DOES COMPETITION SOLVE THE HOLD-UP PROBLEM?*

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Abstract

In an environment in which both buyers and sellers can undertake match specific investments, the presence of market competition for matches may solve hold-up and coordination problems generated by the absence of complete contingent contracts. In particular, this paper shows that when matching is assortative and sellers' investments precede market competition then investments are constrained efficient. One equilibrium is efficient with efficient matches but also there can be equilibria with coordination failures. Different types of efficiency arise when buyers undertake investment before market competition. These inefficiencies lead to buyers' under-investment due to a hold-up problem but, when competition is at its peak, there is a unique equilibrium of the competition game with efficient matches – no coordination failures – and the aggregate hold-up inefficiency is small in a well defined sense, independent of market size.

Keywords: Competition, hold-up problem, matching, specific investments.
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Does Competition Solve the Hold-up Problem?

1. Introduction

A central concern for economists is the extent to which competitive market systems are efficient and, in the idealized Arrow-Debreu model of general equilibrium, efficiency follows under mild conditions, notably the absence of externalities. But in recent years, economists have become interested in studying market situations less idealized than in the Arrow-Debreu set-up and in examining the pervasive inefficiencies that may exist.[1] This paper studies a market situation where there are two potential inefficiencies — these are often referred to as the "hold-up problem" and as "coordination failures". An important part of our analysis will be to examine the connection between, as well as the extent of, the inefficiencies induced by these two problems and whether market competition may solve them.

The hold-up problem applies when a group of agents, e.g. a buyer and a seller, share some surplus from interaction and when an agent making an investment is unable to receive all the benefits that accrue from the investment. The existence of the problem is generally traced to incomplete contracts: with complete contracts, the inefficiency induced by the failure to capture benefits will not be permitted to persist. In the standard set-up of the problem, investments are chosen before agents interact and contracts can be determined only when agents meet. Prior investments will be a sunk cost and negotiation over the division of surplus resulting from an agreement is likely to lead to a sharing of the surplus enhancement made possible by one agent's investment (Williamson 1985, Grout 1984, Grossman and Hart 1986, Hart and Moore 1988).

Coordination failures arise when a group of agents can realise a mutual gain only by a change in behaviour by each member of the group. For instance, a buyer may receive the marginal benefits from an investment when she is matched with any particular seller, so there is no hold-up problem, but she may be inefficiently matched with a seller; the incentive to change the match may not exist because mutual gains may be realised only if the buyer to be displaced is willing to alter her investment to

[1] See Hart (1995) and Holmström (1999) for an extensive discussion of these inefficiencies.

make it appropriate for the new matching.

What happens if agent's interaction is through the marketplace? In an Arrow-Debreu competitive model, complete markets, with price-taking in each market, are assumed; if an agent chooses investment *ex-ante*, every different level of investment may be thought of as providing the agent with a different good to bring to the market (Makowski and Ostroy 1995). If the agent wishes to choose a particular level of investment over some other, and the "buyer" he trades with also prefers to trade with the agent in question, rather than with an "identical" agent with another investment level, then total surplus to be divided must be maximized by the investment level chosen: investment will be efficiently chosen and there is no hold-up problem. In this situation, the existence of complete markets implies that agents know the price that they will receive or pay whatever the investment level chosen: complete markets imply complete contracts. In addition, as long as there are no externalities, coordination failures will not arise as the return from any match is priced in the market and this price is independent of the actions of agents not part to the match.

An unrealistic failure of the Arrow-Debreu set-up is that markets are assumed to exist for every conceivable level of investment, irrespective of whether or not trade occurs in such a market. However, if ex-ante investments are specific to a particular trade in most of these markets there will be no trade. It is then far-fetched to assume that agents will believe that they can trade in inactive markets and, more importantly, that a competitive price will be posted for such markets.

The purpose of this paper is to investigate the efficiency of investments when the trading pattern and terms of trade are determined explicitly by the competition of buyers and sellers. To ensure that there are no inefficiencies resulting from market power, a model of Bertrand competition is analyzed where some agents invest prior to trade; however, this does not rule out the dependence of the pattern of outcomes on the initial investment of any agent and the analysis concentrates on the case of a finite number of traders to ensure this possibility. Contracts are the result of competition in the marketplace and we are interested in the degree to which the hold-up problem and coordination problems are mitigated by contracts that result from Bertrand competition. In this regard, it should be said that we shall not permit Bertrand competition in contingent contracts; in our analysis, contracts take the form of an agreement between a buyer and a seller to trade at a particular price. We are thus investigating the efficiency of contracts implied by a simple trading structure rather than attempting explicitly to devise contracts that help address particular problems (e.g. Aghion, Dewatripont, and Rey 1994, Maskin and Tirole 1999, Segal and Whinston 1998).

We will also restrict attention to markets where the Bertrand competitive outcome is robust to the way that markets are made to clear. Specifically, we assume that buyers and sellers can be ordered by their ability to generate surplus with a complementarity between buyers and sellers. This gives rise to assortative matching in the quality of buyers and sellers. With investment choices, the quality of buyers and/or sellers is assumed to depend on such investments. This set-up has the virtue that, as we will show, the Bertrand outcome is always efficient when investment levels are not subject to choice.

We first consider a world in which only sellers' quality depends on their ex-ante investments, buyers' qualities being exogenously given. In this case we demonstrate that sellers' investment choices are constrained efficient. In particular, for a given equilibrium match, a seller bids just enough to win the right to trade with a buyer and, if he were to have previously enhanced his quality and the value of the trade by extra investment, he would have been able to win the right with the same bid, as viewed by the buyer, and so receive all the marginal benefits of the extra investment. We are able to extend this result to show that, with other agents' behaviour fixed, sellers make efficient investment choices even when they recognise that these actions will lead to a change in match. A consequence of this is that an outcome where all sellers choose efficient investments is an equilibrium in the model.

When the returns of investments in terms of sellers' quality are not too high it is possible that a seller might undertake a high investment with the sole purpose of changing the buyer with whom he will be matched and a byproduct of this will be that another seller is deterred from undertaking investment appropriate to this match.

This may lead to inefficient equilibrium matches. In such an environment, hold-up problems are solved and the only inefficiencies left are due to sellers' pre-emption strategies when choosing their investments — inefficiencies are due to coordination failures. We show that these inefficiencies will not arise if the returns from investments differ enough across sellers.[2]

We then consider a world in which the buyers' quality depends on their ex-ante investments. In this case we indeed show that buyers' investments are inefficient. However, we are able to show that the extent of the inefficiency is limited.

On the one hand, when the competition among sellers for a match is most intense the overall inefficiency in a market is less than that which could result from an under-investment by one (the best) buyer in the market with all other buyers making efficient investments. This result holds irrespective of the number of sellers or buyers in the market. The feature of the Bertrand competition game that determines the intensity of the competition among sellers is the sequential order in which buyers selects their partner to the match. If this order is determined, at an early stage of the game, by the competition among buyers then we demonstrate that, in equilibrium, the order will be such that competition among sellers will be most intense — provided that the returns from buyers' investments differ enough across matches. In other words, competition among buyers lead to a high intensity level of the competition among sellers for a match that limits, in a well defined sense, the inefficiencies generated by the buyers' underinvestment.

On the other hand, surprisingly in this case, when competition among sellers for a match is most intense all coordination problems are solved and the equilibrium matches are the efficient ones: the ordering of the buyers' qualities generated by ex-ante investments coincides with the ordering of buyers' innate qualities. The reason for this is that buyers only reap those gains from an investment that would accrue if they were to be matched with the seller who is the runner-up in the competitive bidding process. Critically, a buyer who through investment changes his place in

[2]For an analysis of how market competition may fail to solve coordination problems see also Hart (1979), Cooper and John (1988) and Makowski and Ostroy (1995).

the quality ranking does not by that change necessarily alter the runner-up and the buyer will ignore gains and losses that come purely from a change of match. Thus, it is the blunted (inefficient) incentives created by a hold-up problem that remove the inefficiencies that come from coordination failures.

The structure of the paper is as follows. After a discussion of related literature in the next section, Section 3 lays down the basic model and the extensive form of the Bertrand competition game between workers (sellers) and firms (buyers). It is then shown in Section 4 that, with fixed investments, the competition game gives rise to an efficient outcome — buyers and sellers match efficiently. Section 5 then investigates the efficiency properties of the model where workers undertake ex-ante investments before competition occurs. We show that workers' investments are efficient given equilibrium matches and that the efficient outcome is always an equilibrium. However, depending on parameters, we show that equilibria with coordination failures may arise that lead to inefficient matches. We then consider in Section 6 the model in which the firms undertake ex-ante investments. We first characterize the inefficient investment choices that will be made. We then show in Section 7 that in equilibrium firms' competition raises the intensity of the workers' competition for a match to its peak. When this is the case the inefficiencies generated by firms' underinvestment are limited in a well defined sense. Section 8 provides concluding remarks.

2. Related Literature

The literature on the hold-up problem has mainly analyzed the bilateral relationship of two parties that may undertake match specific investments in isolation (Williamson 1985, Grout 1984, Grossman and Hart 1986, Hart and Moore 1988). In other words, these papers identify the inefficiencies that the absence of complete contingent contracts may induce in the absence of any competition for the parties to the match.[3]

[3] A notable exception is Bolton and Whinston (1993). This is the first paper to analyze an environment in which an upstream firm (a seller) trades with two downstream firms (two buyers) that undertake ex-ante investments. One of the cases they analyze coincides with the Bertrand competition outcome we identify in our model. However, given that this case of non-integration when only one buyer can be served arises only with an exogenously given probability and that in

This literature identifies the institutional (Grossman and Hart 1986, Hart and Moore 1990, Aghion and Tirole 1997, Rajan and Zingales 1998) or contractual (Aghion, Dewatripont, and Rey 1994, Maskin and Tirole 1999, Segal and Whinston 1998, Che and Hausch 1999) devices that might reduce and possibly eliminate these inefficiencies. We differ from this literature in that we do not alter either the institutional or contractual setting in which the hold-problem arises but rather analyze how competition among different sides of the market may eliminate the inefficiencies associated with such a problem.[4]

The literature on bilateral matching, on the other hand, concentrates on the inefficiencies that arise because of frictions present in the matching process. These inefficiencies may lead to market power (Diamond 1971, Diamond 1982), unemployment (Mortensen and Pissarides 1994) and a class structure (Burdett and Coles 1997, Eeckhout 1999). A recent development of this literature shows how efficiency can be restored in a matching environment thanks to free entry into the market (Roberts 1996, Moen 1997) or Bertrand competition (Felli and Harris 1996). We differ from this literature in that we abstract from any friction in the matching process and focus on the presence of match specific investments by either side of the market.

A small recent literature considers investments in a matching environment. Some of the papers focus on general investments that may be transferred across matches and identify the structure of contracts (MacLeod and Malcomson 1993) or the structure of competition (Holmström 1999) and the market structure (Acemoglu and Shimer 1999, Spulber 2000) that may lead to efficiency. Other papers (Ramey and Watson 1996, Acemoglu 1997) focus on the inefficiencies induced on parties' investments by the presence of an exogenous probability that the match will dissolve. These inefficiencies arise in the presence of incomplete contracts (Ramey and Watson 1996) or even in

case both buyers can be served the gains from trade are equally shared among the seller and the two buyers in equilibrium both buyers under-invest. In other words, the way the surplus is shared in the absence of shortage and the focus on the competition among only two buyers greatly limits the efficiency enhancing effect of competition that is the main focus of our analysis.

[4]It should be said that Che and Hausch (1996) suggests the possibility that competition may enhance parties' incentives to undertake specific investments when involved in a hold-up problem.

the presence of complete but bilateral contracts (Acemoglu 1997).[5] A recent paper by Kranton and Minehart considers, instead, the efficiency of investments in the competitive structure itself (Kranton and Minehart 2000); specifically, markets are limited by the networks that agents create through investment. Finally, two recent papers (Burdett and Coles 1999, Peters and Siow 2000) focus on the efficiency of ex-ante investments in a model in which utility is not transferable across the parties to a match, in other words they analyze marriage problems.

The two papers closest to our analysis are Cole, Mailath, and Postlewaite (2001a) and Cole, Mailath, and Postlewaite (2001b). These are the first papers to provide a detailed analysis of specific investments and market competition for matches. In particular, both papers assume that the two sides of a market first undertake match specific investments and then compete in the market place for a match. The investment choice is modelled as a non-cooperative decision while the matching process is modelled as a cooperative assignment game. Both papers focus on the core of this assignment game. The two sides of the market are assumed to be heterogeneous. In Cole, Mailath, and Postlewaite (2001b) there is a continuum of different types of individuals on both sides of the market. As a result competition for matches occurs among individuals that, before undertaking the investment, are almost perfect substitutes. Conversely, in Cole, Mailath, and Postlewaite (2001a) there is a finite number of different types of individuals on both sides of the market. Hence competition occurs among individuals that in terms of their innate characteristics are, potentially, imperfect substitutes.

Therefore on this dimension Cole, Mailath, and Postlewaite (2001a) is closest to our setting. In such a framework Cole, Mailath and Postlewaite demonstrate the existence of an equilibrium allocation that induces efficient investments as well as allocations that yield inefficiencies. When the numbers of workers (sellers) and firms (buyers) are discrete they are able to uniquely select an equilibrium allocation of

[5] Notice that Ramey and Watson (1996) also consider how matching frictions can alleviate the inefficiencies due to the hold-up problem in the presence of incomplete contracts and match specific investments in an ongoing repeated relationship. See also Ramey and Watson (1997) for a related result.

the matches' surplus yielding efficient investments via a condition defined as 'double-overlapping'. This condition requires the presence of at least two workers (or two firms) with identical innate characteristics; it implies the existence of a perfect substitute for each worker and each firm in the match. In other words competition does not occur among individuals that are fully heterogeneous with respect to their innate characteristics. In this case, both sides to a match obtain exactly their outside option and, at the same time, their most favorable share of the surplus hence efficiency is promoted. In the absence of double-overlapping — therefore when competing individual are fully heterogeneous — equilibrium investments may not be efficient since at least one of the parties to a match is not obtaining the most favorable share of the match surplus. This creates room for equilibria with under-investments though Cole, Mailath, and Postlewaite (2001a) show that, even in the latter case, there exists a sharing rule of the surplus that leads to efficient investments.

Our analysis differs from Cole, Mailath, and Postlewaite (2001a) in that we do not use cooperative game concepts and matching is though a non-cooperative Bertrand competition game. We are also able to analyze the extend of inefficiency under an 'equilibrium' sharing rule. Each firm's outside option is binding for any value of the workers' and firms' innate characteristics. However a worker's outside option is never binding although workers do obtain their most preferred share of the match's surplus. We thus choose a particular model of the competition among fully heterogeneous individuals and thanks to this specific extensive form we are able to provide a bound on the overall inefficiency that arises because of the firms' underinvestments.

Finally de Meza and Lockwood (1998) and Chatterjee and Chiu (1999) also analyze a matching environment with transferable utility in which both sides of the market can undertake match specific investments but focus on a setup that delivers inefficient investments. As a result the presence of asset ownership may enhance welfare (as in Grossman and Hart 1986). In particular, de Meza and Lockwood (1998) consider a repeated production framework and focus on whether one would observe asset trading before or after investment and match formation. Chatterjee and Chiu (1999), on the other hand, analyze a setup in which, as in our case, trade occurs only once. The inefficiency takes the form of the choice of general investments when specific ones would be efficient and arise from the way surplus is shared by the parties to a match when the short side of the market undertakes the investments. They focus on the (possibly adverse) efficiency enhancing effect of ownership of assets. In our setting, given that we obtain efficiency and near-efficiency of investments, we abstract from any efficiency enhancing role of asset ownership.

3. The Framework

We consider a simple matching model: S workers match with T firms, we assume that the number of workers is higher than the number of firms $S > T$.[6] Each firm is assumed to match only with one worker. Workers and firms are labelled, respectively, $s = 1, \ldots, S$ and $t = 1, \ldots, T$. Both workers and firms can make match specific investments, denoted respectively x_s and y_t, incurring costs $C(x_s)$ respectively $C(y_t)$.[7] The cost function $C(\cdot)$ is strictly convex and $C(0) = 0$. The surplus of each match is then a function of the quality of the worker σ and the firm τ involved in the match: $v(\sigma, \tau)$. Each worker's quality is itself a function of the worker innate ability, indexed by the worker's identity s, and the worker specific investment x_s: $\sigma(s, x_s)$. In the same way, we assume that each firm's quality is a function of the firm's innate ability, indexed by the firm's identity t, and the firm's specific investment y_t: $\tau(t, y_t)$.

We assume *complementarity* of the qualities of the worker and the firm involved in a match. In other words, the higher is the quality of the worker and the firm the higher is the surplus generated by the match:[8] $v_1(\sigma, \tau) > 0$, $v_2(\sigma, \tau) > 0$. Further, the marginal surplus generated by a higher quality of the worker or of the firm in the match increases with the quality of the partner: $v_{12}(\sigma, \tau) > 0$. We also assume

[6] We label the two sides of the market workers and firms only for expositional convenience they could be easily re-labelled buyers and sellers without any additional change.

