

**Angelos Dassios and Shanle Wu**

## Brownian excursions outside a corridor and two-sided Parisian options

**Working paper**

**Original citation:**

Dassios, Angelos and Wu, Shanle (2011) Brownian excursions outside a corridor and two-sided Parisian options. Department of Statistics, London School of Economics and Political Science, London, UK.

This version available at: <http://eprints.lse.ac.uk/32045/>

Available in LSE Research Online: March 2011

© 2011 the authors

LSE has developed LSE Research Online so that users may access research output of the School. Copyright © and Moral Rights for the papers on this site are retained by the individual authors and/or other copyright owners. Users may download and/or print one copy of any article(s) in LSE Research Online to facilitate their private study or for non-commercial research. You may not engage in further distribution of the material or use it for any profit-making activities or any commercial gain. You may freely distribute the URL (<http://eprints.lse.ac.uk>) of the LSE Research Online website.

# Brownian excursions outside a corridor and two-sided Parisian options

Angelos Dassios<sup>†</sup>, Shanle Wu<sup>\*</sup>

<sup>†\*</sup>Department of Statistics, London School of Economics  
Houghton Street, London WC2A 2AE

<sup>†</sup>a.dassios@lse.ac.uk

<sup>\*</sup>s.wu3@lse.ac.uk

## Abstract

In this paper, we study the excursion time of a Brownian motion with drift outside a corridor by using a four states semi-Markov model. In mathematical finance, these results have an important application in the valuation of double barrier Parisian options. In this paper, we obtain an explicit expression for the Laplace transform of its price.

**Keywords:** excursion time, four states Semi-Markov model, double barrier Parisian options, Laplace transform.

## 1 Introduction

The concept of Parisian options was first introduced by Chesney, Jeanblanc-Picque and Yor [6]. It is a special case of path dependent options. The owner of a Parisian option will either gain the right or lose the right to exercise the option upon the price reaching a predetermined barrier level  $L$  and staying above or below the level for a predetermined time  $d$  before the maturity date  $T$ .

More precisely, the owner of a *Parisian down-and-out option* loses the option if the underlying asset price  $S$  reaches the level  $L$  and remains constantly below this level for a time interval longer than  $d$ . For a *Parisian down-and-in option* the same event gives the owner the right to exercise the option. For details on the pricing of Parisian options see [6], [13], [15] and [12].

The double barrier Parisian options are a version with two barriers of the standard Parisian options introduced by Chesney, Jeanblanc-Picque and Yor [6]. In contrast to the Parisian options mentioned above, we consider the excursions both below the lower barrier and above the upper barrier, i.e. outside a corridor formed by these two barriers. Let us look at two examples, depending on whether the condition is that the required excursions above the upper barrier and below the lower barrier have to both happen before the maturity date or that either one of them happens before the maturity. In one example, the owner

of a *double barrier Parisian max-out option* loses the option if the underlying asset process  $S$  has both an excursion above the upper barrier for longer than a continuous period  $d_1$  and below lower the barrier for longer than  $d_2$  before the maturity of the option. In the other example, the owner of a *double barrier Parisian min-out option* loses the right to exercise the option if either one of these two events happens before the maturity. Later on, we will derive the Laplace transforms which can be used to price this type of options.

In this paper, we are going to use the same definition for the excursion as in [6] and [7]. Let  $S$  be a stochastic process and  $l_1, l_2, l_1 > l_2$  be the levels of these two barriers. As in [6], we define

$$g_{l_i,t}^S = \sup\{s \leq t \mid S_s = l_i\}, \quad d_{l_i,t}^S = \inf\{s \geq t \mid S_s = l_i\}, \quad i = 1, 2, \quad (1)$$

with the usual conventions,  $\sup\{\emptyset\} = 0$  and  $\inf\{\emptyset\} = \infty$ . Assuming  $d_1 > 0$ ,  $d_2 > 0$ , we now define

