(z)$, where $H'(z) = h(z)$ and $\theta(z) = \exp\{H(z)-z\}$. While our main aim will be to prove the existence of star solutions, we shall also consider the properties of solutions of S generally. Apart from any mathematical interest which this rather unusual system of o.d.e s may possess, it is useful to have some insight into the economic consequences of choosing the 'wrong' solution as the consumption function.

3. PHASE ANALYSIS

(i) *Generalities.*

The present Section gives a preliminary discussion of S followed by a detailed discussion of certain auxiliary systems which define bounds for the motion of S; further details about S are established in Section 4.

To begin with, a brief survey of some properties of S It is necessary to bear in mind that the independent variable is not time but log-capital, but we shall nevertheless slip into much of the usual terminology of forward (z↑) and backward (z↓) motion and limits, stable/unstable or in/out curves which move towards/away from a given point for the *forward* motion, etc We say that a *solution* of S is a pair of functions $(h(z),\theta(z))$, with $\theta \in C^2$, satisfying (0.1) on some maximal interval $I = (z_-,z_+)$, and the corresponding curve $(h(z),\theta(z); z\in I)$ is called a solution path or simply a *path*, (with analogous terminology for other systems introduced below). A solution is called *bounded* on an interval if both h and θ are bounded there, otherwise it is *unbounded* We consider phase diagrams (or rather path diagrams) with θ on the horizontal axis and h on the vertical; thus we speak of motion to the left or right, up or down. Unless otherwise stated or implied, we consider S only in the open half–plane $\{\theta > 0\} = \{(h,\theta)\colon \theta > 0\}$, or possibly its closure $\{\theta \geq 0\}$ (with perhaps a vertical strip $-\epsilon < \theta \leq 0$ when it is necessary to make sense of statements about phase behaviour in full neighbourhoods of points on the vertical axis); the choice of space will usually be clear from the context Plane sets are written with curly brackets, often omitting the argument (h,θ). For plane sets, 'closed', 'open', 'boundary' etc are defined relative to $\Re^2$, whereas 'relatively closed' etc are defined relative to $\{\theta > 0\}$. (For sets in $\{\theta > 0\}$, 'open' and 'relatively open' are equivalent, but a relatively closed set may not be closed in the plane.) The relative closure of a set $\mathscr{A}$ is written $[\![\mathscr{A}]\!]$. Often we denote by $\pi = (h,\theta)$ a point in $\Re^2$ and by $\Pi = (h,\theta,z) = (\pi,z)$ a corresponding point in $\Re^3$; (note the new use of π, for 'πosition'.)

Since the functions F and G are C^1 in (h,θ,z), a unique local solution through a given point $\Pi_\diamond = (\pi_\diamond, z_\diamond) = (h_\diamond, \theta_\diamond, z_\diamond)$ always exists; its position at $z \in I$ may be written $\pi(z;\Pi_\diamond) = (h(z;\Pi_\diamond), \theta(z;\Pi_\diamond))$. If we consider the solution only for $z \geq z_\diamond$, or only for $z \leq z_\diamond$, we refer to the forward or backward solution through $\Pi_\diamond$, and call $\Pi_\diamond$ (or just $\pi_\diamond$ or $z_\diamond$) the *start* of the solution. According to standard results, a solution whose path stays bounded as $z\uparrow$ $(z\downarrow)$ can be continued to $z_+ = \infty$ $(z_- = -\infty)$, see Nemytskii and Stepanov [1960] T 1 21. Here we can do slightly better, as follows:

PROPOSITION 1 If, for a given solution $\pi = (h,\theta)$ of S, $h(z)$ stays bounded as $z\uparrow$ $(z\downarrow)$, then the solution can be continued to $z_+ = \infty$ $(z_- = -\infty)$.

PROOF Suppose that $(h(z),\theta(z))$: $z\in I)$ is a given solution through a point $\Pi_\diamond$ and that $|h(z)| \leq \alpha$ for $z_\diamond < z < z_+$. If $\theta_\diamond = 0$, then $\theta' = (h-1)\theta = 0$ always and there is nothing to show. Let $\theta_\diamond > 0$ and suppose that $z_+ < \infty$. Now $\theta' = (h-1)\theta$ implies $\theta(z) \leq \theta_\diamond \cdot \exp\{\alpha(z_+ - z_\diamond)\}$ for $z_\diamond < z < z_+$. But then $\theta(z_+)$ exists as a finite limit. Applying the preceding inequality together with $|h(z)| \leq \alpha$ to the equation $h' = F(h,\theta,z)$ it is found that $h(z_+)$ also exists as a finite limit, and the usual continuation argument shows that the solution can be continued forward from $(h(z_+),\theta(z_+),z_+)$, contrary to the assumption that $z_+ < \infty$. The argument for $z \downarrow z_-$ is analogous. || It will also appear later that, if $h(z)$ becomes unbounded as $z\uparrow$ $(z\downarrow)$, then $z_+ < \infty$ $(z_- > -\infty)$.

A related question concerns the continuation of *paths*. It follows from the equation $\theta' = (h-1)\theta$ that the motion of S is always to the left $(\theta\downarrow)$ in the region $\{h<1, \theta>0\}$, always to the right $(\theta\uparrow)$ in the region $\{h>1, \theta>0\}$. Consequently, given a solution of S through a point $\Pi_\diamond = (h_\diamond,\theta_\diamond,z_\diamond)$ with $\theta_\diamond > 0$ and $h_\diamond \neq 1$, we may take θ as path parameter, i.e. we may represent the path locally as the solution $h = h(\theta;z_\diamond)$ of the equation $dh/d\theta = F(h,\theta,z(\theta))/(h-1)\theta$, where $z(\theta)$ is the function inverse to $\theta(z;z_\diamond)$. The usual continuation argument then shows *that the representation can be* *continued as* $\theta\downarrow$ *and as* $\theta\uparrow$ *so long as* h–1 *keeps the same (definite) sign and stays finite* In particular, *no path can terminate in the interior of either of the regions* $\{h<1, \theta>0\}$ *or* $\{h>1, \theta>0\}$ *as* $z \uparrow z_+$ *or as* $z \downarrow z_-$.

Difficulties with the system S arise from its being non–linear, non-autonomous, with no stationary point, incomplete (i.e. finite escape levels z_+ or z_- occur), and unstable with respect to perturbation of initial values. This instability is present in particular along paths which converge to one of the boundary values prescribed in Theorem 4A; in fact, the system possesses a version of the 'knife–edge' property found in certain deterministic models of economic growth. To set against these vices there are virtues. All solutions *converge to limits*, finite or infinite, at the endpoints of their intervals of definition. The path map has some simplifying features Motion is always to the left if $h < 1$, $\theta > 0$, to the right if $h > 1$, $\theta > 0$ There are also *order-preserving properties*: loosely speaking, if $\Pi^i_\diamond = (h^i_\diamond, \theta^i_\diamond, z_\diamond)$ are distinct points, $i = 0,1$, an inequality of the form $\{0 < h^1 < h^0, 0 < \theta^1 < \theta^0\}$ is preserved along solutions through these paths as $z\uparrow$ (at least while both solutions remain defined with both co-ordinates positive), while an inequality $\{0 < h^1 < h^0, 0 < \theta^0 < \theta^1\}$ is preserved as $z\downarrow$ (with the same proviso). Closely connected with these ordering properties, it is possible to define autonomous systems which give simple *upper and lower bounds* for the motion of S (with more accurate bounds if only large $|z|$ are considered) Most important, the system is *asymptotically autonomous* for both $z \to \infty$ and $z \to -\infty$; we begin our detailed discussion with this last point

Referring to (0.1) and the definitions of the functions M and A, it is seen that the term $(2/\sigma^2)[M(z)/b - hA(z)]$ tends to zero as $z \to \infty$ and to $(2/\sigma^2)\psi'_0[1/b-h]$ as $z \to -\infty$, so that the function $F(h,\theta,z)$ converges, uniformly on (h,θ)–compacts of $\Re^2$, to the functions

$$(3.1) \qquad F_\infty(h,\theta) = bh^2 + (2/\sigma^2)h[\theta - n + m/b - \tfrac{1}{2}b\sigma^2] - 2m/b\sigma^2$$

$$(3.2) \qquad F_{-\infty}(h,\theta) = bh^2 + (2/\sigma^2)h[\theta - N + (m-\psi'_0)/b - \tfrac{1}{2}b\sigma^2] - 2(m-\psi'_0)/b\sigma^2$$

as $z \to \infty$ and $z \to -\infty$ respectively, taking into account that $N = n + (b-1)\psi_0'/b$ Since $G = (h-1)\theta$ does not depend on z, it follows that the system $S = (F,G)$ is asymptotically autonomous in the sense of Markus [1956], with limiting 'autonomous systems at $\pm\infty$' defined by $S_\infty = (F_\infty,G)$ and $S_{-\infty} = (F_{-\infty},G)$; see also Opial [1960] Some simple but useful properties follow immediately; we state them for the forward motion only.

PROPOSITION 2. (i) If the forward limit set[1] $\Pi^\triangleright$ of a solution π of S is not empty, then $z_+ = \infty$ and $\Pi^\triangleright$ is the union of paths of S_∞. Consequently:

(ii) If π *converges to a point* $\pi^\triangleright = (h^\triangleright,\theta^\triangleright)$, then that point must be a stationary point of S_∞.

(iii) If $\pi^\triangleright$ is a stationary point of S_∞ and the variational equations of this system based on $\pi^\triangleright$ have characteristic values with negative real parts, then there is a neighbourhood $\mathcal{N}$ of $\pi^\triangleright$ and a number $z_\diamond$ such that every solution of S whose path meets $\mathcal{N}$ at some $z > z_\diamond$ converges to $\pi^\triangleright$.

(iv) If $\mathcal{G}$ is an unbounded open region of the (h,θ)–plane from which paths of S and of S_∞ do not escape, and if all paths of S_∞ which enter $\mathcal{G}$ become unbounded, then the same is true of paths of S.

These results allow information about solutions of S to be obtained from corresponding information about the asymptotic systems, whose phase picture is relatively simple In particular it will be found that, for each combination of parameters considered in Theorem 4A, each of the asymptotic systems has at most three stationary points in the half–plane $\{\theta \geq 0\}$, one of which is a saddle while the others are stable or unstable nodes.[2] According to property (ii) above, a star solution must

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[1] A point $\pi^\triangleright = (h^\triangleright,\theta^\triangleright)$ belongs to the forward (or 'omega') limit set if there is a sequence (z_k) such that $h(z_k) \to h^\triangleright$, $\theta(z_k) \to \theta^\triangleright$ as $k \to \infty$. A solution which is bounded for the forward motion obviously has a non–empty forward limit set.

[2] There is a minor qualification in the cases $0 = n < q$ and $0 = N < q$, where there is a saddle–node bifurcation; see below, fn. 4.

converge at each end to one of these points. It turns out that the co–ordinates of the saddles, and only these, satisfy the prescribed conditions. *The problem of proving that a particular b.v.p. has a solution is therefore equivalent to proving the existence of a sort of saddle connection* (but between saddles of the asymptotic systems, not of S)

This way of stating the matter suggests an analysis designed to show that there is a pair of two–dimensional manifolds of integral curves of S, converging respectively to the saddle points of S_∞ ($S_{-\infty}$) as $z\uparrow$ ($z\downarrow$), which intersect transversely in a single curve defining a star solution. This is essentially what we shall do, but in a way which relies as much as possible on elementary methods using phase analysis in the plane.[3]

In addition to the systems $S_{\pm\infty} = (F_{\pm\infty},G)$, we shall need to consider certain other auxiliary two–dimensional autonomous systems which will serve to define bounds for the motion of S. The rest of this Section is concerned with these systems. In order to establish a unified notation we write

(3.3) $\qquad \bar{F} = \bar{F}(h,\theta) = bh^2 + (2/\sigma^2)h(\theta - \bar{Q}) - 2\bar{m}/b\sigma^2$

Thus a triple of parameters $(b,\bar{Q},\bar{m})$ satisfying suitable conditions defines a system $\bar{S} = (\bar{F},G)$, and we label parameters according to the systems to which they belong. Other parameters of importance are the numbers

(3.4) $\qquad \bar{\theta}_1 = \bar{Q} - \frac{1}{2}b\sigma^2 + \bar{m}/b$, defined as the solution of $\bar{F}(1,\theta) = 0$,

(3.5) $\qquad \bar{\theta}_{1/b} = \bar{Q} - \frac{1}{2}\sigma^2 + \bar{m}$, defined as the solution of $\bar{F}(1/b,\theta) = 0$,

(3.6) $\qquad \bar{R} = 2\bar{m}/b\sigma^2$, where $\bar{R} = -\bar{F}(0,\theta)$ by definition.

We shall consider only systems $\bar{S}$ for which either $\bar{\theta}_1 > 0$, called Type 1 Systems, or $\bar{\theta}_{1/b} > 0 \geq \bar{\theta}_1$, called Type 0 Systems. The relation

[3] It is possible to imbed S in an autonomous three–dimensional system $\mathfrak{S}$ in such a way that the stationary points (in particular the saddles) of the asymptotic systems become true stationary points (saddles) of $\mathfrak{S}$. This approach yields results about phase behaviour in neighbourhoods and about stable and unstable manifolds directly, and I hope to pursue it elsewhere. However it involves substantial preliminaries and extra notation, and is not particularly helpful here since it does not avoid the phase analysis needed to show that the stable and unstable manifolds at the relevant saddle points intersect in a suitable way.

(3.6a) $\bar{\theta}_{1/b} - \bar{\theta}_1 = (b-1)(\bar{m}/b+\frac{1}{2}\sigma^2) = \frac{1}{2}\sigma^2(b-1)(1+\bar{R})$

obtained from (4–6) is useful for determining whether a given set of parameter values satisfies one of these conditions. Note that, if $b = 1$, then $\bar{\theta}_1 = \bar{\theta}_{1/b} > 0$, so that only Type 1 systems occur; sometimes we omit special discussion of this case for brevity. Using (0.3–0.6) we obtain

(3.7) for $\bar{F} = F_{\infty}$: $\bar{Q} = Q$, $\bar{m} = m$, $\bar{\theta}_1 = n$, $\bar{\theta}_{1/b} = q$, $\bar{R} = 2m/b\sigma^2$,

(3.8) for $\bar{F} = F_{-\infty}$: $\bar{Q} = Q+\psi_0'$, $\bar{m} = m-\psi_0'$, $\bar{\theta}_1 = N$, $\bar{\theta}_{1/b} = q$, $\bar{R} = 2(m-\psi_0')/b\sigma^2$.

A word here about classification and terminology. We have said previously that a *b.v.p.* of Type 1 arises if $b > 0$ and both $n > 0$ and $N > 0$, i.e. if both S_{∞} and $S_{-\infty}$ are Type 1 *Systems*. A *b.v.p.* of Type 0(i) arises if $b > 1$, $N > 0$ and $q > 0 \geq n$, i.e. if $S_{-\infty}$ is Type 1 and S_{∞} is Type 0; again, a b.v.p. of Type 0(ii) arises if $b < 1$, $n > 0$ and $q > 0 \geq N$, i.e. if S_{∞} is Type 1 and $S_{-\infty}$ is Type 0. Thus the Standing Assumptions require that at least one of $S_{-\infty}$ and S_{∞} be a Type 1 System. Type 1 b.v.p.s are those for which this holds for both systems; Type 0 b.v.p.s are those for which only one of the systems is Type 0, and then the sign of $b-1$ defines the case. These remarks define the main criteria according to which both auxiliary systems and b.v.p.s will be classified: first as Type 1 or 0, then according to the sign of $b-1$, (which affects the analysis of b.v.p.s of both types). The sign of $\bar{m}$ then defines a further criterion for individual systems, while for b.v.p.s there is a classification according to the signs of m and $m - \psi_0'$ (yielding three cases if the borderline values $m = 0$ and $m = \psi_0'$ are left aside, which we shall sometimes do). This classification is reflected in Figures 2–4, which are explained below.

(ii) *Phase Contours and Stationary Points of Systems* $\bar{S} = (\bar{F},G)$.

In the phase diagrams, arrows always relate to the forward motion. We begin by examining the contours (level curves) of the functions G and $\bar{F}$. Evidently the contours of $G = (h-1)\theta$ are rectangular hyperbolae with asymptotes $\theta = 0$ and

$h = 1$; we shall not stop to draw these. In the half–plane $\{\theta > 0\}$ we have $\theta' = G > 0$ when $h > 1$ and $\theta' = G < 0$ when $h < 1$. *Further, since* $\theta' = 0$ *only along* $\{h = 1\}$ *and* $\{\theta = 0\}$, *any stationary point of* $(\bar{F},G)$ *must lie on one of these lines.*

Referring next to (3) and (6), we note that the equation $\bar{F}(h,\theta) = \gamma$, where γ is a constant, may be rewritten as

(3.9) $$[h+(\theta-\bar{Q})/b\sigma^2]^2 - [(\theta-\bar{Q})/b\sigma^2]^2 - (\bar{R}+\gamma)/b = 0.$$

For $\gamma \neq \bar{R}$, this is the equation of a hyperbola with centre at $(h = 0,\ \theta = \bar{Q})$, axes $h = -(\theta-\bar{Q})/b\sigma^2$ and $\theta = \bar{Q}$, and asymptotes $h = -2(\theta-\bar{Q})/b\sigma^2$ and $h = 0$. The hyperbola consists of two distinct curves, one in $\{h > 0\}$ the other in $\{h < 0\}$, which we call the positive and negative contours of $\bar{F}$ at the level γ and denote by $\bar{F}^+(\gamma)$ and $\bar{F}^-(\gamma)$. In case $\gamma = -\bar{R}$, the contours are the asymptotes. See Figure 1. It is clear that the asymptotes define the boundaries of four domains, with contours at level $\gamma > -\bar{R}$ to the 'north–east' and 'south-west' and those for $\gamma < -\bar{R}$ to the 'north–west' and 'south–east'. The slope $s(h,\theta) = dh/d\theta$ along $\bar{F} = \gamma$ (at a point (h,θ) different from the centre of the hyperbola) is given by

(3.9a) $$s(h,\theta) = -(\partial\bar{F}/\partial\theta)/(\partial\bar{F}/\partial h) = -h/(bh\sigma^2+\theta-\bar{Q}).$$

We have $\partial\bar{F}/\partial h = 0$ along the first axis (leaving aside the centre), so that the contours have vertical slope there. Also $\bar{F} = -\bar{R}$ along both asymptotes, so that in particular the motion on the horizontal axis is up/down according as $\bar{R}$ (or $\bar{m}$) is $-/+$.

Contours for $\gamma > -\bar{R}$ have negative slope throughout and both contours cross every vertical line, so that the quadratic equation $\bar{F}(h,0) = \gamma$ has two real solutions h of opposing signs. For $\gamma = -\bar{R}$, one solution is positive, one zero. For $\gamma < -\bar{R}$, the two contours lie on opposite sides of a certain vertical open strip, so that no real solutions exist if the line $\{\theta = 0\}$ lies in this strip; however, if $\bar{F}^+(\gamma)$ meets $\{\theta > 0\}$, there are two distinct positive solutions. We denote the real solutions of $\bar{F}(h,0) = 0$ (when they exist) by $\bar{h}^+$ and $\bar{h}^-$; thus

(3.10) $$b\sigma^2\bar{h}^{\pm} = \bar{Q} \pm[\bar{Q}^2 + 2\bar{m}\sigma^2]^{\frac{1}{2}}.$$

For $\gamma = 0$, these remarks yield the following consequences:

PROPOSITION 3. $\bar{h}^+ > 0 > \bar{h}^-$ iff $\bar{R} > 0$; $\bar{h}^+ > 0 = \bar{h}^-$ iff $\bar{R} = 0$;

$\bar{h}^+ > \bar{h}^- > 0$ iff $\bar{R} < 0$ and $\bar{F}^+(0)$ meets $\{\theta > 0\}$.

Referring now to the definition (4) of $\bar{\theta}_1$, we note that, if $\bar{\theta}_1 > 0$, the contour $\bar{F}^+(0)$ must cut $\{\theta = 0\}$ at a point $\bar{h}^+ > 1$. If in addition $\bar{R} < 0$, there is a second intersection at $\bar{h}^-$ with $\bar{h}^- > 0$, otherwise $\bar{h}^- \leq 0$, and in either case $\bar{h}^+ > 1 > \bar{h}^-$. Again, by (5), if $\bar{\theta}_{1/b} > 0$, then $\bar{F}^+(0)$ must cut $\{\theta = 0\}$ at some $\bar{h}^+ > 1/b$. If in addition $\bar{R} < 0$, there is a second intersection at $\bar{h}^-$ with $\bar{h}^- > 0$, otherwise $\bar{h}^- \leq 0$, and in either case $\bar{h}^+ > 1/b > \bar{h}^-$. *Thus the assumption that* $\bar{\theta}_1 \vee \bar{\theta}_{1/b} > 0$ *ensures that there are always distinct real solutions* $\bar{h}^+$ *and* $\bar{h}^-$ *of* $\bar{F}(h,0) = 0$. Further inequalities which are easily checked from diagrams are set out in the following

PROPOSITION 4. Distinct real solutions $\bar{h}^+$ and $\bar{h}^-$ of the equation $\bar{F}(h,0) = 0$ exist in all cases if $\bar{\theta}_1 \vee \bar{\theta}_{1/b} > 0$, with $\bar{h}^+ > 0$ and $\operatorname{sgn}(\bar{h}^-) = \operatorname{sgn}(-\bar{R})$.

If $\bar{\theta}_1 > 0$, then $\bar{h}^+ > 1 > \bar{h}^-$

If $\bar{\theta}_{1/b} > 0$, then $\bar{h}^+ > 1/b > \bar{h}^-$.

If $\bar{\theta}_1 > 0$ and $b > 1$, then $\operatorname{sgn}(\bar{\theta}_{1/b}) = \operatorname{sgn}(1/b - \bar{h}^-)$.

If $\bar{\theta}_1 > 0$ and $b < 1$, then $\operatorname{sgn}(\bar{\theta}_{1/b}) = \operatorname{sgn}(\bar{h}^+ - 1/b)$.

If $\bar{\theta}_{1/b} > 0 \geq \bar{\theta}_1$ and $b > 1$, then $1 \geq \bar{h}^+ > 1/b > \bar{h}^-$,

and $\bar{h}^+ = 1$ only if $\bar{\theta}_1 = 0$.

If $\bar{\theta}_{1/b} > 0 \geq \bar{\theta}_1$ and $b < 1$, then $\bar{h}^+ > 1/b > \bar{h}^- \geq 1$,

and $\bar{h}^- = 1$ only if $\bar{\theta}_1 = 0$.

These assertions apply in particular to $\bar{F} = F_\infty$ or $\bar{F} = F_{-\infty}$, with parameter values as in (7–8). In these cases we write $\bar{h}^\pm$ as $h^\pm_\infty$ or $h^\pm_{-\infty}$.

Since a stationary point of $\bar{S}$ must satisfy $\bar{F} = G = 0$, (and we consider only points with $\theta \geq 0$), it follows that Type 1 systems (those with $\bar{\theta}_1 > 0$) have precisely three such points, namely

$(1,\bar{\theta}_1)$, $(\bar{h}^+,0)$, $(\bar{h}^-,0)$, while Type 0 systems (those with $\bar{\theta}_{1/b} > 0 \geq \bar{\theta}_1$) have only two, namely

$(\bar{h}^+,0)$ and $(\bar{h}^-,0)$.

The results of the geometric discussion so far are illustrated in Figures 2, which show both Type 1 and Type 0 systems with $\bar{m} < 0$ and $\bar{m} > 0$, distinguishing between Type 0 systems with $b > 1$ and those with $b < 1$. (There is no Fig. 2(vi), because by (6a) the conditions $b < 1$, $\bar{\theta}_{1/b} > 0 \geq \bar{\theta}_1$, $\bar{m} \geq 0$ are inconsistent). It is clear enough from the diagrams that in Type 1 systems the point $(1,\bar{\theta}_1)$ is a saddle, $(\bar{h}^+,0)$ is an unstable node and $(\bar{h}^-,0)$ is a stable node, but appropriate calculations are also given below. Again, in Type 0 systems with $b > 1$, the point $(\bar{h}^+,0)$ is a saddle and $(\bar{h}^-,0)$ is a stable node; while for $b < 1$, the point $(0,\bar{h}^+)$ is an unstable node and $(0,\bar{h}^-)$ is a saddle.[4] For each saddle point we have drawn in the 'stable' manifold (or curve) $\bar{\mathscr{M}}^{\triangleright}$ (labelled $\bar{f}$) and the 'unstable' manifold $\bar{\mathscr{M}}^{\triangleleft}$ (labelled $\bar{g}$), except that in Type 0 systems one of the manifolds lies on the vertical axis; more of these manifolds later.

In the particular cases $F = F_{\pm\infty}$, information about phase behaviour is also shown in Figures 3–4. Fig. 3, comprising six diagrams, illustrates cases where both S_∞ and $S_{-\infty}$ are of Type 1 (corresponding to Type 1 b.v.p.s), classified according to the sign of $b-1$ and then according to the signs of m and $m-\psi_0'$. Fig. 4 has the same information for cases where either $b > 1$ and S_∞ is of Type 0 or $b < 1$ and $S_{-\infty}$ is of Type 0 (except that there is no Fig. 4(vi), because the conditions $b < 1$, $N \leq 0 < n$, $q > 0$, $m \geq \psi_0'$ are inconsistent; to see this, evaluate (6a) with the parameter values in (8)). Each diagram shows (where space permits) the curves $F_\infty = 0$ and $F_{-\infty} = 0$ and the stable and unstable curves at saddle points of S_∞ and $S_{-\infty}$ (as well as other

[4] Once again, there is a qualification, relating to cases with $\theta_{1/b} > \theta_1 = 0$. There is a saddle-node bifurcation at $(1,\bar{\theta}_1) = (1,0)$, but the phase behaviour remains saddle–like in the closed right half–plane and we treat the point as a saddle without special discussion. The stable node remains at $(\bar{h}^-,0)$ if $b > 1$, the unstable node at $(\bar{h}^+,0)$ if $b < 1$. Space does not allow a detailed discussion of this amusing case, but an example is illustrated in Fig. 5.

information to be explained later).

The assertions made above on the basis of geometric arguments can of course be checked and made more precise by calculation, but to save space and tedium we shall only outline selected cases. Thus to *check the existence of real roots* $\bar{h}^+$ and $\bar{h}^-$ in the cases mentioned we first write $\bar{F} = \bar{F}(h,\theta)$ in the alternative forms

(3.11) $$\bar{F} = b(h-1)^2 + (2/\sigma^2)(h-1)(\theta - \bar{Q} + b\sigma^2) + (2/\sigma^2)(\theta-\bar{\theta}_1)$$

(3.12) $$\bar{F} = b(h-1/b)^2 + (2/\sigma^2)(h-1/b)(\theta - \bar{Q} + \sigma^2) + (2/b\sigma^2)(\theta-\bar{\theta}_{1/b}).$$

If $\bar{\theta}_1 > 0$, we solve $\bar{F}(h,0) = 0$ in the first form to obtain

(3.13) $$b\sigma^2(\bar{h}^{\pm} - 1) = \bar{Q} - b\sigma^2 \pm [(\bar{Q} - b\sigma^2)^2 + 2b\sigma^2\bar{\theta}_1]^{\frac{1}{2}},$$

and $\bar{\theta}_1 > 0$ implies the existence of distinct real roots $\bar{h}^+ > 1 > \bar{h}^-$. If $\bar{\theta}_{1/b} > 0$ we solve $\bar{F}(h,0) = 0$ in the second form to obtain

(3.14) $$b\sigma^2(\bar{h}^{\pm} - 1/b) = \bar{Q} - \sigma^2 \pm [(\bar{Q} - \sigma^2)^2 + 2\sigma^2\bar{\theta}_{1/b}]^{\frac{1}{2}},$$

and $\bar{\theta}_{1/b} > 0$ implies the existence of distinct real roots $\bar{h}^+ > 1/b > \bar{h}^-$.

Turning to the *characterisation of the stationary points*, we write the Jacobian matrix of $\bar{S}$ at an arbitrary point, using obvious notation for derivatives, as

(3.15) $$\begin{bmatrix} \bar{F}_h, & \bar{F}_\theta \\ G_h & G_\theta \end{bmatrix} = \begin{bmatrix} (2/\sigma^2)(b\sigma^2 h+\theta-\bar{Q}) & (2/\sigma^2)h \\ \theta & h-1 \end{bmatrix}$$

so that the characteristic roots are given by

(3.16) $$2\lambda_\pm = 2\lambda_\pm(h,\theta) = \bar{F}_h + G_\theta \pm [(\bar{F}_h - G_\theta)^2 + 4\bar{F}_\theta G_h]^{\frac{1}{2}}.$$ [5]

For brevity we give details only for those points which define saddles. Consider first the point $(1,\bar{\theta}_1)$ in case $\bar{\theta}_1 > 0$. Here, using (4) and (6), we have

(3.17) $$\bar{F}_h = b + 2\bar{m}/b\sigma^2 = b + \bar{R}, \quad \bar{F}_\theta = 2/\sigma^2, \quad G_h = \bar{\theta}_1, \quad G_\theta = 0,$$

so that

(3.18) $$2\lambda_\pm(1,\bar{\theta}_1) = b + \bar{R} \pm[(b + \bar{R})^2 + 8\bar{\theta}_1/\sigma^2]^{\frac{1}{2}}, \quad \bar{R} = 2\bar{m}/b\sigma^2$$

Since $\bar{\theta}_1 > 0$, the characteristic roots are real and of opposing sign, confirming that the

[5] Note that we distinguish between the points h^+ and h^- by superscripts, but between characteristic roots at a given point by subscripts.

point is a saddle. Let $h = \bar{f}(\theta)$ and $h = \bar{g}(\theta)$ denote the equations of the 'stable' ($z \uparrow$) and 'unstable' ($z \downarrow$) curves at $(1,\bar{\theta}_1)$, considered as *local* manifolds for the moment. The directions of 'arrival' and 'departure', i.e. the limits of $[h(z)-1]/[\theta(z)-\bar{\theta}_1]$ as $z \to \infty$ and $z \to -\infty$, may be calculated from the linear variational equations about $(1,\bar{\theta}_1)$ as

$$(3.19) \qquad \bar{f}'(\bar{\theta}_1) = \lambda_-/\bar{\theta}_1 < 0, \quad \bar{g}'(\bar{\theta}_1) = \lambda_+/\bar{\theta}_1 > 0.$$

In particular, for $\bar{F} = F_\infty$, we have $\bar{\theta}_1 = n$, which was proposed as the limiting value of $\theta(z)$ as $z \uparrow \infty$ whenever $n > 0$. In this case the characteristic roots $\lambda_\pm(1,n)$ are calculated with $\bar{m} = m$ – see (7). In the same way, for $\bar{F} = F_{-\infty}$ we have $\bar{\theta}_1 = N$, which was proposed as the limiting value of $\theta(z)$ as $z \downarrow -\infty$ whenever $N > 0$. In this case $\lambda_\pm(1,N)$ are calculated with $\bar{m} = m-\psi_0'$, see (8).