[7] For simplicity we take both cost functions to be identical, none of our results depending on this assumption. If the cost functions were type specific we would require the marginal costs to increase with the identity of the worker or the firm.

[8] For convenience we denote with $v_l(\cdot, \cdot)$ the partial derivative of the surplus function $v(\cdot, \cdot)$ with respect to the l-th argument and with $v_{lk}(\cdot, \cdot)$ the cross-partial derivative with respect to the l-th and k-th argument or the second-partial derivatives if $l = k$. We use the same notation for the functions $\sigma(\cdot, \cdot)$ and $\tau(\cdot, \cdot)$ defined below.

that the quality of the worker depends negatively on the worker's innate ability s, $\sigma_1(s, x_s) < 0$ (so that worker $s = 1$ is the highest ability worker) and positively on the worker's specific investment x_s. Similarly, the quality of a firm depends negatively on the firm's innate ability t, $\tau_1(t, y_t) < 0$, (firm $t = 1$ is the highest ability firm) and positively on the firm's investment y_t: $\tau_2(t, y_t) > 0$. Finally we assume that the quality of both the workers and the firms satisfy a *single crossing condition* requiring that the marginal productivity of both workers and firms investments decreases in their innate ability index: $\sigma_{12}(s, x_s) < 0$ and $\tau_{12}(t, y_t) < 0$.

The combination of the assumption of complementarity and the single crossing condition gives a particular meaning to the term specific investments we used for x_s and y_t. Indeed, in our setting the investments x_s and y_t have a use and value in matches other than (s, t); however, these values decrease with the identity of the partner implying that at least one component of this value is specific to the match in question, since we consider a discrete number of firms and workers.

We also assume that the surplus of each match is concave in the workers and firms quality — $v_{11} < 0$, $v_{22} < 0$ — and that the quality of both firms and workers exhibit decreasing marginal returns in their investments: $\sigma_{22} < 0$ and $\tau_{22} < 0$.[9]

In Section 7 below we need stronger assumptions on the responsiveness of firms' investments to both the workers' and firms' identities and on each match surplus function.

The first assumption, labelled *responsive complementarity*, can be described as follows. For a given level of worker's investment x_s, denote $y(t, s)$ firm t' efficient investment when matched with worker s defined as:

$$y(t, s) = \underset{y}{\text{argmax}}\, v(\sigma(s), \tau(t, y)) - C(y) \tag{1}$$

[9]As established in Milgrom and Roberts (1990), Milgrom and Roberts (1994) and Edlin and Shannon (1998) our results can be derived with much weaker assumptions on the smoothness and concavity of the surplus function $v(\cdot, \cdot)$ and the two quality functions $\sigma(\cdot, \cdot)$ and $\tau(\cdot, \cdot)$ in the two investments x_s and y_t.

In other words $y(t,s)$ satisfies:

$$v_2(\sigma(s), \tau(t, y(t,s)))\ \tau_2(t, y(t,s)) = C'(y(t,s)) \tag{2}$$

where $C'(\cdot)$ is the first derivative of the cost function $C(\cdot)$. Then firm t's investment $y(t,s)$ satisfies responsive complementarity if and only if:

$$\frac{\partial}{\partial t}\left(\frac{\partial y(t,s)}{\partial s}\right) > 0. \tag{3}$$

In other words:

$$\frac{\partial}{\partial t}\left(-\frac{v_{12}\ \sigma_1\ \tau_2}{v_{22}(\tau_2)^2 + v_2\ \tau_{22} - C''}\right) > 0 \tag{4}$$

where the first and second order derivatives τ_2 and τ_{22} are computed at $(t, y(t,s))$, the derivatives v_h and v_{hk}, $h, k \in \{1, 2\}$ are computed at $(\sigma(s), \tau(t, y_t(s)))$ and C'' is the second derivative of the cost function $C(\cdot)$ computed at $y(t,s)$.

We label the second assumption *marginal complementarity*. This assumption requires that the marginal surplus generated by a higher firm's quality satisfies:

$$\frac{\partial^2 v_2(\sigma, \tau)}{\partial\sigma\ \partial\tau} > 0. \tag{5}$$

or $v_{122} > 0$. Notice that both responsive and marginal complementarity, and the other conditions that we have imposed, are satisfied by a standard iso-elastic specification of the model.

We analyze different specifications of our general framework.

We first characterize (Section 4 below) the equilibrium of the Bertrand competition game for given vectors of firms' and workers' qualities.

We then move (Section 5 below) to the analysis of the workers' investment choice in a model in which only the workers choose ex-ante match specific investments x_s that determine the quality of each worker $\sigma(s, x_s)$ while firms are of exogenously given qualities: $\tau(t)$.

We conclude (Section 6 and 7 below) with the analysis of the firms' investment choice in the model in which only firms choose ex-ante match specific investments y_t that determine each firm t's quality $\tau(t, y_t)$ while workers are of exogenously given quality $\sigma(s)$.

The case in which both firms and workers undertake ex-ante investments is briefly discussed in the conclusions.

We assume the following extensive forms of the Bertrand competition game in which the T firms and the S workers engage. Workers Bertrand compete for firms. All workers simultaneously and independently make wage offers to every one of the T firms. Notice that we allow workers to make offers to more than one, possibly all firms. Each firm observes the offers she receives and decides which offer to accept. We assume that this decision is taken sequentially in the order of a given permutation $(t_1, \ldots, t_T)$ of the vector of firms' identities $(1, \ldots, T)$. In other words the firm labelled t_1 decides first which offer to accept. This commits the worker selected to work for firm t_1 and automatically withdraws all offers this worker made to other firms. All other firms and workers observe this decision and then firm t_2 decides which offer to accept. This process is repeated until firm t_T decides which offer to accept. Notice that since $S > T$ even firm t_T, the last firm to decide, can potentially choose among multiple offers.

In Sections 5 and 6 below we focus mainly on the case in which firms choose their bids in the decreasing order of their identity (innate ability): $t_n = n$, for all $n = 1, \ldots, T$. We justify this choice in Section 4 below.

We look for the *trembling-hand-perfect* equilibria of our model. Notice that in the extensive form we just described there exists an asymmetry between the timing of workers's bids (they are all simultaneously submitted at the beginning of the Bertrand competition subgame) and the timing of each firm choice of the bid to accept (firms choose their most preferred bid sequentially in a given order). This implies that while in equilibrium it is possible that a firm's choice between two identical bids is uniquely determined this is not any more true following a deviation of a worker whose bid in equilibrium is selected by a firm who gets to choose her most preferred bid at an earlier stage of the subgame. To prevent firms from deviating when choosing among identical bids following a worker's deviation that possibly does not even affect the equilibrium bids submitted to the firm in question we modify the extensive form in the following way. We allow workers, when submitting a bid, to state that they are prepared to bid more if this becomes necessary. In the construction of the trembling-hand-perfect equilibrium we then restrict the totally mixed strategy of each firm to be such that each firm selects bids starting with a higher-order probability on the highest bidders and allocates a lower-order probability of being selected on a bid submitted by a worker that did not specify such a proviso.[10]

4. Bertrand Competition

We now proceed to characterize the equilibria of the model described in Section 3 above solving it backwards. In particular we start from the characterization of the equilibrium of the Bertrand competition subgame. In doing so we take the investments and hence the qualities of both firms and workers for given.

To simplify the analysis below let τ_1 be the quality of firm t_1 that, as described in Section 3 above, is the first firm to choose her most preferred bid in the Bertrand competition subgame. In a similar way, denote τ_n the quality of firm t_n, $n = 1, \ldots, T$, that is the n-th firm to choose her most preferred bid. The vector of firms' qualities is then $(\tau_1, \ldots, \tau_T)$.

We first identify an efficiency property of any equilibrium of the Bertrand competition subgame. All the equilibria of the Bertrand competition subgame exhibit *positive assortative matching*. In other words, for given investments, matches are efficient: the worker characterized by the k-th highest quality matches with the firm characterized by the k-th highest quality.

Lemma 1: *Every equilibrium of the Bertrand competition subgame is such that every pair of equilibrium matches (σ', τ_i) and (σ'', τ_j), $i, j \in \{1, \ldots, T\}$ satisfies the property: If $\tau_i > \tau_j$ then $\sigma' > \sigma''$.*

[10] This modification of the extensive form is equivalent to a Bertrand competition model in which there exists an indivisible smallest possible unit of a bid (a penny) so that each worker can break any tie by bidding one penny more than his opponent if he wishes to do so.

Proof: Assume by way of contradiction that the equilibrium matches are not assortative. In other words, there exist a pair of equilibrium matches (σ'', τ_i) and (σ', τ_j) such that $\tau_i > \tau_j$, and $\sigma' > \sigma''$. Denote $B(\tau_i)$, respectively $B(\tau_j)$, the bids that in equilibrium the firm of quality τ_i, respectively of quality τ_j, accepts.

Consider first the match (σ'', τ_i). For this match to occur in equilibrium we need that it is not convenient for the worker of quality σ'' to match with the firm of quality τ_j rather than τ_i. If worker σ'' deviates and does not submit a bid that will be selected by firm τ_i then two situations may occur depending on whether the firm of quality τ_i chooses her bid before, $(i < j)$, or after $(i > j)$, the firm of quality τ_j. In particular if τ_i chooses her bid before τ_j then following the deviation of the worker of quality σ'' a different worker will be matched with firm τ_i. Then the competition for the firm of quality τ_{i+1} will be won either by the same worker as in the absence of the deviation or, if that worker has already been matched, by another worker who now would not be bidding for subsequent firms. Repeating this argument for subsequent firms we conclude that when following a deviation by worker σ'' it is the turn of the firm of quality τ_j to choose her most preferred bid the set of unmatched workers, excluding worker σ'', is depleted of exactly one worker, if compared with the set of unmatched workers when in equilibrium the firm of quality τ_j chooses her most preferred bid. Hence the maximum bids of these workers $\hat{B}(\tau_j)$ cannot be higher than the equilibrium bid $B(\tau_j)$ of the worker of quality σ': $\hat{B}(\tau_j) \leq B(\tau_j)$.[11]

Therefore for (σ'', τ_i) to be an equilibrium match we need that

$$v(\sigma'', \tau_i) - B(\tau_i) \geq v(\sigma'', \tau_j) - \hat{B}(\tau_j)$$

[11]Notice that we can conclude that following a deviation by worker σ'' the bid accepted by firm τ_j is not higher than $B(\tau_j)$ since — as discussed in Section 3 above — we allow workers to specify in their bid that they are willing to increase such a bid if necessary. Moreover we restrict the totally mixed strategy used by each firm so as to put higher order probabilities on the bids that contain this proviso. In the absence of these restrictions it is possible to envisage a situation in which following a deviation by worker σ'' the firms that select their bid after firm τ_i and before firm τ_j may no longer choose among equal bids the one submitted by the worker with the highest willingness to pay. The result is then that the bid accepted by firm τ_j following a deviation might actually be higher than $B(\tau_j)$. Notice that this problem disappears if we assume that there exists a smallest indivisible unit of a bid (see also Footnote 10 above).

or given that, as argued above, $\hat{B}(\tau_j) \leq B(\tau_j)$ we need that the following necessary condition is satisfied:

$$v(\sigma'', \tau_i) - B(\tau_i) \geq v(\sigma'', \tau_j) - B(\tau_j) \tag{6}$$

Alternatively if τ_i chooses her bid after τ_j then for (σ'', τ_i) to be an equilibrium match we need that worker σ'' does not find convenient to deviate and outbid the worker of quality σ' by submitting bid $B(\tau_j)$. This equilibrium condition therefore coincides with (6) above.

Consider now the equilibrium match (σ', τ_j). For this match to occur in equilibrium we need that the worker of quality σ' does not want to deviate and be matched with the firm of quality τ_i rather than τ_j. As discussed above, depending on whether the firm of quality τ_j chooses her bid before, $(j < i)$, or after, $(j > i)$, the firm of quality τ_i, the following is a necessary condition for (σ', τ_j) to be an equilibrium match:

$$v(\sigma', \tau_j) - B(\tau_j) \geq v(\sigma', \tau_i) - B(\tau_i). \tag{7}$$

The inequalities (6) and (7) imply:

$$v(\sigma'', \tau_i) + v(\sigma', \tau_j) \geq v(\sigma', \tau_i) + v(\sigma'', \tau_j). \tag{8}$$

Condition (8) contradicts the complementarity assumption $v_{12}(\sigma, \tau) > 0$. ■

Notice that, as argued in Section 5 and 6 below, Lemma 1 does not imply that the order of firms' qualities, which are endogenously determined by firms' investments, coincides with the order of firms' identities (innate abilities).

Using Lemma 1 above we can now label workers' qualities in a way that is consistent with the way firms' qualities are labelled. Indeed, Lemma 1 defines an equilibrium relationship between the quality of each worker and the quality of each firm. We can therefore denote σ_n, $n = 1, \ldots, T$ the quality of the worker that in equilibrium matches with firm τ_n. Furthermore, we denote $\sigma_{T+1}, \ldots, \sigma_S$ the qualities of the workers that in equilibrium are not matched with any firm and assume that these qualities are ordered so that $\sigma_i > \sigma_{i+1}$ for all $i = T+1, \ldots, S-1$.

Consider now stage t of the Bertrand competition subgame characterized by the fact that the firm of quality τ_t chooses her most preferred bid. The workers that are still unmatched at this stage of the subgame are the ones with qualities $\sigma_t, \sigma_{t+1}, \ldots, \sigma_S$. We define the *runner-up* worker to the firm of quality τ_t to be the worker, among the ones with qualities $\sigma_{t+1}, \ldots, \sigma_S$, who has the highest willingness to pay for a match with firm τ_t. We denote this worker $r(t)$ and his quality $\sigma_{r(t)}$. Clearly $r(t) > t$.

This definition can be used recursively so as to define the runner-up worker to the firm that is matched in equilibrium with the runner-up worker to the firm of quality τ_t. We denote this worker $r^2(t) = r(r(t))$ and his quality $\sigma_{r^2(t)}$: $r^2(t) > r(t) > t$. In an analogous way we can then denote $r^k(t) = r(r^{k-1}(t))$ for every $k = 1, \ldots, \rho_t$ where $r^k(t) > r^{k-1}(t)$, $r^1(t) = r(t)$ and $\sigma_{r^{\rho_t}(t)}$ is the quality of the last workers in the chain of runner-ups to the firm of quality τ_t.

We have now all the elements to provide a characterization of the equilibrium of the Bertrand competition subgame. In particular we first identify the runner-up worker to every firm and the difference equation satisfied by the equilibrium payoffs to all firms and workers. This is done in the following lemma.

Lemma 2: *The runner-up worker to the firm of quality τ_t, $t = 1, \ldots, T$, is the worker of quality $\sigma_{r(t)}$ such that:*

$$\sigma_{r(t)} = \max\left\{\sigma_i \mid i = t+1, \ldots, S \text{ and } \sigma_i \leq \sigma_t\right\}. \tag{9}$$

Further the equilibrium payoffs to each firm and each worker are such that for every $t = 1, \ldots, T$:

$$\begin{aligned} \pi^W_{\sigma_t} &= [v(\sigma_t, \tau_t) - v(\sigma_{r(t)}, \tau_t)] + \pi^W_{\sigma_{r(t)}} && (10) \\ \pi^F_{\tau_t} &= v(\sigma_{r(t)}, \tau_t) \quad - \quad \pi^W_{\sigma_{r(t)}} && (11) \end{aligned}$$

and for every $i = T+1, \ldots, S$:

$$\pi^W_{\sigma_i} = 0 \tag{12}$$

We present the formal proof of this result in the Appendix. Notice however that equation (9) identifies the runner-up worker of the firm of quality τ_t as the worker — other than the one that in equilibrium matches with firm τ_t — which has the highest quality among the workers with qualities lower than σ_t that are still unmatched at stage t of the Bertrand competition subgame. For any firm of quality τ_t it is then possible to construct a chain of runner-up workers: each one the runner-up worker to the firm that in equilibrium is matched with the runner-up worker that is ahead in the chain. Equation (9) implies that for every firm the last worker in the chain of runner-up workers is the worker of quality σ_{T+1}. This is the highest quality worker among the ones that in equilibrium do not match with any firm. In other words every chain of runner-up workers has at least one worker in common.

Given that workers Bertrand compete for firms, each firm will not be able to capture all the match surplus but only her outside option that is determined by the willingness to pay of the runner-up worker to the firm. This willingness to pay is the difference between the surplus of the match between the runner-up worker and the firm in question and the payoff the runner-up worker obtains in equilibrium if he is not successful in his bid to the firm: the difference equation in (11). Given that the quality of the runner-up worker is lower than the quality of the worker the firm is matched with in equilibrium the share of the surplus each firm is able to capture does not coincide with the entire surplus of the match. The payoff to each worker is then the difference between the surplus of the match and the runner-up worker's bid: the difference equation in (10).