$$\tau_1^S = \inf\{t > 0 \mid \mathbf{1}_{\{S_t > l_1\}}(t - g_{l_1,t}^S) \geq d_1\}, \quad (2)$$

$$\tau_2^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}} \mathbf{1}_{\{g_{l_1,t}^S > g_{l_2,t}^S\}}(t - g_{l_1,t}^S) \geq d_2\right\}, \quad (3)$$

$$\tau_3^S = \inf\left\{t > 0 \mid \mathbf{1}_{\{l_2 < S_t < l_1\}} \mathbf{1}_{\{g_{l_1,t}^S < g_{l_2,t}^S\}}(t - g_{l_2,t}^S) \geq d_3\right\}, \quad (4)$$

$$\tau_4^S = \inf\{t > 0 \mid \mathbf{1}_{\{S_t < l_2\}}(t - g_{l_2,t}^S) \geq d_4\}, \quad (5)$$

$$\tau^S = \tau_1^S \wedge \tau_4^S. \quad (6)$$

We can see that  $\tau_1^S$  is the first time that the length of the excursion of process  $S$  above the barrier  $l_1$  reaches a given level  $d_1$ ;  $\tau_4^S$  corresponds to the one below  $l_2$  with required length  $d_4$ ; and  $\tau^S$  is the smaller of  $\tau_1^S$  and  $\tau_4^S$ . We also see that  $\tau_2^S$  is the first time that the length of the excursion in the corridor reaches given level  $d_2$ , given that the excursion starts from the upper barrier  $l_1$ ;  $\tau_3^S$  corresponds to the one in the corridor starting from the lower barrier  $l_2$ . Our aim is to study the excursion outside the corridor, therefore  $\tau_2^S$  and  $\tau_3^S$  are not of interest here. However we need to use these two stopping times to define our four states semi-Markov model that will be the main tool used for calculation.

Now assume  $r$  is the risk-free rate,  $T$  is the term of the option,  $S_t$  is the price of its underlying asset,  $K$  is the strike price and  $Q$  is the risk neutral measure. If we have a double barrier Parisian min-out call option with the barrier  $l_1$  and  $l_2$ , its price can be expressed as:

$$DP_{min-out-call} = e^{-rT} E_Q \left( \mathbf{1}_{\{\tau^S > T\}} (S_T - K)^+ \right);$$

and the price of a double barrier Parisian min-in put option is:

$$DP_{min-in-put} = e^{-rT} E_Q \left( \mathbf{1}_{\{\tau^S < T\}} (K - S_T)^+ \right).$$

In this paper, we are going to study the excursion time outside the corridor using a semi-Markov model consisting of four states. By applying the model to a Brownian motion, we can get the explicit form of the Laplace transform for the price of double barrier options. One can then invert using techniques as in [13].

In Section 2 we introduce the four states semi-Markov model as well as a new process, the doubly perturbed Brownian motion, which has the same behavior as a Brownian motion except that each time it hits one of the two barriers, it moves towards the other side of the barrier by a jump of size  $\epsilon$ . In Section 3 we obtain the martingale to which we can apply the optional sampling theorem and get the Laplace transform that we can use for pricing later. We give our main results applied to Brownian motion in Section 4, including the Laplace transforms for the stopping times we defined by(2)-(6) for both a Brownian motion with drift, i.e.  $S = W^\mu$ , and a standard Brownian motion, i.e.  $S = W$ . In Section 5, we focus on pricing the double barrier Parisian options.