Turning to stationary points of $(\bar{F},G)$ with $\theta = 0$, we have, using (10),

$$(3.20) \qquad \bar{F}_h = (2/\sigma^2)(b\sigma^2\bar{h}-\bar{Q}) = \pm(2/\sigma^2)[\bar{Q}^2+2\bar{m}\sigma^2]^{\frac{1}{2}},$$

$$\bar{F}_\theta = (2/\sigma^2)\bar{h}, \quad G_h = 0, \quad G_\theta = \bar{h} - 1,$$

where $\bar{h}$ is $\bar{h}^+$ or $\bar{h}^-$ and the sign of the square root in the expression for $\bar{F}_h$ is chosen accordingly.[6] Now (16) yields

$$(3.21) \qquad \lambda_+(\bar{h},0) = \bar{F}_h \vee G_\theta, \quad \lambda_-(\bar{h},0) = \bar{F}_h \wedge G_\theta.$$

If $b > 1$ and $\bar{\theta}_{1/b} > 0 > \bar{\theta}_1$ we select $\bar{h}^+$ and obtain $\lambda_+ = \bar{F}_h > 0$, $\lambda_- = \bar{h}^+ - 1 < 0$ by Prop. 4, confirming the saddle property. In this case the unstable curve is on the vertical axis. The stable curve may again be written locally as $h = \bar{f}(\theta)$ and the limit of $[h(z)-\bar{h}^+]/\theta(z)$ calculated from the linear variational equations about $(\bar{h}^+,0)$. In particular, if $\bar{F} = F_\infty$, we have $\bar{\theta}_{1/b} = q$, $\bar{\theta}_1 = n$, $\bar{m} = m$, $\bar{Q} = Q = q-m+\frac{1}{2}\sigma^2$, see (7) and (0.7), and the values of $\bar{h}^+ = h_\infty^+$ and $\lambda_\pm$ are obtained from (10) and (20–21). Since $1/b < h_\infty^+ < 1$ by Prop. 4, the point $(h_\infty^+,0)$ satisfies the condition (0.13) for the

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[6] The root must be real because $\bar{h}^\pm$ are real, but it can also be checked directly that $\bar{Q}^2 + 2\bar{m}\sigma^2 > 0$ in case either $\bar{\theta}_1 > 0$ or $\bar{\theta}_{1/b} > 0$. If $\bar{m} > 0$, this is obvious. If $\bar{m} \leq 0$, express $\bar{Q}$ in terms of $\bar{\theta}_1$ or $\bar{\theta}_{1/b}$ using (3.4–5) and rearrange to represent $\bar{Q}^2 + 2\bar{m}\sigma^2$ as the sum of perfect squares and a positive term.

limit of an optimal consumption function (and this remains true if $h^+_\infty = 1$) Writing the stable curve as $h = f_\infty(\theta)$ we obtain

$$(3.22) \qquad f'_\infty(0) = 2(1+\lambda_-)/\sigma^2(\lambda_- - \lambda_+), \quad \text{where} \quad \lambda_\pm = \lambda_\pm(h^+_\infty, 0)$$

Similarly, if $b < 1$ and $\bar{\theta}_{1/b} > 0 > \bar{\theta}_1$, we select $\bar{h}^-$ and obtain $\lambda_+ = \bar{F}_h < 0$, $\lambda_+ = \bar{h}^- - 1 > 0$ by Prop 4, again a saddle. This time the stable curve is on the vertical axis and the unstable curve may be written locally as $h = \bar{g}(\theta)$. In particular, if $\bar{F} = F_{-\infty}$ we have $\bar{\theta}_{1/b} = q$, $\bar{\theta}_1 = N$, $\bar{Q} = Q + \psi'_0$, $\bar{m} = m - \psi'_0$, $\bar{h}^- = h^-_{-\infty}$. Since $1/b > h^-_{-\infty} > 1$, the point $(h^-_{-\infty}, 0)$ satisfies (0.14), (also if $h^-_{-\infty} = 1$) Writing the unstable curve as $h = g_{-\infty}(\theta)$ we obtain

$$(3.23) \qquad g'_{-\infty}(0) = 2(1+\lambda_+)/\sigma^2(\lambda_+ - \lambda_-), \quad \text{where} \quad \lambda_\pm = \lambda_\pm(h^-_{-\infty}, 0)$$

If $\bar{\theta}_1 = 0$, with either $b > 1, \bar{h}^+ = 1$, or $b < 1, \bar{h}^- = 1$, the point $(1,0)$ is a saddle–node but the preceding remarks apply with routine changes; see fn. 4

This discussion yields the following important result:

PROPOSITION 5. The points satisfying the boundary conditions at $z = \infty$ and $z = -\infty$ prescribed by Theorem 4A are precisely the saddle points of S_∞ and $S_{-\infty}$ In particular, the condition (0.13) cannot be satisfied by any $h^*(+\infty)$ other than h^+_∞, and (0.14) cannot be satisfied by any $h^*(-\infty)$ other than $h^-_{-\infty}$.

PROOF. A star solution must by definition be a solution of S defined on the whole of $\Re$ and converging to finite limits as $z \to \pm\infty$. According to Prop. 2(ii) the limits must be stationary points of S_∞ and $S_{-\infty}$ For Type 1 b.v.p.s the result is immediate because (0.11) and (0.12) give the precise co-ordinates of the relevant saddle points. In the case of Type 0 b.v.p.s the saddle points $(h^+_\infty, 0)$ and $(h^-_{-\infty}, 0)$ satisfy (0.13) and (0.14) respectively, and Prop. 4 shows that these are the only stationary points of S_∞ and $S_{-\infty}$ in the prescribed intervals. (Once again, 'saddle' here includes 'saddle–node'.) ||

(iii) *Dynamics and Asymptotic Behaviour of Solutions of Systems* $\bar{S} = (\bar{F},G)$.

We next note some properties of the phase behaviour of solutions of $\bar{S}$, in particular their limiting behaviour. Where the assertions are elementary or sufficiently obvious from the Figures we omit formal proofs. The remarks about the choice of phase space, solution of the initial value problem and continuation of solutions and paths made in connection with S apply here also. An obvious additional feature is that the path defined by a solution through a given point $\Pi_\diamond = (h_\diamond, \theta_\diamond, z_\diamond)$ does not depend on the value of $z_\diamond \in (z_-, z_+)$

In order to show that *every path of* $\bar{S}$ *converges to a limit, finite or infinite, as* $z \uparrow$ *or* $z \downarrow$, we review the directions of motion within and between various phase regions. (By a phase region we mean an open connected set of $\{\theta > 0\}$ on which $\bar{F}$ and G both have constant signs, the boundary consisting of arcs of $\bar{F} = 0$ or $G = 0$ or both.) The regions are shown in Figures 2, each with its pair of phase arrows. Cases with $\bar{m} = 0$, and those with $b = 1$ or $\bar{\theta}_1 = 0$ are not depicted but unless stated offer no significant exception to what follows. The only paths never entering any phase region are the stationary points and the paths lying on the vertical axis; these obviously converge and so may be left aside. According to earlier discussion, no path can terminate in the interior of a phase region. Within each phase region a path is monotone in both co-ordinates; thus it is enough to check that each path is ultimately in one of the regions as $z \uparrow z_+$ or $z \downarrow z_-$. In fact, a review of phase transitions yields more: (a) A path which passes through a point on the boundary between two phase regions immediately enters one of them. (b) There is a one-way flow between regions as $z \uparrow$, also as $z \downarrow$. (c) A path which once leaves a region, as $z \uparrow$ or as $z \downarrow$, cannot return to it via a sequence of other regions. The existence of limits follows.

As regards *the values of the limits*, the possibilities in the case of *bounded solutions* are few. (We again leave aside stationary solutions and those with $\theta = 0$.) If a solution is bounded as $z \uparrow$ ($z \downarrow$) then $z_+ = \infty$ ($z_- = -\infty$) – see Prop. 1. The limit must be a stationary point, say $(\bar{h},\bar{\theta})$, and for a forward (backward) limit it must satisfy $\bar{h} \leq 1$ $(\bar{h} \geq 1)$ since otherwise the arrows point the wrong way. Thus in Type 1 systems the only possible finite forward limits are $(1,\bar{\theta}_1)$ and $(\bar{h}^-,0)$, the only backward ones are $(1,\bar{\theta}_1)$ and $(\bar{h}^+,0)$ – see Figs. 2(i) and (ii). There are no finite backward limits in Type 0 systems with $b > 1$, but there are forward limits at $(\bar{h}^{\pm},0)$ – see Figs. 2(iii) and (iv). Similarly there are no finite forward limits in Type 0 systems with $b < 1$, but there are backward limits at $(\bar{h}^{\pm},0)$ – see Fig. 2(v).

As previously mentioned, the Figures are also classified according to the sign of $\bar{m}$. If $\bar{m} > 0$, forward motion on the axis $\{h = 0\}$ is always downward, if $\bar{m} < 0$ it is upward, and if $\bar{m} = 0$ the motion is along the axis, which acts as a barrier. Also, in cases with $\bar{m} > 0$, $\bar{h}^+$ and $\bar{h}^-$ have opposite signs, whereas with $\bar{m} < 0$ they have the same sign; in the latter case all finite limits of paths, forward or backward, have both co–ordinates positive, (non–negative if $\bar{m} \leq 0$).

To see some useful consequences of these remarks, consider the 'invariant' paths at the saddle $(1,\bar{\theta}_1)$ in Figs. 2(i) and (ii), representing Type 1 systems. It has been shown above that the stable manifold $\bar{\mathcal{M}}^{\triangleright}$ can be represented locally by a function $\bar{h} = \bar{f}(\theta)$ with negative slope. This manifold consists of two paths (in addition to the saddle point). Tracing the left path $\bar{\mathcal{M}}^{\triangleright L}$ backward as $z \downarrow$, it is seen that $\theta \downarrow$ and $h \uparrow$ with no possible phase transitions, so the backward limit must be $(\bar{h}^+,0)$ and $z_- = -\infty$. Tracing the right path $\mathcal{M}^{\triangleright R}$ backward, one has $\theta \uparrow$, $h \downarrow$ as long as h stays positive, which it must do forever if $\bar{m} \geq 0$ as in Fig. 2(ii); then the path limit is given by $h = 0$, $\theta = \infty$, and again $z_- = -\infty$ since h stays bounded. If $\bar{m} < 0$ as in Fig. 2(i), the path cannot stay in the domain $\{h > 0\}$ forever but crosses at some $\theta = \bar{\theta}_+$ into $\{h < 0\}$, where it remains and continues moving to the right. Now there are two possibilities, which will be discussed more fully below. One is that the path eventually crosses the curve $\bar{F}^-(0)$, after which the path limit is again $(0,\infty)$, this time approached from below, again with $z_- = -\infty$. The other is that $h \downarrow -\infty$, $\theta \uparrow \infty$ as

$z \downarrow z_-$ (with $z_- > -\infty$, see below, and it turns out that this possibility is open only if $b > 1$). Here we anticipate the result that in all these cases the representation $h = \bar{f}(\theta)$ can be continued for all $\theta \in (0,\infty)$, (global stable manifold), and record that $\bar{f}(\theta) > 0$ for all θ if $\bar{m} \geq 0$, but $\bar{f}(\theta) < 0$ for θ greater than some $\bar{\theta}_+ < \infty$ if $\bar{m} < 0$. A further useful remark is that the curve $\bar{\mathscr{M}}^{\triangleright} = \{h = \bar{f}(\theta)\}$ separates two open 'half–spaces' within $\{\theta > 0\}$, say

(3.24) $$\bar{\mathscr{U}}^{\triangleright} = \{h > \bar{f}(\theta)\}, \quad \bar{\mathscr{B}}^{\triangleright} = \{h < \bar{f}(\theta)\}.$$

Paths in the lower half–space $\bar{\mathscr{B}}^{\triangleright}$ are bounded for the forward motion and converge to $(\bar{h}^-,0)$ as $z \uparrow$, with $z_+ = \infty$, while paths in $\bar{\mathscr{U}}^{\triangleright}$ pass ultimately into the region $\{F>0, G>0\} = \{F>0, h>1\}$ and so become unbounded (with $z_+ < \infty$, see below).

Consider now the unstable manifold $\bar{\mathscr{M}}^{\triangleleft}$, which is represented locally by a function $h = \bar{g}(\theta)$ with positive slope. If the left branch $\bar{\mathscr{M}}^{\triangleleft L}$ is traced forward as $z \uparrow$, it is seen that $\theta \downarrow$ and $h \downarrow$ as long as the path remains in $\{h \geq 0\}$, which goes on for all $\theta > 0$ if $\bar{m} \leq 0$; but if $\bar{m} > 0$ the path must cross into $\{h < 0\}$ at some $\theta = \bar{\theta}_- > 0$ and then eventually pass into a region with $\bar{F} > 0$, after which h increases again while remaining negative. In either case the path limit is $(\bar{h}^-,0)$ and $z_+ = \infty$. As to the right path $\bar{\mathscr{M}}^{\triangleleft R}$, this passes immediately into the region $\{\bar{F}>0, G>0\}$ and eventually becomes unbounded with $h \uparrow \infty$, $\theta \uparrow \infty$ (and $z_+ < \infty$). Once again, the representation can be continued for all $\theta > 0$ (global unstable manifold), with $\bar{g}(\theta) > 0$ for all θ if $\bar{m} \leq 0$ but $\bar{g}(\theta) < 0$ for θ less than some $\bar{\theta}_-$ if $\bar{m} > 0$. In particular, *whatever* $\bar{m}$, *one of the curves* $\bar{f}$ *and* $\bar{g}$ *always stays positive on* $\{\theta > 0\}$ (but both have this property only if $\bar{m} = 0$). The curve $\bar{\mathscr{M}}^{\triangleleft} = \{h = \bar{g}(\theta)\}$ separates two open half-spaces

(3.25) $$\bar{\mathscr{U}}^{\triangleleft} = \{h < \bar{g}(\theta)\}, \quad \bar{\mathscr{B}}^{\triangleleft} = \{h > \bar{g}(\theta)\}$$

in $\{\theta > 0\}$, with paths in the 'upper' half–space $\bar{\mathscr{B}}^{\triangleleft}$ bounded for the backward motion and converging to $(\bar{h}^+,0)$ as $z \downarrow -\infty$, and paths in the 'lower' half–space $\bar{\mathscr{U}}^{\triangleleft}$ following one of the two possibilities indicated above for $\bar{\mathscr{M}}^{\triangleright R}$, with $\theta \to \infty$

anyway as $z \downarrow z_-$.

A similar, but rather simpler, analysis can be carried out in the remaining cases. Suffice it to say that in cases with $b > 1$ and $\bar\theta_{1/b} > 0 \geq \bar\theta_1$, the stable manifold at $(\bar h^+,0)$ has only one branch lying in $\{\theta > 0\}$, which we denote by $\bar{\mathscr{M}}^{\triangleright}$. Its representation by $h = \bar f(\theta)$ can be continued for all $\theta > 0$, with a negative slope while h remains positive. This goes on for all $\theta > 0$ if $\bar m \geq 0$ as in Fig. 2(iv); but if $m < 0$ as in Fig. 2(iii) then h becomes negative for θ greater than some finite $\bar\theta_+$, with alternative behaviour thereafter as described above. Once again, all paths in $\bar{\mathscr{B}}^{\triangleright}$ go to $(\bar h^-,0)$ as $z \uparrow z_+ = \infty$ while paths in $\bar{\mathscr{U}}^{\triangleright}$ become unbounded upward. *The unstable manifold is on the vertical axis so that the function* $\bar g$ *is undefined, and we set* $\bar{\mathscr{U}}^{\triangleleft} = \{\theta > 0\}$. In case $b < 1$ and $\bar\theta_{1/b} > 0 \geq \bar\theta_1$ as in Fig. 2(v), the stable manifold at $(\bar h^-,0)$ has only one branch lying in $\{\theta > 0\}$, which is denoted by $\bar{\mathscr{M}}^{\triangleleft}$. Its representation by $h = \bar g(\theta)$ can be continued for all $\theta > 0$, but this time the only admissible possibility is that $\bar m < 0$; thus the curve $\bar g$ has positive slope for all θ and becomes unbounded as $z \uparrow$, with $h \uparrow \infty$, $\theta \uparrow \infty$ (and $z_+ < \infty$). For $z \downarrow$, the paths in $\bar{\mathscr{B}}^{\triangleleft}$ converge to $(\bar h^+,0)$ and those in $\bar{\mathscr{U}}^{\triangleleft}$ to $(0,\infty)$. *Now the stable manifold is on the vertical axis, the function* f^V *is undefined, and we set* $\bar{\mathscr{U}}^{\triangleright} = \{\theta > 0\}$. [7]

It remains to give a brief account of the *asymptotic behaviour of unbounded solutions of* $\bar S$. The main result needed below is that *a path corresponding to such a solution can always be continued as* $\theta \uparrow \infty$, more precisely that for a solution which is

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[7] Some pedantic distinctions concerning definitions should be stated once and for all. The *term* 'stable manifold' denotes (unless the context indicates otherwise) only that part of the manifold as usually defined which lies in $\{\theta \geq 0\}$; in particular, the saddle point itself is included. However the *notation* $\bar{\mathscr{M}}^{\triangleright}$ is reserved for the part of the manifold lying in $\{\theta > 0\}$ and we write $\bar{\mathscr{M}}^{\triangleright} = \{h = \bar f(\theta)\}$, so that the function $\bar f$ is 'properly' defined only for $0 < \theta < \infty$. Nevertheless we often write the limiting values of $\bar f$ at $\theta = \infty$ and $\theta = 0$ as $\bar f(\infty)$ and $\bar f(0)$, and when convenient treat $\bar f$ as defined at these endpoints. Similarly for $\bar g$ and $\bar{\mathscr{M}}^{\triangleleft}$. Also, we set $\bar\theta_+ = \infty$ if $\bar f(\theta) > 0$ for all $\theta > 0$, and $\bar\theta_- = 0$ if $\bar g(\theta) > 0$ for all $\theta > 0$. Analogous conventions for other systems.

unbounded for the forward (backward) motion we have $\theta(z) \uparrow \infty$ on a final interval as $z \uparrow z_+$ (or $z \downarrow z_-$). (However, the details are not needed for later proofs and it is possible to skip to Prop 6). Consider first a *solution* (h,θ) *which is unbounded as* $z \uparrow$ As the diagrams and previous discussion show, the corresponding path is in $\mathscr{U}^{-\triangleright}$ and we may assume that it is ultimately in the phase region $\{F>0, G>0\}$, say for $z_\diamond \leq z < z_+$ In this region, $h(z)$ and $\theta(z)$ are always increasing. We first show that $z_+ < \infty$. Indeed, if we had $z_+ = \infty$, it would follow from

$$\theta'(z)/\theta(z) = h(z) - 1 > h(z_\diamond) - 1 > 0$$

that $\theta(z) \uparrow \infty$, and hence from (0.1) that $h'(z) > bh^2(z)$ for large enough z, say for $z > z_1$; however the solution of the equation $y' = by^2$ with initial condition $y(z_1) = h(z_1)$ explodes at some finite $z_2 > z_1$, implying that $h(z)$ also explodes at some $z_+ \leq z_2$, contrary to assumption. So $z_+ < \infty$. From this it further follows (Prop. 1) that $h(z)$ is not bounded, so $h(z) \uparrow \infty$ as $z \uparrow z_+$, and the (finite or infinite) limit $\theta(z_+)$ exists.

To investigate this limit further, it is convenient to introduce a new variable

$$\zeta(z) \;=\; [h(z) - 1]/\theta(z)$$

defined for $z_\diamond \leq z < z_+$. In the region considered, $\zeta(z) > 0$ and $\theta(z) \uparrow$ Write $\theta(z_\diamond) = \theta_\diamond$, $\zeta(z_\diamond) = \zeta_\diamond$. Taking θ as path parameter we have

$$\theta\, d\zeta/d\theta \;=\; \theta^{-1}[\theta\, dh/d\theta - h + 1] \;=\; \bar{F}/G - \zeta,$$

and writing $\bar{F}$ as a function of $h-1$ as in (3.11), dividing by $G = \theta' = (h-1)\theta = \zeta\theta^2$ and simplifying we get

(3.26) $$\theta\, d\zeta/d\theta \;=\; (b-1)\zeta + (2/\sigma^2)[1 - (\bar{Q}-b\sigma^2)/\theta + (\theta-\bar{\theta}_1)/\zeta\theta^2]$$

The term in square brackets is bounded, for $\theta > \theta_\diamond$, by $1 \pm \gamma$, where γ is a suitable constant. Now the equation

(3.27) $$\theta\, d\zeta/d\theta \;=\; (b-1)\zeta + (2/\sigma^2)(1 \pm \gamma)$$

with $\theta > 0$ has the solution

(3.28) $$\zeta(\theta) \;=\; C\theta^{b-1} \;+\; (2/\sigma^2)(1 \pm \gamma)/(1-b) \qquad \text{if } b \neq 1,$$

(3.29) $\zeta(\theta) = C + (2/\sigma^2)(1 \pm \gamma) \ln \theta$ if $b = 1$,

see Kamke [1943] p.311, eq. 1.94., where $C = C(\gamma)$ is a constant to be determined from the initial condition $(\zeta_\diamond, \theta_\diamond)$. Since this solution may be continued up to $\theta = \infty$ for each γ, the same must be true of the solution of (26), so that $\theta(z_+) = \infty$. It follows that γ can be made arbitrarily small by choosing $z_\diamond$ close enough to z_+, hence $h_\diamond$ and $\theta_\diamond$ large enough (along the given path). The asymptotic slope of the path can now be calculated informally as follows. For $b < 1$, let $\theta \to \infty$ in (28), followed by $\gamma \downarrow 0$ (corresponding to $\theta_\diamond \uparrow \infty$) to obtain $\zeta(z_+) = 2/(1-b)\sigma^2$. For $b = 1$, divide both sides of (29) by $\ln \theta$, let $\theta \to \infty$ and then $\gamma \downarrow 0$ to obtain $\zeta(\theta)/\ln \theta \to 2/\sigma^2$ as $z \uparrow z_+$, so $\zeta(\theta) \to \infty$. For $b > 1$, we have $\zeta(\theta) \to \infty$ as $\theta \to \infty$; dividing both sides of (28) by θ^{b-1} and letting $\theta \to \infty$ one gets $\zeta/\theta^{b-1} \sim C$, but no limit of ζ/θ^{b-1} independent of the initial conditions is obtained.

Consider now *solutions which are unbounded as* $z \downarrow$. It is clear from the phase diagrams (with the arrows reversed) that the corresponding paths are those in $\bar{\mathscr{U}}^{\triangleleft}$ and that the following types of asymptotic behaviour are possible:

(a) If $\bar{m} > 0$, as in Figs. 2(ii) and (iv), there are paths which are ultimately in $\bar{\mathscr{U}}^{\triangleleft} \cap \{F>0, h>0\}$, and then the limiting behaviour as $z \downarrow$ is $\theta \uparrow \infty, h \downarrow 0$, hence $z_- = -\infty$.

(b) If $\bar{m} < 0$, as in Figs. 2(i), (iii) and (v), there are paths which are ultimately in $\bar{\mathscr{U}}^{\triangleleft} \cap \{F<0, h<0\}$, and then the limiting behaviour as $z \downarrow$ is $\theta \uparrow \infty, h \uparrow 0$, hence $z_- = -\infty$.

(c) If $\bar{m} = 0$, both the preceding possibilities are open, as well as $\theta \uparrow \infty, h \equiv 0$, $z_- = -\infty$

So much is fairly obvious and we omit formal proof. The interesting question is whether an additional possibility exists, namely:

(d) Some paths are ultimately in $\bar{\mathscr{U}}^{\triangleleft} \cap \{F>0, h<0\}$, in which case $h \downarrow -\infty$ and $\theta \uparrow \infty$ as $z \downarrow z_-$. The diagrams suggest that, of the paths which are in this region at *some* z, those with a relatively flat negative slope will in due course pass into $\{F<0, h<0\}$, while those (if any) whose slope is sufficiently steep will remain in the region as $z \downarrow z_-$. We shall now make this argument more precise.

Suppose that a given solution passes through $(\pi_\diamond, z_\diamond) = (h_\diamond, \theta_\diamond, z_\diamond)$ with $\pi_\diamond \in \mathscr{U}^{\triangleleft}$, $\bar{F}(h_\diamond, \theta_\diamond) > 0$ and $h_\diamond < 0$. This time we introduce the variable

$$\eta(z) = [1-h(z)]/\theta(z),$$

defined for $z_\diamond \geq z > z_-$. On this interval, $\eta(z) > 0$ and $\theta(z) \uparrow$ (but it is not clear in advance whether z_- is finite). Taking θ as path parameter, we calculate $\theta\, d\eta/d\theta$ in the same way as for $\theta\, d\zeta/d\theta$ above (paying attention to changes of sign) and arrive at

$$\theta\, d\eta/d\theta = (b-1)\eta + (2/\sigma^2)[-1 + (\bar{Q}-b\sigma^2)/\theta + (\theta-\bar{\theta}_1)/\eta\theta^2]. \tag{3.30}$$

It can be checked as above that the solution of this equation can be continued as $\theta \uparrow \theta(z_-) = \infty$. Now the asymptotic slope of the contour $\bar{F}^{-}(0)$ as θ increases is $-2/b\sigma^2$ – see (3.9a) and Figs. 1 and 2 – so that the path through $(h_\diamond, \theta_\diamond)$ will pass out of the region $\{\bar{F}>0, h<0\}$ if $\eta < 2/b\sigma^2$ ultimately as $\theta \uparrow$ $(z \downarrow)$. *Suppose first that* $b \leq 1$. If the path stays in the region, then $\eta\theta = 1-h \uparrow \infty$ and $\eta\theta^2 \uparrow \infty$ as $\theta \uparrow \infty$, so ultimately $\theta\, d\eta/d\theta < -1/\sigma^2$, implying $\eta < \theta^{-1/\sigma^2} \times$ const., hence $\eta < 2/b\sigma^2$ for large θ, contrary to assumption. *So in this case the region is transient and* $z_- = -\infty$ *always*.

Now let $b > 1$. For each $\theta > \bar{\theta}_-$, there is a number $\eta_1 = \eta_1(\theta)$ such that the points $(h, \theta) = (1-\theta\eta, \theta)$ belong to the region $\{\bar{F}>0, h<0\}$ for all $\eta > \eta_1$; (the condition $\theta > \bar{\theta}_-$ ensures that the points also belong to $\mathscr{U}^{\triangleleft}$, cf. Fig. 2(ii)). Now choose $\theta_\diamond > \bar{\theta}_-$ so large that the expression in square brackets on the right of (30) is in the interval $[-2,0]$ whenever $\theta \geq \theta_\diamond$ and $\eta \geq 2/b\sigma^2$, and then choose $\eta_\diamond > \eta_1(\theta_\diamond) \vee 4/(b-1)\sigma^2$. Then $(h_\diamond, \theta_\diamond)$ is in the region, and the value of $\theta\, d\eta/d\theta$ at this point is at least $(b-1)\eta_\diamond - 4/\sigma^2$, which is positive. Following the path through $(h_\diamond, \theta_\diamond)$ as $\theta \uparrow$ it is seen that, since η increases initially, the expression in square brackets in (30) remains within $[-2,0]$ and the inequality

$$\theta\, d\eta/d\theta \geq (b-1)\eta - 4/\sigma^2 > (b-1)\eta_\diamond - 4/\sigma^2 > 0 \tag{3.31}$$

remains in force *The path therefore remains in the region as* η *increases without bound*. Note further that the preceding inequalities imply

$$d\ln \eta / d\ln \theta \; = \; (\theta/\eta) d\eta/d\theta \; > \; b - 1 - 4/\sigma^2\eta \; > \; b - 1 - 4/\sigma^2\eta_\diamond \; > \; 0.$$

Writing $\epsilon = 4/\sigma^2\eta_\diamond$ and integrating, one has $\eta > D^{b-1-\epsilon}$, where $D > 0$ is a constant. Reverting now to z as the path parameter and using $\theta' = (h-1)\theta$ and the definition of η, we have

$$-d\theta/dz \; = \; (1-h)\theta \; = \; \theta^2\eta \; > \; D\,\theta^{b+1-\epsilon},$$

hence, passing to $-dz/d\theta$ and integrating,

$$-z \; < \; \Delta - D^{-1}(b-\epsilon)^{-1}\,\theta^{\epsilon-b},$$

where Δ is another constant, and since $\epsilon - b < -1$ the last expression $\uparrow \Delta$ as $\theta \uparrow \infty$ It follows that $z_- > -\infty$. Arguing informally as above, it is further seen that the asymptotic path behaviour of η is analogous to that of ζ in the case $b > 1$, i.e $\eta(z) \to \infty$ as $z \downarrow z_-$ and $\eta/\theta^{b-1} \sim C$, where C depends on the initial conditions

To sum up: Paths passing through points of the region $\bar{\mathcal{U}}^{\triangleleft} \cap \{\bar{F}>0, h<0\}$ as $z \downarrow$ correspond to solutions of two types: those for which h remains bounded and which pass out of the region (after which $\theta \to \infty$, $h \to 0$ and $z_- = -\infty$) and those for which h, θ and $\eta = (1-h)/\theta$ become unbounded and which remain in the region up to some $z_- > -\infty$. If $b \leq 1$, only the first type occurs. If $b > 1$, it appears that both possibilities are open, for either sign of $\bar{m}$ [8]

The following proposition sets out properties of the functions $\bar{f}$ and $\bar{g}$ which will be needed later:

[8] Without going into further details, it seems clear that the two types lie on opposite sides of some separating path, and in fact this path belongs to the set remaining in $\{\bar{F}>0\}$, because the points of $\bar{F}^{-}(0)$ defined by paths crossing from $\{\bar{F}>0\}$ to $\{\bar{F}<0\}$ as $z \downarrow$ form a relatively open subset of $\bar{F}^{-}(0)$.