The characterization of the equilibrium of the Bertrand competition subgame is summarized in the following proposition.

Proposition 1: *For any given vector of firms' qualities* $(\tau_1, \ldots, \tau_T)$ *and corresponding vector of workers' qualities* $(\sigma_1, \ldots, \sigma_S)$, *the unique equilibrium of the Bertrand* *competition subgame is such that every pair of equilibrium matches* (σ_i, τ_i) *and* (σ_j, τ_j), $i, j \in \{1, \ldots, T\}$, *is such that:*

$$\text{If} \quad \tau_i > \tau_j \qquad \text{then} \qquad \sigma_i > \sigma_j. \tag{13}$$

Further, the equilibrium shares of the match surplus that each worker of quality σ_t *and each firm of quality* τ_t, $t = 1, \ldots, T$, *receive are such that:*

$$\begin{aligned} \pi^W_{\sigma_t} = \ & [v(\sigma_t, \tau_t) \ - \ v(\sigma_{r(t)}, \tau_t)] \ + \\ & + \sum_{k=1}^{\rho_t} \left[v(\sigma_{r^k(t)}, \tau_{r^k(t)}) - v(\sigma_{r^{k+1}(t)}, \tau_{r^k(t)}) \right] \end{aligned} \tag{14}$$

$$\pi^F_{\tau_t} = v(\sigma_{r(t)}, \tau_t) - \sum_{k=1}^{\rho_t} \left[v(\sigma_{r^k(t)}, \tau_{r^k(t)}) - v(\sigma_{r^{k+1}(t)}, \tau_{r^k(t)}) \right] \tag{15}$$

where $r^{\rho_t}(t) = T + 1$ *and* $v(\sigma_{r^{\rho_t}(t)}, \tau_{r^{\rho_t}(t)}) = v(\sigma_{r^{\rho_t+1}(t)}, \tau_{r^{\rho_t}(t)}) = 0$.

Proof: Condition (13) is nothing but a restatement of Lemma 1. The proof of (14) and (15) follows directly from Lemma 2. In particular, solving recursively (10), using (12), we obtain (14); then substituting (14) into (11) we obtain (15). ■

We now analyze the unique equilibrium of the Bertrand competition subgame in the case in which the order in which firms select their most preferred bid is the decreasing order of their qualities: $\tau_1 > \ldots > \tau_T$ and $\sigma_1 > \ldots > \sigma_S$. From Lemma 2 — condition (9) — this also implies that the runner-up worker to the firm of quality τ_t is the worker of quality σ_{t+1} for every $t = 1, \ldots, T$. The following proposition characterizes the equilibrium of the Bertrand competition subgame in this case.

Proposition 2: *For any given ordered vector of firms' qualities* $(\tau_1, \ldots, \tau_T)$ *and corresponding vector of workers' qualities* $(\sigma_1, \ldots, \sigma_S)$ *the unique equilibrium of the Bertrand competition subgame is such that the equilibrium matches are* (σ_k, τ_k), $k = 1, \ldots, T$ *and the shares of the match surplus that each worker of quality* σ_t *and* *each firm of quality τ_t receive are such that:*

$$\pi^W_{\sigma_t} = \sum_{h=t}^{T} [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] \tag{16}$$

$$\pi^F_{\tau_t} = v(\sigma_{t+1}, \tau_t) - \sum_{h=t+1}^{T} [v(\sigma_h, \tau_h) - v(\sigma_{h+1}, \tau_h)] \tag{17}$$

Proof: This result follows directly from Lemma 1, Lemma 2 and Proposition 1 above. In particular, (9) implies that when $(\tau_1, \ldots, \tau_T)$ and $(\sigma_1, \ldots, \sigma_S)$ are ordered vectors of qualities $\sigma_{r(t)} = \sigma_{t+1}$ for every $t = 1, \ldots, T$. Then substituting the identity of the runner-up worker in (14) and (15) we obtain (16) and (17). ■

The main difference between Proposition 2 and of Proposition 1 can be described as follows. Consider the subgame in which the firm of quality τ_t chooses among her bids and let $(\tau_1, \ldots, \tau_T)$ be an ordered vector of qualities as in Proposition 2. This implies that $\sigma_t > \sigma_{t+1} > \sigma_{t+2}$. The runner-up worker to the firm with quality τ_t is then the worker of quality σ_{t+1} and the willingness to pay of this worker (hence the share of the surplus accruing to firm τ_t) is, from (11) above:

$$v(\sigma_{t+1}, \tau_t) - \pi^W_{\sigma_{t+1}}. \tag{18}$$

Notice further that since the runner-up worker to firm τ_{t+1} is σ_{t+2} from (10) above the payoff to the worker of quality σ_{t+1} is:

$$\pi^W_{\sigma_{t+1}} = v(\sigma_{t+1}, \tau_{t+1}) - v(\sigma_{t+2}, \tau_{t+1}) + \pi^W_{\sigma_{t+2}}. \tag{19}$$

Substituting (19) into (18) we obtain that the willingness to pay of the runner-up worker σ_{t+1} is then:

$$v(\sigma_{t+1}, \tau_t) - v(\sigma_{t+1}, \tau_{t+1}) + v(\sigma_{t+2}, \tau_{t+1}) - \pi^W_{\sigma_{t+2}}. \tag{20}$$

Consider now a new vector of firms qualities $(\tau_1, \ldots, \tau'_{t-1}, \tau_t, \tau'_{t+1}, \ldots, \tau_T)$ where the qualities τ_i for every i different from $t-1$ and $t+1$ are the same as the ones in the ordered vector $(\tau_1, \ldots, \tau_T)$. Assume that $\tau'_{t-1} = \tau_{t+1} < \tau_t$ and $\tau'_{t+1} = \tau_{t-1} > \tau_t$. This assumption implies that the vector of workers' qualities $(\sigma'_1, \ldots, \sigma'_S)$ differs from the ordered vector of workers qualities $(\sigma_1, \ldots, \sigma_S)$ only in its $(t-1)$-th and $(t+1)$-th components that are such that: $\sigma'_{t-1} = \sigma_{t+1} < \sigma_t$ and $\sigma'_{t+1} = \sigma_{t-1} > \sigma_t$. From (9) above we have that the runner-up worker for firm τ_t is now worker σ_{t+2} and the willingness to pay of this worker is:

$$v(\sigma_{t+2}, \tau_t) - \pi^W_{\sigma_{t+2}}. \tag{21}$$

Comparing (20) with (21) we obtain, using the complementarity assumption $v_{12} > 0$, that

$$v(\sigma_{t+1}, \tau_t) - v(\sigma_{t+1}, \tau_{t+1}) + v(\sigma_{t+2}, \tau_{t+1}) > v(\sigma_{t+2}, \tau_t).$$

In other words, the willingness to pay of the runner-up worker to firm τ_t in the case considered in Proposition 2 is strictly greater than the willingness to pay of the runner-up worker to firm τ_t in the special case of Proposition 1 we just considered. The reason is that in the latter case there is one less worker σ_{t+1} to actively compete for the match with firm τ_t. This comparison is generalized in the following proposition proved in the Appendix.

Proposition 3: *Let $(\tau_1, \ldots, \tau_T)$ be an ordered vector of firms qualities such that $\tau_1 > \ldots > \tau_T$ and $(\tau'_1, \ldots, \tau'_T)$ be any permutation (other than the identity one) of the vector $(\tau_1, \ldots, \tau_T)$ with the same t-th element: $\tau'_t = \tau_t$. Denote $(\sigma_1, \ldots, \sigma_T)$ and $(\sigma'_1, \ldots, \sigma'_T)$ the corresponding vectors of workers' qualities. Then firm τ_t's payoff, as in (17), is greater than firm τ'_t's payoff, as in (15):*

$$\begin{aligned} v(\sigma_{t+1}, \tau_t) \quad - \quad \sum_{h=t+1}^{T} [v(\sigma_h, \tau_h) \quad - \quad v(\sigma_{h+1}, \tau_h)] \quad > \\ > \quad v(\sigma'_{r(t)}, \tau'_t) \quad - \quad \sum_{k=1}^{\rho'_t} \left[v(\sigma'_{r^k(t)}, \tau'_{r^k(t)}) \quad - \quad v(\sigma'_{r^{k+1}(t)}, \tau'_{r^k(t)}) \right] \end{aligned} \tag{22}$$

Proposition 3 allow us to conclude that when firms select their preferred bid in the decreasing order of their qualities competition among workers for each match is maximized.[12] This is apparent when we consider the case in which the order in which firms select their most preferred bid in the increasing order of their qualities: $\tau_1 < \ldots < \tau_T$. In this case, according to (9) above, the runner-up worker to each firm has quality σ_{T+1}. This implies that the payoff to each firm $t = 1, \ldots, T$ is:

$$\pi^F_{\tau_t} = v(\sigma_{T+1}, \tau_t) \tag{23}$$

In this case only two workers — the worker of quality σ_t and the worker of quality σ_{T+1} — actively compete for the match with firm τ_t and firms' payoffs are at their minimum.

Given that in our analysis we stress the role of competition in solving the inefficiencies due to match-specific investments in what follows we mainly focus on the case in which firms choose their most preferred bid in the decreasing order of their innate ability. Notice that this does not necessarily mean that firms choose their most preferred bid in the decreasing order of their qualities $\tau_1 > \ldots > \tau_T$ and hence competition among workers is at its peak. Indeed, firms' qualities are endogenously determined in the analysis that follows. However, in Section 6 below we show that firms will choose their investments so that the order of their innate abilities coincides with the order of their qualities. Hence Proposition 2 applies in this case.

We conclude this section by observing that from Proposition 1 above, the worker's equilibrium payoff $\pi^W_{\sigma_t}$ is the sum of the social surplus produced by the equilibrium match $v(\sigma_t, \tau_t)$ and an expression $\mathcal{W}_{\sigma_t}$ that does not depend on the quality σ_t of the worker involved in the match. In particular this implies that $\mathcal{W}_{\sigma_t}$ does not depend on the match-specific investment of the worker of quality σ_t:

$$\pi^W_{\sigma_t} = v(\sigma_t, \tau_t) + \mathcal{W}_{\sigma_t}. \tag{24}$$

[12] Notice that trembling-hand-perfection implies that all unmatched workers with a strictly positive willingness to pay for the match with a given firm submit their bids in equilibrium.

Moreover, from (15), each firm's equilibrium payoff $\pi^F_{\tau_t}$ is also the sum of the surplus generated by the inefficient (if it occurs) match of the firm of quality τ_t with the runner-up worker of quality $\sigma_{r(t)}$ and an expression $\mathcal{P}_{\tau_t}$ that does not depend on the match-specific investment of the firm of quality τ_t:

$$\pi^F_{\tau_t} = v(\sigma_{r(t)}, \tau_t) + \mathcal{P}_{\tau_t}. \tag{25}$$

Of course when firms select their bids in the decreasing order of their qualities the runner-up worker to firm t is the worker of quality σ_{t+1}, as from (9) above. Therefore equation (25) becomes:

$$\pi^F_{\tau_t} = v(\sigma_{t+1}, \tau_t) + \mathcal{P}_{\tau_t}. \tag{26}$$

These conditions play a crucial role when we analyze the efficiency of the investment choices of both workers and firms.

5. Workers' Investments

In this section we analyze the model under the assumption that the quality of firms is exogenously give $\tau(t)$ while the quality of workers depends on both the workers' identity (innate ability) and their match specific investments $\sigma(s, x_s)$.

We consider first the case in which firms choose their preferred bids in the decreasing order of their innate abilities. In this contest since firms' qualities are exogenously determined this assumption coincides with the assumption that firms choose their preferred bid in the decreasing order of their qualities $\tau_1 > \ldots > \tau_T$. Hence, Proposition 2 provides the characterization of the unique equilibrium of the Bertrand competition subgame in this case.

We proceed to characterize the equilibrium of the workers' investment game. We first show that an equilibrium of this simultaneous move investment game always exist and that this equilibrium is efficient: the order of the induced qualities $\sigma(s, x_s)$, $s = 1, \ldots, S$, coincides with the order of the workers' identities s, $s = 1, \ldots, S$. We then show that an inefficiency may arise, depending on the distribution of firms' qualities and workers' innate abilities. This inefficiency takes the form of additional inefficient equilibria, such that the order of the workers' identities differs from the order of their induced qualities.

Notice first that each worker's investment choice is efficient given the equilibrium match the worker is involved in. Indeed, the Bertrand competition game will make each worker residual claimant of the surplus produced in his equilibrium match. Therefore, the worker is able to appropriate the marginal returns from his investment and hence his investment choice is efficient given the equilibrium match.

Assume that the equilibrium match is the one between the s worker and the t firm, from equation (24) worker s's optimal investment choice $x_s(t)$ is the solution to the following problem:

$$x_s(t) = \underset{x}{\operatorname{argmax}}\ \pi^W_{\sigma(s,x)} - C(x) = v(\sigma(s,x), \tau_t) - \mathcal{W}_{\sigma(s,x)} - C(x). \tag{27}$$

This investment choice is defined by the following necessary and sufficient first order conditions of problem (27):

$$v_1(\sigma(s, x_s(t)), \tau_t)\ \sigma_2(s, x_s(t)) = C'(x_s(t)). \tag{28}$$

Notice that (28) follows from the fact that $\mathcal{W}_{\sigma(s,x)}$ does not depend on worker s's quality $\sigma(s,x)$, and hence on worker s's match specific investment x. The following two lemmas derive the properties of worker s's investment choice $x_s(t)$ and his quality $\sigma(s, x_s(t))$.

Lemma 3: *For any given equilibrium match $(\sigma(s, x_s(t)), \tau_t)$ worker s's investment choice $x_s(t)$, as defined in (28), is constrained efficient.*

Proof: Notice first that if a central planner is constrained to choose the match between worker s and firm t worker s's constrained efficient investment is the solution to the following problem:

$$x^*(s,t) = \underset{x}{\operatorname{argmax}}\, v(\sigma(s,x), \tau_t) - C(x). \tag{29}$$

This investment $x^*(s,t)$ is defined by the following necessary and sufficient first order conditions of problem (29):

$$v_1(\sigma(s,x^*(s,t)),\tau_t)\ \sigma_2(s,x^*(s,t)) = C'(x^*(s,t)). \tag{30}$$

The result then follows from the observation that the definition of the constrained efficient investment $x^*(s,t)$, equation (30), coincides with the definition of worker s's optimal investment $x_s(t)$, equation (28) above. ■

Lemma 4: *For any given equilibrium match $(\sigma(s,x_s(t)),\tau_t)$ worker s's optimally chosen quality $\sigma(s,x_s(t))$ decreases both in the worker's identity s and in the firm identity t:*

$$\frac{d\,\sigma(s,x_s(t))}{d\,s} < 0, \qquad \frac{d\,\sigma(s,x_s(t))}{d\,t} < 0.$$

Proof: The result follows from condition (28) that implies:

$$\frac{d\,\sigma(s,x_s(t))}{d\,s} = \frac{\sigma_1\ v_1\ \sigma_{22} - \sigma_1\ C'' - v_1\ v_2\ \sigma_{12}}{v_{11}\ (\sigma_2)^2 + v_1\ \sigma_{22} - C''} < 0,$$

and

$$\frac{d\,\sigma(s,x_s(t))}{d\,t} = \frac{v_{12}\ (\sigma_2)^2}{v_{11}\ (\sigma_2)^2 + v_1\ \sigma_{22} - C''} < 0,$$

where the functions σ_h and σ_{hk}, $h,k \in \{1,2\}$, are computed at $(s,x_s(t))$; the functions v_h and v_{hk}, $h,k \in \{1,2\}$, are computed at $(\sigma(s,x_s(t)),\tau_t)$ and the second derivative of the cost function C'' is computed at $x_s(t)$. ■

We define now an equilibrium of the workers' investment game. Let $(s_1,\ldots,s_S)$ denote a permutation of the vector of workers' identities $(1,\ldots,S)$. An equilibrium of the workers' investment game is then a vector of investment choices $x_{s_i}(i)$, as defined in (28) above, such that the resulting workers' qualities have the same order as the identity of the associated firms:

$$\sigma(s_i,x_{s_i}(i)) = \sigma_i < \sigma(s_{i-1},x_{s_{i-1}}(i-1)) = \sigma_{i-1} \qquad \forall i = 2,\ldots,S, \tag{31}$$

where σ_i denotes the i-th element of the equilibrium ordered vector of qualities $(\sigma_1, \dots, \sigma_S)$.[13]

Notice that this equilibrium definition allows for the order of workers' identities to differ from the order of their qualities and therefore from the order of the identities of the firms each worker is matched with.

We can now proceed to show the existence of the efficient equilibrium of the worker investment game. This is the equilibrium characterized by the coincidence of the order of workers' identities and the order of their qualities. From Lemma 1 the efficient equilibrium matches are $(\sigma(t, x_t(t)), \tau_t)$, $t = 1, \dots, T$.

