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A note: orthogonality in linear congruence duality

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A NOTE: ORTHOGONALITY IN LINEAR CONGRUENCE DUALITY

by

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Abstract

A duality result for linear congruences analogous to that for Linear Programming is extended to given an analogous orthogonality result as well.

Keywords: Linear Congruences, Linear Programming, Duality, Orthogonality, Lattice Theory.

1. INTRODUCTION

In [2] Williams defines two optimization problems involving Linear Congruences and shows that they can be regarded as duals of each other in an analogous fashion to dual Linear Programming (LP) models. The homogeneous primal problem is defined as:

Find the maximum integer Mp such that:

$$P: \forall x_j \left| \sum_{j=1}^{n} a_{ij}x_j \equiv 0 (\text{mod } b_i),\ i = 1, 2, \ldots, m \rightarrow \sum_{j=1}^{n} c_j x_j \equiv 0 (\text{mod} M_p) \right|$$

where a_{ij}, b_i c_j are given integers. x_j are integer variables

The associated dual problem is defined as:

Find the minimum integer $M_D > 0$ such that

$$D: \exists y_i \left| \sum_{i=1}^{m} a_{ij}y_i \equiv c_j (\text{mod} M_D), j = 1, 2, \ldots, n, b_i y_i \equiv 0 (\text{mod} M_D) \right|$$

where y_i are integer variables.

The following theorem is proved in [2]

Theorem (a) Weak Duality. M_D/M_P ('/' stands for 'divides')

(b) Strong Duality. $M_D = M_p$

In this note we demonstrate a relationship between the solutions (values of x_j, y_i) of P and D analogous to the orthogonality result for LP. Hence we extend further the correspondence between duality in a continuous system and duality in a discrete system.

2. PRIMAL AND DUAL SOLUTIONS

In [2] some discussion is given to the meaning of a "solution" of P in terms of variables x_j since the quantifier "$\forall x_j$" applies. The solution is defined to be a set of x_j, j = 1,2,...,n such that

In figure 1 the set of x_j satisfying the congruences

$$\sum_{j=1}^{n} a_{ij}x_j \equiv 0 \pmod{b_i} \quad i = 1, 2, \ldots, m$$

give the <u>lattice</u> of points marked. Problem P is to find the maximum difference between the values of two expressions of the family

$$\sum_{j=1}^{n} c_j x_j$$

between which none of the lattice points lie. Two such lines in this family are marked in figure 1. Clearly there is the "maximum gap" between them with no lattice points lying in between. Taking the difference in coordinates between a point on one line and a point on the other gives a solution.

For the purposes of this note we present both the Primal and Dual problems entirely in equation form as

Find the maximum integer M_P such that

$$\sum_{j=1}^{n} a_{ij}x_j - b_i u_i = 0 \quad i = 1, 2, \ldots, m \tag{1}$$

P'

$$\sum_{j=1}^{n} c_j x_j = M_P \tag{2}$$

where u_i are additional integer variables (analogous to slack variables in LP).

Find the maximum integer M_D such that

$$\sum_{i=1}^{m} a_{ij}y_i - M_D v_j = c_j \quad j = 1, 2, \ldots, n \tag{3}$$

D'

$$b_i y_i - M_D w_i = 0 \qquad i = 1, 2, \ldots, m \tag{4}$$

where v_j , w_i are additional "slack" integer variables.

i.e. $\sum_{j=1}^{n} x_j(c_j + M_D v_j) - \sum_{i=1}^{m} u_i M_D w_i = 0$ from (3) and (4)

i.e. $M_P + M_D\left(\sum_{j=1}^{n} x_j v_j - \sum_{i=1}^{m} u_i w_i\right) = 0$ from (2) .

But from the Duality Theorem in [2] $M_P = M_D$.

Therefore $-\sum_{j=1}^{n} x_j v_j + \sum_{i=1}^{m} u_i w_i = 1$ □

Note that the above proof also demonstrates directly the weak duality result M_D/M_P .

The above result also furnishes a "proof of optimality" analogous to the use of duality in LP to demonstrate optimality. If (x_j, u_i) and (y_i, v_j, w_i) were feasible solutions to P' and D' respectively satisfying equation (5) then we could demonstrate that $M_D = M_P$. Since M_D/M_P we have the maximum M_D (and maximum M_P) .

4. A NUMERICAL EXAMPLE

Find the maximum integer M_P such that

P'

$$5x_1 + x_2 - 6u_1 = 0$$

$$13x_1 + 7x_2 - 20u_2 = 0$$

$$2x_1 + 13x_2 = M_P$$

An optimal solution is

$x_1 = 158,\ x_2 = -22,\ u_1 = 128,\ u_2 = 95,\ M_P = 8$.

Find the maximum integer M_D such that

D'

$$5y_1 + 13y_2 - M_D v_1 = 2$$

$$y_1 + 7y_2 - M_D v_2 = 13$$

$$6y_1 - M_D w_1 = 0$$

$$20y_2 - M_D w_2 = 0$$