PROPOSITION 6 (Stable and unstable curves for three–parameter systems).

(i) If $\bar{\theta}_1 > 0$, the system $\bar{S} = (\bar{F},G)$ has a saddle point at $(1,\bar{\theta}_1)$. The stable and unstable manifolds at this point are represented by functions $\bar{f}$ and $\bar{g}$ defined and continuous for $\theta \in (0,\infty)$, (with limits at $\theta = 0$ and $\theta = \infty$, all limits at $\theta = 0$ being finite).

$\bar{f}$ is positive and strictly decreasing on an interval $[0,\bar{\theta}_+)$, with

(3.32) $\qquad \bar{\theta}_+ \leq \infty$ and $\bar{f}(\bar{\theta}_+) = 0$ in all cases;

(3.32a) $\qquad \bar{\theta}_+ = \infty$ iff $\bar{m} \geq 0$.

Thus $\bar{f}(\infty) = +0$ if $\bar{\theta}_+ = \infty$; but $\bar{f}$ is negative on $(\bar{\theta}_+,\infty)$ if this interval is not empty, and then either $\bar{f}(\infty) = -0$ or $\bar{f}(\infty) = -\infty$ (only the former case arising if $b \leq 1$).

$\bar{g}$ is positive and strictly increasing on an interval $(\bar{\theta}_-,\infty)$, with $g(\infty) = \infty$, and negative on $(0,\bar{\theta}_-)$ if this interval is not empty. We have

(3.33) $\qquad \bar{\theta}_- \geq 0$, $\bar{g}(\bar{\theta}_-) \geq 0$ and $\bar{\theta}_- \cdot \bar{g}(\bar{\theta}_-) = 0$ in all cases,

(3.34) $\qquad \bar{\theta}_- = 0$ iff $\bar{m} \leq 0$; $\bar{g}(\bar{\theta}_-) = 0$ iff $\bar{m} \geq 0$.

The following inequalities hold:

(3.35) $\qquad \bar{h}^+ = \bar{f}(0) > \bar{f}(\bar{\theta}_1) = 1,$

(3.36) $\qquad \bar{h}^- = \bar{g}(0) < \bar{g}(\bar{\theta}_1) = 1.$

(ii) If $\bar{\theta}_{1/b} > 0 \geq \bar{\theta}_1$, $b > 1$, then $\bar{S}$ has a saddle point at $(\bar{h}^+,0)$. The stable manifold is represented by a function $\bar{f}$ defined and continuous for $\theta \in [0,\infty)$. The properties of $\bar{f}$ and $\bar{\theta}_+$ are as in (i) except that (35–36) are replaced by

(3.37) $\qquad 1 \geq \bar{h}^+ = \bar{f}(0) > 1/b > \bar{h}^-.$

(iii) If $\bar{\theta}_{1/b} > 0 \geq \bar{\theta}_1$, $b < 1$, then $\bar{S}$ has a saddle point at $(\bar{h}^-,0)$. The unstable manifold is represented by a function $\bar{g}$ defined and continuous for $\theta \in [0,\infty)$. The properties of $\bar{g}$ and $\bar{\theta}_-$ are as in (i) except that only the case $\bar{m} < 0$, $\bar{\theta}_- = 0$ is admissible and (35–36) are replaced by

(3.38) $\qquad 1 \leq \bar{h}^- = \bar{g}(0) < 1/b < \bar{h}^+.$

(iv) *Five–parameter autonomous systems* So far we have considered auxiliary autonomous systems $\bar{S} = (\bar{F},G)$ where $\bar{F}$ is defined by three parameters $(b, \bar{Q}, \bar{m})$, see (3–6) above. In order to set bounds for the motion of S it is also necessary to consider autonomous systems with $\bar{F}$ defined by two formulas of the form (3), with the same value of b but possibly different values of $\bar{Q}$ and $\bar{m}$; one formula applies for values of (h,θ) above a certain line, which is either $\{h = 0\}$ or $\{h = 1/b\}$, the other below the same line, the values on the line being equal so that a continuous $\bar{F}$ is defined overall. The values of $\bar{\theta}_1$, $\bar{\theta}_{1/b}$ and $\bar{R}$ are still uniquely defined since they depend only on the values of $\bar{F}$ on the lines with $h = 1$, $1/b$ and 0, (provided that in (4–6) one takes the appropriate values of $\bar{m}$ and $\bar{Q}$). The definition of systems as Type 1 or Type 0 therefore still makes sense, and all systems considered will be of one of these types.

We shall need to consider pairs of inequalities, each pair defining an upper or lower bound for F and hence a certain five–parameter system $\bar{S}$. Some of the inequalities will be global, applying for all values of z, some only for far right or far left values For each inequality we shall tabulate values of $\bar{Q}$ and $\bar{m}$ which define $\bar{F}$ on the part of the phase space for which that inequality holds The phase analysis of the various systems is very similar to that of the three–parameter systems considered above if the derivative discontinuity of $\bar{F}$ along one horizontal line is taken into account. In particular, Props 3–4 apply. The number of cases to be considered is rather large and various details will be omitted.

Upper and lower bounds for $F(h,\theta,z)$ are obtained from the properties of A(z) and M(z) – see (0.2–0.4) For brevity we write $F = F(h,\theta,z)$, $F^\Lambda = F^\Lambda(h,\theta)$, $F^V = F^V(h,\theta)$ etc Starting with *inequalities and bounds which are valid for all* $z \in \Re$ *and for* $h \in \Re$ *and* $\theta \geq 0$, we have

$$(3.39a)\qquad F > F^\Lambda \doteq \begin{cases} F_{-\infty} - 2\psi_0'/b\sigma^2 & \text{if } h \geq 0 \quad [\bar{Q} = Q + \psi_0',\ \bar{m} = m] \\ F_\infty & \text{if } h \leq 0 \quad [\bar{Q} = Q,\ \bar{m} = m] \end{cases}$$

$$\text{(3.39b)} \qquad \bar{\theta}_1 = N + \psi_O'/b = n + \psi_O', \ \bar{\theta}_{1/b} = q + \psi_O',$$
$$\bar{R} \doteq -F^{\Lambda}(0,\theta) = 2m/b\sigma^2 = R_{\infty}.$$

The expressions following the left curly brackets in (39a) define F^{Λ} on the domains $\{h \geq 0\}$ and $\{h \leq 0\}$, while the quantities in square brackets are the values of $\bar{Q}$ and $\bar{m}$ to be chosen in order to represent F^{Λ} as a function $\bar{F}$ of the type defined in (3) on the appropriate domain. It can be checked that F^{Λ} is continuous on $\{h = 0\}$. The values of $\bar{\theta}_1$ and $\bar{\theta}_{1/b}$ in (39b) are calculated as in (4) and (5) from the values of F^{Λ} along the lines $\{h = 1\}$ and $\{h = 1/b\}$. Thus F^{Λ}, or equivalently the system $S^{\Lambda} = (F^{\Lambda},G)$, may be regarded as defined by five parameters: b, and two values of $\bar{Q}$ and $\bar{m}$. When necessary we write $\bar{\theta}_1 = \theta_1^{\Lambda}$, $\bar{R}_{1/b} = R^{\Lambda}$ etc. for the parameters defined in (39b), with similar notation for other systems defined below. *The number* θ_1^{Λ} *plays a special part in what follows and is denoted by* ν.

In the same way, and omitting detailed explanations, we have

$$\text{(3.40a)} \qquad F < F^{V} \doteq \begin{cases} F_{\infty} & \text{if } h \geq 1/b \quad [\bar{Q} = Q, \qquad \bar{m} = m] \\ F_{-\infty} & \text{if } h \leq 1/b \quad [\bar{Q} = Q + \psi_O', \bar{m} = m - \psi_O'] \end{cases}$$

$$\text{(3.40b)} \qquad \bar{\theta}_1 = n \text{ if } b > 1; \ \bar{\theta}_1 = N \text{ if } b \leq 1; \ \bar{\theta}_{1/b} = q;$$
$$\bar{R} \doteq -F^{V}(0,\theta) = 2(m-\psi_O')/b\sigma^2 = R_{-\infty}.$$

According to our Standing Assumptions, we have $N > 0$ with $n \vee q > 0$ if $b \geq 1$ and $n > 0$ with $N \vee q > 0$ if $b \leq 1$. It follows that $\theta_1^{\Lambda} = \nu > 0$ *so that* $S^{\Lambda} = (F^{\Lambda},G)$ *is always a Type 1 system*. On the other hand, $S^{V} = (F^{V},G)$ *is of the same Type* (1 *or* 0) *as* S_{∞} *if* $b \geq 1$ *and of the same Type as* $S_{-\infty}$ *if* $b \leq 1$.

The various possible combinations of S^{Λ} and S^{V}, classified according to the Type of S^{V}, the sign of $b-1$ and the signs of m and $m - \psi_O'$ are illustrated (apart from certain borderline cases) in Figs. 3–4. Fig. 3 has Type 1 cases, Fig. 4 has Type 0. In each Figure, the first three diagrams relate to $b > 1$, the last three to $b < 1$, taking in turn the cases $m < 0$, $0 < m < \psi_O'$ and $m > \psi_O'$ (except that there is no Fig. 4(vi)). Cases with $b = 1$ may be assimilated to $b < 1$, Type 1. Cases with

$m = 0$ or $m = \psi'_0$ will be dealt with as we go along All diagrams are drawn with $q > 0$, even where $q \leq 0$ is consistent with our text.[9] These reservations apart, the diagrams exhaust the possibilities. Some remarks about the phase pictures follow.

The phase picture of S^Λ is similar to that of Type 1 systems $\bar{S}$ discussed above, making allowance for the break at $\{h = 0\}$ There is a saddle at $(1,\nu)$, an unstable node at a point $(h^{\Lambda+}, 0)$, with $h^{\Lambda+} > (h^+_\infty \vee h^+_{-\infty})$, and a stable node at a point $(h^{\Lambda-}, 0)$ with $h^{\Lambda-} \leq (h^-_\infty \wedge h^-_{-\infty})$. At the saddle there is a stable manifold $\mathscr{M}^{\Lambda^\triangleright}$ represented by a continuous function $h = f^\Lambda(\theta)$ defined for $\theta \in (0,\infty)$, satisfying $1 = f^\Lambda(\nu)$, decreasing as long as $f^\Lambda(\theta) > 0$ and with a left limit $h^{\Lambda+} = f^\Lambda(0)$. If $m \geq 0$, then $f^\Lambda > 0$ on the whole axis $\theta > 0$, but if $m < 0$ then $f^\Lambda = 0$ at some finite θ^Λ_+ and thereafter remains negative and behaves in one of the ways described earlier as $\theta \uparrow \infty$. Again, there is an unstable manifold $\mathscr{M}^{\Lambda^\triangleleft}$ represented by a continuous function $h = g^\Lambda(\theta)$ defined for $\theta \in (0,\infty)$, satisfying $1 = g^\Lambda(\nu)$, increasing when $g^\Lambda(\theta) > 0$ with $g^\Lambda(\infty) = \infty$ and with a left limit $g^\Lambda(0) = h^{\Lambda-}$. If $m > 0$, then $h^{\Lambda-} = h^-_\infty < 0$ and g^Λ is negative on some initial interval $(0,\theta^\Lambda_-)$ and positive on $(\theta^\Lambda_-,\infty)$; but if $m \leq 0$ then $h^{\Lambda-} \geq 0$ and g^Λ is positive on the whole axis $\theta > 0$. *Thus in each case either* f^Λ *or* g^Λ *is positive on the whole axis,* (but both are positive only if $m = 0$).

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[9] The main effect on Figs. 3–4 of setting $q > 0$ is as follows. Since $\bar{\theta}_{1/b} = q$ for each of S_∞, $S_{-\infty}$ and S^V, it follows from Prop. 4 that, for each of these systems, $\bar{\theta}_{1/b} > 0$ implies $\bar{h}^- < 1/b$ if $b \geq 1$, $\bar{h}^+ > 1/b$ if $b \leq 1$. It then follows from the definition (3.40) of F^V that $h^{V-} = h^-_{-\infty} < 1/b$ if $b \geq 1$, $h^{V+} = h^+_\infty > 1/b$ if $b \leq 1$. In several cases, the condition $q > 0$ follows from the Standing Assumptions and so does not impose additional restrictions on the diagrams Thus, in Figs. 4, either S_∞ or $S_{-\infty}$ is of Type 0, so $q > 0$ is assumed. In Figs. 3, both systems are of Type 1, i.e. $n > 0$ and $N > 0$. If $b = 1$, then $q = n = N > 0$. If $b > 1$, choose the parameters for S_∞ given by (3.7), and conclude from (3.6a) that $q \leq 0$ is inadmissible if $m \geq -\frac{1}{2}b\sigma^2$, as in Figs. 3(ii)–(iii). If $b < 1$, choose the parameters for $S_{-\infty}$ given by (3.8), and conclude from (3.6a) that $q \leq 0$ is inadmissible if $m \leq \psi'_0 - \frac{1}{2}b\sigma^2$. In other cases it can apparently happen that $q \leq 0$ and either $h^{V-} = h^-_\infty \geq 1/b$ if $b > 1$, or $h^{V+} = h^+_{-\infty} \leq 1/b$ if $b < 1$.

Note also the bounds which S^{Λ} defines for the motion S (and for every $\bar{S}$) Since $F > F^{\Lambda}$, *the motion is always upward for* $z \uparrow$ *if* $F^{\Lambda} \geq 0$, *in particular* $F(h,\theta) > 0$ *for all* $\theta > 0$ *when* $h \geq h^{\Lambda+}$. Further, *the stable curve* $\{h = f^{\Lambda}(\theta)\}$ *forms a barrier to downward motion for* S, *i.e it can be crossed only from below as* $z \uparrow$ *or from above as* $z \downarrow$, (where 'below' and 'above' refer to the half–spaces defined by the curve). In the same way, *the unstable curve* $\{h = g^{\Lambda}(\theta)\}$ *can be crossed only from below as* $z \uparrow$, *from above as* $z \downarrow$

Consider now $S^V = (F^V,G)$, starting with $b > 1, N > 0$ and S_∞ of Type 1, so that $n > 0$. The phase picture above the line $\{h = 1/b\}$ is obviously the same as for S_∞; thus there is a saddle at $(1,n)$, with stable and unstable curves f^V and g^V which coincide with f_∞ and g_∞ as long as they lie above $\{h = 1/b\}$ Similarly, the unstable node is defined by $h^{V+} = h^+_\infty$, but for the stable node we have $h^{V-} = h^-_\infty$ only in case $h^-_\infty \geq 1/b$ (i.e. in case $q \leq 0$), otherwise $h^{V-} = h^-_{-\infty} < 1/b$ (see fn. 9).

The curve f^V is defined for $\theta > 0$, it satisfies $f^V(n) = 1$ and (as a limit) $f^V(0) = h^{V+}$, and it is decreasing as long as it is positive The behaviour of f^V below the line $\{h = 1/b\}$ is determined by $F_{-\infty}$; thus if $m - \psi'_0 \geq 0$, then $f^V > 0$ on the whole axis, but if $m - \psi'_0 < 0$ the curve becomes negative at some θ^V_+, etc Again, g^V is defined for $\theta > 0$, it satisfies $g^V(n) = 1$ and (as a limit) $g^V(0) = h^{V-}$, it is increasing as long as it is positive, and $g^V(\theta) > g^{\Lambda}(\theta)$ for each θ. If $m - \psi'_0 > 0$, it is found that $h^{V-} < 0$ so that $g^V(\theta) < 0$ on some interval $(0,\theta^V_-)$ and positive thereafter; but if $m - \psi'_0 \leq 0$, then $g^V > 0$ for all $\theta > 0$ *Thus in each case either* f^V *or* g^V *is positive on the whole axis* (but both are positive only if $m - \psi'_0 = 0$). Since $F < F^V$, *the motion* S *is always downward for* $z \uparrow$ *if* $F^V \leq 0$ *The stable curve* $\{h = f^V(\theta)\}$ *can be crossed only from above as* $z \uparrow$, *and the unstable curve* $\{h = g^V(\theta)\}$ *can be crossed only from below as* $z \downarrow$.

Taking the phase pictures for S^{Λ} and S^V together (Type 1, $b > 1$), it is seen that $f^{\Lambda}(\theta) > f^V(\theta)$ for all $\theta \in [0,\infty)$. (This follows, for example, because the saddle point of S^V is in the lower half–space defined by f^Λ, and $F^\Lambda < F^V$ implies that f^V cannot cross f^Λ from below) Consequently $\theta_+^V \leq \theta_+^\Lambda$, with a strict inequality if either of these numbers is finite, i.e. if $m < \psi_0'$. The curves f^V and f^Λ form the lower and upper boundaries of a 'tube'

(3.41) $$\mathscr{C}^{\triangleright} = \{(h,\theta): f^V(\theta) < h < f^\Lambda(\theta),\ \theta > 0\},$$

an open plane set from which paths of S *can exit as* $z \uparrow$ *but not enter.* Similarly, $g^\Lambda(\theta) < g^V(\theta)$ for $\theta \in [0,\infty)$, so that $\theta_-^V \leq \theta_-^\Lambda$, with a strict inequality if either of these numbers is positive, i.e. if $m > 0$. The curves g^Λ and g^V define a 'tube'

(3.42) $$\mathscr{C}^{\triangleleft} = \{(h,\theta): g^\Lambda(\theta) < h < g^V(\theta),\ \theta > 0\},$$

an open plane set from which paths of S *can exit as* $z \downarrow$ *but not enter.* (Moreover, any path of S which reaches the relative boundary of $\mathscr{C}^{\triangleright}$ or $\mathscr{C}^{\triangleleft}$ crosses immediately).

Systems with $b > 1$ and S_∞ of Type 0 are simpler and we shall be brief. Now $N > 0 \geq n$ and $q > 0$. The phase picture for S^V above $\{h = 1/b\}$ is the same as for S_∞, with a saddle at $(h^{V+},0)$ and $h^{V+} = h_\infty^+ \in (1/b,1]$. The (one-sided) stable curve f^V starts at $f^V(0) = h^{V+}$ and is decreasing, coinciding with f_∞ as long as the latter lies above $\{h = 1/b\}$, and its behaviour thereafter is the same as with Type 1. The curves f^V and f^Λ form a tube $\mathscr{C}^{\triangleright}$ with the properties mentioned for Type 1. However, *the unstable curve now lies on the vertical axis and so fails to define a useful bound for the motion* S^V; this is the main reason why we shall introduce alternative bounds below. Since $q > 0$, the unstable node for S^V lies below $\{h = 1/b\}$, and so $h^{V-} = h_{-\infty}^-$ (see fn. 9).

The discussion of cases with $b \leq 1$ is largely symmetrical with that for $b > 1$. No more need be said about S^Λ. If $N > 0$, S^V is of Type 1. This time the saddle is at $(1,N)$, the stable node is at $(h^{V-},0)$ and both points satisfy $h \leq 1/b$ (even $h < 1/b$ if $b < 1$), so the interesting part of the phase map is the same as for $S_{-\infty}$ and we have $h^{V-} = h_{-\infty}^- \leq 1/b$. If $q > 0$, the unstable node is defined by $h^{V+} = h_\infty^+ > 1/b$ (see fn. 9), otherwise $h^{V+} = h_{-\infty}^+ \leq 1/b$. The remarks about the curves f^V and g^V, including the definitions of 'tubes', continue to apply.

If $b < 1$ and $q > 0 \geq N$, S^V is of Type 0. The saddle is at $(h^{V-},0)$ with $h^{V-} = h^-_{-\infty} \in [1,1/b)$, and so is again in the region where the phase map is the same as for $S_{-\infty}$. Since $q > 0$ in this case, the unstable node is defined by $h^{V+} = h^+_{\infty} > 1/b$. The unstable curve g^V starts at $g^V(0) = h^{V-}$ and is increasing, coinciding with $g_{-\infty}$ while the latter lies below $\{h = 1/b\}$ and becoming unbounded thereafter. Also $R^V = R_{-\infty}$, which for $b < 1$ and $S_{-\infty}$ of Type 0 must be negative, so that only cases with $m - \psi_0' < 0$ are admissible here. We have $g^\Lambda < g^V$ and these curves form a tube $\mathscr{C}^\triangleleft$ as before. *This time it is the stable curve which lies on the vertical axis.*

Sharper bounds for F *can be defined at far left or far right values of* z. Given any $\delta > 0$, $\delta < \psi_0'$, we can choose z^δ so far left that

(3.43) $$0 < \psi_0' - A(z) < \psi_0' - M(z) < \delta < \psi_0' \qquad \text{for } z \in (-\infty, z^\delta],$$

and then, for these values of z and for $h \in \Re$ and $\theta \geq 0$, we have

(3.44a) $$F > F^{\Lambda\delta} \doteq \begin{cases} F_{-\infty} - 2\delta/b\sigma^2 & \text{if } h \geq 0 \quad [\bar{Q} = Q+\psi_0', \quad \bar{m} = m-\psi_0'+\delta] \\ F_{-\infty} - 2\delta(1-bh)/b\sigma^2 & \text{if } h \leq 0 \quad [\bar{Q} = Q+\psi_0'-\delta, \ \bar{m} = m-\psi_0'+\delta] \end{cases}$$

(3.44b) $$\bar{\theta}_1 = N + \delta/b, \quad \bar{\theta}_{1/b} = q + \delta,$$

$$\bar{R} \doteq -F^{\Lambda\delta}(0,\theta) = 2(m-\psi_0' + \delta)/b\sigma^2 = R_{-\infty} + 2\delta/b\sigma^2.$$

Also,

(3.45a) $$F < F^{V\delta} \doteq \begin{cases} F_{-\infty}+2\delta(bh-1)/b\sigma^2 & \text{if } h \geq 1/b \quad [\bar{Q} = Q+\psi_0'-\delta, \ \bar{m} = m-\psi_0'+\delta] \\ F_{-\infty} & \text{if } h \leq 1/b \quad [\bar{Q} = Q+\psi_0', \quad \bar{m} = m-\psi_0'] \end{cases}$$

(3.45b) $$\bar{\theta}_1 = N-\delta(b-1)/b \text{ if } b > 1; \ \bar{\theta}_1 = N \text{ if } b \leq 1; \ \bar{\theta}_{1/b} = q;$$

$$\bar{R} \doteq -F^{V\delta}(0,\theta) = 2(m-\psi_0')/b\sigma^2 = R_{-\infty}.$$

The resulting systems $S^{\Lambda\delta} = (F^{\Lambda\delta},G)$ and $S^{V\delta} = (F^{V\delta},G)$ will be needed only for cases with $b > 1$, and then for small $\delta > 0$ the phase pictures look roughly like that for $S_{-\infty}$, with saddle points slightly to the right and left of the saddle of $S_{-\infty}$ at (1,N) and unstable curves $g^{\Lambda\delta}$ and $g^{V\delta}$ slightly below and above $g_{-\infty}$. More precisely, comparison among the systems S^{Λ}, $S^{\Lambda\delta}$, $S_{-\infty}$, $S^{V\delta}$ shows that

(3.46a) $\quad F^{\Lambda} < F^{\Lambda\delta} < F_{-\infty} \leq F^{V\delta}$

and that the values of $\bar{\theta}_1$ decrease along the sequence – explicitly,

(3.46b) $\quad \nu = N+\psi_0'/b > N+\delta/b > N > N-\delta(b-1)/b.$

Since all these numbers are positive for small δ, they define the positions of the saddle points of the systems along $\{h=1\}$. In particular, $S^{\Lambda\delta}$ *and* $S^{V\delta}$ *will be of Type 1 in all cases considered.* Next, $h^{-}_{-\infty} < 1 < h^{+}_{-\infty}$ because $N > 0$ (Prop.4), and in case $q > 0$ we also have $h^{-}_{-\infty} < 1/b < h^{+}_{-\infty}$; the corresponding inequalities also hold with $h^{\pm}_{-\infty}$ replaced by $h^{\Lambda\delta\pm}$ or by $h^{V\delta\pm}$. Corresponding to $g_{-\infty}(0) = h^{-}_{-\infty}$ we have $g^{\Lambda\delta}(0) = h^{\Lambda\delta-}$ and $g^{V\delta}(0) = h^{V\delta-}$, with $g^{\Lambda\delta}(0) < g_{-\infty}(0) \leq g^{V\delta}(0)$; also $g^{\Lambda}(0) < g^{\Lambda\delta}(0)$. Taking into account (46a–b), it follows easily that $g^{\Lambda} < g^{\Lambda\delta} < g_{-\infty} \leq g^{V\delta}$ for $\theta \in [0,\infty)$. Further, we know that if $m - \psi_0' < 0$ then $R_{-\infty} < 0$ and $g_{-\infty}$ is positive and increasing on $(0,\infty)$; and so for small δ both $R^{\Lambda\delta}$ and $R^{V\delta}$ are negative and both $g^{\Lambda\delta}$ and $g^{V\delta}$ are positive and increasing on $(0,\infty)$. On the other hand, if $m - \psi_0' \geq 0$, then $R_{-\infty} \geq 0$ and $g_{-\infty}$ is positive only on an interval (θ_{-},∞) and is increasing there; and so for small δ both $R^{\Lambda\delta}$ and $R^{V\delta}$ are ≥ 0, while $g^{\Lambda\delta}$ and $g^{V\delta}$ are positive only on intervals $(\theta^{\Lambda\delta}_{-},\infty)$ and $(\theta^{V\delta}_{-},\infty)$ resp. and are increasing on these intervals. (The weak inequalities allow for the case $m = \psi_0'$.) Clearly $\theta^{V\delta}_{-} \leq \theta_{-} \leq \theta^{\Lambda\delta}_{-}$, and $\theta^{V\delta}_{-} < \theta^{\Lambda\delta}_{-}$ if one of these numbers is positive, i.e. if $m - \psi_0' \geq 0$. In any case, $g^{\Lambda\delta}_{-}(\infty) = g^{V\delta}_{-}(\infty) = \infty$.

The stable curves $f^{\Lambda\delta}$ and $f^{V\delta}$ will not be of particular interest in cases with $b > 1$. However, referring to the discussion of S^{Λ} and S^{V}, we recall that f^{Λ} and f^{V} are both positive on the whole axis $(0,\infty)$ in case $m - \psi_0' \geq 0$, whereas for $m - \psi_0' < 0$ the functions are positive only on intervals $(0,\theta^{\Lambda}_{+})$ and $(0,\theta^{V}_{+})$ with $0 < \theta^{V}_{+} < \theta^{\Lambda}_{+} \leq \infty$. Thus, *in all cases with* $b \geq 1$, *one of the pairs of functions* f^{Λ}, f^{V} *and* $g^{\Lambda\delta}$, $g^{V\delta}$ *is always positive on the whole axis for small* δ. Henceforth it is assumed without special mention that (43) and other properties requiring a small δ are satisfied.