PROPOSITION 4: *The equilibrium of the workers' investment game characterized by $s_i = i$, $i = 1, \dots, S$ always exists and is efficient.*

The formal proof of this result is presented in the Appendix. However the intuitive argument behind this proof is simple to describe. The payoff to worker i, $\pi_i^W(\sigma) - C(x(i,\sigma))$, changes expression as worker i increases his investment so as to improve his quality and match with a higher quality firm.[14] This payoff however is continuous at any point, such as σ_{i-1}, in which in the continuation Bertrand game the worker matches with a different firm, but has a kink at such points.[15]

However, if the equilibrium considered is the efficient one — $s_i = i$ for every $i = 1, \dots, S$ — the payoff to worker i is monotonic decreasing in any interval to the right of the $(\sigma_{i+1}, \sigma_{i-1})$ and increasing in any interval to the left. Therefore, this payoff has a unique global maximum. Hence worker i has no incentive to deviate and change his investment choice.

[13] Recall that since $\tau_1 > \dots > \tau_T$ Lemma 1 and the notation defined in Section 4 above imply that $\sigma_1 > \dots > \sigma_S$.

[14] The level of investment $x(i,\sigma)$ is defined, as in the Appendix: $\sigma(i,x) \equiv \sigma$.

[15] Indeed, from (A.20) and (A.21) we get that $\frac{\partial\left[\pi_i^W(\sigma_{i-1}^-) - C\left(x(i,\sigma_{i-1}^-)\right)\right]}{\partial\sigma} = v_1(\sigma_{i-1}, \tau_i) - \frac{C'(x(i,\sigma_{i-1}))}{\sigma_2(i,x(i,\sigma_{i-1}))}$ and $\frac{\partial\left[\pi_i^W(\sigma_{i-1}^+) - C\left(x(i,\sigma_{i-1}^+)\right)\right]}{\partial\sigma} = v_1(\sigma_{i-1}, \tau_{i-1}) - \frac{C'(x(i,\sigma_{i-1}))}{\sigma_2(i,x(i,\sigma_{i-1}))}$. Therefore, from $v_{12} > 0$, we conclude that $\frac{\partial\left[\pi_i^W(\sigma_{i-1}^+) C\left(x(i,\sigma_{i-1}^+)\right)\right]}{\partial\sigma} > \frac{\partial\left[\pi_i^W(\sigma_{i-1}^-) - C\left(x(i,\sigma_{i-1}^-)\right)\right]}{\partial\sigma}$.

If instead we consider an inefficient equilibrium — an equilibrium where $s_1, \ldots, s_S$ differs from $1, \ldots, S$ — then the payoff to worker i is still continuous at any point, such as $\sigma(s_i, x_{s_i}(i))$, in which in the continuation Bertrand game the worker gets matched with a different firm. However, this payoff is not any more monotonic decreasing in any interval to the right of the $(\sigma(s_{i+1}, x_{s_{i+1}}(i+1)), \sigma(s_{i-1}, x_{s_{i-1}}(i-1)))$ and increasing in any interval to the left. In particular, this payoff is increasing at least in the right neighborhood of the switching points $\sigma(s_h, x_{s_h}(h))$ for $h = 1, \ldots, i-1$ and decreasing in the left neighborhood of the switching points $\sigma(s_k, x_{s_k}(k))$ for $k = i+1, \ldots, N$.

This implies that depending on the values of parameters these inefficient equilibria may or may not exist. We show below that for given firms' qualities it is possible to construct inefficient equilibria if two workers' qualities are close enough. Alternatively, for given workers' qualities inefficient equilibria do not exist if the firms qualities are close enough.

Proposition 5: *Given any ordered vector of firms' qualities $(\tau_1, \ldots, \tau_T)$, it is possible to construct an inefficient equilibrium of the workers' investment game such that there exists at least an i such that $s_i < s_{i-1}$.*

Moreover, given any vector of workers' quality functions $(\sigma(s_1, \cdot), \ldots, \sigma(s_S, \cdot))$, it is possible to construct an ordered vector of firms' qualities $(\tau_1, \ldots, \tau_T)$ such that there does not exist any inefficient equilibrium of the workers' investment game.

We present the formal proof of this proposition in the Appendix. We describe here the intuition of why such result holds. The continuity of each worker's payoff implies that, when two workers have similar innate abilities, exactly as it is not optimal for each worker to deviate when he is matched efficiently it is also not optimal for him to deviate when he is inefficiently assigned to a match. Indeed, the difference in workers' qualities is almost entirely determined by the difference in the qualities of the firms they are matched with rather than by the difference in workers' innate ability. This implies that when the worker of low ability has undertaken the high investment, at the purpose of being matched with the better firm, it is not worth any more for the worker of immediately higher ability to try to outbid him. The willingness to pay of the lower ability worker for the match with the better firm is in fact enhanced by this higher investment. Therefore the gains from outbidding this worker are not enough to justify the high investment of the higher ability worker. Indeed, in the Bertrand competition game each worker is able to capture just the difference between the match surplus and the willingness to pay for the match of the runner-up worker that in this outbidding attempt would be the low ability worker that undertook the high investment.

Conversely, if firms' qualities are similar then the difference in workers qualities is almost entirely determined by the difference in workers' innate abilities implying that it is not possible to construct an inefficient equilibrium of the workers' investment game. The reason being that the improvement in the worker's incentives to invest due to a match with a better firm are more than compensated by the decrease in the worker's incentives induced by a lower innate ability of the worker. Hence it is not optimal for two workers of decreasing innate abilities to generate increasing qualities so as to be matched with increasing quality firms.

We then conclude that when workers are undertaking ex-ante match specific investments and then Bertrand compete for a match with a firm investments are constrained efficient. If workers are similar in innate ability inefficiencies may arise that take the form of additional equilibria characterized by inefficient matches. However, the higher is the degree of specificity due to the workers' characteristics with respect to the specificity due to the firms' characteristics the less likely is this inefficiency.

We conclude this section by discussing the general case in which firms choose their most preferred bid in the (not necessarily decreasing) order of any vector of qualities $(\tau_1, \ldots, \tau_T)$.[16] In this case we can prove the following corollary.

Corollary 1: *Propositions 4 and 5 hold in the general case in which firms choose their most preferred bid in the order of any vector of firms qualities* $(\tau_1, \ldots, \tau_T)$.

[16]Recall that firm τ_1 chooses her most preferred bid first, followed by firm τ_2 and so on till firm τ_T chooses her most preferred.

The proof is presented in the Appendix and follows from the observation that neither Proposition 4 nor Proposition 5 depend on how intensely workers compete for firms.

6. Firms' investments

We move now to the model in which the qualities of workers are exogenously given by the following ordered vector $(\sigma_{(1)}, \ldots, \sigma_{(S)})$, where $\sigma_{(s)} = \sigma(s)$, while the qualities of firms are a function of firms' ex-ante match specific investments y and the firm's identity t: $\tau(t, y)$. In this model we show that firms' investments are not constrained efficient. Firms under-invest since their marginal incentives to undertake investments are determined by their outside option that depends on the surplus of the match between the firm and the immediate competitor to the worker the firm is matched with in equilibrium (this match yields a strictly lower surplus than the equilibrium one).[17] However, a central result is that we are able to show that equilibrium matches are always efficient: the order of firms innate abilities coincides with the order of their derived qualities. In other words, all coordination problems are solved.

All these results crucially depend on the amount of competition in the market. Therefore in this section we almost exclusively focus on the case in which firms select their preferred bid in the decreasing order of their innate ability. In the next section, a model of endogenous ordering is used to justify this assumption.

Notice that in characterizing the equilibrium of the firms' investment game we cannot bluntly apply Proposition 2 as the characterization of the equilibrium of the Bertrand competition subgame. Indeed, the order in which firms choose among bids in this subgame is determined by the firms' innate abilities rather than by their qualities. This implies that unless firms' qualities (which are endogenously determined) have the same order of firms' innate abilities it is possible that firms do not choose among bids in the decreasing order of their marginal contribution to a match (at least off the equilibrium path).

[17]We determine the size of this inefficiency in Section 7 below.

Proposition 6: *If firms select their most preferred bid in the decreasing order of their innate abilities the unique equilibrium of the firms' investment game is such that firm t chooses investment $y(t, t+1)$, as defined in (2).*

The formal proof is presented in the Appendix. However, we discuss here the intuition behind this result.

The nature of the Bertrand competition game is such that each firm is not able to capture all the match surplus but only the outside option that is determined by the willingness to pay of the runner-up worker for the match. Since the match between a firm and her runner-up worker yields a match surplus that is strictly lower than the equilibrium surplus produced by the same firm the share of the surplus the firm is able to capture does not coincide with the entire surplus of the match. This implies that firms will under-invest rendering the equilibrium investment choice inefficient.

Corollary 2: *When firms undertake ex-ante investments and choose their most preferred bid in the decreasing order of their innate abilities then each firm $t = 1, \ldots, T$ chooses an inefficient investment level $y(t, t+1)$. Indeed, $y(t, t+1)$ is strictly lower than the investment $y(t, t)$ that would be efficient for firm t to choose given the equilibrium match of worker t with firm t.*

Proof: The result follows from Proposition 6, the definition of efficient investment (1) when worker t matches with firm t, and condition (A.38) in the Appendix. ■

In contrast with the case in which workers undertake ex-ante investments, in this framework the equilibrium of the Bertrand competition game is unique and characterized by efficient matches.

Corollary 3: *When firms undertake ex-ante investments the unique equilibrium of the Bertrand competition game is characterized by efficient matches between worker t and firm t, $t = 1, \ldots, T$.*

Proof: The result follows immediately from Proposition 6 above. ■

Two features of the model may explain why equilibria with inefficient matches do not exist. First, as argued above, each firm's payoff is completely determined by the firm outside option and hence independent of the identity and quality of the worker the firm is matched with. Second, firms choose their bid in the decreasing order of their innate abilities hence this order is independent of firms' investments. These two features of the model together with positive assortative matching (Lemma 1 above) imply that when a firm chooses an investment that yields a quality higher than the one of the firm with a lower identity (higher innate ability) it modifies the set of unmatched workers, and hence of bids among which the firm chooses, only of the bid of the worker the firm will be matched with in equilibrium. Hence this change will not affect the outside option and therefore the payoff of this firm implying that the optimal investment cannot exceed the optimal investment of the firm with higher innate ability. Therefore an equilibrium with inefficient investment does not exist.

An interesting issue is whether this uniqueness is preserved if we modify the extensive form of the Bertrand competition game and in particular the order in which firms choose their most preferred bid.

Notice first that the intuition we just described does not hold if firms choose their bid in the decreasing order of their qualities and not of their innate abilities. In this case the order in which firms choose their most preferred bid is endogenously determined. An argument similar to the one used in the analysis of the workers' investment game (Proposition 4 above) will then show that equilibrium with efficient matches always exist. However there may exist multiple equilibria that exhibit inefficient matches.

Consider now the general case in which firms choose their bid in the order of the permutation $(t_1, \ldots, t_T)$. For simplicity we focus on the case in which firms choose their bids in the increasing order of their innate ability: $t_1 = T, \ldots, t_T = 1$.[18] Notice first that an efficient equilibrium exists in which firms qualities have the same order

[18] Using Propositions 1 and 3 above this analysis can be generalized to the case in which firms choose their most preferred bid in the order of any permutation $(t_1, \ldots, t_T)$.

of firms' innate abilities. Consider such an equilibrium of the firms' investment game. As argued in Section 4 above, in this case the runner-up worker to every firm is the highest quality worker that does not match with any firm in equilibrium. This implies that each firm t's payoff is

$$v(\sigma_{(T+1)}, \tau(t, y)). \tag{32}$$

Therefore each firm's net payoff function $v(\sigma_{(T+1)}, \tau(t, y)) - C(y)$ has a unique maximum at $y(t, T+1)$. This implies that firms' equilibrium investments and hence firms' qualities have the same order of firms' innate abilities.

Notice however that inefficient equilibria may arise as well. The logic behind these equilibria can be described as follows. Consider firm t and assume that this firm chooses a level of investment yielding a quality higher than the one chosen by firm $k < t$. Notice now that, from Lemma 1, in the case in question this change in investment affects the equilibrium matches of all the workers with identities between t and k that are un-matched when it is firm t's turn to choose a bid. This implies that the outside option of firm t will also be affected by this increase in investment creating the conditions for an equilibrium characterized by inefficient matches.

7. The Inefficiency of Firms' Investments

7.1. The Intensity of Competition

The analysis above shows that if, in the Bertrand competition game, firms select their most preferred bid in the decreasing order of their innate ability, as opposed to any other order, competition among workers for matches is at its peak.

Notice that Proposition 1 has demonstrated that, for given firms' investments, an individual firm's payoff in the Bertrand competition game is highest if the order in which firms select their bid is the decreasing order of their innate ability. Also, Lemma 2 has shown that the quality of the runner-up worker to each firm is highest when firms follow the decreasing order of their innate abilities in choosing their most preferred bid. From (2) this implies that, if we restrict attention to the equilibria of the firms' investment game that exhibit efficient matches,[19] each firm will choose the highest investment level when the order in which firms select their most preferred bid is the decreasing order of their innate abilities. In other words, the intensity of workers' competition for matches is highest when firms select their bid in the decreasing order of their innate abilities.

The question we ask in this section is whether we expect such a high intensity of competition to arise when we endogenize the order in which firms select their most preferred bid. We do this by allowing firms to compete for the order in which they select their bid at an ex-ante stage of the game that precedes the firms' investment decision. We are able to show that there is an equilibrium of this ex-ante firms' competition game in which the equilibrium order is the decreasing order of the firms' innate abilities. Moreover, if firms innate abilities are sufficiently far apart, this is the unique equilibrium of the ex-ante firms' competition game.

As discussed at the end of Section 6 above, when firms select their most preferred bid in any other order but the decreasing order of their innate abilities, multiple equilibria may arise in the firms' investment game. These equilibria are characterized by inefficient matches. In our analysis below, whatever the order in which firms select their most preferred bid, we restrict attention to the equilibrium of the firms' investment game with efficient matches. Adapting the arguments presented in the proof of Propositions 4 and 5 it is possible to show, as argued in Section 6 above, that this equilibrium always exists and it is unique if the firms' innate abilities are sufficiently far apart. In other words, we restrict ourselves to the case in which firms' innate abilities are sufficiently far apart so that the unique equilibrium of the firms' investment game is the one that exhibits efficient matches whatever the order in which firms select their most preferred bid. The equilibrium of the ex-ante firms' competition game is then summarized by the order of the vector of firms' qualities $(\tau_1, \ldots, \tau_T)$ where, as discussed in Section 4 above, the firm of quality τ_1 is the first firm to select her most preferred bid while τ_T is the last firm to select her most preferred bid.

[19] These equilibria always exist as argues in Section 6 above.

Consider now an equilibrium of this ex-ante firms' competition game $(\tau_1, \ldots, \tau_T)$ that differs from the decreasing order of the firms' innate abilities or the decreasing order of their qualities (given the equilibrium of the firms' investment game). Then we can find a t^* such that

$$\tau_{t^*} < \tau_{t^*+1} > \tau_{t^*+2} > \ldots > \tau_T \tag{33}$$

where $t^* \in \{1, \ldots, T-1\}$. Now, it is clear that there exists a firm of quality $\tau_{t'}$ such that $t' \in \{t^*+1, \ldots, T\}$ and

$$\tau_{t'} > \tau_{t^*} > \tau_{t'+1}$$

We want to compare the equilibrium payoffs to firms t^* and t' if the equilibrium order of the ex-ante firms' competition game is the one described in (33) above with the one if the two firms t^* and t' swap position in the order in which they choose their most preferred bid. Notice that in the latter case all firms can be expected to choose a different investment level and hence will be associated with a different quality level. We denote the quality levels chosen when firms select their most preferred bid according to this new order $\tilde{\tau}_t$.

Notice first that from Lemma 1, given that we restrict attention to equilibria of the firms' investment game with efficient matches, if firms follow this new order in selecting their bid each firm will be matched in equilibrium with the same worker. Secondly, Lemma 2 and Proposition 1 (in particular the firms's payoffs in (15) above) imply that firms $\{\tilde{\tau}_{t'+1}, \ldots, \tilde{\tau}_T\}$ will have the same runner-up worker, get the same payoff and choose the same investment as in the order described in (33) above: $\tau_t = \tilde{\tau}_t$ for every $t \in \{t'+1, \ldots, T\}$. Further, Lemma 2 implies that the firm of quality $\tilde{\tau}_{t^*}$ has the same runner-up worker and, from (15) above, the same payoff as the firm of quality τ_{t^*}. In other words,

$$\tau_{t^*} = \tilde{\tau}_{t^*}$$

and

$$\pi^F_{t^*} = \tilde{\pi}^F_{t^*}$$

where we denote $\pi^F_{t^*}$ the equilibrium payoff to the firm of quality τ_{t^*} and $\tilde{\pi}^F_{t^*}$ the equilibrium payoff of the firm of quality $\tilde{\tau}_{t^*}$. Thus, firm t^* obtains the same payoff if the equilibrium order of the ex-ante firms' competition game is the one described in (33) above or the one in which firm t^* and t' swap position.