## 2 Definitions

From the description above, it is clear that we are actually considering four states, the state when the stochastic process is above the barrier  $l_1$  the state when it is below  $l_2$  and two states when it is between  $l_1$  and  $l_2$  depending on whether it comes into the corridor through  $l_1$  or  $l_2$ . For each state, we are interested in the time the process spends in it. We introduce a new process

$$Z_t^S = \begin{cases} 1, & \text{if } S_t > l_1 \\ 2, & \text{if } l_1 > S_t > l_2 \text{ and } g_{l_1,t}^S > g_{l_2,t}^S \\ 3, & \text{if } l_1 > S_t > l_2 \text{ and } g_{l_1,t}^S < g_{l_2,t}^S \\ 4, & \text{if } S_t < l_2 \end{cases}.$$

We can now express the variables defined above (see definitions (1)-(5)) in terms of  $Z_t^S$ :

$$g_{l_i,t}^S = \sup \{s \leq t \mid Z_s^S \neq Z_t^S\}, \quad (7)$$

$$d_{l_i,t}^S = \inf \{s \geq t \mid Z_s^S \neq Z_t^S\}, \quad (8)$$

$$\tau_1^S = \inf \left\{ t > 0 \mid \mathbf{1}_{\{Z_t^S=1\}} (t - g_{l_1,t}^S) \geq d_1 \right\}, \quad (9)$$

$$\tau_2^S = \inf \left\{ t > 0 \mid \mathbf{1}_{\{Z_t^S=2\}} (t - g_{l_1,t}^S) \geq d_2 \right\}, \quad (10)$$

$$\tau_3^S = \inf \left\{ t > 0 \mid \mathbf{1}_{\{Z_t^S=3\}} (t - g_{l_2,t}^S) \geq d_3 \right\}, \quad (11)$$

$$\tau_4^S = \inf \left\{ t > 0 \mid \mathbf{1}_{\{Z_t^S=4\}} (t - g_{l_2,t}^S) \geq d_4 \right\}. \quad (12)$$

We then define

$$V_t^S = t - \max(g_{l_1,t}^S, g_{l_2,t}^S), \quad (13)$$

the time  $Z_t^S$  has spent in the current state. It is easy to see that  $(Z_t^S, V_t^S)$  is a Markov process.  $Z_t^S$  is therefore a semi-Markov process with the state space  $\{1, 2, 3, 4\}$ , where 1 stands for the state when the stochastic process  $S$  is above the barrier  $l_1$ ; 4 corresponds to the state below the barrier  $l_2$ ; 2 and 3 represent the state when  $S$  is in the corridor given that it comes into it through  $l_1$  and  $l_2$  respectively.

For  $Z_t^S$ , define the transition intensities  $\lambda_{ij}(u)$  by

$$P(Z_{t+\Delta t}^S = j, i \neq j \mid Z_t^S = i, V_t^S = u) = \lambda_{ij}(u)\Delta t + o(\Delta t), \quad (14)$$

$$P(Z_{t+\Delta t}^S = i \mid Z_t^S = i, V_t^S = u) = 1 - \sum_{i \neq j} \lambda_{ij}(u)\Delta t + o(\Delta t). \quad (15)$$

Define

$$\bar{P}_i(\mu) = \exp \left\{ - \int_0^\mu \sum_{i \neq j} \lambda_{ij}(v) dv \right\}, \quad p_{ij}(\mu) = \lambda_{ij}(\mu) \bar{P}_i(\mu).$$

Notice that

$$P_i(\mu) = 1 - \bar{P}_i(\mu)$$

is the distribution function of the excursion time in state  $i$ , which is a random variable  $U_i$  defined as

$$U_i = \inf_{s>0} \{Z_s^S \neq i \mid Z_0^S = i, V_0^S = 0\}.$$

Note that because the process is time homogeneous this has the same distribution as

$$\inf_{s>0} \{Z_{t+s}^S \neq i \mid Z_t^S = i, V_t^S = 0\}.$$

for any time  $t$ . We have therefore

$$p_{ij}(\mu) = \lim_{\Delta\mu \rightarrow 0} \frac{P(U_i \in (\mu, \mu + \Delta\mu), Z_{U_i}^S = j)}{\Delta\mu}.$$