Since $f^\Lambda > f^V$ and $g^\Lambda < g^{\Lambda\delta} < g^{V\delta}$, it follows from

(3.46c) $\qquad \theta_1^\Lambda = \nu$ and $f^\Lambda(\nu) = g^\Lambda(\nu) = 1$

that

(3.47) $\qquad f^V(\nu) < f^\Lambda(\nu) < g^{\Lambda\delta}(\nu) < g^{V\delta}(\nu).$

We also know that, if $b > 1$ and $n > 0$, so that S^V and $S^{V\delta}$ are both of Type 1, then $f^V(0) = h^{V+}$ and $g^{V\delta}(0) = h^{V\delta-}$ lie on opposite sides of $\{h=1\}$, hence

(3.48) $\qquad f^\Lambda(0) > f^V(0) > 1 > g^{V\delta}(0) > g^{\Lambda\delta}(0) \quad$ if $n > 0$

On the other hand, if $q > 0 \geq n$, so that S^V is of Type 0, then $\theta^V_{1/b} = \theta^{V\delta}_{1/b} = q$ implies $f^V(0) = h^{V+} > 1/b$ and $g^{V\delta}(0) = h^{V\delta-} < 1/b$ (Prop. 4), hence

(3.49) $\qquad f^\Lambda(0) > f^V(0) > 1/b > g^{V\delta}(0) > g^{\Lambda\delta}(0) \quad$ if $q > 0 \geq n.$

Another point to note is that, since $F^{\Lambda\delta} < F < F^{V\delta}$ for $z \leq z^\delta$, the motion of S as $z\downarrow$ is then always downward when $F^{\Lambda\delta} \geq 0$ and upward when $F^{V\delta} \leq 0$. *Further, for* $z \leq z^\delta$ *the curve* $g^{\Lambda\delta}$ *can be crossed by a path of* S *only from above as* $z\downarrow$ *and* $g^{V\delta}$ *can be crossed only from below. Thus the curves define a 'tube'*

(3.50) $\qquad \mathscr{C}^{\triangleleft\delta} = \{(h,\theta): g^{\Lambda\delta}(\theta) < h < g^{V\delta}(\theta),\ \theta > 0\},$

(*or simply* $\mathscr{C}^\delta$) *from which paths of* S *can exit as* $z\downarrow, z \leq z^\delta$, *but not enter.* The preceding statements are illustrated in Figs. 3–4.

Now consider far right values of z. Given $\rho > 0$, $\rho < \psi_0'$, one can choose z^ρ so that

(3.51) $\qquad 0 < M(z) < A(z) < \rho < \psi_0' \qquad$ for $z \in [z^\rho,\infty)$,

and then, for these values of z and for $h \in \Re$ and $\theta \geq 0$, we have

$$(3.52a) \qquad F > F^{\Lambda\rho} \doteq \begin{cases} F_\infty - 2\rho h/\sigma^2 & \text{if } h \geq 0 \quad [\bar{Q} = Q+\rho,\ \bar{m} = m+\rho] \\ F_\infty & \text{if } h \leq 0 \quad [\bar{Q} = Q,\quad \bar{m} = m] \end{cases}$$

(3.52b) $\qquad \bar{\theta}_1 = n+\rho, \quad \bar{\theta}_{1/b} = q + \rho(b-1)/b,$

$\qquad \bar{R} \doteq -F^{\Lambda\rho}(0,\theta) = 2m/b\sigma^2 = R_\infty$

Also,

$$(3.53a) \qquad F < F^{V\rho} \doteq \begin{cases} F_\infty & \text{if } h \geq 1/b \quad [\bar{Q} = Q,\quad \bar{m} = m] \\ F_\infty + 2\rho(1-bh)/b\sigma^2 & \text{if } h \leq 1/b \quad [\bar{Q} = Q+\rho,\ \bar{m} = m-\rho] \end{cases}$$

(3.53b) $\bar\theta_1 = \mathrm{n}$ if $\mathrm{b} \geq 1$; $\bar\theta_1 = \mathrm{n} + \rho(\mathrm{b}-1)/\mathrm{b}$ if $\mathrm{b} \leq 1$; $\bar\theta_{1/\mathrm{b}} = \mathrm{q}$;

$$\bar{\mathrm{R}} \doteq -\mathrm{F}^{\mathrm{V}\rho}(0,\theta) = 2(\mathrm{m}-\rho)/\mathrm{b}\sigma^2 = \mathrm{R}_\infty - 2\rho/\mathrm{b}\sigma^2.$$

The resulting systems $\mathrm{S}^{\Lambda\varrho} = (\mathrm{F}^{\Lambda\varrho},\mathrm{G})$ and $\mathrm{S}^{\mathrm{V}\rho} = (\mathrm{F}^{\mathrm{V}\rho},\mathrm{G})$ will be needed only for cases with $\mathrm{b} \leq 1$, and then for small $\rho > 0$ the phase pictures look roughly like that for S_∞, with saddle points slightly to the right and left of the saddle of S_∞ at (1,n) and stable curves $\mathrm{f}^{\Lambda\varrho}$ and $\mathrm{f}^{\mathrm{V}\rho}$ slightly above and below f_{∞}. More precisely, comparison among the systems S^Λ, $\mathrm{S}^{\Lambda\varrho}$, S_∞, $\mathrm{S}^{\mathrm{V}\rho}$ shows that

(3.54a) $\mathrm{F}^\Lambda \leq \mathrm{F}^{\Lambda\varrho} \leq \mathrm{F}_\infty \leq \mathrm{F}^{\mathrm{V}\rho}$

and that the values of $\bar\theta_1$ decrease along the sequence — explicitly,

(3.54b) $\nu = \mathrm{n}+\psi_0' > \mathrm{n}+\rho > \mathrm{n} > \mathrm{n}+\rho(\mathrm{b}-1)/\mathrm{b}.$

Since all these numbers are positive for small ρ, they define the positions of the saddle points along $\{\mathrm{h}=1\}$. In particular, $\mathrm{S}^{\Lambda\varrho}$ *and* $\mathrm{S}^{\mathrm{V}\rho}$ *will be of Type 1 in all cases considered.* Next, $\mathrm{h}_\infty^- < 1 < \mathrm{h}_\infty^+$ because $\mathrm{n} > 0$ (Prop.4), and in case $\mathrm{q} > 0$ we also have $\mathrm{h}_\infty^- < 1/\mathrm{b} < \mathrm{h}_\infty^+$; the corresponding inequalities also hold with $\mathrm{h}_\infty^\pm$ replaced by $\mathrm{h}^{\Lambda\varrho\pm}$ or by $\mathrm{h}^{\mathrm{V}\rho\pm}$. Corresponding to $\mathrm{f}_\infty(0) = \mathrm{h}_\infty^+$ we have $\mathrm{f}^{\Lambda\varrho}(0) = \mathrm{h}^{\Lambda\varrho+}$ and $\mathrm{f}^{\mathrm{V}\rho}(0) = \mathrm{h}^{\mathrm{V}\rho+}$, with $\mathrm{f}^{\Lambda\varrho}(0) > \mathrm{f}_\infty(0) \geq \mathrm{f}^{\mathrm{V}\rho}(0)$; also $\mathrm{f}^\Lambda(0) > \mathrm{f}^{\Lambda\varrho}(0)$. Taking into account (54a–b), it follows that $\mathrm{f}^\Lambda > \mathrm{f}^{\Lambda\varrho} > \mathrm{f}_\infty \geq \mathrm{f}^{\mathrm{V}\rho}$ for $\theta \in [0,\infty)$. Further, we know that if $\mathrm{m} > 0$ then $\mathrm{R}_\infty > 0$ and f_∞ is positive and increasing on $(0,\infty)$; and so for small ρ both $\mathrm{R}^{\Lambda\varrho}$ and $\mathrm{R}^{\mathrm{V}\rho}$ are positive and both $\mathrm{f}^{\Lambda\varrho}$ and $\mathrm{f}^{\mathrm{V}\rho}$ are positive and decreasing on $(0,\infty)$. On the other hand, if $\mathrm{m} \leq 0$, then $\mathrm{R}_\infty \leq 0$ and f_∞ is positive only on an interval $(0,\theta_+)$ and is decreasing there; and so for small ρ both $\mathrm{R}^{\Lambda\varrho}$ and $\mathrm{R}^{\mathrm{V}\rho}$ are ≤ 0, while $\mathrm{f}^{\Lambda\varrho}$ and $\mathrm{f}^{\mathrm{V}\rho}$ are positive only on intervals $(0,\theta_+^{\Lambda\varrho})$ and $(\theta_+^{\mathrm{V}\rho},\infty)$ resp. and are decreasing on these intervals. (Here the weak inequalities allow for $\mathrm{m} = 0$). Clearly $\theta_+^{\mathrm{V}\rho} \leq \theta_+ \leq \theta_+^{\Lambda\varrho} \leq \infty$, and $\theta_+^{\mathrm{V}\rho} < \theta_+^{\Lambda\varrho}$ if one of these numbers is finite, i.e. if $\mathrm{m} \leq 0$. In any case, $\mathrm{f}^{\Lambda\varrho}(\theta_+^{\Lambda\varrho}) = \mathrm{f}^{\mathrm{V}\rho}(\theta_+^{\mathrm{V}\rho}) = 0$.

The unstable curves $\mathrm{g}^{\Lambda\varrho}$ and $\mathrm{g}^{\mathrm{V}\rho}$ will not be of particular interest in cases with $\mathrm{b} \leq 1$. However, we recall that g^Λ and g^V are both positive on the whole axis

$(0,\infty)$ in case $m \leq 0$, whereas for $m > 0$ the functions are positive only on intervals $(\theta_-^\Lambda,\infty)$ and (θ_-^V,∞) with $0 \leq \theta_-^V < \theta_-^\Lambda < \infty$. Thus, *in all cases with* $b \leq 1$, *one of the pairs of functions* $f^{\Lambda\varrho}$, $f^{V\rho}$ *and* g^Λ, g^V *is always positive on the whole axis for small* ρ Henceforth it is assumed without special mention that (51) and other properties requiring a small ρ are satisfied

Clearly (46c) remains in force for $b \leq 1$, and (47–9) remain valid if n, f^Λ, f^V, $g^{V\delta}$, $g^{\Lambda\delta}$ are replaced therein by N, $f^{\Lambda\varrho}$, $f^{V\rho}$, g^V, g^Λ. Also, since $F^{\Lambda\varrho} < F < F^{V\rho}$ for $z \geq z^\varrho$, the motion of S as $z\uparrow$ is then always upward when $F^{\Lambda\varrho} \geq 0$ and downward when $F^{V\rho} \leq 0$. *Further, for* $z \geq z^\varrho$ *the curve* $f^{\Lambda\varrho}$ *can be crossed by a path of* S *only from below as* $z\downarrow$, *and* $f^{V\rho}$ *only from above. Thus the curves define a 'tube'*

(3.55) $$\mathscr{C}^{\triangleright\rho} = \{(h,\theta): f^{\Lambda\varrho}(\theta) < h < f^{V\rho}(\theta),\ \theta > 0\}$$

(*or simply* $\mathscr{C}^{\rho}$) *from which paths of* S *can exit as* $z\uparrow$, $z \geq z^\varrho$, *but not enter.*

Collecting results from the preceding discussion, and referring to Figs. 3–4, we state

PROPOSITION 7 (Stable and unstable curves for five–parameter systems)

(i) Let $b > 1$, $N > 0$ and $n \vee q > 0$ and choose $\delta > 0$ as in (3.43) ff. The functions f^Λ, f^V and $g^{V\delta}$, $g^{\Lambda\delta}$ are defined and continuous for $\theta \in [0,\infty)$, (including finite limits at $\theta = 0$). They satisfy

(3.56) $$f^\Lambda(\theta) > f^V(\theta), \qquad g^{V\delta}(\theta) > g^{\Lambda\delta}(\theta) \qquad \text{for } \theta \in [0,\infty)$$

and the inequalities (3.47–49).

The functions f^Λ, f^V are positive and strictly decreasing on intervals $[0,\theta_+^\Lambda)$, $[0,\theta_+^V)$, and negative on intervals $(\theta_+^\Lambda,\infty)$, (θ_+^V,∞) if these intervals are not empty. Limits at $\theta = \infty$ are as in Prop. 6(i). We have

(3.57) $$0 < \theta_+^V \leq \theta_+^\Lambda \leq \infty \quad \text{and} \quad f^V(\theta_+^V) = f^\Lambda(\theta_+^\Lambda) = 0 \quad \text{in all cases;}$$

(3.57a) $$\theta_+^V = \infty \text{ iff } m \geq \psi_0'; \quad \theta_+^\Lambda = \infty \text{ iff } m \geq 0.$$

The functions $g^{V\delta}$, $g^{\Lambda\delta}$ are positive and strictly increasing on intervals $(\theta_-^{V\delta},\infty)$,

$(\theta_-^{\Lambda\delta},\infty)$, with $g^{V\delta}(\infty) = g^{\Lambda\delta}(\infty) = \infty$, and negative on $(0,\theta_-^{V\delta})$, $(0,\theta_-^{\Lambda\delta})$ if these intervals are not empty. We have

(3.58) $\quad 0 \le \theta_-^{V\delta} \le \theta_-^{\Lambda\delta} < \infty \quad$ and $\quad \theta_-^{V\delta}\cdot g^{V\delta}(\theta_-^{V\delta}) = \theta_-^{\Lambda\delta}\cdot g^{\Lambda\delta}(\theta_-^{\Lambda\delta}) = 0 \quad$ in all cases;

(3.59) $\quad \theta_-^{V\delta} = 0$ iff $m \le \psi_0'$; $\quad \theta_-^{\Lambda\delta} = 0$ iff $m < \psi_0'$;

$\quad g^{V\delta}(\theta_-^{V\delta}) = 0$ iff $m \ge \psi_0'$; $\quad g^{\Lambda\delta}(\theta_-^{\Lambda\delta}) = 0$ iff $m \ge \psi_0'$.

(ii) Let $b \le 1$, $n > 0$, $N \vee q > 0$ and choose $\rho > 0$ as in (3.51) ff. The assertions under (i) remain valid if

(3.60) $\quad N, n, f^\Lambda, f^V, g^{\Lambda\delta}, g^{V\delta}, \theta_+^\Lambda, \theta_+^V, \theta_-^{\Lambda\delta}, \theta_-^{V\delta}$

are replaced by

(3.61) $\quad n, N, f^{\Lambda\varrho}, f^{V\rho}, g^\Lambda, g^V, \theta_+^{\Lambda\varrho}, \theta_+^{V\rho}, \theta_-^\Lambda, \theta_-^V,$

including replacements in (3.47–49), with the following exceptions: In place of (57a) and (59) we have

(3.62) $\quad \theta_+^{V\rho} = \infty$ iff $m > 0$; $\quad \theta_+^{\Lambda\varrho} = \infty$ iff $m \ge 0$;

(3.63) $\quad \theta_-^V = 0$ iff $m \le \psi_0'$; $\quad \theta_-^\Lambda = 0$ iff $m \le 0$;

$\quad g^V(\theta_-^V) = 0$ iff $m \ge \psi_0'$; $\quad g^\Lambda(\theta_-^\Lambda) = 0$ iff $m \ge 0$.

If $q > 0 \ge n$, then only cases with $m - \psi_0' < 0$ are admissible.

(iii) Condition (3.46c) holds in all cases.

4 EXISTENCE PROOFS

By virtue of Prop. 5, Theorem 4A can be restated as follows:

THEOREM 4B (Saddle Connection).

In all cases consistent with the Standing Assumptions, the system $S = (F,G)$ defined by (0.1) has one and only one solution $(h^*,\theta^*) = (h^*(z), \theta^*(z); z \in \Re)$ which converges as $z \to \infty$ to the saddle point of S_∞ and as $z \to -\infty$ to the saddle point of $S_{-\infty}$.

We recall that the saddle point of S_∞ is at $(1,n)$ if $n > 0$ (in particular, if $b \leq 1$), and at $(h^+_\infty,0)$ if $b \geq 1$, $q > 0 \geq n$. The saddle point of $S_{-\infty}$ is at $(1,N)$ if $N > 0$ (in particular, if $b > 1$), and at $(h^-_{-\infty},0)$ if $b \leq 1$, $q > 0 \geq N$.

A solution (h,θ) of S with $\theta > 0$ which is defined on some interval $[z_\diamond,\infty)$ and converges as $z \to \infty$ to the saddle point of S_∞ will be called a *forward special solution (f.s.s.)*, similarly a solution defined on some $(-\infty,z_\diamond]$ and converging as $z \to -\infty$ to the saddle point of $S_{-\infty}$ is called a *backward special solution (b.s.s.)*. Thus a solution defined for all $z \in \Re$ is a star solution iff it is both a f.s.s. and a b.s.s., and Theorem 4 asserts the existence and uniqueness of such a solution. The proof rests on several Lemmas, some of which are also of independent interest. The details differ according to the values of parameters, and sometimes we shall spell out only selected cases. In particular, proofs for cases with $b \leq 1$ are similar to those for $b > 1$ but usually slightly simpler, so we shall concentrate on the latter.

In this Section, the following notation for subsets of $\{(h,\theta): \theta > 0\}$ will be used:

(4.1a) $\qquad \mathscr{U}^{\triangleright} = \{h > f^{\Lambda}(\theta)\}, \quad \mathscr{M}^{\Lambda\triangleright} = \{h = f^{\Lambda}(\theta)\};$

also, if $f^V(\theta)$ is defined – in particular, if $b > 1$ –

(4.1b) $\qquad \mathscr{B}^{\triangleright} = \{h < f^V(\theta)\}, \quad \mathscr{M}^{V\triangleright} = \{h = f^V(\theta)\}, \quad \mathscr{C}^{\triangleright} = \{f^V(\theta) < h < f^{\Lambda}(\theta)\},$

cf. (3.24) and (3.41). For brevity we write $\mathscr{U}^{\triangleright}$, $\mathscr{B}^{\triangleright}$ rather than $\mathscr{U}^{\Lambda\triangleright}$, $\mathscr{B}^{V\triangleright}$. If f^{Λ} and f^V are replaced by $f^{\Lambda\varrho}$ and $f^{V\rho}$ – see (3.51) ff – we denote the corresponding sets by $\mathscr{U}^{\varrho}$, $\mathscr{M}^{\Lambda\varrho}$, $\mathscr{B}^{\varrho}$, $\mathscr{M}^{V\rho}$, $\mathscr{C}^{\varrho}$, (omitting the superscript $\triangleright$). Again, we write

(4.2a) $\qquad \mathscr{U}^{\triangleleft} = \{h < g^{\Lambda}(\theta)\}, \quad \mathscr{M}^{\Lambda\triangleleft} = \{h = g^{\Lambda}(\theta)\};$

also, if g^V is defined – in particular, if $b \leq 1$ –

(4.2b) $\mathscr{B}^{\triangleleft} = \{h > g^V(\theta)\}$, $\mathscr{M}^{V\triangleleft} = \{h = g^V(\theta)\}$, $\mathscr{C}^{\triangleleft} = \{g^{\Lambda}(\theta) < h < g^V(\theta)\}$,

cf. (3.25) and (3.42) If g^{Λ} and g^V are replaced by $g^{\Lambda\delta}$ and $g^{V\delta}$ – see (3.43) ff – we write $\mathscr{U}^{\delta}$, $\mathscr{M}^{\Lambda\delta}$, $\mathscr{B}^{\delta}$, $\mathscr{M}^{V\delta}$, $\mathscr{C}^{\delta}$. The conventions stated in fn. 7 apply.

PROPOSITION 8 (Bounds for Solutions).

Let (h,θ) be a solution of S defined and finite for $z \in [z_a,z_b]$, $-\infty \leq z_a < z_b \leq \infty$, (values at $\pm\infty$, if relevant, being defined as limits).

(i) If $h(z_a) > 0$ and $h(z_b) > 0$, then $h(z) \geq 0$ on $[z_a,z_b]$.

(ii) If $\theta(z_a) < \nu$ and $\theta(z_b) < \nu$, then $\theta(z) \leq \nu$ on $[z_a,z_b]$.

PROOF (i) If the assertion were false, there would be $z_a < z_\alpha < z_\beta < z_b$ such that $h(z)$ passes from positive to non–positive values at z_α as $z \uparrow$ and from negative to non–negative values at z_β. This implies

$$2[M(z_\alpha)-m]/b\sigma^2 = F[0,\theta(z_\alpha),z_\alpha] \leq 0 \leq F[0,\theta(z_\beta),z_\beta] = 2[M(z_\beta)-m]/b\sigma^2$$

hence $M(z_\alpha) \leq M(z_\beta)$, contrary to the fact that $M(z)$ is strictly decreasing. (It is not difficult to sharpen the assertion to '$h(z) > 0$ on $[z_a,z_b]$', but we omit this.) ||

(ii) If the assertion were false, there would be $z_a < z_\alpha < z_\beta < z_b$ such that $\theta(z)$ crosses the line $\{\theta = \nu\}$ from left to right at $z = z_\alpha$ and from right to left at $z = z_\beta$. This implies $\theta'(z_\alpha) = h(z_\alpha)-1 \geq 0$, $\theta'(z_\beta) = h(z_\beta)-1 < 0$, (taking into account that $F[1,\nu,z] > F^{\Lambda}[1,\nu] = 0$, so that the case $h(z_\beta)-1 = 0$ is ruled out). But then $h(z_\alpha) \geq 1 = f^{\Lambda}(\nu) = f^{\Lambda}(\theta(z_\alpha)) = f^{\Lambda}(\theta(z_\beta)) > h(z_\beta)$, and since no passage is possible from $\{h \geq f^{\Lambda}(\theta)\}$ to $\{h < f^{\Lambda}(\theta)\}$ as $z\uparrow$ we have a contradiction. ||

COROLLARY 8. (i) Let (h^*,θ^*) be a forward (backward) special solution. Then

$h^*(z) > 0$ and $\theta^*(z) < \nu$ eventually as $z \uparrow$ $(z\downarrow)$.

If $h^*(z_\diamond) > 0$ for some $z_\diamond$, then $h^*(z) \geq 0$ for all $z > z_\diamond$ $(z < z_\diamond)$

If $\theta^*(z_\diamond) < \nu$ for some $z_\diamond$, then $\theta^*(z) \leq \nu$ for all $z > z_\diamond$ $(z < z_\diamond)$.

(ii) Let (h^*,θ^*) be a star solution. Then

$h^*(z) > 0$ and $\theta^*(z) < \nu$ for all $z \in \Re$

PROOF (i) A f.s.s must satisfy $h^*(\infty) > 0$ and $\theta^*(\infty) < \nu$; similarly for a b.s.s. with ∞ replaced by $-\infty$. The assertions then follow from Prop. 8 (ii) This follows because a star solution is both a f.s.s. and a b.s.s. ||

PROPOSITION 9 (Ordering Lemma).

Let $\Pi^i_\diamond = (h^i_\diamond, \theta^i_\diamond, z_\diamond)$, $i = 0,1$, be points with $\theta^i_\diamond > 0$, (not necessarily distinct, but with the same $z_\diamond$). Let $\pi^i = \pi^i(z) = \pi\, z; \Pi^i_\diamond)$ with components $h^i(z)$, $\theta^i(z)$ denote the solution of $S = (F,G)$ through $\Pi^i_\diamond$. Similarly, let π^{iv}, $\pi^{i\Lambda}$ denote the corresponding solutions of $S^V = (F^V,G)$ and $S^\Lambda = (F^\Lambda,G)$

(α) If $h^1_\diamond \leq h^0_\diamond$ and $0 < \theta^1_\diamond \leq \theta^0_\diamond$, then, on any interval of the form $z_\diamond < z < \bar{z} < \infty$,

(i) $h^1(z) < h^0(z)$ and $\theta^1(z) < \theta^0(z)$,

(ii) $h^1(z) < h^{0v}(z)$ and $\theta^1(z) < \theta^{0v}(z)$,

(iii) $h^0(z) > h^{1\Lambda}(z)$ and $\theta^0(z) > \theta^{1\Lambda}(z)$,

provided that the following hold: in each line both solutions exist on $[z_\diamond,\bar{z}]$; for at least one of the solutions, the h–coordinate remains positive on this interval in case (i), non–negative in cases (ii) and (iii); and, in case (i), the points $\Pi^i_\diamond$ are distinct

If ρ and z^ρ are chosen as in (3.51) ff. and $z_\diamond \geq z^\rho$, then $\pi^{1\Lambda}$, π^{0v} may be replaced throughout by $\pi^{1\Lambda\rho}$, $\pi^{0v\rho}$, defined as the solutions of $S^{\Lambda\rho}$, $S^{V\rho}$ respectively.

(β) If $h^1_\diamond \leq h^0_\diamond$ and $0 < \theta^0_\diamond \leq \theta^1_\diamond$, then, on any interval of the form $z_\diamond > z > \bar{z} > -\infty$,

(i) $h^1(z) < h^0(z)$ and $\theta^1(z) > \theta^0(z)$,

(ii) $h^0(z) > h^{1v}(z)$ and $\theta^0(z) < \theta^{1v}(z)$,

(iii) $h^1(z) < h^{0\Lambda}(z)$ and $\theta^1(z) > \theta^{0\Lambda}(z)$,

provided that the following hold: in each line both solutions exist on $[\bar{z},z_\diamond]$; for at least one of the solutions, the h–co-ordinate remains positive on this interval in case (i), non–negative in cases (ii) and (iii); and, in case (i), the points $\Pi^i_\diamond$ are distinct.

If δ and z^δ are chosen as in (3.43) ff. and $z_\diamond \leq z^\delta$, then $\pi^{1\Lambda}$, π^{0v} may be replaced throughout by $\pi^{1\Lambda\delta}$, $\pi^{0v\delta}$, defined as the solutions of $S^{\Lambda\delta}$, $S^{V\delta}$ respectively.

PROOF (α) (i). Suppose first that $h^0_\diamond > h^1_\diamond > 0$ and $\theta^0_\diamond > \theta^1_\diamond > 0$. Then $h^0(z) > h^1(z)$ for z in a right neighbourhood of $z_\diamond$, and since $(d/dz)(\ln \theta^0 - \ln \theta^1) = h^0 - h^1$ it follows that $\theta^0(z) > \theta^1(z)$ in this neighbourhood; moreover, the latter inequality persists as long as $h^0(z) > h^1(z)$ for increasing z. If some $\tilde{z} \in (z_\diamond, \bar{z})$ were reached with $h = h^0(\tilde{z}) = h^1(\tilde{z}) > 0$, then (for variables evaluated at $\tilde{z}$) we should have $(d/dz)(h^0 - h^1) = F(h,\theta^0,\tilde{z}) - F(h,\theta^1,\tilde{z}) = (2/\sigma^2)h(\theta^0 - \theta^1) > 0$, so that in fact the inequality $h^0 > h^1$ would persist, contrary to assumption. Now suppose that $h^0_\diamond = h^1_\diamond > 0$ and $\theta^0_\diamond > \theta^1_\diamond > 0$. Then $(d/dz)(h^0 - h^1) > 0$ for z to the right of $z_\diamond$ and the rest of the argument proceeds as before. ||

(α) (ii) Suppose initially that $h^0_\diamond > h^1_\diamond > 0$ and $\theta^0_\diamond > \theta^1_\diamond > 0$. Then it follows as before that $h^{0v}(z) > h^1(z)$ for z to the right of $z_\diamond$, and as long as this inequality persists it follows that $\theta^{0v}(z) > \theta^1(z)$ also. If some $\tilde{z} \in (z_\diamond, \bar{z})$ were reached with $h = h^{0v}(\tilde{z}) = h^1(\tilde{z}) > 0$, then we should have, at $\tilde{z}$,

$$(4.3) \qquad (d/dz)(h^{0v} - h^1) = F^v(h,\theta^{0v},\tilde{z}) - F(h,\theta^1,\tilde{z}) > F(h,\theta^{0v},\tilde{z}) - F(h,\theta^1,\tilde{z}) > 0,$$

because $F^V > F$ and $\partial F/\partial\theta = (2/\sigma^2)h > 0$, so that once again the inequality $h^0 > h^1$ would persist. This last assertion remains true even if $h = h^{0v}(\tilde{z}) = h^1(\tilde{z}) = 0$, since only the second strict inequality in (3) need be replaced by a weak one. Supposing now that $h^0_\diamond = h^1_\diamond \geq 0$ and $\theta^0_\diamond > \theta^1_\diamond > 0$, the preceding argument with $z = \tilde{z}$ replaced by $z = z_\diamond$ shows that $h^0_\diamond(z) > h^1_\diamond(z)$ in a right neighbourhood of $z_\diamond$, and the rest of the argument proceeds as before. ||

The proofs of the remaining assertions under (α), and those under (β), are analogous.

REMARKS: (1) The requirement that one of the h–coordinates in each line remain positive (or non–negative in cases (ii) and (iii)) seems to be essential.

(2) Part (α) of the Lemma is essentially a version of a theorem of Kamke [1932] T 6 or [1943] A 23.2 on what are now called 'co-operative' systems, see Hirsch [1984], Smith [1988] and [1995] for surveys. Briefly, the system $S = (F,G)$ is 'co–operative' at z if the off–diagonal elements of the Jacobian matrix are positive, which here means that $\partial F/\partial\theta = (2/\sigma^2)h > 0$ and $\partial G/\partial h = \theta > 0$. It is however more

efficient to give a direct proof for the present model.

In general, the inequalities (α) (i–iii) in Prop. 9 cannot be extended to limits as $z \to \infty$ even if both solutions in question are defined on $[z_\diamond,\infty)$, nor can (β) (i–iii) be extended to limits as $z \to -\infty$. For example, Prop 2(iii) shows that the stable node of S_∞ attracts solutions of S starting (for large enough $z_\diamond$) in a whole neighbourhood of points $\pi_\diamond$. Nevertheless, such an extension is sometimes possible in the case of special solutions. Rather than set out the relevant comparisons between *limits* at this stage, it is convenient to state some of the results in terms of comparisons between *starts* of solutions converging to saddle points of asymptotic systems:

PROPOSITION 10 (Uniqueness Lemma for Special Starts).

Let $\Pi^i_\diamond = (h^i_\diamond, \theta^i_\diamond, z_\diamond)$, $i = 0,1$, be points with $\theta^i_\diamond > 0$, and either $h^0_\diamond \geq 0$ or $h^1_\diamond \geq 0$. As in Prop 9, let $\pi^i(z)$ and $\pi^{iv}(z)$ denote the solutions of $S = (F,G)$ and $S^V = (F^V,G)$ through $\Pi^i_\diamond$.