Consider now firm t'. The payoff function in (15) is identical in the two potential orders with the sole exception that, from Lemma 2 above, when t^* and t' swap positions, the runner-up worker to firm t' is the worker that in equilibrium matches with firm t^*. Therefore from (15) above we obtain:

$$\tilde{\pi}^F_{t'} - \pi^F_{t'} = v(\sigma_{t^*}, \tilde{\tau}_{t'}) - v(\sigma_{t^*}, \tilde{\tau}_{t^*}) + v(\sigma_{t'+1}, \tilde{\tau}_{t^*}) - v(\sigma_{t'+1}, \tau_{t'}) \tag{34}$$

Notice that, as $\tau_{t'} > \tilde{\tau}_{t^*}$ and $\sigma_{t^*} > \sigma_{t'+1}$ by the complementarity assumption, $v_{12} > 0$, we can conclude that if $\tau_{t'} = \tilde{\tau}_{t'}$ then the difference in (34) is strictly positive. But since the runner up worker of the firms of quality $\tilde{\tau}_{t'}$ is of higher quality σ_{t^*} than the runner-up worker of quality $\sigma_{t'+1}$ of the firm of quality $\tau_{t'}$ when firms t^* and t' swap their positions, firm t' will choose a higher investment: $\tilde{\tau}_{t'} > \tau_{t'}$. Hence, the difference in (34) is further magnified and strictly positive. In other words, firm t' strictly gains from swapping position in the order in which firms select their most preferred bid with firm t^*.

In essence, firm t^*'s runner-up worker has not changed as a consequence of the swap and hence firm t^* does not lose out from the change. At the same time, by swapping position, firm t' improves the quality of her runner-up worker, increasing in this way her payoff, and further gains by being able to exploit this better potential match at the investment stage.

We can now summarize our findings in the following proposition.

Proposition 7: *If firms select their most preferred bid in any order other than the decreasing order of their innate abilities then there always exists a pair of firms who gain, one weakly and one strictly, by swapping their position in the order in which they select their bid.*

Notice that Proposition 7 above implies that there cannot exist an equilibrium order of the ex-ante firms' competition game that differs from the decreasing order of the firms' innate abilities. Indeed, a minimal requirement for any model of competition for a position (a widget) is that there does not exist a pair of competitors that strictly gains by swapping position.

Therefore, when the intensity of workers' competition for firms is endogenized, the unique equilibrium is such that competition will be at its peak. This result provides us with a justification for restricting attention, in the reminder of this section, to the properties of the equilibrium of the model in which firms select their bid in the decreasing order of their innate abilities.

7.2. The Size of the Inefficiency

In this section we evaluate the size of the inefficiency generated by firms' under-investment and characterized in Section 6 above. In particular we argue that when competition among workers for firms is maximized this inefficiency is small in a well defined sense. In particular, we show that when firms choose their most preferred bid in the decreasing order of their innate abilities the overall inefficiency generated by firms' equilibrium under-investment is strictly lower than the inefficiency induced by the under-investment of one firm (the best one) if it matches in isolation with the best worker.

Denote $\omega(s,t)$ the net surplus function when worker t matches with firm t and the firm's investment is the one, defined in (1) above, that maximize the surplus of the match between worker s and firm t.

$$\omega(s,t) = v(\sigma(t), \tau(t, y(t,s))) - C(y(s,t)). \tag{35}$$

Clearly, in definition (35), the investment $y(t,s)$ maximizes the net surplus of a match (between worker s and firm t) that might differ from the match with worker t in which firm t is involved.

Further recall that we assume that $v(\cdot,\cdot)$ and $\tau(\cdot,\cdot)$ satisfy both the responsive complementarity and the marginal complementarity assumptions as stated in (3) and (4) above.

From Corollary 2 above we know that each firm will under-invest and choose an investment $y(t,t+1) < y(t,t)$. Hence the inefficiency associated with each firm t's investment decision is characterized by the difference between the match surplus generated by the efficient investment $y(t,t)$ and the match surplus generated by the equilibrium investment $y(t,t+1)$:

$$\omega(t,t) - \omega(t,t+1).$$

Therefore the inefficiency of the equilibrium investments by all firms is given by

$$L = \sum_{t=1}^{T} \omega(t,t) - \sum_{t=1}^{T} \omega(t,t+1). \tag{36}$$

How large is this loss L? First, notice that the difference between the efficient investment $y(t,t)$ and the equilibrium investment $y(t,t+1)$ is approximately proportional to the difference in characteristics between worker t and $t+1$ (given that $y(t,s)$ as defined in (1) is differentiable in s). On the other hand, as $y(t,t)$ solves (2), the difference between the efficient surplus $\omega(t,t)$ and the equilibrium surplus $\omega(t,t+1)$ will be approximately proportional to the *square* of the difference between $y(t,t)$ and $y(t,t+1)$ which will be small if worker t and worker $t+1$ have similar characteristics. To give an example of how this affects L, consider a situation where the characteristics of a worker are captured by a real number c with workers 1 through S having characteristics which are evenly spaced between $\overline{c}$ and $\underline{c}$. How is L affected by the size of the market T? The difference between $y(t,t)$ and $y(t,t+1)$ is approximately proportional to $[(\overline{c} - \underline{c})/T]$ and the difference between $\omega(t,t)$ and $\omega(t,t+1)$ will be approximately proportional to $[(\overline{c} - \underline{c})/T]^2$. Summing over t then gives a total loss L that is proportional to $[(\overline{c} - \underline{c})^2/T]$: in large markets the aggregate inefficiency created by firms' investments will be arbitrarily small.[20]

[20]See Kamecke (1992).

This is a result that changes the degree of specificity of the firms' investment choices. Increasing the number and hence the density of workers evenly spaced in the interval $[\bar{c}, \underline{c}]$ is equivalent to introducing workers with closer and closer characteristics. This is equivalent to reducing the loss in productivity generated by the match of a firm that choose an investment so as to be matched with the worker that is immediately below in characteristics levels. Hence, there is a sense in which this result is not fully satisfactory since we know that if each firm's investment is general in nature the investment choices are efficient.

Therefore, in the rest of this section, we identify an upper-bound on the aggregate inefficiency present in the economy that is independent of the number of firms and does not alter the specificity of the workers investment choices. Whatever the size of T, it is possible to get a precise upper-bound on the loss L. Indeed, the inefficiency created by the firms' equilibrium under-investment is less than that which could be created by the under-investment of only one firm (the best firm 1) in a match with only one worker (the best one labelled 1).

Proposition 8: *Assume that there are at least two firms ($T \geq 2$). Let M be the efficiency loss resulting from firm 1 choosing an investment level given by $y(1, T+1)$, as defined in (1):*

$$M = \omega(1,1) - \omega(1, T+1). \tag{37}$$

If both responsive complementarity, as in (3), and marginal complementarity, as in (5), are satisfied then

$$L < M. \tag{38}$$

The formal proof is presented in the Appendix, while the intuition of Proposition 8 can be described as follows. As a result of the Bertrand competition game firms have incentive to invest in match specific investments with the purpose of improving their outside option: the maximum willingness to pay of the runner-up worker to the firm. This implies that the under-investment of each firm is relatively small. The total inefficiency is then obtained by aggregating these relatively small under-investments. Given the decreasing returns to investment and the assumptions on how optimal firms' investments change across different matches, the sum of the loss in surplus generated by these almost optimal investments is clearly dominated by the loss in surplus generated by the unique under-investment of the best firm matched with the best worker. Indeed, the firm's investment choice in the latter case is very far from the optimal level (returns from a marginal increase of investment are very high).

8. Concluding Remarks

When workers and firms can undertake match specific investments, Bertrand competition for matches may help solve the hold-up and coordination problems generated by the absence of fully contingent contracts. In this paper, we have uncovered a number of characterization results that highlight how competition may solve, or at least attenuate, the impact of these problems.

When workers choose investments that precede Bertrand competition then the workers' investment choices are constrained efficient. However, coordination failure inefficiencies may arise that take the form of multiple equilibria and only one of these equilibria is characterized by efficient matches: there may exist inefficient equilibria that exhibit matches such that workers with lower innate ability invest more than better workers at the sole purpose of being matches with a higher quality firm.

If instead firms choose investments that precede the Bertrand competition game a different set of inefficiencies may arise. When buyers are competed for in decreasing order of innate ability then the equilibrium of the Bertrand competition game is unique and involves efficient matches. However, firms choose an inefficient level of investment given the equilibrium match they are involved in. In this case, however, we are able to show that the aggregate inefficiency due to firms' under-investments is low in the sense that is bounded above by the inefficiency that would be induced by the sole under-investment of the best firm matched with the best worker. In other words the inefficiencies due to the hold-up problem do not cumulate in the presence of workers' competition for matches.

Consider now what will happen in this environment if both firms and workers undertake ex-ante investments. Workers' investments will still be constrained efficient while firms' investments, although inefficient, can still be near efficient (when competition is in the decreasing order of buyers' innate ability and the appropriate equilibrium is selected). However, if both firms and workers undertake ex-ante investments then the inefficiency that takes the form of multiple equilibria, some of them characterized by inefficient matches, can still arise.

We conclude with the observation that the extensive form of the Bertrand competition game we use in the paper coincides with a situation in which firms are sequentially auctioned off to workers. Our result can then be re-interpreted as applying to a model of perfect information sequential auctions in which workers' valuations for each firm and the value of each auctioned-off firm can be enhanced by ex-ante investments.

APPENDIX

Proof of Lemma 2: We concentrate on the case where all firms and all workers have different induced quality. Bids are made as part of a trembling-hand-perfect equilibrium. We will prove the result by induction on the number of firms still to be matched. Without any loss in generality, we take $S = T + 1$. Consider the (last) stage T of the Bertrand competition game. In this stage only two workers are unmatched and from Lemma 1 have qualities σ_T and σ_{T+1}. Clearly in this case the only possible runner-up to firm T is the worker of quality σ_{T+1}, and given that by Lemma 1 $\sigma_T > \sigma_{T+1}$ the quality of this worker satisfies (9) above.

Further this stage of the Bertrand competition game is a simple decision problem for firm T that has to choose between the bids submitted by the two workers with qualities σ_T and σ_{T+1}. Let $B(\sigma_T)$, respectively $B(\sigma_{T+1})$, be their bids. Firm T clearly chooses the highest of these two bids.

Worker of quality σ_{T+1} generates surplus $v(\sigma_{T+1}, \tau_T)$ if selected by firm T while the worker of quality σ_T generates surplus $v(\sigma_T, \tau_T)$ if selected. This implies that $v(\sigma_{T+1}, \tau_T)$ is the maximum willingness to bid of the runner-up worker σ_{T+1}, while $v(\sigma_T, \tau_T)$ is the maximum willingness to bid of the worker of quality σ_T. Notice that from $\sigma_T > \sigma_{T+1}$ and $v_1 > 0$ we have:

$$v(\sigma_T, \tau_T) > v(\sigma_{T+1}, \tau_T).$$

Worker σ_T therefore submits a bid equal to the minimum necessary to outbid worker σ_{T+1}. In other words the equilibrium bid of worker σ_T coincides with the equilibrium bid of worker σ_{T+1}:

$B(\sigma_T) = B(\sigma_{T+1})$. Worker σ_{T+1}, on his part, has an incentive to deviate and outbid worker σ_T for any bid $B(\sigma_T) < v(\sigma_{T+1}, \tau_T)$. Therefore the unique equilibrium is such that both workers' equilibrium bids are:[21]

$$B(\sigma_T) = B(\sigma_{T+1}) = v(\sigma_{T+1}, \tau_T)$$

Consider now the stage $t < T$ of the Bertrand competition game. The induction hypothesis is that the runner-up worker for every firm of quality $\tau_{t+1}, \ldots, \tau_T$ is defined in (9) above. Further, the shares of surplus accruing to the firms of qualities τ_j, $j = t+1, \ldots, T$ and to the workers of qualities σ_j, $j = t+1, \ldots, S$ are:

$$\hat{\pi}^W_{\sigma_j} = [v(\sigma_j, \tau_j) - v(\sigma_{r(j)}, \tau_j)] + \hat{\pi}^W_{\sigma_{r(j)}} \tag{A.1}$$

$$\hat{\pi}^F_{\tau_j} = v(\sigma_{r(j)}, \tau_j) - \hat{\pi}^W_{\sigma_{r(j)}}. \tag{A.2}$$

From Lemma 1 the worker of quality σ_t will match with the firm of quality τ_t which implies that the runner-up worker for firm τ_t has to be one of the workers with qualities $\sigma_{t+1}, \ldots, \sigma_{T+1}$. Each worker will bid an amount for every firm which gives him the same payoff as he receives in equilibrium. To prove that the quality of the runner-up worker satisfies (9) we need to rule out that the quality of the runner-up worker is $\sigma_{r(t)} > \sigma_t$ and, if $\sigma_{r(t)} \leq \sigma_t$, that there exist an other worker of quality $\sigma_i \leq \sigma_t$ such that $i > t$ and $\sigma_i > \sigma_{r(t)}$.

Assume first by way of contradiction that $\sigma_{r(t)} > \sigma_t$. Then the willingness to pay of the runner-up worker for the match with firm τ_t is the difference between the surplus generated by the match of the runner-up worker of quality $\sigma_{r(t)}$ and the firm of quality τ_t minus the payoff that the worker would get according to the induction hypothesis by moving to stage $r(t)$ of the Bertrand competition game:

$$v(\sigma_{r(t)}, \tau_t) - \hat{\pi}^W_{\sigma_{r(t))}}. \tag{A.3}$$

From the induction hypothesis, (A.1), we get that the payoff $\hat{\pi}^W_{\sigma_{r(t))}}$ is:

$$\hat{\pi}^W_{\sigma_{r(t)}} = v(\sigma_{r(t)}, \tau_{r(t)}) - v(\sigma_{r^2(t)}, \tau_{r(t)}) + \hat{\pi}^W_{\sigma_{r^2(t)}} \tag{A.4}$$

where, from the induction hypothesis, $\sigma_{r^2(t)} < \sigma_{r(t)}$. Substituting (A.4) into (A.3) we get that the willingness to pay of a runner-up worker of quality $\sigma_{r(t)}$ for the match with the firm of quality τ_t

[21] This is just one of a whole continuum of subgame perfect equilibria of this simple Bertrand game *but* the unique trembling-hand-perfect equilibrium. Trembling-hand-perfection is here used in a completely standard way to insure that worker σ_{T+1} does not choose an equilibrium bid (not selected by firm T) in excess of his maximum willingness to pay.

can be written as:

$$v(\sigma_{r(t)}, \tau_t) - v(\sigma_{r(t)}, \tau_{r(t)}) + v(\sigma_{r^2(t)}, \tau_{r(t)}) - \hat{\pi}^W_{\sigma_{r^2(t)}}. \tag{A.5}$$

Consider now the willingness to pay of the worker of quality $\sigma_{r^2(t)}$ for the match with the same firm of quality τ_t. This is

$$v(\sigma_{r^2(t)}, \tau_t) - \hat{\pi}^W_{\sigma_{r^2(t)}}. \tag{A.6}$$

By definition of runner-up worker the willingness to pay of the worker of quality $\sigma_{r(t)}$, as in (A.5), must be greater or equal than the willingness to pay of the worker of quality $\sigma_{r^2(t)}$ as in (A.6). This inequality is satisfied if and only if:

$$v(\sigma_{r(t)}, \tau_t) + v(\sigma_{r^2(t)}, \tau_{r(t)}) \geq v(\sigma_{r(t)}, \tau_{r(t)}) + v(\sigma_{r^2(t)}, \tau_t). \tag{A.7}$$

Since $\sigma_{r(t)} > \sigma_t$ then from Lemma 1 $\tau_{r(t)} > \tau_t$. The latter inequality together with $\sigma_{r(t)} > \sigma_{r^2(t)}$ allow us to conclude that (A.7) is a contradiction to the complementarity assumption $v_{12} > 0$.