Moreover, in the definition of  $Z^S$ , we deliberately ignored the situation when  $S_t = l_i$ ,  $i = 1, 2$ . The reason is that we only consider the processes, which

$$\int_0^t \mathbf{1}_{\{S_u = l_i\}} du = 0, \quad i = 1, 2, \quad a.s.$$

Also, when  $l_1$  and  $l_2$  are the regular points of the process (see [5] for definition), we have to deal with the degeneration of  $p_{ij}$ . Let us take a Brownian Motion as an example. Assume  $W_t^\mu = \mu t + W_t$  with  $\mu \geq 0$ , where  $W_t$  is a standard Brownian Motion. Setting  $x_0$  to be its starting point, we know its density for the first hitting time of level  $l_i$ ,  $i = 1, 2$  is

$$p_{x_0} = \frac{|l_i - x_0|}{\sqrt{2\pi t^3}} \exp \left\{ - \frac{(l_i - x_0 - \mu t)^2}{2t} \right\}$$

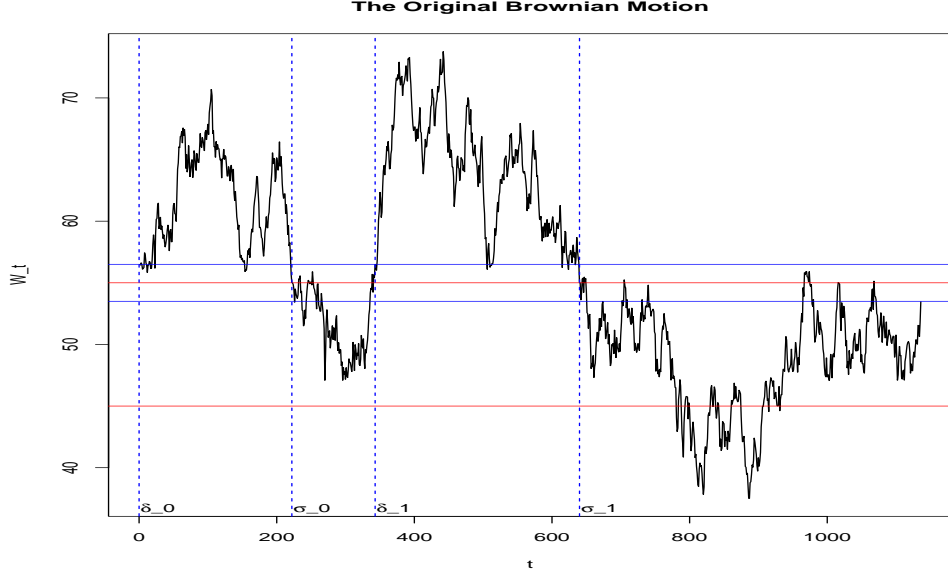


Figure 1: A Sample Path of  $W_t$

(see [4]). According to the definition of the transition density,  $p_{12}(t) = p_{21}(t) = p_{l_1}(t) = 0$  and  $p_{34}(t) = p_{43}(t) = p_{l_2}(t) = 0$ , for  $t > 0$ .

In [9] in order to solve the similar problem, we introduced the perturbed Brownian motion  $X_t^{(\epsilon)}$  with the respect to the barrier we are interested in. We apply the same idea here, and construct a new process the *doubly perturbed Brownian motion*,  $Y_t^{(\epsilon)}$ ,  $l_1 - l_2 > \epsilon > 0$ , with the respect to barriers  $l_1$  and  $l_2$ . Assume  $W_0^\mu = l_1 + \epsilon$ . Define a sequence of stopping times

$$\begin{aligned} \delta_0 &= 0, \\ \sigma_n &= \inf\{t > \delta_n \mid W_t^\mu = l_1\}, \\ \delta_{n+1} &= \inf\{t > \sigma_n \mid W_t^\mu = l_1 + \epsilon\}, \end{aligned}$$

where  $n = 0, 1, \dots$  (see Figure 1). Now define

$$\begin{cases} X_t^{(\epsilon)} = W_t^\mu & \text{if } \delta_n \leq t < \sigma_n \\ X_t^{(\epsilon)} = W_t^\mu - \epsilon & \text{if } \sigma_n \leq t < \delta_{n+1} \end{cases}.$$