(α) (i) If the $\Pi^i_\diamond$ are distinct, and if $\pi^0(z)$ and $\pi^1(z)$ are defined on $[z_\diamond,\infty)$ and both converge to the saddle point of S_∞, or

(ii) if $b > 1$ and if $\pi^1(z)$ and $\pi^{0v}(z)$ are defined on $[z_\diamond,\infty)$ and both converge to the saddle point of S_∞ (which for $b > 1$ is also the saddle point of S^V), then

$$(\theta^0_\diamond - \theta^1_\diamond)(h^0_\diamond - h^1_\diamond) < 0.$$

(β) (i) If the $\Pi^i_\diamond$ are distinct, and if both $\pi^0(z)$ and $\pi^1(z)$ are defined on $(-\infty,z_\diamond]$ and both converge to the saddle point of $S_{-\infty}$, or

(ii) if $b \leq 1$ and both $\pi^1(z)$ and $\pi^{0v}(z)$ are defined on $(-\infty,z_\diamond]$ and both converge to the saddle point of $S_{-\infty}$ (which for $b \leq 1$ is also the saddle point of S^V), then

$$(\theta^0_\diamond - \theta^1_\diamond)(h^0_\diamond - h^1_\diamond) > 0.$$

PROOF. (α) (i) Suppose, contrary to hypothesis, that $h^1_\diamond \leq h^0_\diamond$, $0 < \theta^1_\diamond \leq \theta^0_\diamond$ and $h^0_\diamond \geq 0$. By Prop. 8(i), we have $h^0(z) > 0$ on $(z_\diamond,\infty)$, and then, according to Prop 9(i), the inequalities $h^1(z) < h^0(z)$, $\theta^1(z) < \theta^0(z)$ hold on $(z_\diamond,\infty)$. Since $h^1(z)$ and $h^0(z)$ go to the same positive limit, both are eventually positive for large z, and we may as well assume from the outset that $0 < h^1(z) < h^0(z)$ and $0 < \theta^1(z) < \theta^0(z)$ on $[z_\diamond,\infty)$.

Suppose first that both solutions converge to $(1,n)$ with $n > 0$. Then $\theta^0(z)/\theta^1(z) \to 1$, $ln[\theta^0(z)/\theta^1(z)] \to 0$. On the other hand, $\theta^0(z_\diamond)/\theta^1(z_\diamond) > 1$, and since $d\,ln\,\theta/dz = h-1$ it follows from $h^0(z) > h^1(z)$ that $ln\,\theta^0(z) - ln\,\theta^1(z)$ increases on $[z_\diamond,\infty)$, leading to contradiction.

Now suppose that $q > 0 \geq n$ and that both solutions converge to $(h^+_\infty,0)$. For arbitrary $z > z_\diamond$, an application of the mean value theorem to the difference $F^0(z)-F^1(z)$, where $F^i(z) = F(h^i(z),\theta^i(z),z)$, gives, in abridged notation,

(4.4a) $$(d/dz)(h^0 - h^1) = F^0 - F^1 = (h^0 - h^1)F^\eta_h + (\theta^0 - \theta^1)F^\eta_\vartheta\,;$$

here $F_h = \partial F/\partial h$, $F_\vartheta = \partial F/\partial \theta$, and the superscript η indicates that the derivatives are evaluated at some point

(4.4b) $$(h^\eta,\theta^\eta) = ((1-\eta)h^1+\eta h^0),\ (1-\eta)\theta^1+\eta\theta^0)),\quad 0 < \eta = \eta(z) < 1,\ h^i=h^i(z),\ \theta^i=\theta^i(z)$$

Now $F^\eta_\vartheta = (2/\sigma^2)h^\eta > 0$, and

$$F^\eta_h = 2bh^\eta + (2/\sigma^2)[\theta^\eta - Q - A(z)] \to (2/\sigma^2)[b\sigma^2 h^+_\infty - Q] = (2/\sigma^2)[Q^2 + 2m\sigma^2]^{\frac{1}{2}} > 0$$

as $z \to \infty$ by the definition of h^+_∞, see (3.7), (3.10) and fn.6. On dividing (4a) by $h^0(z)-h^1(z)$ and recalling that $\theta^0 - \theta^1 > 0$ and $h^0 - h^1 > 0$ for $z_\diamond < z < \infty$, we obtain

$$(d/dz)ln(h^0 - h^1) > F^\eta_h \to (2/\sigma^2)[b\sigma^2 h^+_\infty - Q] > 0, \quad z \to \infty.$$

Thus h^0-h^1 is positive and increasing for large z, contrary to the assumption that $h^0-h^1 \to 0$.|| The proof of (β) (i) is analogous.

(α) (ii) We proceed as before, assuming the assertion to be false and replacing π^0 by π^{0v}, but without the assumption that the starts are distinct. The argument up to and including the proof that both solutions cannot converge to $(1,n)$ then proceeds as before. Now suppose that $q > 0 \geq n$ and that both solutions converge to $(h^+_\infty,0)$. For $b > 1$, this implies that both h^1 by h^{1v} are $> 1/b$ for large z. Recall that in the region $\{h \geq 1/b\}$ we have $F^V = F_\infty$, see (3.40a), and that

$$F(h,\theta,z) = F_\infty(h,\theta) + (2/\sigma^2)[M/b - hA], \quad \text{see (0.1) and (3.1)}$$

Writing $F^1_\infty(z) = F_\infty(h(z),\theta(z),z)$, similarly $F^{0v}_\infty(z)$, the first equation in (4a) is replaced by

$$(d/dz)(h^{0v}-h^1) = F_\infty^{0v} - F_\infty^1 - (2/\sigma^2)[M/b - h^1A].$$

The term $-(2/\sigma^2)[M/b - h^1A]$ is positive for large z since $A > M$ and $h^1 > 1/b$, and so can be dropped in the rest of the proof, which proceeds as before with F_∞ in place of F. || A similar argument works for β (ii), allowing for changes of direction and sign.

COROLLARY 10.1. For given $z_\diamond$ and $\theta_\diamond > 0$, there is at most one $h_\diamond \geq 0$ defining a start of a f.s.s., and if there is one such $h_\diamond \geq 0$ then it is the only one of either sign. Again, there is for given $z_\diamond$ and $h_\diamond \geq 0$ at most one $\theta_\diamond > 0$ defining a start of a f.s.s. Similarly for b.s.s.

These results follow immediately from Prop. 10(α)(i) and (β)(i). They will guarantee the uniqueness of star solutions.

COROLLARY 10.2. (α)(i) For $b > 1$, a path of S which passes through a point of $[\![\mathscr{B}^{\triangleright}]\!]$ cannot converge to the saddle point of S_∞ as $z \to \infty$.

(β)(i) For $b > 1$ and δ, z^δ chosen as in (3.43) ff, a path of S which passes through a point of $[\![\mathscr{B}^\delta]\!]$ at some $z_\diamond \leq z^\delta$ cannot converge to the saddle point of $S_{-\infty}$ as $z \to -\infty$.

(α)(ii) For $b \leq 1$ and ρ, z^ρ chosen as in (3.51) ff, a path of S which passes through a point of $[\![\mathscr{B}^\rho]\!]$ at some $z_\diamond \geq z^\rho$ cannot converge to the saddle point of S_∞ as $z \to \infty$.

(β)(ii) For $b \leq 1$, a path of S which passes through a point of $[\![\mathscr{B}^{\triangleleft}]\!]$ cannot converge to the saddle point of $S_{-\infty}$ as $z \to -\infty$.

PROOF. (α)(i) Let $\pi^1(z)$ again be a solution of S which at some $z_\diamond$ passes through a point $(h_\diamond^1, \theta_\diamond^1) \in [\![\mathscr{B}^{\triangleright}]\!] = \mathscr{B}^{\triangleright} \cup \mathscr{M}^{V\triangleright}$; then $h_\diamond^1 \leq f^V(\theta_\diamond^1)$. Setting $h_\diamond^{0v} = f^V(\theta_\diamond^1)$ and $\theta_\diamond^{0v} = \theta_\diamond^1$ defines the start of a solution π^{0v} of S^V which *does* converge to the saddle point of S_∞ as $z \to \infty$ and whose path is part or all of $\mathscr{M}^{V\triangleright}$. Since $F < F^V$ and $\mathscr{M}^{V\triangleright}$ is an invariant set for S^V, the inequality $h^1(z) \leq f^V[\theta^1(z)]$ persists for $z \in (z_\diamond, \infty)$. We may further assume that $h^1(z) > 0$ for large z (since otherwise π^1 cannot go to the saddle point of S_∞ anyway), and so assume wlog that $h_\diamond^1 \geq 0$; but then the assertion follows from Prop. 10(α)(i). || The proof of (β)(ii) is analogous, with f^V replaced by g^V, $h^1 \leq f^V(\theta^1)$ by $h^1 \geq g^V(\theta^1)$ etc.

As to (α)(ii), note that (1,n) is exterior to $\mathscr{B}^{\varrho}$, so that the assertion follows directly from the fact that a path once in $[\![\mathscr{B}^{\varrho}]\!]$ at some $z_{\diamond} \geq z^{\varrho}$ cannot leave that set as $z \uparrow$; in this case Prop. 10 need not be invoked. Similarly for (β)(i). ||

PROPOSITION 11 (Convergence Lemma).
Every solution of $S = (F,G)$ which is bounded for the forward (backward) motion converges to a finite limit as $z \to \infty$ $(z \to -\infty)$. More precisely:

(α)(i) For $b > 1$, a solution of S becomes unbounded as $z \uparrow$ iff its path is ever in $\mathscr{U}^{\triangleright}$. Otherwise it converges as $z \to \infty$ to the stable node of S_{∞} iff its path is ultimately in $\mathscr{B}^{\triangleright}$, and to the saddle point of S_{∞} iff its path lies entirely in $\mathscr{C}^{\triangleright}$. (These statements also apply for $b \leq 1$ if $N > 0$.)

(β)(i) For $b > 1$, and δ, z^{δ} chosen as in (3.43) ff, a solution starting at $z_{\diamond} \leq z^{\delta}$ becomes unbounded as $z \downarrow$ iff its path is ever in $\mathscr{U}^{\delta}$. Otherwise it converges as $z \to -\infty$ to the unstable node of $S_{-\infty}$ iff its path is ultimately in $\mathscr{B}^{\delta}$, and to the saddle point of $S_{-\infty}$ iff its path remains in $\mathscr{C}^{\delta}$.

(α)(ii) For $b \leq 1$ and ρ, z^{ϱ} chosen as in (3.51) ff, a solution starting at $z_{\diamond} \geq z^{\varrho}$ becomes unbounded as $z \uparrow$ iff its path is ever in $\mathscr{U}^{\varrho}$. Otherwise it converges as $z \to \infty$ to the stable node of S_{∞} iff its path is ultimately in $\mathscr{B}^{\varrho}$, and to the saddle point of S_{∞} iff its path remains in $\mathscr{C}^{\varrho}$.

(β)(ii) For $b \leq 1$, a solution becomes unbounded as $z \downarrow$ iff its path is ever in $\mathscr{U}^{\triangleleft}$. Otherwise it converges as $z \to -\infty$ to the unstable node of $S_{-\infty}$ iff its path is ultimately in $\mathscr{B}^{\triangleleft}$, and to the saddle point of $S_{-\infty}$ iff its path lies entirely in $\mathscr{C}^{\triangleleft}$. (These statements also apply for $b \geq 1$ if $n > 0$.)

REMARK. Solutions of S which are unbounded as $z \to z_{+}$ $(z \to z_{-})$ also converge, the possible limits being those identified in Section 3 for solutions of S_{∞} $(S_{-\infty})$. This can be proved by 'comparison' arguments like those used for Props. 9 and 10, but since such solutions cannot solve a b.v.p. we shall not go into details.

PROOF. For solutions with $\theta(z) = 0$ at some (and therefore all) z the Proposition is obvious, and we assume $\theta(z) > 0$ without special mention.

Let $\pi = \pi(z) = (h(z),\theta(z): z_{-}<z<z_{+})$ be an arbitrary solution and let $\Pi_{\diamond} = (\pi_{\diamond},z_{\diamond})$ with $\pi_{\diamond} = (h_{\diamond},\theta_{\diamond})$ be a point through which the solution passes.

(α)(i) *Forward Motion*, $b > 1$. Referring to (4.1 a–b), we note that the sets defined there partition the half-plane $\{\theta > 0)\}$. Of these sets, $\mathscr{U}^{\triangleright}$ is an unbounded region from which paths of S and S_{∞} cannot escape as $z \uparrow$ and (as shown earlier) all paths of S_{∞} which enter this region become unbounded. According to Prop. 2(iv), the same is true of paths of S. Any path which reaches $\mathscr{M}^{\Lambda\triangleright}$ passes immediately into $\mathscr{U}^{\triangleright}$. Also, any path which reaches $\mathscr{M}^{V\triangleright}$ passes immediately into $\mathscr{B}^{\triangleright}$ and cannot escape as $z \uparrow$. Thus a path is ultimately in one of $\mathscr{U}^{\triangleright}$, $\mathscr{B}^{\triangleright}$ or $\mathscr{C}^{\triangleright}$.

If a given solution π never enters $\mathscr{U}^{\triangleright}$, it is bounded for $z \geq z_{\diamond}$. Indeed, the path is bounded above and to the right by the graph of f^{Λ}; and if h(z) assumes negative values they are bounded because of the term bh^2 in F and the fact that $\theta(z)$ is then decreasing – see Figs 3–4. Thus $z_{+} = \infty$, and the forward limit set $\Pi^{\triangleright}$ is not empty and is contained in $\{(h,\theta)\colon h \leq f^{\Lambda}(\theta),\ \theta \geq 0\}$. We wish to characterise this set.

There are three possibilities a priori concerning the ultimate path behaviour:

(I) $\theta(z)$ is non–decreasing and $h(z) \geq 1$ for all z large enough, say for $z \geq z_{\diamond}$.

(II) $\theta(z)$ is non–increasing and $h(z) \leq 1$ for all z large enough, say for $z \geq z_{\diamond}$.

(III) $\theta(z)$ is not (weakly) monotonic and $h(z)-1$ does not have constant definite sign for large z

In case (I), $\theta(z) \uparrow$ some θ_{∞}, and since the path cannot terminate in the interior of $\{h \geq 1\}$, and by assumption does not pass into $\mathscr{U}^{\triangleright}$ or into $\{h < 1\}$, we must have $\theta_{\infty} \leq \nu$ and $h(z) \to 1$; but then $(1,\theta_{\infty})$ must be a stationary point of S_{∞} by Prop. 2(ii), which is possible only if $\theta_{\infty} = n \geq 0$. If $n < 0$, case (I) cannot occur for bounded solutions.

In case (II), $\theta(z) \downarrow$ some θ_{∞}, and since the path cannot terminate in the interior of $\{h \leq 1\}$ and does not leave this set we have either a limit (1,n) with $n > 0$, or $\theta_{\infty} = 0$. Consider the latter case. Any point of $\Pi^{\triangleright}$ must be of the form $(\bar{h},0)$ with

$\bar{h} \leq 1$, and for such a point there must be a sequence $z_k \uparrow \infty$ with $h(z_k) \to \bar{h}$ Consequently, if there are two points in $\Pi^{\triangleright}$, say with $\bar{h} = h_\alpha$ and $\bar{h} = h_\beta$, $h_\alpha < h_\beta$, then all $\bar{h}$ in the interval $[h_\alpha, h_\beta]$ must also define points $(\bar{h}, \theta_\infty)$ of $\Pi^{\triangleright}$; thus $\Pi^{\triangleright}$ has the form $I \times \{0\}$, where I is an interval bounded above by 1 On the other hand, $\Pi^{\triangleright}$ must be the union of complete paths of S_∞ by Prop 2(ii) Now, the paths of S_∞ lying on the vertical axis are as follows: the stationary points $(h_\infty^+, 0)$ and $(h_\infty^-, 0)$, and the intervals $I_- = \{(h,0): h < h_\infty^-\}$, $I_0 = \{(h,0): h_\infty^- < h < h_\infty^+\}$ and $I_+ = \{(h,0): h \geq h_\infty^+\}$. It is impossible for the whole of I_- to define limit points because $h(z)$ is bounded below Also, I_+ may be left aside since it is not bounded above by 1 Further, if $n > 0$, then $h_\infty^+ > 1$, so $(h_\infty^+, 0)$ and I_0 do not satisfy the stated bound; thus in this case $\Pi^{\triangleright}$ consists of the single point $(h_\infty^-, 0)$, the stable node of S_∞, and this point is the required limit. Now suppose $n \leq 0$, so that $h_\infty^+ \leq 1$. It is impossible for every point of I_0 to be a limit point of the solution because $F_\infty(h,0) < 0$ for $h \in (h_\infty^-, h_\infty^+)$, and since $F \to F_\infty$ uniformly on (h,θ)-compacts it follows that, for $\epsilon > 0$ small enough, there is $z(\epsilon)$ such that $F(h(z),\theta(z),z) < -\epsilon$ for $h \in (h_\infty^- + \epsilon, h_\infty^+ - \epsilon)$ and $z > z(\epsilon)$, so that eventually the interval $(h_\infty^- + \epsilon, h_\infty^+ - \epsilon)$ can be traversed only in the downward direction. Thus if $n \leq 0$ the only possible limit points are $(h_\infty^-, 0)$ and $(h_\infty^+, 0)$, the stable node and saddle point of S_∞, and one of these must be the required limit.

Consider now case (III) There must be a sequence (z_k) such that, for each $k = 1,2,\ldots$, we have $h(z)-1 \geq 0$ on (z_{2k-1}, z_{2k}), $h(z)-1 \leq 0$ on (z_{2k}, z_{2k+1}), and moreover $h(z)-1$ does not vanish identically on any of these intervals or on any neighbourhood of the z_k. Note that $\theta(z)$ is non-decreasing on each (z_{2k-1}, z_{2k}), non-increasing on each (z_{2k}, z_{2k+1}) Writing $F(z) = F(h(z),\theta(z),z)$ and $\delta(z) = A(z) - M(z)/b$, it follows from (0.1) that if $h(z) = 1$ then $F(z)$ has the same sign as $\theta(z) - n - \delta(z)$. Now $F(z_{2k-1}) \geq 0 \geq F(z_{2k})$, and since $\delta(z) \to 0$ as $z \to \infty$ we have $\theta(z_{2k-1}) \to n$ and $\theta(z_{2k}) \to n$. It then follows from the monotonicity of $\theta(z)$ on each interval that $\theta(z) \to n$ as $z \to \infty$. Consequently the forward limit set must have the form $\{(h,n): h \in I\}$ where I is an interval, and this set must be a union of complete paths of S_∞. If

$n > 0$, this is possible only if the set reduces to the singleton $(1,n)$. If $n = 0$, an argument like that given in case (II) yields the same conclusion; (in fact we must have $h^+_\infty = 1$). If $n < 0$, case (III) cannot occur.

So far we have shown that, for $b > 1$, a path of S becomes unbounded if it is ever in $\mathscr{U}^{\triangleright}$, otherwise it is in $\mathscr{B}^{\triangleright}$ or $\mathscr{C}^{\triangleright}$ for large z and converges either to the stable node or to the saddle point of S_∞. If a path is ever in $\mathscr{B}^{\triangleright}$, then according to Cor. 10.2(α)(i), it cannot converge to the saddle point of S_∞ and so converges to the stable node of S_∞. Also, as noted earlier – see (3.41) – paths may leave but not enter $\mathscr{C}^{\triangleright}$ as $z \uparrow$, so that if a path converges to the saddle point of S_∞ it must be in $\mathscr{C}^{\triangleright}$ for all $z \in \mathfrak{R}$. Obviously each of the three ocurrences of 'if' in this paragraph may be replaced by 'iff'. || A slightly modified proof applies if $b \le 1$ and $N > 0$.

(β)(i) *Backward Motion*, $b > 1$. It is tempting to imitate the argument for the forward motion, replacing the functions f^Λ, f^V and sets $\mathscr{U}^{\triangleright}$, $\mathscr{B}^{\triangleright}$, $\mathscr{C}^{\triangleright}$ with g^Λ, g^V and $\mathscr{U}^{\triangleleft}$, $\mathscr{B}^{\triangleleft}$, $\mathscr{C}^{\triangleleft}$. This works well enough if $n > 0$, but if $q > 0 \ge n$ the function g^V is undefined and the argument fails. In order to have a unified argument we choose δ and z^δ as in (3.43) ff., hence define $S^{\Lambda\delta}$ and $S^{V\delta}$, consider the backward motion only for $z \le z_\diamond \le z^\delta$ and work with the functions $g^{\Lambda\delta}$, $g^{V\delta}$ and sets $\mathscr{U}^\delta$, $\mathscr{B}^\delta$, $\mathscr{C}^\delta$.

The proof procedure is then more or less analogous with that for the forward motion. For any solution which is defined at z^δ the path passes ultimately into one of $\mathscr{U}^\delta$, $\mathscr{B}^\delta$ or $\mathscr{C}^\delta$. If the path enters $\mathscr{U}^\delta$, it becomes unbounded (and it is not hard to see that it ultimately enters $\mathscr{U}^{\triangleleft}$). Contrariwise, the path stays bounded as $z \downarrow$ if it never enters $\mathscr{U}^\delta$ (and hence never enters $\mathscr{U}^{\triangleleft}$). Indeed, such a path is bounded below and to the right by the graph of g^Λ; and if $h(z)$ assumes values > 1 they are bounded, because then $\theta(z)$ is decreasing as $z \downarrow$ and $h(z)$ is decreasing as long as the path is in $\{F^\Lambda > 0\}$ – see Figs. 3–4.

Restricting attention now to bounded backward solutions, there are again three possibilities regarding ultimate path behaviour: (I) $\theta(z)$ is non–decreasing and $h(z) \le 1$ as $z \downarrow$. (II) $\theta(z)$ is non–increasing and $h(z) \ge 1$ as $z \downarrow$. (III) $\theta(z)$ is not (weakly) monotonic However the analysis is simpler than in the forward case because, with $N > 0$, only Type 1 systems $S_{-\infty}$, $S^{\Lambda\delta}$ and $S^{V\delta}$ need be considered, see above, (3.46) ff It is shown as before that, in case (I), the backward limit set reduces to the single point (1,N), the saddle point of $S_{-\infty}$. In case (II) there are two possible limits, namely (1,N) and the unstable node of $S_{-\infty}$ at $(h^+_{-\infty},0)$. In case (III), the argument is again similar to that in the forward case. Note that, when $h(z) = 1$, $F(z)$ has the same sign as

$$\theta(z) - n - A(z) + M(z)/b = \theta(z) - N + (\psi_0' - A(z)) - (\psi_0' - M(z))/b,$$

and then, since $A(z)$ and $M(z)$ tend to ψ_0' as $z \to -\infty$, we find first that $\theta(z) \to N$ and then that the only limit point is (1,N).

It then remains to note that, by Cor. 10 2(β)(i), a path of S which is ever in $\mathscr{B}^\delta$ for $z \leq z^\delta$ cannot converge to the saddle point of $S_{-\infty}$ and so must go to the unstable node. Finally paths may enter but not leave $\mathscr{C}^\delta$ as $z \downarrow$, leading to the conclusion that the paths which go to the saddle are precisely those which are in $\mathscr{C}^\delta$ for $z \leq z^\delta$. ||

(α&β)(ii) Let $b \leq 1$. The arguments are similar to those for $b > 1$ and will not be set out in detail. Loosely speaking, the *backward* argument for $b \leq 1$ is symmetrical with the forward argument for $b > 1$, with g^Λ, g^V, $\mathscr{U}^\triangleleft$, $\mathscr{B}^\triangleleft$, $\mathscr{C}^\triangleleft$ replacing f^Λ, f^V, $\mathscr{U}^\triangleright$, $\mathscr{B}^\triangleright$, $\mathscr{C}^\triangleright$ and the limits (1,N), $(h^-_{-\infty},0)$ and $(h^+_{-\infty},0)$ replacing (1,n), $(h^+_\infty,0)$ and $(h^-_\infty,0)$ However the details regarding characterisation of bounded solutions, direction of motion and solutions of $F(1,\theta,z) = 0$ are similar to those for the backward argument for $b > 1$. Again, the *forward* argument for $b \leq 1$ is symmetrical with the backward argument for $b > 1$ in that the function f^V is undefined if $q > 0 \geq N$, so that we consider the forward motion only for $z \geq z_\diamond \geq z^\varrho$ and work with the functions $f^{\Lambda\varrho}$, $f^{V\rho}$ and sets $\mathscr{U}^\varrho$, $\mathscr{B}^\varrho$, $\mathscr{C}^\varrho$; but most details are similar to those for the forward motion with $b > 1$, with the simplification that for $n > 0$ only Type 1 systems S_∞, $S^{\Lambda\varrho}$ and $S^{V\rho}$ need be considered, see (3.54) ff. ||

COROLLARY 11. (α) Let (h^*,θ^*) be a forward special solution. Then

(i) $h^*(z) < f^\Lambda[\theta^*(z)] < h^{\Lambda+}$ for all z

(ii) $h^*(z) > f^V[\theta^*(z)]$ for all z if $N > 0$, in particular, if $b \geq 1$.

(iii) If $b \leq 1$, and ρ, z^ϱ are chosen as in (3.51) ff., then

$$f^{V\rho}[\theta^*(z)] < h^*(z) < f^{\Lambda\varrho}[\theta^*(z)] \quad \text{for } z > z^\varrho$$

(β) Let (h^*,θ^*) be a backward special solution. Then

(i) $h^*(z) > g^\Lambda[\theta^*(z)]$ for all z.

(ii) $h^*(z) < g^V[\theta^*(z)]$ for all z if $n > 0$, in particular, if $b \leq 1$.

(iii) If $b > 1$, and δ, z^δ are chosen as in (3.43) ff, then

$$g^{\Lambda\delta}[\theta^*(z)] < h^*(z) < g^{V\delta}[\theta^*(z)] \quad \text{for all } z < z^\delta.$$

REMARK. This Corollary is just a restatement of points established previously. It yields some bounds for a star solution in addition to those stated in Corollary 8. In particular, such a solution must lie in the compact $\{0 \leq h \leq h^{\Lambda+}, 0 \leq \theta \leq \nu\}$

We turn now to the main Lemmas on which Theorem 4A depends.

PROPOSITION 12 (Existence Lemma for Special Starts, $b > 1$).

(α) For each fixed $z_\diamond \in \Re$, and each $\theta = \theta_\diamond > 0$, there is at least one $h = h_\diamond$ such that $(h_\diamond,\theta_\diamond,z_\diamond)$ is the start of a forward special solution. The values of $\theta_\diamond$ for which there is a positive $h_\diamond$ with this property form an interval $(0,\theta_+)$, where $\theta_+ = \theta_+(z_\diamond)$. For $\theta_\diamond$ in this interval, $h_\diamond$ is unique and the function $h_\diamond = f(\theta_\diamond,z_\diamond)$, or simply $h = f(\theta)$, is continuous, strictly decreasing and satisfies

(4.5) $\qquad f^\Lambda(\theta) > f(\theta) > f^V(\theta)$ Also

(4.5a) $\qquad 0 < \theta_+^V \leq \theta_+ \leq \theta_+^\Lambda \leq \infty$ and $f(\theta_+) = 0$ in all cases;

$$\theta_+ = \infty \text{ if } m \geq \psi_0'; \quad \theta_+ \leq \infty \text{ if } 0 \leq m < \psi_0'; \quad \theta_+ < \infty \text{ if } m < 0.$$

(β) For each fixed $\delta > 0$ and $z^\delta = z_\diamond$ chosen as in (3.43) ff., and each $\theta = \theta_\diamond > 0$, there is at least one $h = h_\diamond$ such that $(h_\diamond,\theta_\diamond,z_\diamond)$ is the start of a backward special solution. The values of $\theta_\diamond$ for which there is a positive $h_\diamond$ with this property form an interval (θ_-,∞), where $\theta_- = \theta_-(z_\diamond)$. For $\theta_\diamond$ in this interval, $h_\diamond$ is unique and the function $h_\diamond = g(\theta_\diamond, z_\diamond)$, or simply $h = g(\theta)$, is continuous, strictly increasing with $g(\infty) = \infty$ and satisfies

(4.6) $\quad g^{\wedge\delta}(\theta) < g(\theta) < g^{\vee\delta}(\theta)$ Also

(4.6a) $\quad 0 \le \theta_-^{\vee\delta} \le \theta_- \le \theta_-^{\wedge\delta}$ and $\theta_- \cdot g(\theta_-) = 0$ in all cases;

$\theta_- > 0$ if $m > \psi_0'$; $\theta_- \ge 0$ if $m = \psi_0'$; $\theta_- = 0$ if $m < \psi_0'$;

$g(\theta_-) = 0$ if $m \ge \psi_0'$; $g(\theta_-) > 0$ if $m < \psi_0'$.