Assume now by way of contradiction that the $\sigma_{r(t)} \leq \sigma_t$ but there exists another worker of quality $\sigma_i \leq \sigma_t$ such that $i > t$ and $\sigma_i > \sigma_{r(t)}$. The definition of runner-up worker implies that his willingness to pay, as in (A.3), for the match with the firm of quality τ_t is greater than the willingness to pay $v(\sigma_i, \tau_t) - \hat{\pi}^W_{\sigma_i}$ of the worker of quality σ_i, for the same match:

$$v(\sigma_{r(t)}, \tau_t) - \hat{\pi}^W_{\sigma_{r(t)}} \geq v(\sigma_i, \tau_t) - \hat{\pi}^W_{\sigma_i}. \tag{A.8}$$

Moreover, for $(\sigma_{r(t)}, \tau_{r(t)})$ to be an equilibrium match worker $\sigma_{r(t)}$ should have no incentive to be matched with firm τ_i instead. This implies, using an argument identical to the one presented in the proof of Lemma 1, that the following necessary condition needs to be satisfied:

$$\hat{\pi}^W_{\sigma_{r(t)}} = v(\sigma_{r(t)}, \tau_{r(t)}) - B(\tau_{r(t)}) \geq v(\sigma_{r(t)}, \tau_i) - B(\tau_i); \tag{A.9}$$

where $B(\tau_{r(t)})$ and $B(\tau_i)$ are the equilibrium bids accepted by firm $\tau_{r(t)}$, respectively τ_i. Further, the equilibrium payoff to worker σ_i is:

$$\hat{\pi}^W_{\sigma_i} = v(\sigma_i, \tau_i) - B(\tau_i). \tag{A.10}$$

Substituting (A.9) and (A.10) into (A.8) we obtain that for (A.8) to hold the following necessary condition needs to be satisfied:

$$v(\sigma_{r(t)}, \tau_t) + v(\sigma_i, \tau_i) \geq v(\sigma_i, \tau_t) + v(\sigma_{r(t)}, \tau_i). \tag{A.11}$$

Since by assumption $\sigma_t \geq \sigma_i$ from Lemma 1 $\tau_t > \tau_i$. The latter inequality together with $\sigma_i > \sigma_{r(t)}$

imply that (A.11) is a contradiction to the complementarity assumption $v_{12} > 0$. This concludes the proof that the quality of the runner-up worker for firm τ_t satisfies (9).

An argument similar to the one used in the analysis of stage T of the Bertrand competition subgame concludes the proof of Lemma 2 by showing that the worker of quality σ_t submits in equilibrium a bid equal to the willingness to pay of the runner-up worker to firm τ_t as in (A.3). This bid is the equilibrium payoff to the firm of quality τ_t and coincides with (11). The equilibrium payoff to the worker of quality σ_t is then the difference between the match surplus $v(\sigma_t, \tau_t)$ and the equilibrium bid in (A.3) as in (10). ■

LEMMA A.1: *Given any ordered vector of firms' qualities $(\tau_1, \ldots, \tau_T)$ and the corresponding vector of workers' qualities $(\sigma_1, \ldots, \sigma_S)$ we have that for every $t = 1, \ldots, T-1$ and every $m = 1, \ldots, T-t$:*

$$v(\sigma_{t+1}, \tau_t) - \sum_{h=1}^{m} [v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] > v(\sigma_{t+m}, \tau_t) \tag{A.12}$$

PROOF: We prove this result by induction. In the case $m = 1$ inequality (A.12) becomes:

$$v(\sigma_{t+1}, \tau_t) - v(\sigma_{t+1}, \tau_{t+1}) + v(\sigma_{t+2}, \tau_{t+1}) > v(\sigma_{t+2}, \tau_t)$$

which is satisfied by the complementarity assumption $v_{12} >$, given that $\sigma_{t+1} > \sigma_{t+2}$ and $\tau_t > \tau_{t+1}$. Assume now that for every $1 \leq n < m$ the following condition holds:

$$v(\sigma_{t+1}, \tau_t) - \sum_{h=1}^{n} [v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] > v(\sigma_{t+n}, \tau_t) \tag{A.13}$$

We need to show that (A.12) holds for $m = n + 1$. Inequality (A.12) can be written as:

$$\begin{aligned} v(\sigma_{t+1}, \tau_t) \quad &- \quad \sum_{h=1}^{n} [v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] \quad - \\ &- \quad [v(\sigma_{t+n+1}, \tau_{t+n+1}) - v(\sigma_{t+n+2}, \tau_{t+n+1})] > v(\sigma_{t+n+1}, \tau_t) \end{aligned} \tag{A.14}$$

Substituting the induction hypothesis (A.13) into (A.14) we obtain:

$$\begin{aligned} v(\sigma_{t+1}, \tau_t) \quad &- \quad \sum_{h=1}^{n} [v(\sigma_{t+h}, \tau_{t+h}) - v(\sigma_{t+h+1}, \tau_{t+h})] \quad - \\ &- \quad [v(\sigma_{t+n+1}, \tau_{t+n+1}) - v(\sigma_{t+n+2}, \tau_{t+n+1})] \quad > \\ &> \quad v(\sigma_{t+n+1}, \tau_t) - v(\sigma_{t+n+1}, \tau_{t+n+1}) + v(\sigma_{t+n+2}, \tau_{t+n+1}) \end{aligned} \tag{A.15}$$

Notice now that the complementarity assumption $v_{12} > 0$ and the inequalities $\sigma_{t+n+1} > \sigma_{t+n+2}$,

$\tau_t > \tau_{t+n+1}$ imply:

$$v(\sigma_{t+n+1}, \tau_t) - v(\sigma_{t+n+1}, \tau_{t+n+1}) + v(\sigma_{t+n+2}, \tau_{t+n+1}) > v(\sigma_{t+n+2}, \tau_t) \tag{A.16}$$

Substituting (A.16) into (A.15) we conclude that (A.12) holds for $m = n + 1$. ■

Proof of Proposition 3: Consider the vectors of subsequent runner-up workers $(\sigma_t, \ldots, \sigma_{T+1})$ and $(\sigma'_t, \sigma'_{r(t)}, \ldots, \sigma'_{r^{\rho'_t}(t)})$. From Lemma 1 and the assumption $\tau'_t = \tau_t$ we get that $\sigma_t = \sigma'_t$. Moreover from (9) we have that $\sigma_{T+1} = \sigma'_{r^{\rho'_t}(t)}$ and there exist an index $\ell(r^k(t)) \in \{t+1, \ldots, T+1\}$ such that

$$\sigma_{\ell(r^k(t))} = \sigma'_{r^k(t)}$$

for every $k = 0, \ldots, \rho'_t$, where $r^0(t) = t$. In other words, the characterization of the runner-up worker (9) implies that the elements of the vector $(\sigma'_t, \sigma'_{r(t)}, \ldots, \sigma'_{r^{\rho'_t}(t)})$ are a subset of the elements of the vector $(\sigma_t, \sigma_{t+1}, \ldots, \sigma_{T+1})$. Lemma 1 then implies that

$$\tau_{\ell(r^k(t))} = \tau'_{r^k(t)}$$

for every $k = 0, \ldots, \rho'_t$. Therefore we can rewrite the payoff to firm τ'_t, as in (15), in the following way:

$$v(\sigma_{\ell(r(t))}, \tau_{\ell(t)}) \; - \; \sum_{k=1}^{\rho'_t} \left[v(\sigma_{\ell(r^k(t))}, \tau_{\ell(r^k(t))}) \; - \; v(\sigma_{\ell(r^{k+1}(t))}, \tau_{\ell(r^k(t))}) \right]. \tag{A.17}$$

Define now δ_k be an integer number such that $\ell(r^k(t)) + \delta_k = \ell(r^{k+1}(t))$. Then Lemma A.1 implies that:

$$\begin{aligned} v(\sigma_{\ell(r^k(t))+1}, \tau_{\ell(r^k(t))}) \; - \; \sum_{h=1}^{\delta_k - 1} \left[v(\sigma_{\ell(r^k(t))+h}, \tau_{\ell(r^k(t))+h}) - v(\sigma_{\ell(r^k(t))+h+1}, \tau_{\ell(r^k(t))+h}) \right] \; > \\ > \; v(\sigma_{\ell(r^{k+1}(t))}, \tau_{\ell(r^k(t))}) \end{aligned} \tag{A.18}$$

for every $k = 0, \ldots, \rho'_t - 1$. Substituting (A.18) into (A.17) we obtain (22). ■

Proof of Proposition 4: We prove this result in three steps. We first show that the workers' equilibrium qualities $\sigma(i, x_i(i))$ associated with the equilibrium $s_i = i$ satisfy condition (31). We then show that the net payoff to worker i associated with any given quality σ of this worker is continuous in σ. This result is not obvious since, from Lemma 1 — given the investment choices of other workers — worker i can change his equilibrium match by changing his quality σ. Finally, we show that this net payoff has a unique global maximum and this maximum is such that the corresponding quality σ is in the interval in which worker i is matched with firm i. These steps clearly imply that each worker i has no incentive to deviate and choose an investment different from the one that maximizes his net payoff and yields an equilibrium match with firm i.

Let $\pi_i^W(\sigma) - C(x(i,\sigma))$ be the net payoff to worker i where $x(i,\sigma)$ denotes worker i's investment level associated with quality σ:

$$\sigma(i, x(i,\sigma)) \equiv \sigma. \tag{A.19}$$

Step 1: Worker i's equilibrium quality $\sigma(i, x_i(i))$ is such that:

$$\sigma(i, x_i(i)) = \sigma_i < \sigma(i-1, x_{i-1}(i-1)) = \sigma_{i-1}, \qquad \forall i = 2, \ldots, S.$$

The proof follows directly from Lemma 4 above.

Step 2: The net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is continuous in σ.

Let $(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_S)$ be the given ordered vector of the qualities of the workers, other than i. Notice that if $\sigma \in (\sigma_{i-1}, \sigma_{i+1})$ by Lemma 1 worker i is matched with the firm of quality τ_i. Then by Proposition 2 and the definition of $v(\cdot,\cdot)$, $C(\cdot)$, $\sigma(\cdot,\cdot)$ and (A.19) the payoff function $\pi_i^W(\sigma) - C(x(i,\sigma))$ is continuous in σ.

Consider now the limit for $\sigma \to \sigma_{i-1}^-$ from the right of the net payoff to worker i when it is matched with the firm of quality τ_i, $\sigma \in (\sigma_{i+1}, \sigma_{i-1})$. From (16) this limit is

$$\begin{aligned} \pi_i^W(\sigma_{i-1}^-) \;\; - \;\; C(x(i,\sigma_{i-1}^-)) \;=\; & v(\sigma_{i-1},\tau_i) - v(\sigma_{i+1},\tau_i) \;+ \\ + \;\; \sum_{h=i+1}^{T} & [v(\sigma_h,\tau_h) - v(\sigma_{h+1},\tau_h)] - C(x(i,\sigma_{i-1})). \end{aligned} \tag{A.20}$$

Conversely, if $\sigma \in (\sigma_{i-1}, \sigma_{i-2})$ then by Lemma 1 worker i is matched with the firm of quality $\tau_{i-1}$and the payoff is continuous in this interval. Then from (16) the limit for $\sigma \to \sigma_{i-1}^+$ from the left of the net payoff to worker i when matched with the firm of quality τ_{i-1} is

$$\begin{aligned} \pi_i^W(\sigma_{i-1}^+) \;\; - \;\; C(x(i,\sigma_{i-1}^+)) \;=\; & v(\sigma_{i-1},\tau_{i-1}) - v(\sigma_{i-1},\tau_{i-1}) \;+ \\ + \;\; & v(\sigma_{i-1},\tau_i) - v(\sigma_{i+1},\tau_i) \;+ \\ + \;\; \sum_{h=i+1}^{T} & [v(\sigma_h,\tau_h) - v(\sigma_{h+1},\tau_h)] - C(x(i,\sigma_{i-1})). \end{aligned} \tag{A.21}$$

In this case while the worker of quality σ is matched with the firm of quality τ_{i-1} the worker of quality σ_{i-1} is matched with the firm of quality τ_i.

Equation (A.20) coincides with equation (A.21) since the first two terms of the left-hand-side of equation (A.21) are identical. A similar argument shows continuity of the net payoff function at $\sigma = \sigma_h$, $h = 1, \ldots, i-2, i+1, \ldots, N$.

Step 3: The net surplus function $\pi_i^W(\sigma) - C(x(i,\sigma))$ has a unique global maximum in the interval $(\sigma_{i+1}, \sigma_{i-1})$.

Notice first that in the interval $(\sigma_{i+1}, \sigma_{i-1})$, by Lemma 1 and Proposition 2, the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ takes the following expression.

$$\begin{aligned} \pi_i^W(\sigma) - C(x(i,\sigma)) &= v(\sigma,\tau_i) - v(\sigma_{i+1},\tau_i) \; + \\ &+ \sum_{h=i+1}^{T} [v(\sigma_h,\tau_h) - v(\sigma_{h+1},\tau_h)] - C(x(i,\sigma)). \end{aligned} \tag{A.22}$$

This expression, and therefore the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$, is strictly concave in σ (by strict concavity of $v(\cdot,\tau_i)$, $\sigma(i,\cdot)$ and strict convexity of $C(\cdot)$) in the interval $(\sigma_{i+1}, \sigma_{i-1})$ and reaches a maximum at $\sigma_i = \sigma(i, x_i(i))$ as defined in (28) above.

Notice, further, that in the right adjoining interval $(\sigma_{i-1}, \sigma_{i-2})$, by Lemma 1 and Proposition 2, the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ takes the following expression — different from (A.22).

$$\begin{aligned} \pi_i^W(\sigma) \; - \; & C(x(i,\sigma)) \;=\; v(\sigma,\tau_{i-1}) - v(\sigma_{i-1},\tau_{i-1}) \; + \\ + \; & v(\sigma_{i-1},\tau_i) - v(\sigma_{i+1},\tau_i) \; + \\ + \; & \sum_{h=i+1}^{T} [v(\sigma_h,\tau_h) - v(\sigma_{h+1},\tau_h)] - C(x(i,\sigma)). \end{aligned} \tag{A.23}$$

This new expression of the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is also strictly concave (by strict concavity of $v(\cdot,\tau_{i-1})$, $\sigma(i,\cdot)$ and strict convexity of $C(\cdot)$) and reaches a maximum at $\sigma(i, x_i(i-1))$. From Lemma 4 above we know that

$$\sigma(i, x_i(i-1)) < \sigma_{i-1} = \sigma(i-1, x_{i-1}(i-1)).$$

This implies that in the interval $(\sigma_{i-1}, \sigma_{i-2})$ the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is strictly decreasing in σ.

A symmetric argument shows that the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is strictly decreasing in σ in any interval (σ_h, σ_{h-1}) for every $h = 2, \ldots, i-2$.

Notice, further, that in the left adjoining interval $(\sigma_{i+2}, \sigma_{i+1})$, by Lemma 1 and Proposition 2, the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ takes the following expression — different from (A.22) and (A.23).

$$\begin{aligned}\pi_i^W(\sigma) \quad - \quad & C(x(i,\sigma)) \;=\; v(\sigma,\tau_{i+1}) - v(\sigma_{i+2},\tau_{i+1}) \;+ \\ + \quad & \sum_{h=i+2}^{T} [v(\sigma_h,\tau_h) - v(\sigma_{h+1},\tau_h)] - C(x(i,\sigma)).\end{aligned} \tag{A.24}$$

This new expression of the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is also strictly concave in σ (by strict concavity of $v(\cdot,\tau_{i+1})$, $\sigma(i,\cdot)$ and strict convexity of $C(\cdot)$) and reaches a maximum at $\sigma(i,x_i(i+1))$ that from Lemma 4 is such that

$$\sigma_{i+1} = \sigma(i+1, x_{i+1}(i+1)) < \sigma(i, x_i(i+1)).$$

This implies that in the interval $(\sigma_{i+2},\sigma_{i+1})$ the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is strictly increasing in σ.

A symmetric argument shows that the net payoff $\pi_i^W(\sigma) - C(x(i,\sigma))$ is strictly increasing in σ in any interval (σ_{k+1},σ_k) for every $k = i+2,\ldots,T-1$. ■

Proof of Proposition 5: First, for a given ordered vector of firms' qualities $(\tau_1,\ldots,\tau_T)$ we construct an inefficient equilibrium of the workers' investment game such that there exist one worker, labelled s_j, $j \in \{2,\ldots,S\}$, such that $s_j < s_{j-1}$.

To show that a vector $(s_1,\ldots,s_j,\ldots,s_S)$ is an equilibrium of the workers' investment game we need to verify that condition (31) holds for every $i = 2,\ldots,S$ and no worker s_i has an incentive to deviate and choose an investment x different from $x_{s_i}(i)$, as defined in (27).

Notice first that for every worker, other than s_j and s_{j-1}, Proposition 4 above applies and hence it is an equilibrium for each worker to choose investment level $x_{s_i}(i)$, as defined in (27), such that (31) is satisfied.