PROOF. (α) We fix $z_\diamond$ throughout this proof and consider only the forward motion, for $z > z_\diamond$, defined by S Often we omit $z_\diamond$ from the notation, also the superscript $\triangleright$ from the symbols in (1). For given $z > z_\diamond$, we denote by $S_z^\triangleright = S_z$ the transformation

$$\pi_\diamond \mapsto \pi(z;\pi_\diamond,z_\diamond), \text{ where } \pi_\diamond = (h_\diamond,\theta_\diamond), \ \pi(z) = (h(z),\theta(z)),$$

whenever this is defined, (i.e. for z less than the value $z_+(\pi_\diamond,z_\diamond)$ of z_+ for the solution starting at $(\pi_\diamond,z_\diamond)$; bear in mind here that a solution whose path enters $\mathscr{U}$ stays there up to explosion at z_+, while for other solutions $z_+ = \infty$). Restricting S_z to $\mathscr{C}$, let

(4.7) $\quad W^u(z) = \{\pi_\diamond \in \mathscr{C}: S_\zeta\pi_\diamond \in \mathscr{U} \text{ for some } \zeta \in (z_\diamond, z]\}, \quad W^b(z) = \{\pi_\diamond \in \mathscr{C}: S_z\pi_\diamond \in \mathscr{B}\}$

$$W^c(z) = \{\pi_\diamond \in \mathscr{C}: S_z\pi_\diamond \in [\![\mathscr{C}]\!]\} = \{\pi_\diamond \in [\![\mathscr{C}]\!] : S_z\pi_\diamond \in [\![\mathscr{C}]\!]\}$$

(where $W^u(z) = W^{u\triangleright}(z;z_\diamond)$ etc). The replacement of $\mathscr{C}$ by $[\![\mathscr{C}]\!]$ in the last equality above is permissible because relative boundary points of $\mathscr{C}$ are mapped into points of $\mathscr{U}$ or $\mathscr{B}$. Of course, $\mathscr{C}$ is the union of the three sets in (7). We note that $W^u(z)$ is open. Indeed, $\mathscr{U}$ is open, therefore so is $S_z^{-1}\mathscr{U} = \mathscr{U} \cup \mathscr{M}^u \cup W^u(z)$, a union of disjoint sets, and since $\mathscr{U} \cup \mathscr{M}^u = [\![\mathscr{U}]\!]$ is closed the assertion follows. Similarly $W^b(z)$ is open. Obviously $W^u(z)$ and $W^b(z)$ are disjoint, and in view of the one-way passage across boundaries these sets are not empty; (more details later). On the other hand, $W^c(z) = S_z^{-1}[\![\mathscr{C}]\!]$ is relatively closed, connected and non–empty because $[\![\mathscr{C}]\!]$ has these properties. Now, if we let $z \uparrow$ (still keeping $z_\diamond$ fixed) it follows from the one-way passage across boundaries that the open, disjoint sets $W^u(z)$ and $W^b(z)$ grow, hence converge to open, disjoint limit sets $W^u_\infty = W^u_\infty(z_\diamond)$ and $W^b_\infty = W^b_\infty(z_\diamond)$. The sets $W^c(z)$ decrease to a relatively closed, connected limit set $W^c_\infty = W^c_\infty(z_\diamond)$, (and this set also is not empty since it has the form $\mathscr{C}\backslash\{W^u_\infty \cup W^b_\infty\}$ with $\mathscr{C}$, W^u_∞ and W^b_∞

all open, connected and non-empty and W^u_∞, W^b_∞ disjoint)

Consider now the sections of these various sets at a fixed $\theta_\diamond > 0$ We have $\mathscr{C}(\theta_\diamond) = (f^V(\theta_\diamond), f^\Lambda(\theta_\diamond))$, an open interval of positive length For given z, the section $W^u(z,\theta_\diamond)$ is open in $\Re$ (as the section of an open set), and it is contained in the open interval $\mathscr{C}(\theta_\diamond)$. It further follows from $F > F^\Lambda$ and the continuity of the various functions that a solution starting at $(h_\diamond,\theta_\diamond,z_\diamond)$ with $h_\diamond \in \mathscr{C}(\theta_\diamond)$ will pass into $\mathscr{U}$ before z if $h_\diamond$ is close enough to $f^\Lambda(\theta_\diamond)$, so $W^u(z;\theta_\diamond)$ contains an interval of the form $(h^u(z), f^\Lambda(\theta_\diamond))$, where $h^u(z) = h^u(z;\theta_\diamond,z_\diamond)$ Similarly $W^b(z;\theta_\diamond)$ is open in $\Re$ and contains an interval $(f^V(\theta_\diamond), h^b(z))$, and $W^u(z;\theta_\diamond)$ and $W^b(z;\theta_\diamond)$ are disjoint Since

(4.8) $$W^c(z;\theta_\diamond) = \mathscr{C}(\theta_\diamond) \setminus \{W^u(z;\theta_\diamond) \cup W^b(z;\theta_\diamond)\}$$

this set must be closed in $\Re$ and non–empty Letting $z \uparrow \infty$, it follows from the monotonicity of the various convergences that, for each of the sets $W^u(z)$, $W^b(z)$, $W^c(z)$ the limit of the section at $\theta_\diamond$ is the section at $\theta_\diamond$ of the limit In particular, $W^u(z;\theta_\diamond)$ and $W^b(z;\theta_\diamond)$ increase to sets $W^u_\infty(\theta_\diamond)$ and $W^b_\infty(\theta_\diamond)$ which are disjoint and open in $\Re$ and in $\mathscr{C}(\theta_\diamond)$ with upper and lower endpoints $f^\Lambda(\theta_\diamond)$ and $f^V(\theta_\diamond)$ of $\mathscr{C}(\theta_\diamond)$ respectively, so $W^c_\infty(\theta_\diamond)$ is a non–empty, closed set in $\Re$ and in $\mathscr{C}(\theta_\diamond)$. *By Prop* 11($\alpha$)(i), *a point* $\Pi_\diamond = (h_\diamond,\theta_\diamond,z_\diamond)$ *with* $h_\diamond \in W^c_\infty(\theta_\diamond;z_\diamond)$ *is the start of a forward special solution, so we have shown that for each* $(\theta_\diamond;z_\diamond)$ *there is at least one such start*.

If, for fixed $\theta_\diamond$, there is one start of a forward special solution with $h_\diamond \geq 0$, then according to Cor 10.1 this $h_\diamond$ is *unique*. Since $f^V(\theta) < h < f^\Lambda(\theta)$ for $h \in \mathscr{C}(\theta)$, this will be the case at least for $\theta = \theta_\diamond$ in the interval $(0,\theta^V_+)$ where f^V is positive, and here we may write $h_\diamond$ as a function $h = f(\theta,z_\diamond) = f(\theta)$ We consider f first as a function on a closed interval $[\theta_1,\theta_2]$ with $0 < \theta_1 < \theta_2 < \theta^V_+$, and note that on this interval the graph of the function is $W^c_\infty \cap \{\theta_1 \leq \theta \leq \theta_2\}$, which is a closed plane set. The function is therefore *continuous* on the interval and, letting $\theta_1 \downarrow 0$, $\theta_2 \uparrow \theta^V_+$, is seen to be continuous on $(0,\theta^V_+)$. Obviously the function is *positive* on this interval, and the fact that it is *decreasing* is a consequence of Prop. 10(α)(i).

If $m \geq \psi'_0$, then $\theta^V_+ = \theta^\Lambda_+ = \infty$ by (3 57a), so f is defined on $(0,\theta_+)$ with

$\theta_+ = \theta_+^V = \infty$, and $f(\infty) = 0$ by (3.57) since $f^\Lambda > f > f^V$; *in this case, the proof is complete.* Suppose that $m < \psi_0'$ and $\theta_+^V < \infty$. It remains true that $W_\infty^c(z_\diamond)$ is the graph of a continuous simple curve, say of the form $f(h,\theta,z_\diamond) = 0$, with at least one solution h for each $\theta > 0$. This follows (for example) from the facts that every forward special solution eventually enters the strip $\{0 < \theta < \theta_+^V\}$, and that the map S_z defines for each $z > z_\diamond$ a homeomorphism from $W_\infty^c(z_\diamond)$ to $W_\infty^c(z)$. Explicitly: if $[\theta_a,\theta_b]$ is a closed, finite sub–interval of $[\theta_+^V,\infty)$, then for large enough z we have $W_\infty^c(z_\diamond) \cap \{\theta_a \leq \theta_b\} = S_z^{-1}[W_\infty^c(z) \cap \{\tilde\theta_a \leq \tilde\theta_b\}]$ for some $0 < \tilde\theta_a < \tilde\theta_b < \theta_+^V$, so that the set on the left of the equality is the homeomorphic image of a segment of a continuous, decreasing curve. Further, Prop. 10(α)(i) with Cor. 10.1 ensures that values of θ for which $W_\infty^c(z_\diamond)$ contains only negative values of h are not succeeded by greater values of θ with positive h; thus the values of θ for which $W_\infty^c(z_\diamond)$ contains precisely one $h > 0$ form an interval $0 < \theta < \theta_+$ and the continuous, positive decreasing function f can be extended to this interval. Since $f^\Lambda > f > f^V$ on the interval, it follows that $\theta_+^V \leq \theta_+ \leq \theta_+^\Lambda$. Moreover $f(\theta_+) = 0$; indeed, we have *either* $\theta_+ = \infty$ and then the assertion follows as above, *or* $\theta_+ < \infty$ and then it follows from continuity and monotonicity of f. Finally, it follows from (3.57a) that $\theta_+ < \infty$ if $m < 0$, but it seems that in general both possibilities are open if $0 \leq m < \psi_0'$ || Note that the preceding argument is stated in a way which avoids the need to distinguish between Type 1 and Type 0 systems.

(β) The main part of the proof is similar. Briefly, we consider only solutions defined for $z < z_\diamond = z^\delta$. The forward motion $S^\triangleright$ is replaced by the backward motion $S^\triangleleft$, $\mathscr{C} = \mathscr{C}^\triangleright$ is replaced by $\mathscr{C}^\delta$ and similarly for $\mathscr{B}$ and $\mathscr{U}$, the locus of special starts W_∞^c is replaced by $W_{-\infty}^c$ etc, and the roles of f^V, θ_+^V, f^Λ, θ_+^Λ, are taken over by $g^{\Lambda\delta}$, $\theta_-^{\Lambda\delta}$, $g^{V\delta}$, $\theta_-^{V\delta}$. The proof then follows much the same lines up to the point where it is established that $W_{-\infty}^c$ can be represented, at least on $(\theta_-^{\Lambda\delta},\infty)$, by a continuous, positive, increasing function g with $g^{\Lambda\delta} < g < g^{V\delta}$, hence $g(\infty) = \infty$. (Minor changes are needed to allow for differences between properties of f^V and $g^{\Lambda\delta}$, θ_+^V and $\theta_-^{\Lambda\delta}$ etc resulting from Props. 6 and 7).

In the last paragraph of the proof, the roles of the inequalities involving $m - \psi_0'$ are interchanged. More precisely, if $m < \psi_0'$, then $\theta_-^{V\delta} = \theta_-^{\Lambda\delta} = 0$ and $g^{V\delta}(0) > g^{\Lambda\delta}(0) > 0$ by (3.58–59), so that g is defined on $(0,\infty)$ with a limit $g(0) > 0$; in this case, the proof is complete. If $m > \psi_0'$, then $0 < \theta_-^{V\delta} < \theta_-^{\Lambda\delta}$ and $g^{V\delta}(\theta_-^{V\delta}) = g^{\Lambda\delta}(\theta_-^{\Lambda\delta}) = 0$ by (3.58–59). The graph of $W_{-\infty}^c$ is again a continuous simple curve, say $g(h,\theta,z^\delta) = 0$ with at least one h for each $\theta > 0$, and the representation $h = g(\theta)$ with $g > 0$ can be extended to a maximal interval (θ_-,∞) satisfying $0 < \theta_-^{V\delta} < \theta_- < \theta_-^{\Lambda\delta}$ and $g(\theta_-) = 0$. The case $m = \psi_0'$ needs separate consideration and one apparently has only $0 = \theta_-^{V\delta} \leq \theta_- < \theta_-^{\Lambda\delta}$ and $g(\theta_-) = 0$.||

PROPOSITION 13 (Existence Lemma for Special Starts, $b \leq 1$).

(α) For each fixed $\rho > 0$ and $z^\varrho = z_\diamond$ chosen as in (3.51) ff., and each $\theta = \theta_\diamond > 0$, there is at least one $h = h_\diamond$ such that $(h_\diamond,\theta_\diamond,z_\diamond)$ is the start of a forward special solution. The values of $\theta_\diamond$ for which there is a positive $h_\diamond$ with this property form an interval $(0,\theta_+)$, where $\theta_+ = \theta_+(z_\diamond)$. For $\theta_\diamond$ in this interval, $h_\diamond$ is unique and the function $h_\diamond = f(\theta_\diamond,z_\diamond)$, or simply $h = f(\theta)$, is continuous, strictly decreasing and satisfies

(4.9) $\quad f^{V\rho}(\theta) < f(\theta) < f^{\Lambda\varrho}(\theta)$. Also

(4.9a) $\quad 0 < \theta_+^{V\rho} \leq \theta_+ \leq \theta_+^{\Lambda\varrho} \leq \infty$ and $f(\theta_+) = 0$ in all cases;

$\theta_+ = \infty$ if $m > 0$; $\theta_+ \leq \infty$ if $m = 0$; $\theta_+ < \infty$ if $m < 0$.

(β) For each fixed $z_\diamond \in \mathfrak{R}$, and each $\theta = \theta_\diamond > 0$, there is at least one $h = h_\diamond$ such that $(h_\diamond,\theta_\diamond,z_\diamond)$ is the start of a backward special solution. The values of $\theta_\diamond$ for which there is a positive $h_\diamond$ with this property form an interval (θ_-,∞), where $\theta_- = \theta_-(z_\diamond)$. For $\theta_\diamond$ in this interval, $h_\diamond$ is unique and the function $h_\diamond = g(\theta_\diamond,z_\diamond)$, or simply $h = g(\theta)$, is continuous, strictly increasing and satisfies

(4.10) $\quad g^\Lambda(\theta) < g(\theta) < g^V(\theta)$. Also

(4.10a) $\quad 0 \leq \theta_-^V \leq \theta_- \leq \theta_-^\Lambda$ and $\theta_- \cdot g(\theta_-) = 0$, in all cases;

$\theta_- > 0$ if $m > \psi_0'$; $\theta_- \geq 0$ if $0 < m \leq \psi_0'$; $\theta_- = 0$ if $m \leq 0$

$g(\theta_-) = 0$ if $m \geq \psi_0'$; $g(\theta_-) \geq 0$ if $0 \leq m < \psi_0'$; $g(\theta_-) > 0$ if $m < 0$.

PROOF. This is analogous to the proof of Prop 12 and will not be set out in detail. The best symmetry is obtained if the backward and forward proofs for $b \leq 1$ follow the forward and backward proofs for $b > 1$ respectively. Thus the main part of the proof of Prop. 13(β) is like that of 12(α), replacing $S^{\triangleright}$ by $S^{\triangleleft}$, $\mathscr{C}^{\triangleright}$ by $\mathscr{C}^{\triangleleft}$ etc, the roles of f^{V}, θ^{V}_{+}, f^{Λ}, θ^{Λ}_{+} being taken over by g^{Λ}, θ^{Λ}_{-}, g^{V}, θ^{V}_{-} (with minor changes taking into account Props 6 and 7). In the last paragraph of the proof, the distinction between cases with $m \geq \psi'_0$, $m < \psi'_0$ is replaced by a distinction between $m \leq 0$, $m > 0$. (If $m \leq 0$, then $\theta^{\Lambda}_{-} = \theta^{V}_{-} = 0$ and the proof is completed with $\theta_{-} = 0$ immediately, taking into account Prop. 7; but if $m > 0$ an extension argument is needed.)

Again, the main part of the proof of 13(α) is like that of 12(β), with $S^{\triangleright}$ considered only for $z > z^{\varrho}$, $\mathscr{C}^{\delta}$ etc replaced by $\mathscr{C}^{\varrho}$ etc, and the roles of $g^{\Lambda\delta}$, $\theta^{\Lambda\delta}_{-}$, $g^{V\delta}$, $\theta^{V\delta}_{-}$ taken over by $f^{V\rho}$, $\theta^{V\rho}_{+}$, $f^{\Lambda\varrho}$, $\theta^{\Lambda\varrho}_{+}$. In the last paragraph, the distinctions among cases with $m < \psi'_0$, $m > \psi'_0$ and $m = \psi'_0$ are replaced by $m > 0$, $m < 0$, $m = 0$. (If $m > 0$, then $\theta^{V\rho}_{+} = \theta^{\Lambda\varrho}_{+} = \infty$, and one gets $\theta_{+} = \infty$ immediately, otherwise an extension argument is needed; the 'borderline' case $m = 0$ needs special consideration.) ||

PROOF OF THEOREM 4

This can now be completed fairly trivially. Let $b > 1$, choose δ and $z_{\diamond} = z^{\delta}$ as in (3.43) ff and define f, θ_{+}, g, θ_{-} as in Prop. 12. Suppose first that $m \geq \psi'_0$. In this case, Prop. 12(α), eqs (5–5a), yield $\theta_{+} = \infty$, $f(\infty) = 0$, with $f\downarrow$ on $(0,\infty)$; also $f(\nu) < f^{\Lambda}(\nu)$ using (3.46c). On the other hand, eqs (6–6a) yield $\theta_{-} \geq 0$, $g(\theta_{-}) = 0$ with $g\uparrow$ on (θ_{-},∞), also $g(\nu) > g^{\Lambda\delta}(\nu) > g^{\Lambda}(\nu) = 1$ using (3.46c), hence $\theta_{-} < \nu$. It follows that

$$f(\theta_{-}) > g(\theta_{-}) = 0, \quad f(\theta) < g(\theta) \text{ for } \nu \leq \theta < \infty,$$

and since $f\downarrow$ and $g\uparrow$ there is precisely one intersection of the curves in the interval (θ_{-},∞). This intersection defines a point (h^*,θ^*) satisfying

(4.11) $\qquad h^* = f(\theta^*) = g(\theta^*) > 0,$

and clearly $\theta_- < \theta^* < \nu$. This point is the start of both a forward and a backward special solution and so is a star solution. It is the only point with these properties, since for $\theta \leq \theta_-$ any point (h,θ) which is the start of a b.s.s. has $h \leq 0$ while any point which is the start of a f.s.s. has $h > 0$. This completes the proof for $m \geq \psi_0'$.

Now let $m < \psi_0'$. Here (5–5a) yield $\theta_+ \leq \infty$, $f(\theta_+) = 0$ with $f \downarrow$ on $(0,\theta_+)$. On the other hand, (6–6a) yield $\theta_- = 0$, $g(\theta_-) > 0$ with $g \uparrow$ on $(0,\infty)$, also $g(\nu) > g^{\Lambda}(\nu) = 1$. Now (3.48–9) together with $g(0) \leq g^{V\delta}(0)$ and $f^{V}(0) \leq f(0)$ yields $g(0) < f(0)$ in all cases. Also $g(\theta_+) > 0 = f(\theta_+)$, so that there is precisely one intersection of the curves f and g in the interval $(0,\theta_+)$, defining a point (h^*,θ^*) which satisfies (11). (Moreover $\theta^* < \nu$ as before; if $\theta_+ \leq \nu$ this is obvious, and if $\nu \leq \theta_+$ it follows from $f(\nu) < f^{\Lambda}(\nu) = 1 < g^{\Lambda\delta}(\nu) < g(\nu)$ as before.) Once again this point is the start of both a f.s.s. and a b.s.s.; and it is the only such point, since for $\theta > \theta_+$ a point which is the start of a f.s.s. has $h \leq 0$, while a point which is the start of a b.s.s. has $h > 0$.

If $b \leq 1$, we choose ρ and $z_{\diamond} = z^{\varrho}$ as in (3.51) ff. and try to imitate the preceding argument with f, θ_+, g, θ_- as in Prop. 13 and other replacements as in (3.60–61); for brevity, we shall merely note some additional minor changes. We now have $f(\theta_+) = 0$ in all cases by Prop. 13. If $m \geq \psi_0'$, Prop. 13 further yields $\theta_+ = \infty$, $\theta_- \geq 0$, $g(\theta_-) = 0$, and the proof is like that for $b > 1$, $m \geq \psi_0'$. If $m \leq 0$, Prop. 13 yields $\theta_+ \leq \infty$, $\theta_- = 0$, $g(\theta_-) \geq 0$, and the proof is like that for $b > 1$, $m < \psi_0'$. In particular, the inequality $g(0) < f(0)$ now follows from (3.48–49) with the substitutions (3.60–61) together with $g(0) < g^{V}(0)$, $f^{V\rho}(0) < f(0)$. Finally, if $0 < m < \psi_0'$, Prop. 13 yields $\theta_+ = \infty$, $\theta_- \geq 0$, $g(\theta_-) \geq 0$ and $\theta_- \cdot g(\theta_-) = 0$. It is then necessary to distinguish between cases with $\theta_- > 0$, $g(\theta_-) = 0$ and those with $\theta_- = 0$, $g(\theta_-) \geq 0$; in the former case, the proof is as for $m \geq \psi_0'$, in the latter as for $m \leq 0$.||

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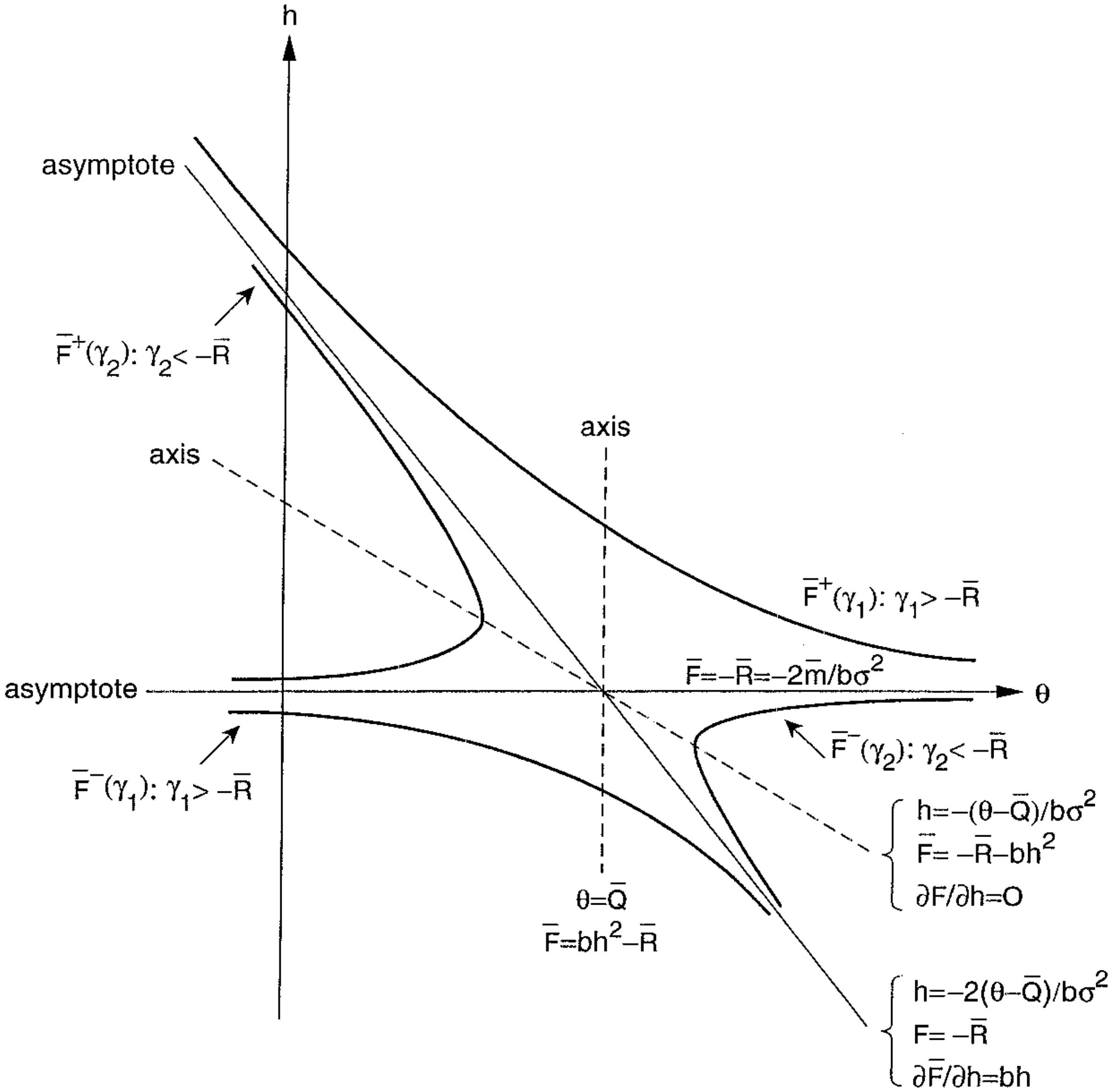

Figure 1: Contours of $\bar{F}(h,\theta)$

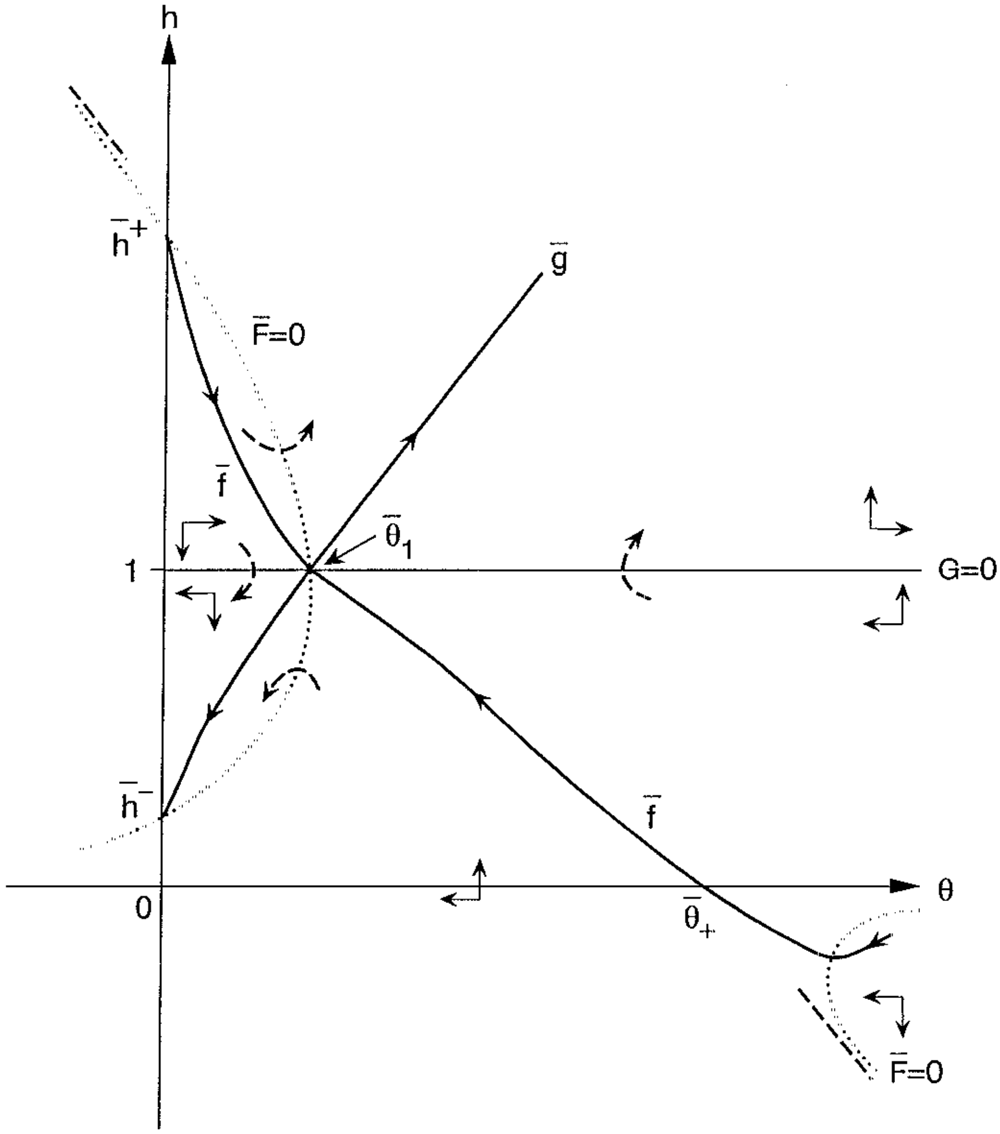

Figure 2(i): 3–Parameter System, Type 1

$\bar{\theta}_1 > 0,\ \bar{m} < 0,\ \bar{h}^+ > 1 > \bar{h}^- > 0$

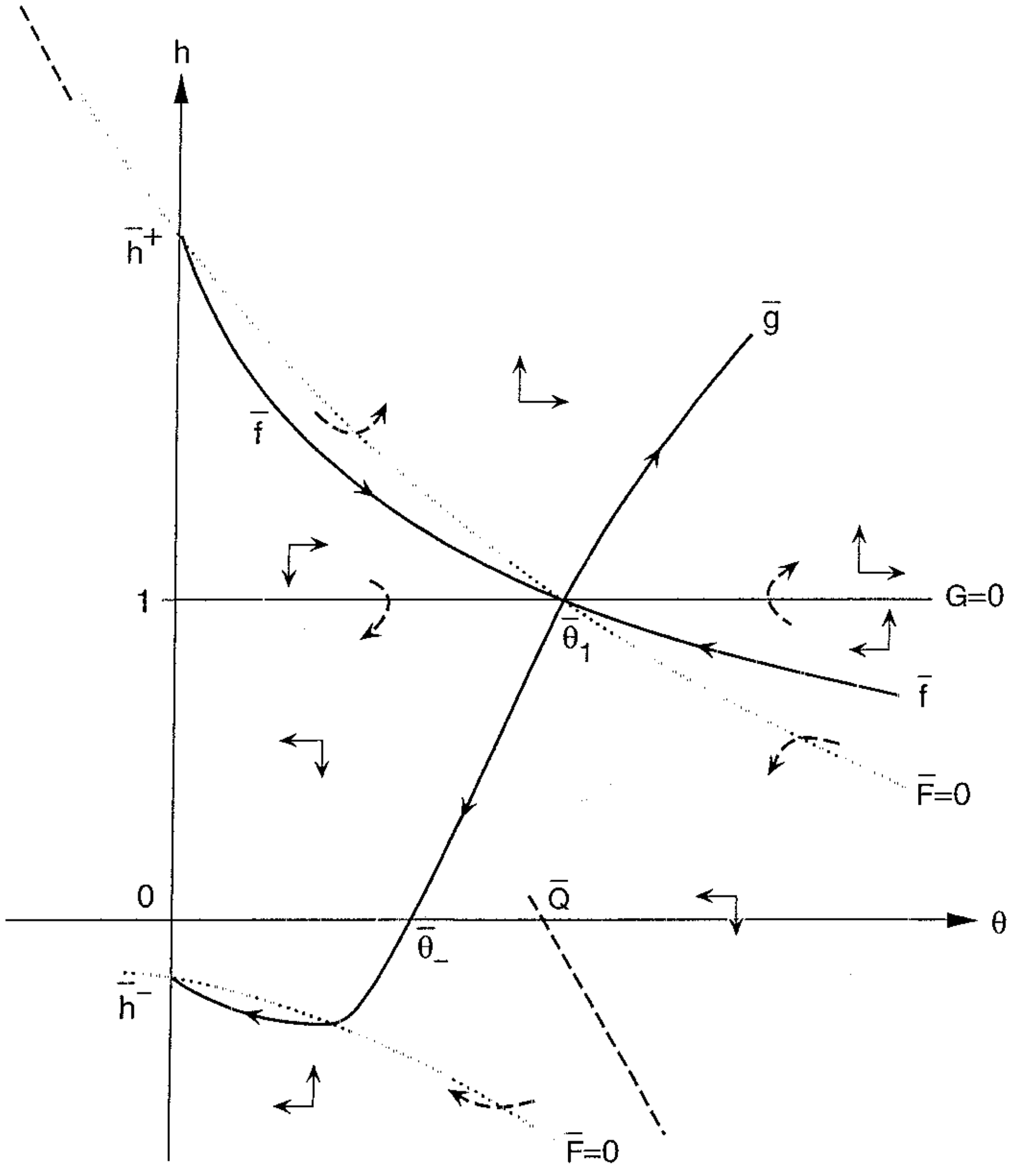

Figure 2(ii): 3–Parameter System, Type 1

$\bar{\theta}_1 > 0,\ \bar{m} > 0,\ \bar{h}^+ > 1 > 0 > \bar{h}^-$

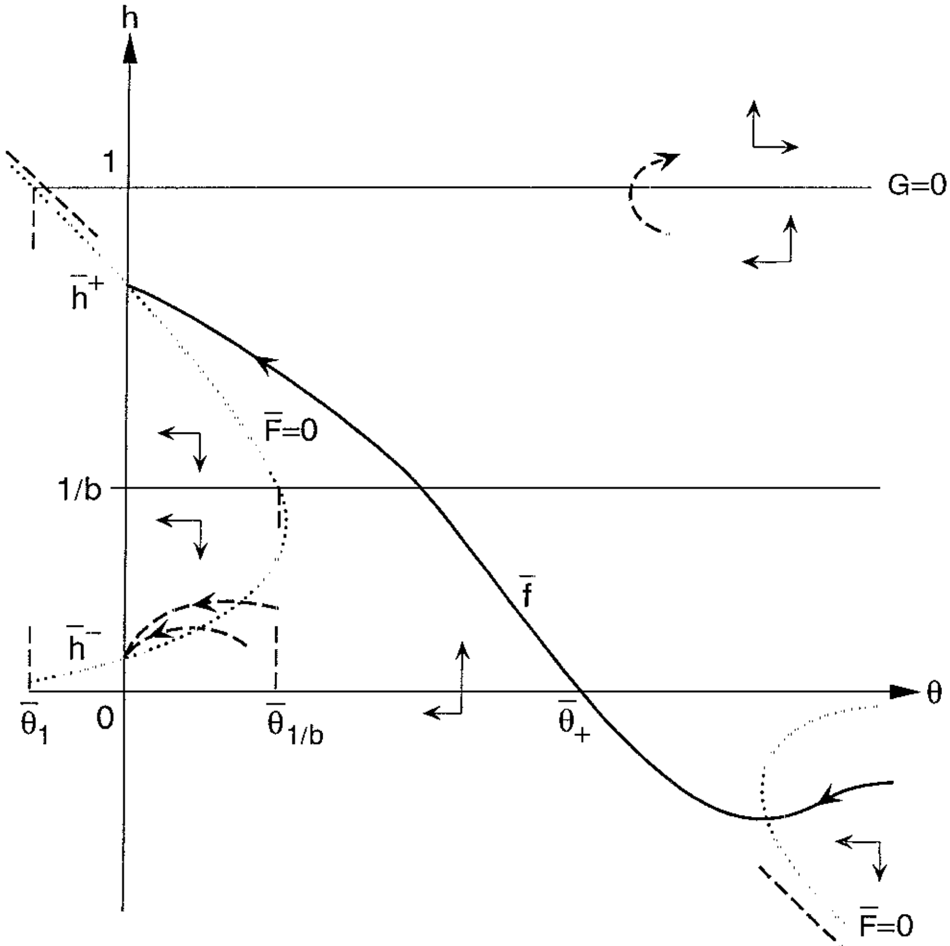

Figure 2(iii): 3–Parameter System, Type 0

$\bar{\theta}_{1/b} > 0 > \bar{\theta}_1$, $b > 1$, $\bar{m} < 0$, $1 > \bar{h}^+ > 1/b > \bar{h}^- > 0$

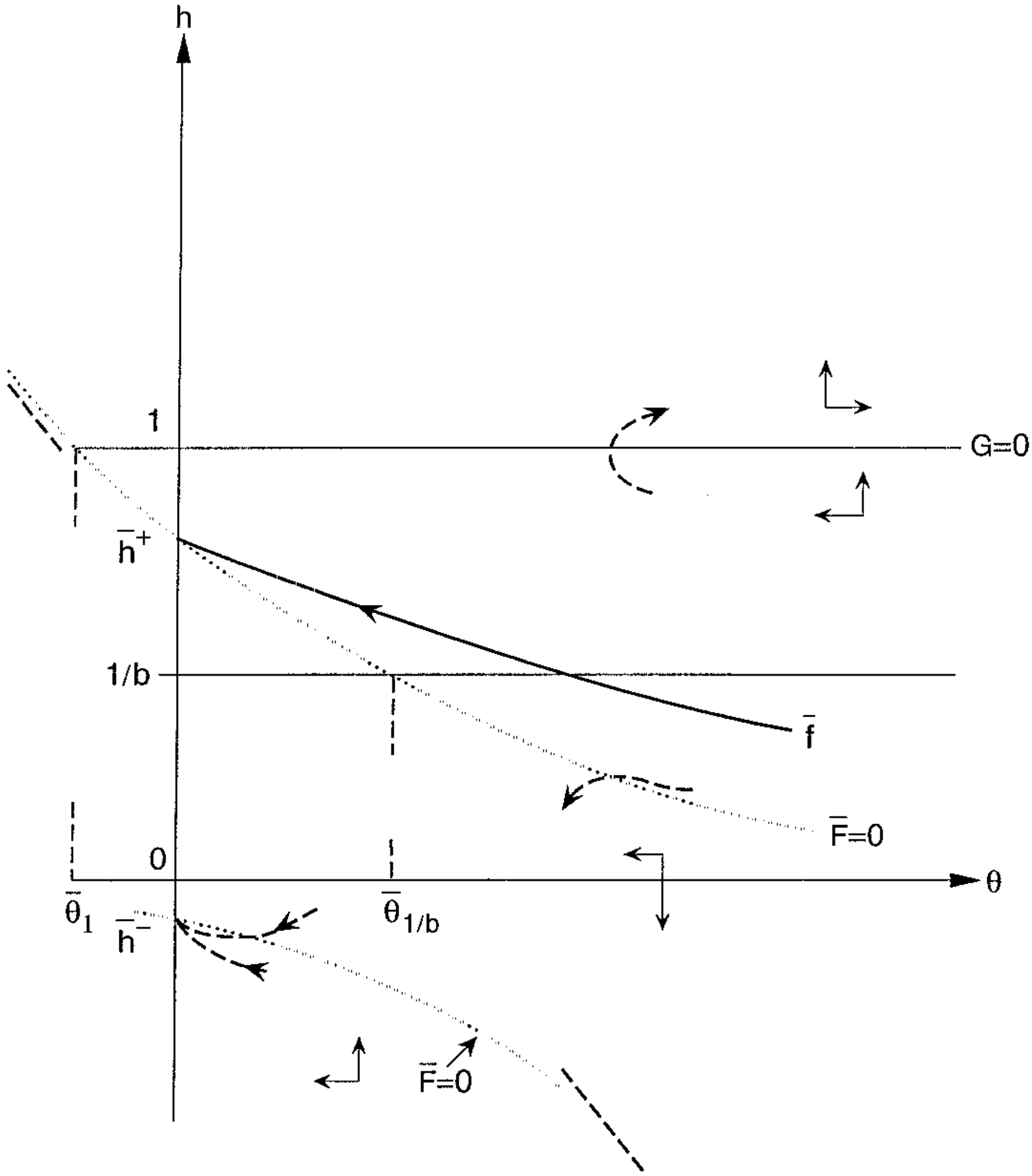

Figure 2(iv): 3–Parameter System, Type 0

$\bar{\theta}_{1/b} > 0 > \bar{\theta}_1,\ b > 1, \bar{m} > 0, 1 > \bar{h}^+ > 1/b > 0 > \bar{h}^-$

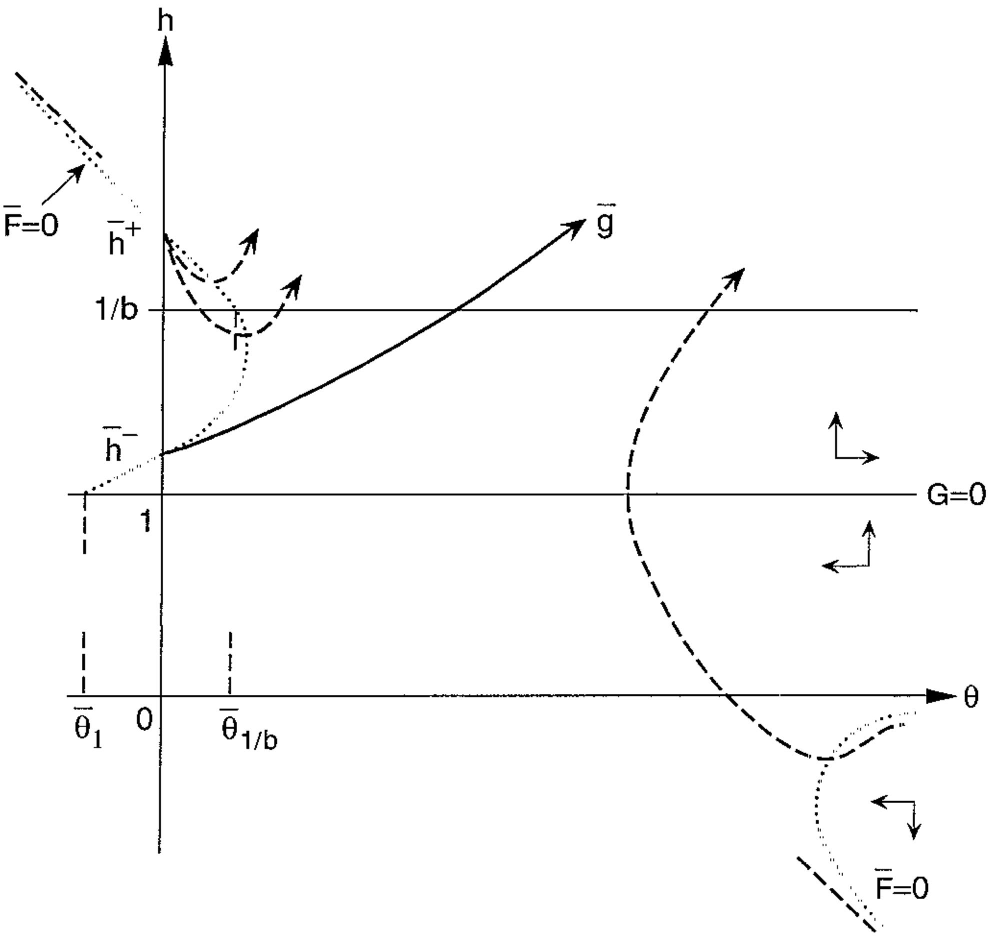

Figure 2(v): 3–Parameter System, Type 0

$\bar{\theta}_{1/b} > 0 > \bar{\theta}_1, \; b < 1, \bar{m} < 0, \bar{h}^+ > 1/b > \bar{h}^- > 1$