We can therefore restrict attention on worker s_j and s_{j-1}. In particular we need to consider a worker s_{j-1} of a quality arbitrarily close to the one of worker s_j. This is achieved by considering a sequence of quality functions $\sigma^n(s_{j-1},\cdot)$ that converges uniformly to $\sigma(s_j,\cdot)$.[22] Then from definition (27), the continuity and strict concavity of $v(\cdot,\tau)$ and $\sigma(s,\cdot)$, the continuity and strict convexity of $C(\cdot)$ and the continuity of $v_1(\cdot,\tau)$, $\sigma_2(s,\cdot)$ and $C'(\cdot)$ for any given $\varepsilon > 0$ there exists an index n_ε

[22]The sequence $\sigma^n(s_{j-1},\cdot)$ converges uniformly to $\sigma(s_j,\cdot)$ if and only if

$$\lim_{n\to\infty}\sup_x |\sigma^n(s_{j-1},x) - \sigma(s_j,x)| = 0.$$

such that from every $n > n_\varepsilon$:

$$\left|\sigma^n(s_{j-1}, x_{s_{j-1}}(j-1)) - \sigma(s_j, x_{s_j}(j-1))\right| < \varepsilon. \tag{A.25}$$

From Lemma 4 and the assumptions $s_j > s_{j-1}$ we also know that for every $n > n_\varepsilon$:

$$\sigma^n(s_{j-1}, x_{s_{j-1}}(i-1)) < \sigma(s_j, x_{s_j}(j-1)). \tag{A.26}$$

While from the assumption $\tau_j < \tau_{j-1}$ we have that:

$$\sigma(s_j, x_{s_j}(j)) < \sigma(s_j, x_{s_j}(j-1)). \tag{A.27}$$

Inequalities (A.25), (A.26) and (A.27) imply that for any worker s_{j-1} characterized by the quality function $\sigma^n(s_{j-1}, \cdot)$ where $n > n_\varepsilon$, the equilibrium condition (31) is satisfied:

$$\sigma(s_j, x_{s_j}(j)) < \sigma^n(s_{j-1}, x_{s_{j-1}}(j-1)). \tag{A.28}$$

To conclude that $(s_1, \ldots, s_j, \ldots, s_S)$ is an equilibrium of the workers' investment game we still need to show that neither worker s_j nor worker s_{j-1} want to deviate and choose an investment different from $x_{s_j}(j)$ and $x_{s_{j-1}}(j-1)$, where the quality function associated with worker s_{j-1} is $\sigma^n(s_{j-1}, \cdot)$ for $n > n_\varepsilon$.

Consider the net payoff to worker s_j: $\pi^W_{s_j}(\sigma) - C(x(s_j, \sigma))$. An argument symmetric to the one used in Step 2 of Proposition 4 shows that this payoff function is continuous in σ. Moreover, from the notation of σ_j in Section 4 above, Lemma 4, (A.26) and (A.28) we obtain that

$$\sigma_j < \sigma^n_{j-1} \ < \sigma(s_j, x_{s_j}(j-1)) \ < \sigma_{j-2}.$$

Then using an argument symmetric to the one used in Step 3 of the proof of Proposition 4 we conclude that this net payoff function has two local maxima at σ_j and $\sigma(s_j, x_{s_j}(j-1))$ and a kink at σ^n_{j-1}. We then need to show that there exist at least an element of the sequence σ^n_{j-1} such that the net payoff $\pi^W_{s_j}(\sigma) - C(x(s_j, \sigma))$ reaches a global maximum at σ_j. Therefore when the quality function of worker s_{j-1} is $\sigma^n(s_{j-1}, \cdot)$ worker s_j has no incentive to deviate and choose a different investment.

From (16) the net payoff $\pi^W_{s_j}(\sigma) - C(x(s_j, \sigma))$ computed at σ_j is greater than the same net payoff computed at $\sigma(s_j, x_{s_j}(j-1))$ if and only if

$$\begin{aligned} v(\sigma_j, \tau_j) \quad - \quad & C\left(x(s_j, \sigma_{(j)})\right) \geq \\ \geq \quad & v(\sigma(s_j, x_{s_j}(j-1)), \tau_{j-1}) \quad - \quad v(\sigma^n_{j-1}, \tau_{j-1}) + \\ + \quad & v(\sigma^n_{j-1}, \tau_j) - C\left(x\left(s_j, \sigma(s_j, x_{s_j}(j-1))\right)\right) \end{aligned} \tag{A.29}$$

Inequality (A.25) above and the continuity of $v(\cdot, \tau_{j-1})$, $\sigma(s_j, \cdot)$ and $C(\cdot)$ imply that for any given $\varepsilon > 0$ there exist a ξ_ε and a n_{ξ_ε} such that for every $n > n_{\xi_\varepsilon}$

$$\left|v(\sigma(s_j, x_{s_j}(j-1)), \tau_{j-1}) - v(\sigma^n_{j-1}, \tau_{j-1})\right| < \xi_\varepsilon$$

and

$$\left|C\left(x\left(s_j, \sigma(s_j, x_{s_j}(j-1))\right)\right) - C\left(x(s_j, \sigma^n_{j-1})\right)\right| < \xi_\varepsilon$$

These two inequalities imply that a necessary condition for (A.29) to be satisfied is

$$v(\sigma_j, \tau_j) - C\left(x(s_j, \sigma_j)\right) \geq v(\sigma^n_{j-1}, \tau_j) - C\left(x(s_j, \sigma^n_{j-1})\right) + 2\xi_\varepsilon. \tag{A.30}$$

We can now conclude that there exist an $\varepsilon > 0$ such that for every $n > n_{\xi_\varepsilon}$ condition (A.30) is satisfied with strict inequality. This is because (by strict concavity of $v(\cdot, \tau_j)$, $\sigma(s_j, \cdot)$ and strict convexity of $C(\cdot)$) the function $v(\sigma, \tau_j) - C\left(x(s_j, \sigma)\right)$ is strictly concave and has a unique interior maximum at σ_j.

Consider now the net payoff to worker s_{j-1}: $\pi^W_{s_{j-1}}(\sigma) - C(x(s_{j-1}, \sigma))$. An argument symmetric to the one used above allow us to prove that this payoff function is continuous in σ. Further, from the notation of σ_j in Section 4 above, Lemma 4, and (A.28) we have that

$$\sigma_{j+1} < \sigma^n(s_{j-1}, x_{s_{j-1}}(j)) \; < \sigma_j < \sigma^n_{j-1}.$$

Therefore we conclude that the net surplus function $\pi^W_{s_{j-1}}(\sigma) - C(x(s_{j-1}, \sigma))$ has two local maxima at σ^n_{j-1} and $\sigma^n(s_{j-1}, x_{s_{j-1}}(j))$ and a kink at σ_j. We still need to prove that there exist at least an element of the sequence σ^n_{j-1} such that the net payoff $\pi^W_{s_{j-1}}(\sigma) - C(x(s_{j-1}, \sigma))$ reaches a global maximum at σ^n_{j-1} which implies that when the quality function of worker s_{j-1} is $\sigma^n(s_{j-1}, \cdot)$ this worker has no incentive to deviate and choose a different investment.

From (16) the net payoff $\pi^W_{s_{j-1}}(\sigma) - C(x(s_{j-1}, \sigma))$ computed at σ^n_{j-1} is greater than the same net payoff computed at $\sigma^n(s_{j-1}, x_{s_{j-1}}(j))$ if and only if

$$\begin{aligned} v(\sigma^n_{j-1}, \tau_{j-1}) \quad - \quad & v(\sigma_j, \tau_{j-1}) + v(\sigma_j, \tau_j) - C\left(x(s_{j-1}, \sigma^n_{j-1})\right) \geq \\ \geq \quad & v(\sigma^n(s_{j-1}, x_{s_{j-1}}(j)), \tau_j) - C\Big(x\left(s_{j-1}, \sigma^n(s_{j-1}, x_{s_{j-1}}(j))\right)\Big) \end{aligned} \tag{A.31}$$

Definition (27), the continuity and strict concavity of $v(\cdot, \tau_j)$ and $\sigma(s_{j-1}, \cdot)$, the continuity and strict convexity of $C(\cdot)$ and the continuity of $v_1(\cdot, \tau_j)$, $\sigma_2(s_j, \cdot)$ and $C'(\cdot)$ imply that for given $\varepsilon' > 0$ there exists a $n_{\varepsilon'}$, a $\xi_{\varepsilon'}$ and a $n_{\xi_{\varepsilon'}}$ such that from every $n > n_{\varepsilon'}$:

$$\left|\sigma^n(s_{j-1}, x_{s_{j-1}}(j)) - \sigma_j\right| < \varepsilon';$$

while for every $n > n_{\xi_{\varepsilon'}}$

$$\left|v(\sigma_j, \tau_j) - v(\sigma^n(s_{j-1}, x_{s_{j-1}}(j)), \tau_j)\right| < \xi_{\varepsilon'}$$

and

$$\left|C\left(x(s_{j-1}, \sigma_j)\right) - C\Big(x\left(s_{j-1}, \sigma^n(s_{j-1}, x_{s_{j-1}}(j))\right)\Big)\right| < \xi_{\varepsilon}.$$

The last two inequalities imply that a necessary condition for (A.31) to be satisfied is

$$v(\sigma^n_{j-1}, \tau_{j-1}) - C\left(x(s_{j-1}, \sigma^n_{j-1})\right) \geq v(\sigma_j, \tau_{j-1}) - C\left(x(s_{j-1}, \sigma_j)\right) + 2\xi_{\varepsilon'}. \tag{A.32}$$

We can now conclude that there exists a $\varepsilon' > 0$ such that for every $n > n_{\xi_{\varepsilon'}}$ condition (A.32) is satisfied with strict inequality. This is because (by strict concavity of $v(\cdot, \tau_{j-1})$, $\sigma^n(s_{j-1}, \cdot)$ and strict convexity of $C(\cdot)$) the function $v(\sigma, \tau_{j-1}) - C\left(x(s_{j-1}, \sigma)\right)$ is strictly concave and has a unique interior maximum at σ^n_{j-1}.

This concludes the construction of the inefficient equilibrium of the workers' investment game.

We need now to show that for any given vector of workers' quality functions $(\sigma(s_1, \cdot), \ldots, \sigma(s_S, \cdot))$ it is possible to construct an ordered vector of firms qualities $(\tau_1, \ldots, \tau_T)$ such that no inefficient equilibrium exist.

Assume, by way of contradiction, that an inefficient equilibrium exists for any ordered vector of firms' qualities $(\tau_1, \ldots, \tau_T)$. Consider first the case in which this inefficient equilibrium is such that there exist only one worker s_j such that $s_j < s_{j-1}$. Let τ^n_{j-1} be a sequence of quality levels of firm $(j-1)$ such that $\tau^n_{j-1} > \tau_j$ and τ^n_{j-1} converges to τ_j.

From Lemma 4 and the assumption $s_j > s_{j-1}$ we have that

$$\sigma(s_j, x_{s_j}(j)) > \sigma(s_{j-1}, x_{s_{j-1}}(j)) \tag{A.33}$$

where $x_{s_j}(j)$ and $x_{s_{j-1}}(j)$ are defined in (27). Further, denote $x^n_{s_{j-1}}(j-1)$ the optimal investment defined, as in (28), by the following set of first order conditions:

$$v_1(\sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)), \tau^n_{j-1})\; \sigma_2(s_{j-1}, x^n_{s_{j-1}}(j-1)), \tau^n_{j-1}) = C'(x^n_{s_{j-1}}(j-1)).$$

Then from Lemma 4 we have that

$$\sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)) > \sigma(s_{j-1}, x_{s_{j-1}}(j)). \tag{A.34}$$

Further, continuity of the functions $v(\sigma, \cdot)$, $v_1(\sigma, \cdot)$, $\sigma(s, \cdot)$, $\sigma_2(s, \cdot)$, $C(\cdot)$ and $C'(\cdot)$ imply that for given $\hat{\varepsilon} > 0$ there exist an $n_{\hat{\varepsilon}}$ such that for every $n > n_{\hat{\varepsilon}}$

$$\left|\sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)) - \sigma(s_{j-1}, x_{s_{j-1}}(j))\right| < \hat{\varepsilon}. \tag{A.35}$$

Then from (A.33), (A.34) and (A.35) there exists an $\hat{\varepsilon} > 0$ and hence an $n_{\hat{\varepsilon}}$ such that for every $n > n_{\hat{\varepsilon}}$

$$\sigma(s_j, x_{s_j}(j)) > \sigma(s_{j-1}, x^n_{s_{j-1}}(j-1)). \tag{A.36}$$

Inequality (A.36) clearly contradicts the necessary condition (31) for the existence of the inefficient equilibrium.

A similar construction leads to a contradiction in the case the inefficient equilibrium is characterized by more than one worker s_j such that $s_j < s_{j-1}$. ■

Proof of Corollary 1: Notice first that the proofs of Lemma 3 and Lemma 4 hold unchanged in the case firms choose their bids in the order of any vector of firms' qualities $(\tau_1, \ldots, \tau_T)$.

The proof of Proposition 4 also holds in this general case provided one substitutes the payoff in (16) with the payoffs in (14). Moreover we need to reinterpret the workers' qualities σ_{i-1}, σ_i and σ_{i-1} to be the qualities of three subsequent workers in the chain of runner-up workers. In particular σ_i is the quality of the runner-up worker to the firm that in equilibrium is matched with the worker of quality σ_{i-1}, while σ_{i+1} is the quality of the runner-up worker to the firm that in equilibrium is matched with the worker of quality σ_i. We do not repeat here the details of the proof.

Finally, the proof of Proposition 5 can also be modified to apply to the general case in which firm choose their bids in the order of the vector of firms' qualities $(\tau_1, \ldots, \tau_T)$. We need to substitute the payoff in (16) with the payoff in (14). Moreover, we need to reinterpret the worker's identity s_j as the identity of the runner-up worker to the firm that in equilibrium matches with the worker s_{j-1}. Once again we do not repeat here the details of the proof. ■

Proof of Proposition 6: We prove this result in two steps. We first show that if firms choose investments $y(t, t+1)$, for $t = 1, \ldots, T$, (labelled *simple* investments, for convenience) then the order of firms' identities coincides with the order of firms' qualities. Hence, Proposition 2 applies and the shares of the surplus accruing to each worker and each firm are the ones defined in (16) and (17) above. We then conclude the proof by showing that the unique equilibrium of the firms' investment subgame is for firm t to choose the simple investment $y(t, t+1)$, $t = 1, \ldots, T$.

Step 1: If each firm t chooses the simple investment $y(t, t+1)$, as defined in (1), then

$$\tau_1 = \tau(1, y(1,2)) > \ldots > \tau_T = \tau(T, y(T, T+1)).$$

The proof follows from the fact that from (2) we obtain:

$$\frac{\partial \tau(t, y(t,s))}{\partial t} = \frac{v_2\ \tau_1\ \tau_{22} - \tau_1\ C'' - v_2\ \tau_2\ \tau_{12}}{v_{22}(\tau_2)^2 + v_2\ \tau_{22} - C''} < 0 \tag{A.37}$$

and

$$\frac{\partial \tau(t, y(t,s))}{\partial s} = \frac{v_{12}(\tau_2)^2}{v_{22}(\tau_2)^2 + v_2\ \tau_{22} - C''} < 0 \tag{A.38}$$

where (with an abuse of notation) we denote with τ_h and τ_{hk}, $h, k \in \{1, 2\}$ the first and second order derivatives of the quality functions $\tau(\cdot, \cdot)$ computed at $(t, y(t, s))$. Moreover the first and second order derivative (v_h and v_{hk}, $h, k \in \{1, 2\}$) of the functions $v(\cdot, \cdot)$ are computed at $(\sigma_s, \tau(t, y(t, s)))$.

Step 2: The unique equilibrium of the firms' investment subgame is such that firm t chooses the simple investment $y(t, t+1)$ for every $t = 1, \ldots, T$.

We prove this result starting from firm T. In the T-th (the last) matching of the Bertrand competition game all firms, but firm T, have selected a worker's bid. Denote τ_T the quality of this firm.

Assume for simplicity that $S = T + 1$. We use the same notation as in the proof of Proposition 2 above. In particular since we want to show that firm T chooses a simple investment independently from the investment choice of the other firms we denote $\alpha_{(T)}$ and $\alpha_{(T+1)}$ the qualities of the two workers that are still un-matched in the T-th subgame, such that $\alpha_{(T)} > \alpha_{(T+1)}$. Indeed, from Lemma 1 the identity of the two workers left will depend on the order of firms' qualities and therefore on the investment choices of the other $(T-1)$ firms.

From Lemma 1 above we have that the worker of quality $\alpha_{(T)}$ matches with firm T. Firm T's payoff is $v(\alpha_{(T+1)}, \tau_T)$ while the payoff of the worker of quality $\alpha_{(T)}$ is $\left[v(\alpha_{(T)}, \tau_T) - v(\alpha_{(T+1)}, \tau_T)\right]$ and the payoff of the worker of quality $\alpha_{(T+1)}$ is zero.

Denote now $a_{(T)}$, respectively $a_{(T+1)}$, the identity of the workers of quality $\alpha_{(T)}$, respectively $\alpha_{(T+1)}$: $a_{(T)} < a_{(T+1)}$. Firm T's optimal investment y_T is then defined as follows

$$y_T = \operatorname*{argmax}_y v(\alpha(T+1), \tau(T, y)) - C(y).$$

This implies that the optimal investment of firm T is the simple investment $y_T = y(T, a_{(T+1)})$, as defined in (2), whatever is the pair of workers left in the T-th subgame. If all other firms undertake a simple investment then from Step 1: $a_{(T)} = T$ and $a_{(T+1)} = T+1$. Hence firm T's optimal investment is $y(T, T+1)$.