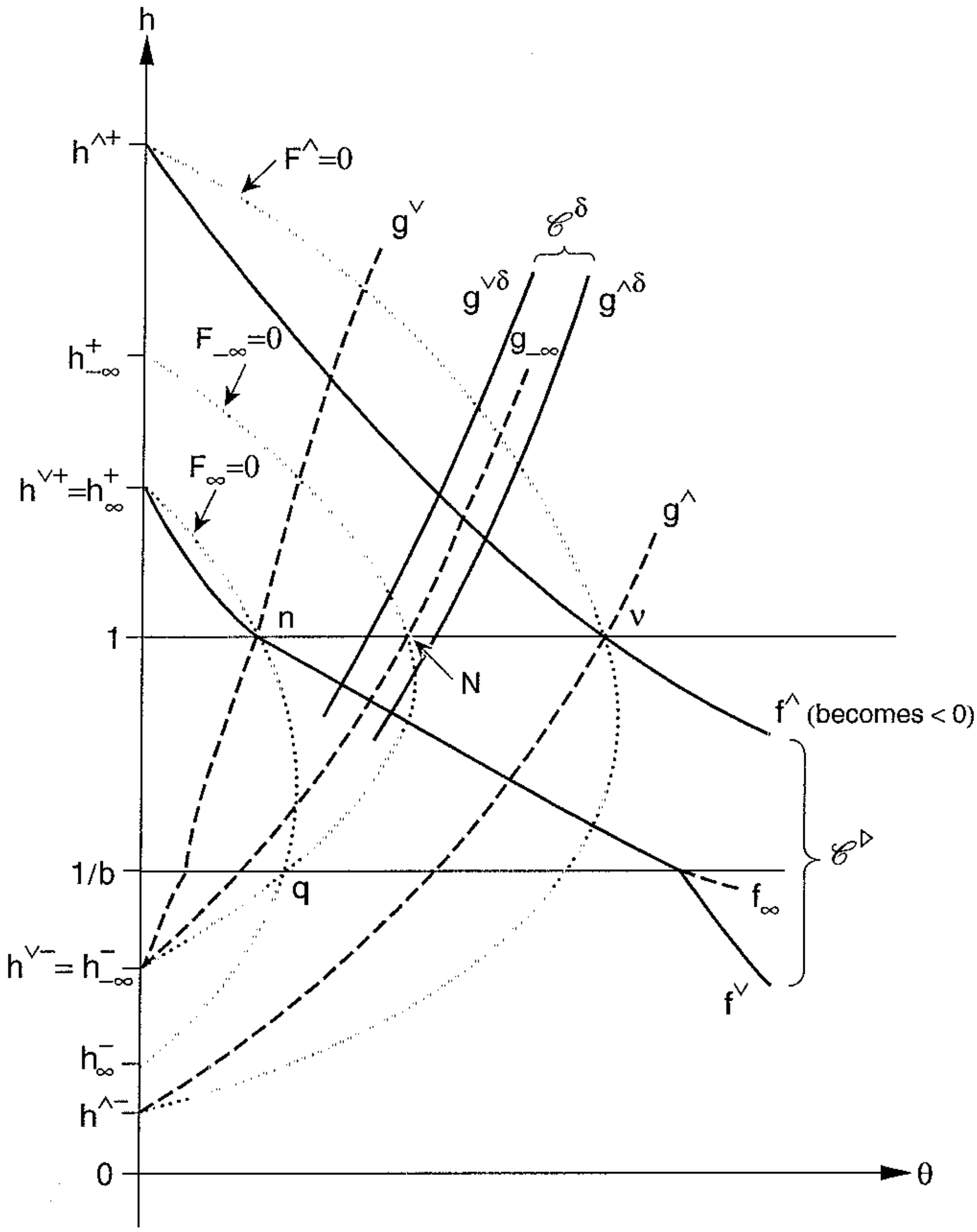

Figure 3(i): 5–Parameter System, Type 1

$N > n > 0,\ b > 1,\ m < 0 < \psi'_o$

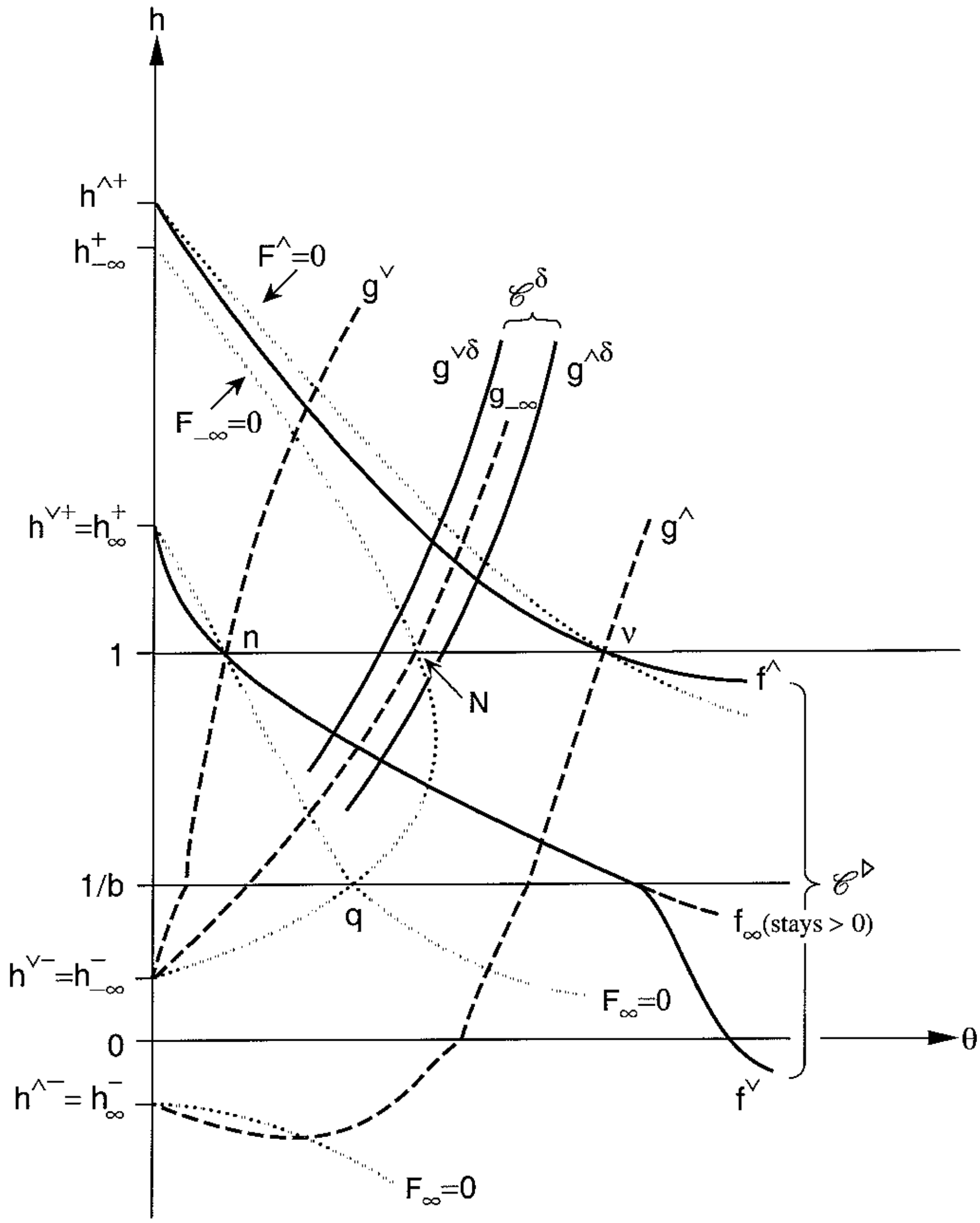

Figure 3(ii): 5–Parameter System, Type 1

$N > n > 0,\ b > 1,\ 0 < m < \psi_o'$

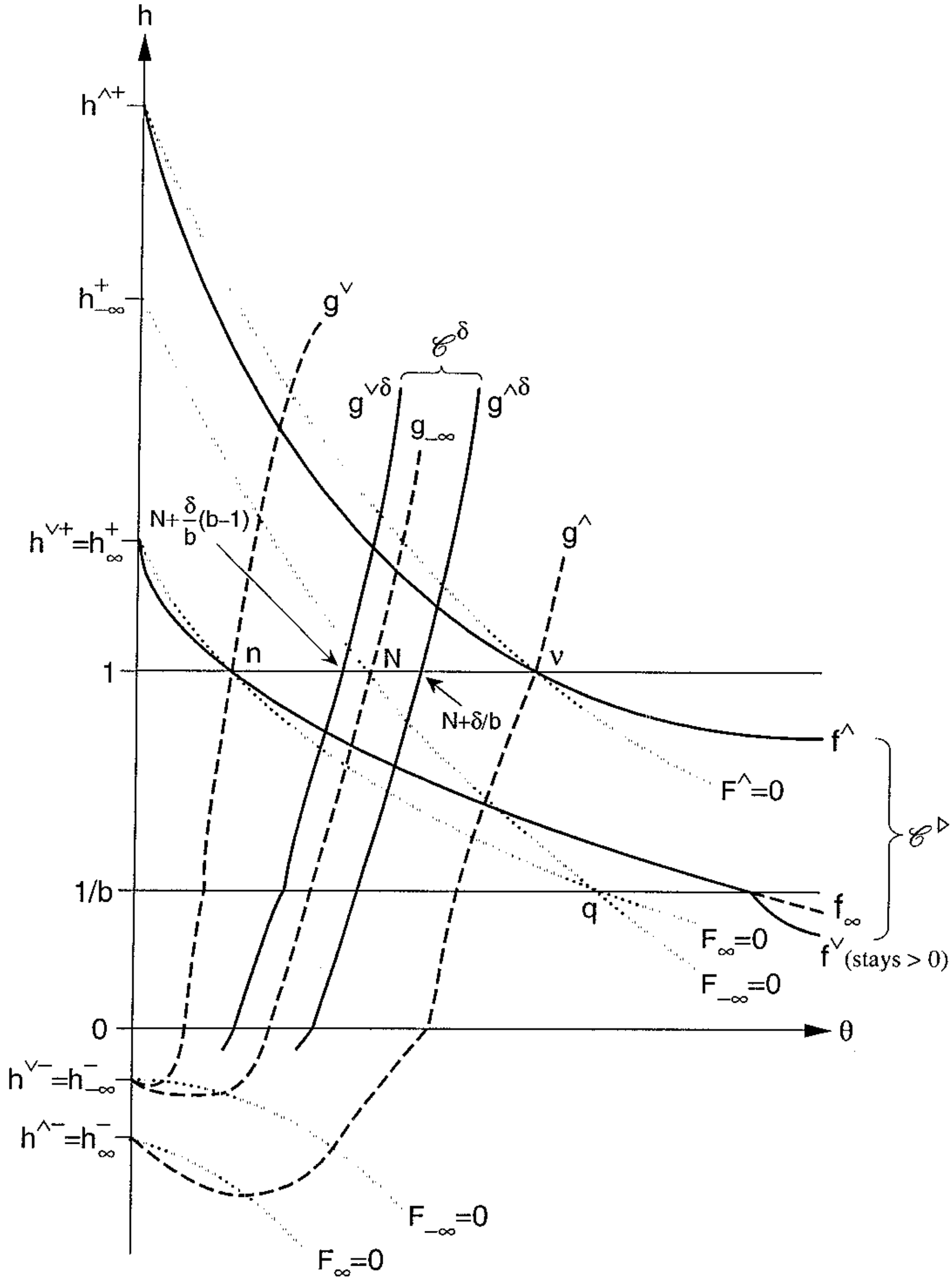

Figure 3(iii): 5–Parameter System, Type 1

$N > n > 0,\ b > 1,\ 0 < \psi_o' < m$

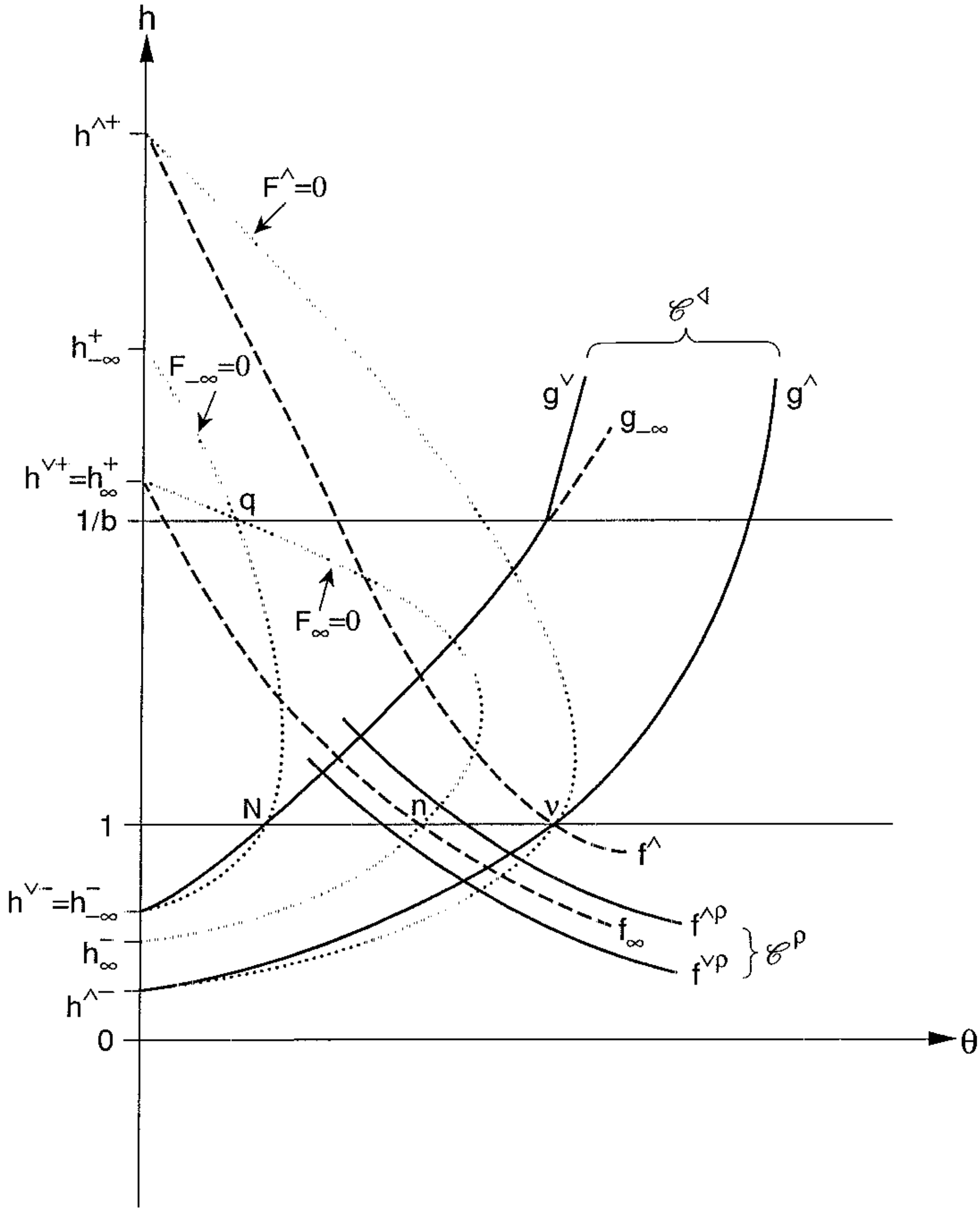

Figure 3(iv): 5–Parameter System, Type 1

$n > N > 0,\ b < 1,\ m < 0 < \psi_o'$

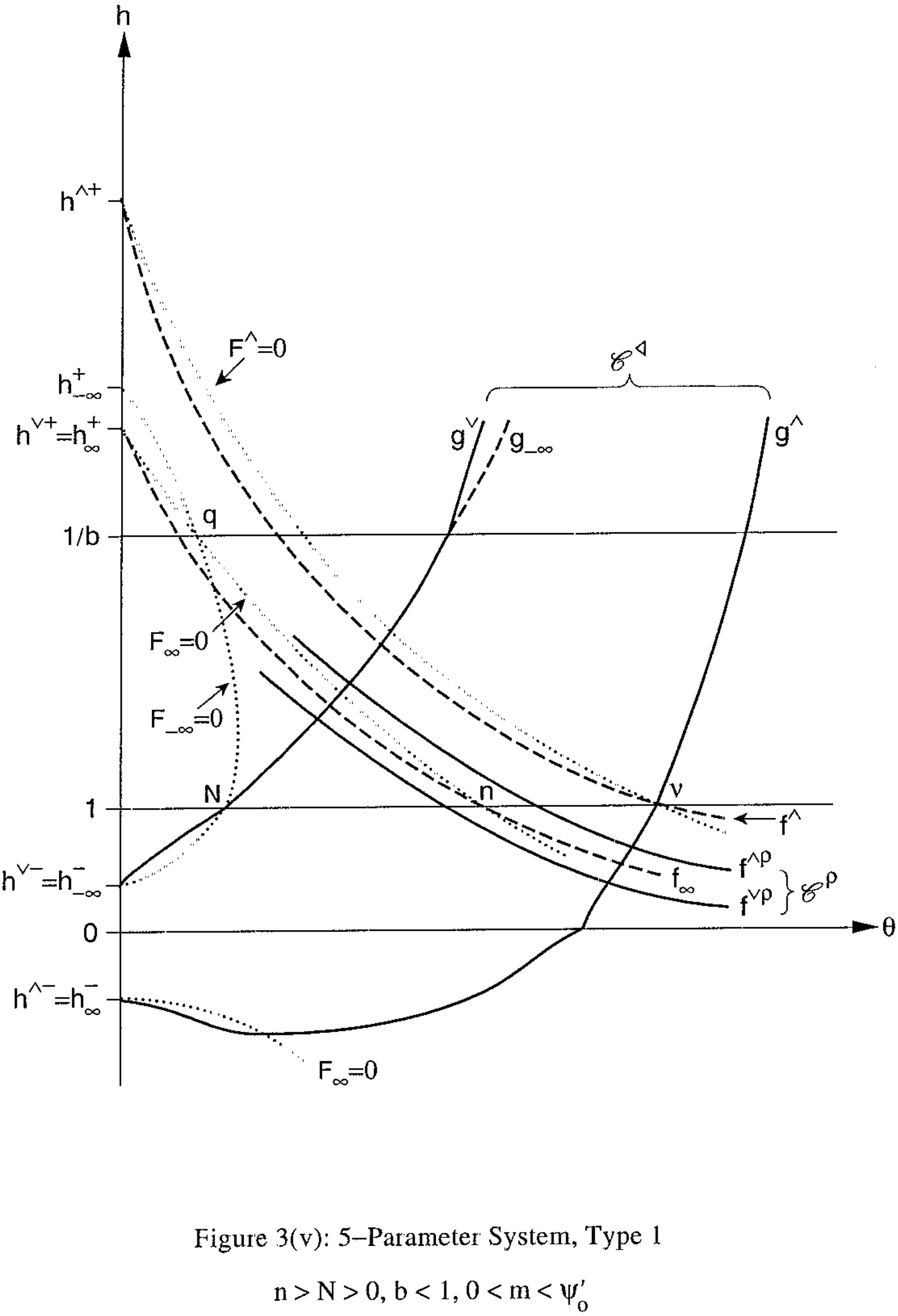

Figure 3(v): 5–Parameter System, Type 1

$n > N > 0,\ b < 1,\ 0 < m < \psi_o'$

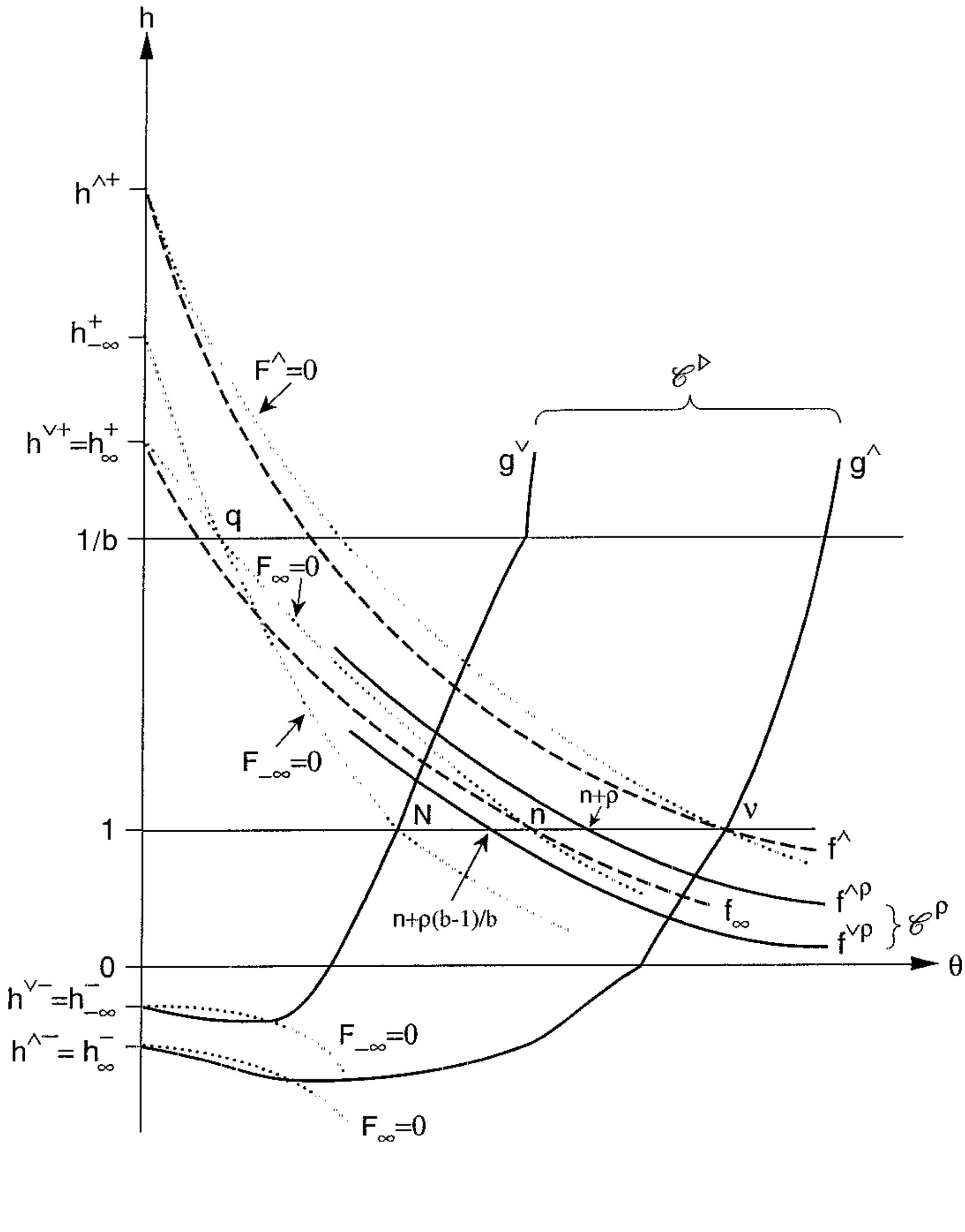

Figure 3(vi): 5–Parameter System, Type 1

$n > N > 0,\ b < 1,\ 0 < \psi_o' < m$

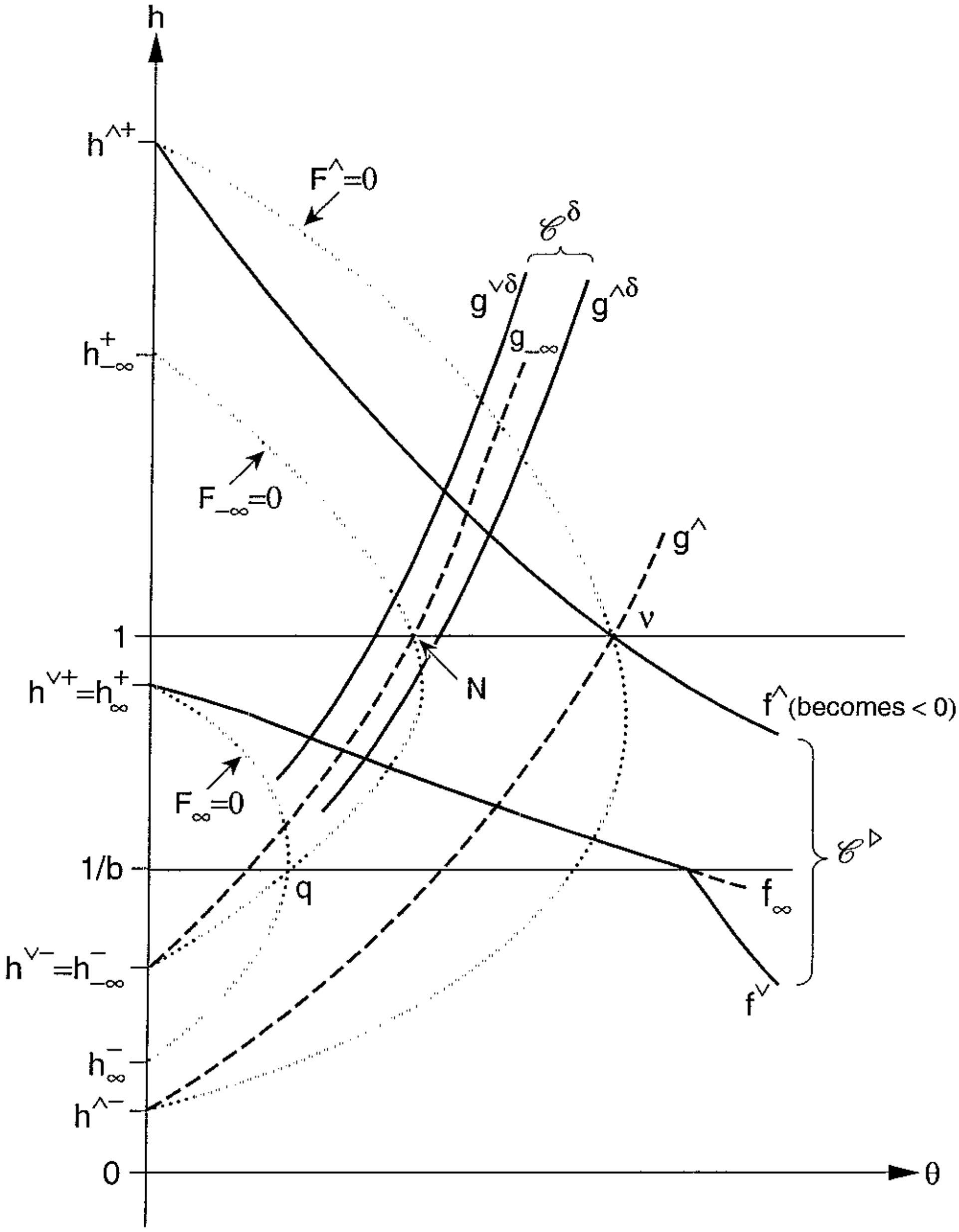

Figure 4(i): 5–Parameter System, Type 0

$N > 0 > n,\ q > 0,\ b > 1,\ m < 0 < \psi'_0$

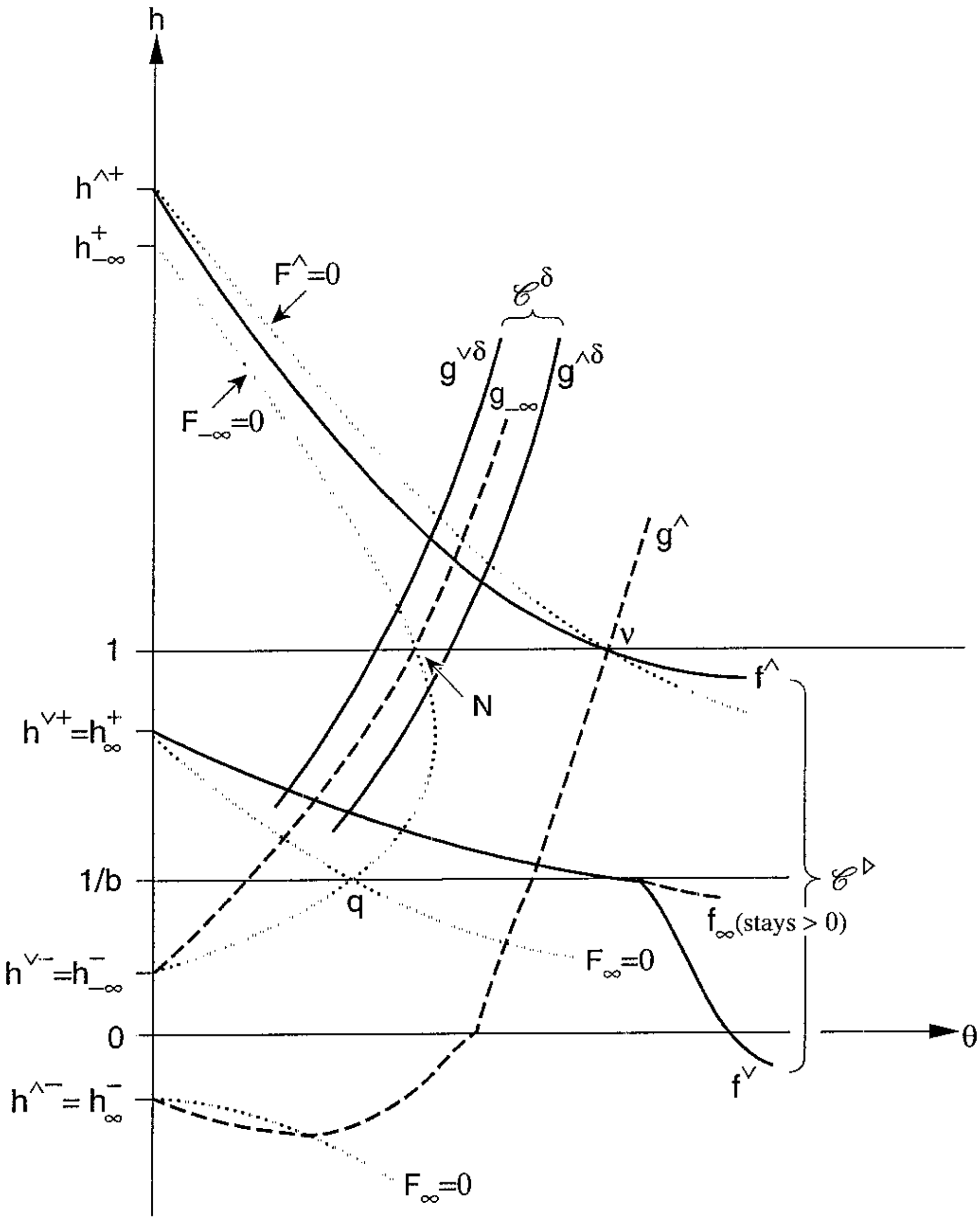

Figure 4(ii): 5–Parameter System, Type 0

$N > 0 > n,\ q > 0,\ b > 1,\ 0 < m < \psi'_o$

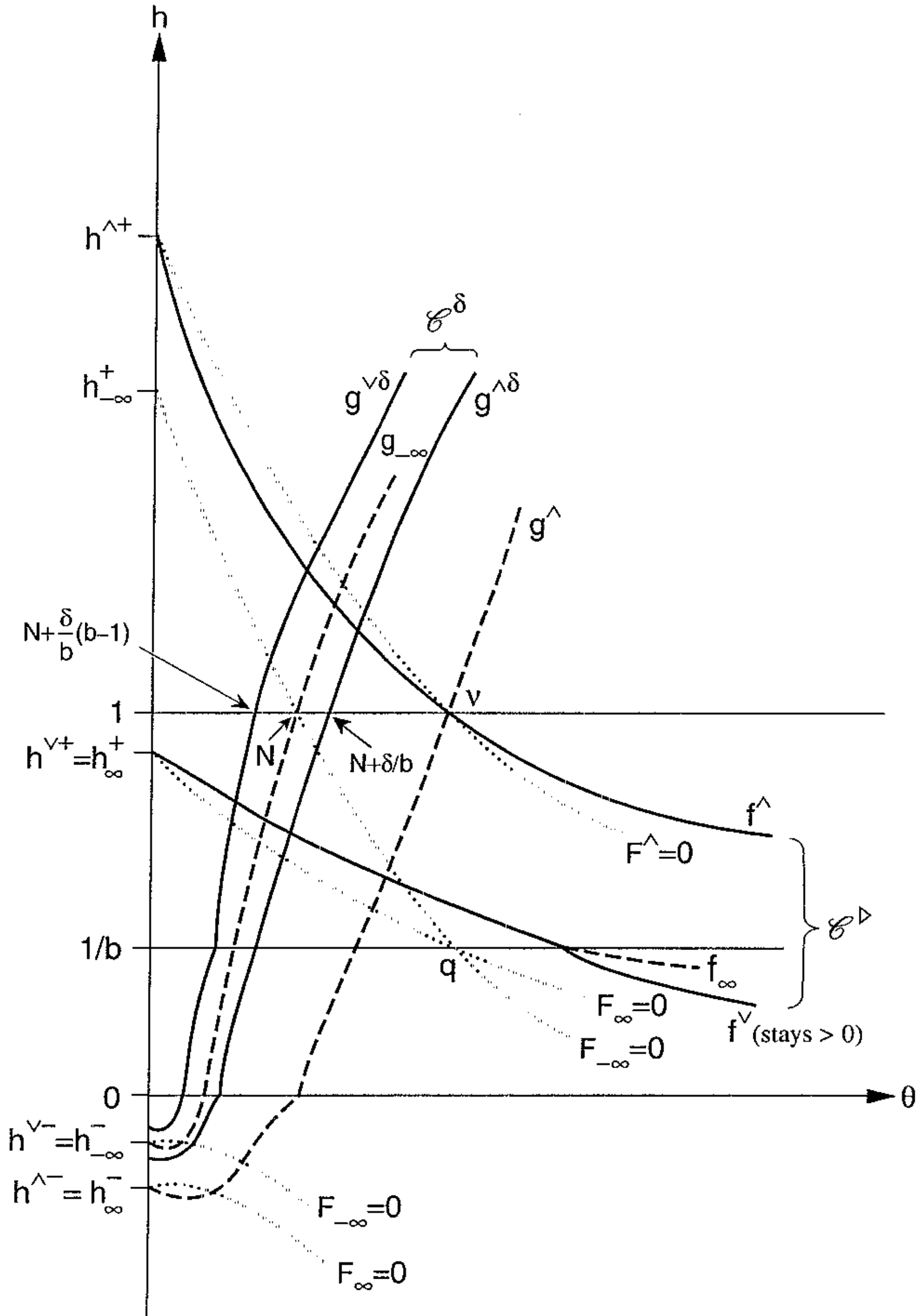

Figure 4(iii): 5–Parameter System, Type 0

$N > 0 > n, q > 0, b > 1, 0 < \psi_o' < m$

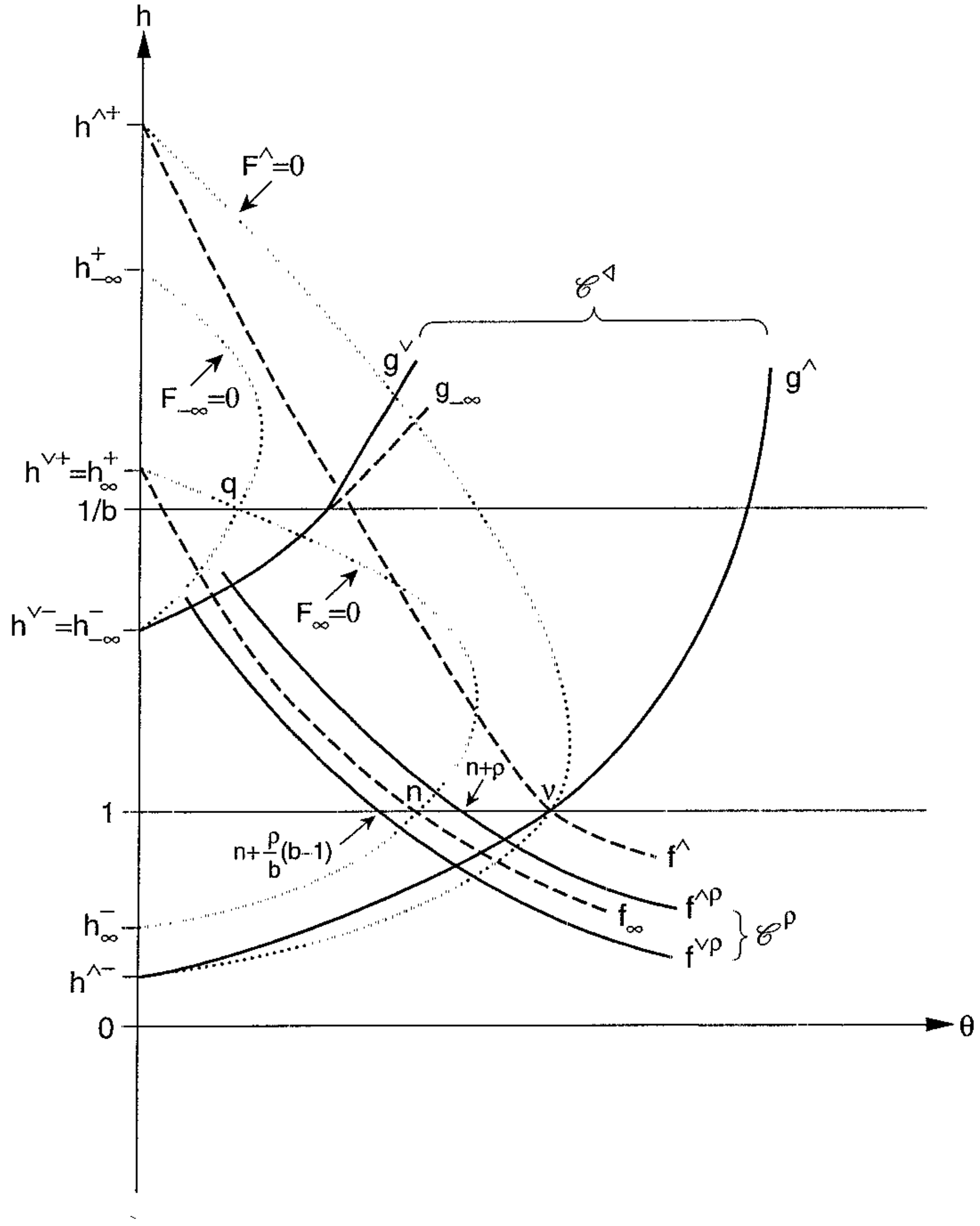

Figure 4(iv): 5–Parameter System, Type 0

$n > 0 > N, q > 0, b < 1, \ m < 0 < \psi_o'$

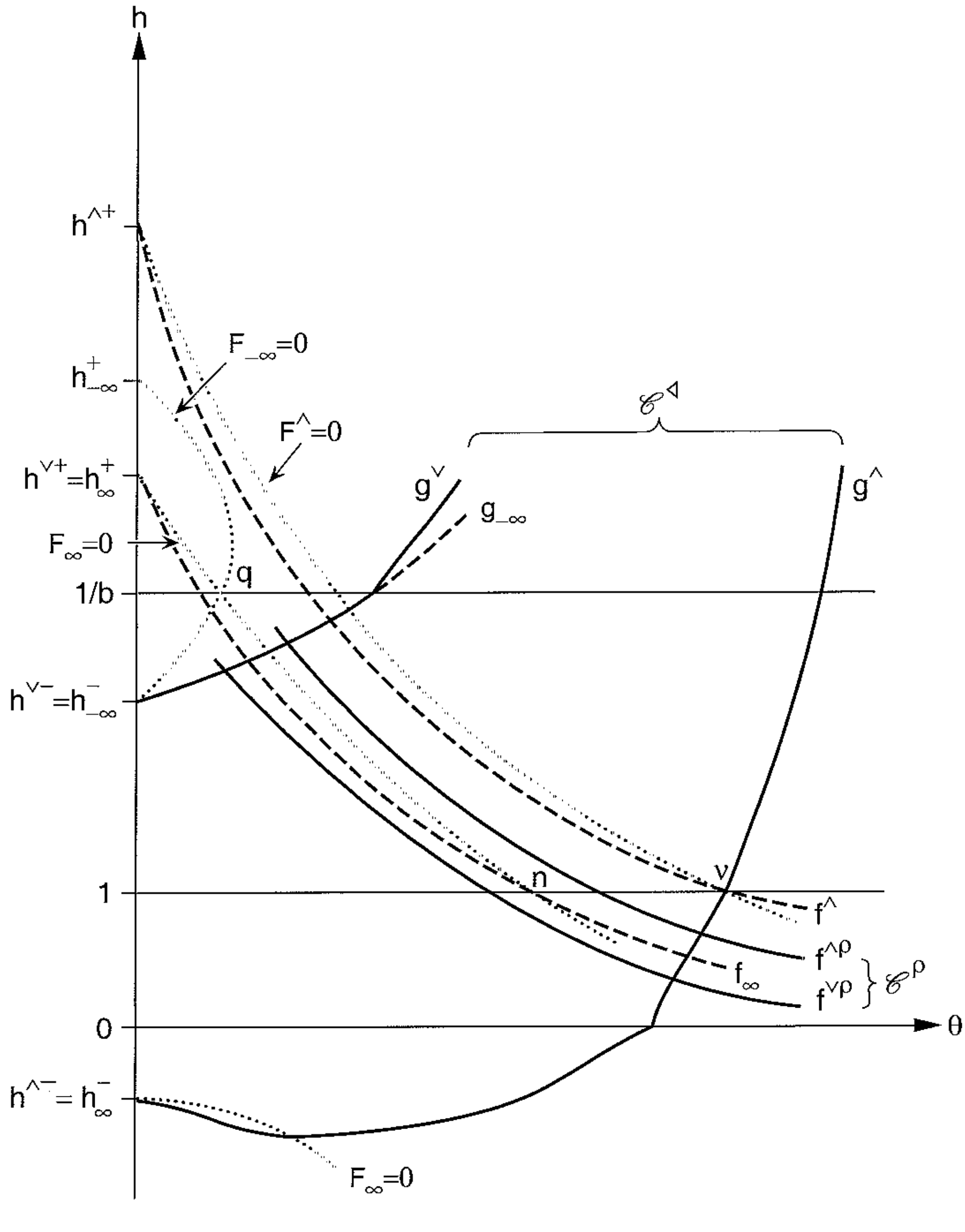

Figure 4(v): 5–Parameter System, Type 0

$n > 0 > N,\ q > 0,\ b < 1,\ 0 < m < \psi'_o$

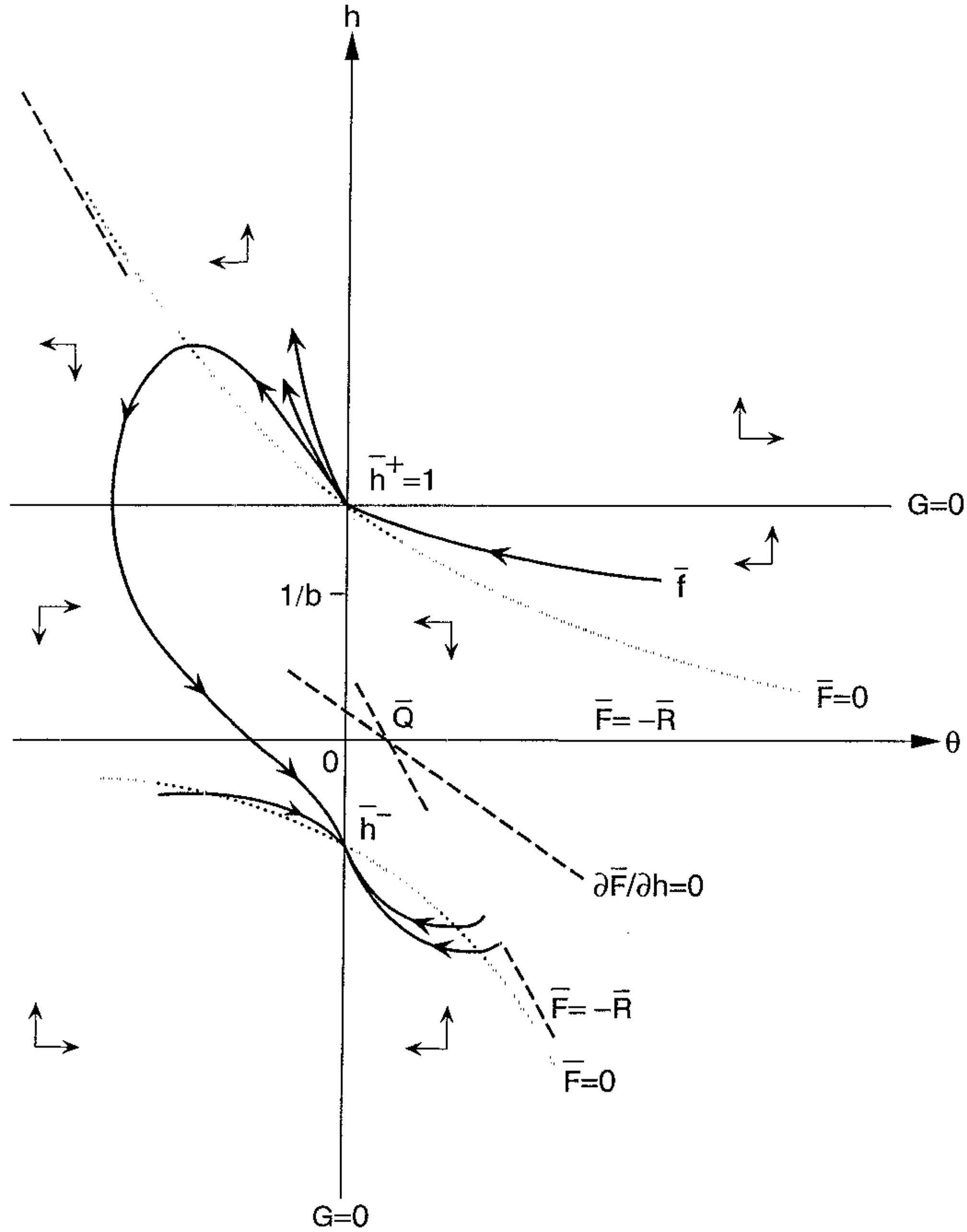

Figure 5: 3–Parameter System, Saddle–Node at (1,0)

$\bar{\theta}_1 = 0$, $b > 1$, $\bar{R} = 2\bar{m}/b\sigma^2 > 0$