Denote now $t+1$, $(t < T)$, the last firm that undertakes a simple investment $y(t+1, t+2)$. We then show that also firm t will choose a simple investment $y(t, t+1)$. Consider the t-th subgame in which firm t has to choose among the potential bids of the remaining $(T-t+2)$ workers labelled $a_{(t)} < \ldots < a_{(T+1)}$, with associated qualities $\alpha_{(t)} > \ldots > \alpha_{(T+1)}$, respectively.[23] From the assumption that every firm $j = t+1, \ldots, T$ undertakes a simple investment $y(j, a_{(j+1)})$ and Step 1 we obtain that $\tau_{t+1} > \ldots > \tau_T$. We first show that the quality associated with firm t is such that $\tau_t > \tau_{t+1}$.

Assume by way of contradiction that firm t chooses investment y^* that yields a quality τ^* such that $\tau_{j+1} \leq \tau^* \leq \tau_j$ for some $j \in \{t+1, \ldots, T-1\}$. Then from Lemma 1 and (17) we have that firm t matches with worker $a_{(j)}$ and firm t's payoff is:

$$\Pi^F_{\tau^*} = v(\alpha_{(j+1)}, \tau(t, y^*)) - \sum_{h=j+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right] \tag{A.39}$$

where $\tau(t, y^*) = \tau^*$. From (A.39) we obtain that y^* is then the solution to the following problem:

$$y^* = \operatorname*{argmax}_y v(\alpha(j+1), \tau(t, y)) - C(y). \tag{A.40}$$

From the assumption that each firm $j \in \{t+1, \ldots, T\}$ undertakes a simple investment and definition (1) we also have that firm j's investment choice $y(j, a_{(j+1)})$ is defined as follows:

$$y(j, a_{(j+1)}) = \operatorname*{argmax}_y v(\alpha(j+1), \tau(j, y)) - C(y). \tag{A.41}$$

Notice further that the payoff to firm t in (A.39) is continuous in τ^*. Indeed the limit for τ^* that converges from the right to τ_j is equal to

$$\Pi^F_{\tau_j} = v(\alpha_{(j+1)}, \tau_j) - \sum_{h=j+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right]. \tag{A.42}$$

[23] Once again we want to show that firm t undertakes a simple investment independently of the investment choice of firms $1, \ldots, t-1$ that, from Lemma 1, determines the exact identities of the un-matched workers in the t-th subgame of the Bertrand competition game.

If instead $\tau_j < \tau^* \leq \tau_{j-1}$ then from (17) the payoff to the firm with quality τ^* is

$$\begin{aligned}\Pi^F_{\tau^*} &= v(\alpha_{(j)}, \tau^*) - v(\alpha_{(j)}, \tau_j) + \\ &+ v(\alpha_{(j+1)}, \tau_j) - \sum_{h=j+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right].\end{aligned}$$

Therefore the limit for τ^* that converges to τ_j from the left is, from (8), equal to $\Pi^F_{\tau_j}$ in (A.42). This proves the continuity in τ^* of the payoff function in (A.39).

Continuity of the payoff function in (A.39) together with definitions (A.40), (A.41) and condition (A.37) imply that $y* > y(j, a_{(j+1)})$ or $\tau^* > \tau_j$ a contradiction to the hypothesis $\tau^* \leq \tau_j$.

We now show that firm t will choose a simple investment $y(t, a_{(t+1)})$. From the result we just obtained $\tau_t > \tau_{t+1} > \ldots > \tau_T$ and the assumption that $\alpha_{(t)} > \ldots > \alpha_{(S)}$ are the qualities of the unmatched workers in the t-th subgame of the Bertrand competition game we conclude, using (17) above, that the payoff to firm t is:

$$\Pi^F_{\tau_t} = v(\alpha_{(t+1)}, \tau_t) - \sum_{h=t+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h) \right] \tag{A.43}$$

Firm t's investment choice is then the simple investment $y(t, a_{(t+1)})$ defined as follows:

$$y(t, a_{(t+1)}) = \underset{y}{\operatorname{argmax}}\, v(\alpha(t+1), \tau(t, y)) - C(y). \tag{A.44}$$

To conclude that a simple investment $y(t, a_{(t+1)})$ is the unique equilibrium choice for firm t in the firms' investment game we still need to show that firm t has no incentive to deviate and choose an investment y^*, and hence a quality τ^*, that exceeds the quality τ_k of one of the $(t-1)$ firms that are already matched at the t-th subgame of the Bertrand competition game: $k < t$. The reason why this choice of investment might be optimal for firm t is that it changes the pool of workers $a_{(t)}, \ldots, a_{(S)}$ unmatched in subgame t. Of course this choice will change the simple nature of firm t's investment only if $\tau_k > \tau_{t+1}$. Indeed we already showed that if $\tau_k < \tau_{t+1}$ then $\tau_t > \tau_k$ and from (A.44) firm t's investment choice is $y_t(a_{(t+1)})$ a simple investment for any given set of unmatched workers.

Consider the following deviation by firm t: firm t chooses an investment $y^* > y(t, a_{(t+1)})$ that yields quality $\tau^* > \tau_k > \tau_{t+1}$. Recall that Lemma 1 implies that the ranking of each firm in the ordered vector of firms' qualities determines the worker each firm is matched with. Hence, firm t's deviation changes the ranking and the matches of all firms whose quality τ is smaller than τ^* and greater than τ_{t+1}. However, this deviation does not alter the ranking of the $T-t$ firms with identities $(t+1, \ldots, T)$ and qualities $(\tau_{t+1}, \ldots, \tau_T)$. Therefore, the only difference between the equilibrium set of un-matched workers in the t-th subgame and the set of un-matched workers in the same subgame following firm t's deviation is the identity and quality of the worker that matches with firm t.[24] The remaining set of workers' identities and qualities $(\alpha_{(t+1)}, \ldots, \alpha_{(S)})$ is unchanged.

Hence, following firm t's deviation the un-matched workers' qualities are $\alpha^* > \alpha_{(t+1)} > \ldots > \alpha_{(T)}$, where α^* is the quality of the worker that according to Lemma 1 is matched with firm t when the quality of this firm is τ^*. Equation (17) implies that firm t's payoff following this deviation is then:

$$\Pi^F_{\tau^*} = v(\alpha_{(t+1)}, \tau^*) - \sum_{h=t+1}^{T} \left[v(\alpha_{(h)}, \tau_h) - v(\alpha_{(h+1)}, \tau_h)\right] \tag{A.45}$$

Continuity of the payoff function in (A.44) together with (A.45) imply that firm t's net payoff is maximized at $y(t, a_{(t+1)})$. Hence, firm t cannot gain from choosing an investment $y^* > y(t, a_{(t+1)})$. This proves that firm t will choose a simple investment $y(t, a_{(t+1)})$. This argument holds for every $t < T$ implying that all firm choose a simple investment. Therefore $a_{(t)} = t$ and firm t's equilibrium investment choice is $y_t = y(t, t+1)$. ■

Proof of Proposition 8: Notice first that L and M can be written as

$$L = \sum_{t=1}^{T} \omega(t, t) - \sum_{t=1}^{T} \omega(t, t+1) \tag{A.46}$$

$$M = \sum_{t=1}^{T} \omega(1, t) - \sum_{t=1}^{T} \omega(1, t+1) \tag{A.47}$$

so that

$$M - L = \sum_{t=1}^{T} \left\{ \Big[\omega(1, t) - \omega(t, t)\Big] - \Big[\omega(1, t+1) - \omega(t, t+1)\Big] \right\} \tag{A.48}$$

From (A.48), it is clear that, as $T > 1$, each bracketed term in the summation will be positive with some strictly positive if

$$\frac{\partial^2 \omega(t, s)}{\partial s\, \partial t} > 0. \tag{A.49}$$

From the definition (35) of $\omega(t, s)$ we have:

$$\frac{\partial^2 \omega(t, s)}{\partial s\, \partial t} = \frac{\partial}{\partial t} \left[(v_2 - \tilde{v}_2) \tau_2 \frac{\partial y(t, s)}{\partial s} \right]. \tag{A.50}$$

[24] Indeed all other firms with identities $(k, \ldots, t-1)$ whose match changed because of the deviation are already matched in the t-th subgame of the Bertrand competition game.

Notice that from $v_{12} > 0$ we have $(v_2 - \tilde{v}_2) > 0$ if $s > t$; while from (2) we have:

$$\frac{\partial y(t,s)}{\partial s} = -\frac{\tilde{v}_{12}\,\sigma_1\,\tau_2}{\tilde{v}_{22}(\tau_2)^2 + \tilde{v}_2\,\tau_{22} - C''} < 0. \tag{A.51}$$

In both expressions (A.50) and (A.51) the derivatives v_h and v_{hk}, $h,k \in \{1,2\}$, are evaluated at $(\sigma(t), \tau(t, y(t,s)))$, while $\tilde{v}_h$ and $\tilde{v}_{hk}$, $h,k \in \{1,2\}$, are evaluated at $(\sigma(s), \tau(t, y(t,s)))$, τ_2 is evaluated at $(t, y(t,s))$ and C'' is evaluated at $y(t,s)$.

From (A.50) the cross partial derivative of $\omega(t,s)$ then takes the following expression:

$$\begin{aligned}
\frac{\partial^2 \omega(t,s)}{\partial s\,\partial t} &= \left[v_{12} + (v_{22} - \tilde{v}_{22})\,\frac{\partial\,\tau(t,y(t,s))}{\partial t}\right]\tau_2\,\frac{\partial y(t,s)}{\partial s} \quad + \\
&+ \quad (v_2 - \tilde{v}_2)\,\frac{\partial y(t,s)}{\partial s}\,\frac{\partial\,\tau_2(t,y(t,s))}{\partial t} \quad + \\
&+ \quad (v_2 - \tilde{v}_2)\,\tau_2\,\frac{\partial^2 y(t,s)}{\partial s\,\partial t}.
\end{aligned} \tag{A.52}$$

To investigate the actual sign of $(\partial^2\omega(t,s)/\partial s\,\partial t)$, we must identify the sing of $(v_{22} - \tilde{v}_{22})$, of the partial derivative $(\partial\tau_2(t,y(t,s))/\partial t)$ and of the cross derivative $(\partial^2 y(t,s)/\partial s\,\partial t)$.

Notice first that the marginal complementarity assumption $v_{122} > 0$ implies that if $s > t$

$$(v_{22} - \tilde{v}_{22}) > 0. \tag{A.53}$$

Second, from the definition (2) of $y(t,s)$ we have that:

$$\frac{\partial\tau_2(t,y(t,s))}{\partial t} = \frac{\tau_{12}\tilde{v}_{22}(\tau_2)^2 - C''\tau_{12} - \tau_{22}\tilde{v}_{22}\tau_1\tau_2}{\tilde{v}_{22}(\tau_2)^2 + \tilde{v}_2\tau_{22} - C''} < 0. \tag{A.54}$$

Finally the responsive complementarity assumption (3) implies that:

$$\frac{\partial^2 y(t,s)}{\partial s\,\partial t} > 0. \tag{A.55}$$

Conditions (A.53), (A.54) and (A.55) imply — together with (A.51) and $(v_2 - \tilde{v}_2) > 0$ if $s > t$ — that all three terms in (A.52) are strictly positive. Thus $(\partial^2\omega(t,s)/\partial s\,\partial t)$ is positive: every term in the summation of (A.48) is positive and $M > L$. The overall efficiency loss in the market is less than that which is induced by the under-investment of the best firm. ■

References

Acemoglu, D. (1997): "Training and Innovation in an Imperfect Labor Market," *Review of Economic Studies*, 64, 445–464.

Acemoglu, D., and R. Shimer (1999): "Holdups and Efficiency with Search Frictions," *International Economic Review*, 40, 827–49.

Aghion, P., M. Dewatripont, and P. Rey (1994): "Renegotiation Design with Unverifiable Information," *Econometrica*, 62, 257–82.

Aghion, P., and J. Tirole (1997): "Formal and Real Authority in Organizations," *Journal of Political Economy*, 105, 1–29.

Bolton, P., and M. Whinston (1993): "Incomplete Contracts, Vertical Integration, and Supply Assurance," *Review of Economic Studies*, 60, 121–48.

Burdett, K., and M. G. Coles (1997): "Marriage and Class," *Quarterly Journal of Economics*, 112, 141–68.

——— (1999): "Transplants and Implants: The Economic of Self-Improvement," Discussion Paper 99/47, University of Essex.

Chatterjee, K., and Y. S. Chiu (1999): "When Does Competition Lead to Efficient Investments?," mimeo.

Che, Y.-K., and D. Hausch (1996): "Cooperative Investments and the Value of Contracting: Coase vs. Williamson," SSRI Working Paper No. 9608, University of Wisconsin, Madison.

——— (1999): "Cooperative Investments and the Value of Contracting," *American Economic Review*, 89, 125–47.

Cole, H. L., G. J. Mailath, and A. Postlewaite (2001a): "Efficient Non-Contractible Investments in Finite Economies," *The B.E. Journals of Theoretical Economics*, forthcoming.

——— (2001b): "Efficient Non-Contractible Investments in Large Economies," *Journal of Economic Theory*, forthcoming.

Cooper, R., and A. John (1988): "Coordinating Coordination Failures in Keynesian Models," *Quarterly Journal of Economics*, 103, 441–63.

de Meza, D., and B. Lockwood (1998): "The Property-Rights Theory of the Firm with Endogenous Timing of Asset Purchase," mimeo.

Diamond, P. (1971): "A Model of Price Adjustment," *Journal of Economic Theory*, 3, 156–68.

——— (1982): "Wage Determination and Efficiency in Search Equilibrium," *Review of Economic Studies*, 49, 217–27.

Edlin, A., and C. Shannon (1998): "Strict Monotonicity in Comparative Statics," *Journal of Economic Theory*, 81, 201–19.

Eeckhout, J. (1999): "Bilateral Search and Vertical Heterogeneity," *International Economic Review*, 40, 869–87.

Felli, L., and C. Harris (1996): "Learning, Wage Dynamics, and Firm-Specific Human Capital," *Journal of Political Economy*, 104, 838–68.

Grossman, S. J., and O. D. Hart (1986): "The Costs and Benefits of Ownership: A Theory of Vertical and Lateral Integration," *Journal of Political Economy*, 94, 691–719.

Grout, P. (1984): "Investment and Wages in the Absence of Binding Contracts: A Nash Bargaining Solution," *Econometrica*, 52, 449–460.

Hart, O. D. (1979): "Monopolistic Competition in a Large Economy with Differentiated Commodities," *Review of Economic Studies*, 46, 1–30.

——— (1995): *Firms, Contracts, and Financial Structure*. Oxford: Oxford University Press.

Hart, O. D., and J. Moore (1988): “Incomplete Contracts and Renegotiation,” *Econometrica*, 56, 755–85.

——— (1990): “Property Rights and the Nature of the Firm,” *Journal of Political Economy*, 98, 1119–58.

Holmström, B. (1999): “The Firm as a Subeconomy,” *Journal of Law Economics and Organization*, 15, 74–102.

Kamecke, U. (1992): “On the Uniqueness of the Solution to a Large Linear Assignement Problem,” *Journal of Mathematical Economics*, 21, 509–21.

Kranton, R., and D. Minehart (2000): “A Theory of Buyer-Seller Networks,” *American Economic Review*, forthcoming.

MacLeod, B., and J. Malcomson (1993): “Investments, Holdup and the Form of Market Contracts,” *American Economic Review*, 83, 811–37.

Makowski, L., and J. Ostroy (1995): “Appropriation and Efficiency: A Revision of the First Theorem of Welfare Economics,” *American Economic Review*, 85, 808–27.

Maskin, E., and J. Tirole (1999): “Unforseen Contingencies, and Incomplete Contracts,” *Review of Economic Studies*, 66, 83–114.

Milgrom, P., and J. Roberts (1990): “Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities,” *Econometrica*, 58, 1255–77.

——— (1994): “Comparing Equilibria,” *American Economic Review*, 84, 441–59.

Moen, E. (1997): “Competitive Search Equilibrium,” *Journal of Political Economy*, 105, 385–411.

Mortensen, D., and C. Pissarides (1994): “Job Creation and Job Destruction in the Theory of Unemployment,” *Review of Economic Studies*, 61, 397–416.

Peters, M., and A. Siow (2000): “Competing Pre-marital Investments,” mimeo.

Rajan, R., and L. Zingales (1998): "Power in a Theory of the Firm," *Quarterly Journal of Economics*, 113, 387–432.

Ramey, G., and J. Watson (1996): "Bilateral Trade and Opportunism in a Matching Market," Discussion Paper 96-08, University of California, San Diego.

——— (1997): "Contractual Fragility, Job Destruction, and Business Cycles," *Quarterly Journal of Economics*, 112, 873–911.

Roberts, K. (1996): "Thin Market Externalities and the Size and Density of Markets," paper presented at the Morishima Conference, Siena 1996.

Segal, I., and M. Whinston (1998): "The Mirrlees Approach to Implementation and Renegotiation: Theory and Applications to Hold-Up and Risk Sharing," mimeo.

Spulber, D. F. (2000): "Market Microstructure and Incentives to Invest," mimeo.

Williamson, O. (1985): *The Economic Institutions of Capitalism.* New York: Free Press.