Similarly, we then define another sequence of stopping times with the respect to process  $X_t^{(\epsilon)}$  and barrier  $l_2$

$$\begin{aligned} \zeta_0 &= 0, \\ \eta_n &= \inf\{t > \zeta_n \mid X_t^{(\epsilon)} = l_2\}, \\ \zeta_{n+1} &= \inf\{t > \eta_n \mid X_t^{(\epsilon)} = l_2 + \epsilon\}, \end{aligned}$$

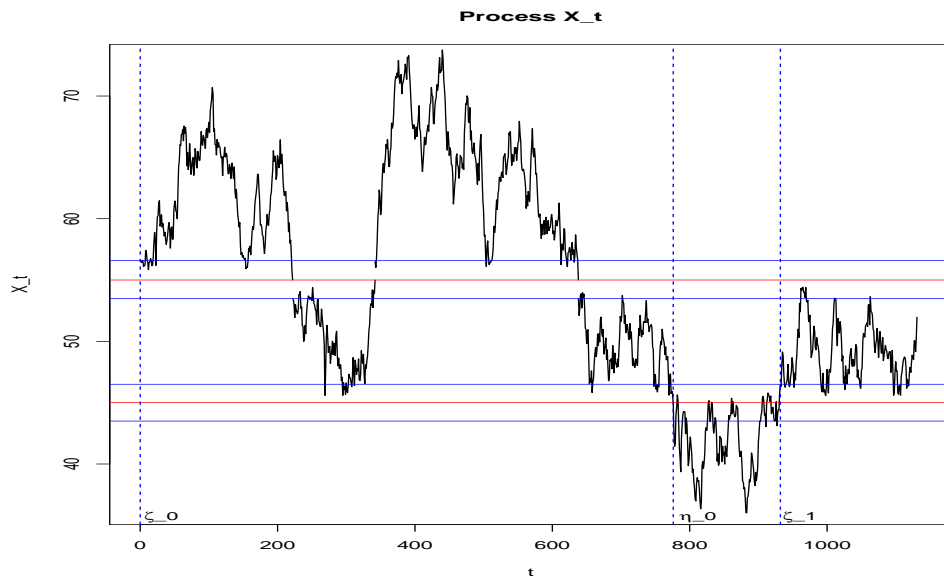


Figure 2: A Sample Path of  $X_t^{(\epsilon)}$

where  $n = 0, 1, \dots$  (see Figure 2). Then define

$$\begin{cases} Y_t^{(\epsilon)} = X_t^{(\epsilon)} & \text{if } \zeta_n \leq t < \eta_n \\ Y_t^{(\epsilon)} = X_t^{(\epsilon)} - \epsilon & \text{if } \eta_n \leq t < \zeta_{n+1} \end{cases} .$$

It is actually a process which starts from  $l_1 + \epsilon$  and has the same behavior as the related Brownian Motion expect that each time it hits the barrier  $l_1$  or  $l_2$ , it will jump towards the opposite side of the barrier with size  $\epsilon$  (see Figure 3).

From the definition, it is clear that  $l_1$  and  $l_2$  become irregular points for  $Y_t^{(\epsilon)}$ . Also  $Y_t^{(\epsilon)}$  converges to  $W_t^\mu$  with  $W_0^\mu = l_1$  almost surely for all  $t$ . Therefore as we saw in [9], the Laplace transforms of the variables defined based on  $Y_t^{(\epsilon)}$  converge to those based on  $W_t^\mu$ . As a result, we can obtain the results for the Brownian Motion by carrying out the calculation for  $Y_t^{(\epsilon)}$  and take the limit as  $\epsilon \rightarrow 0$ .

For  $Y_t^{(\epsilon)}$ , we can define  $Z^Y$ ,  $\tau_1^Y$ ,  $\tau_2^Y$  and  $\tau^Y$  as above (we suppress  $(\epsilon)$  on the

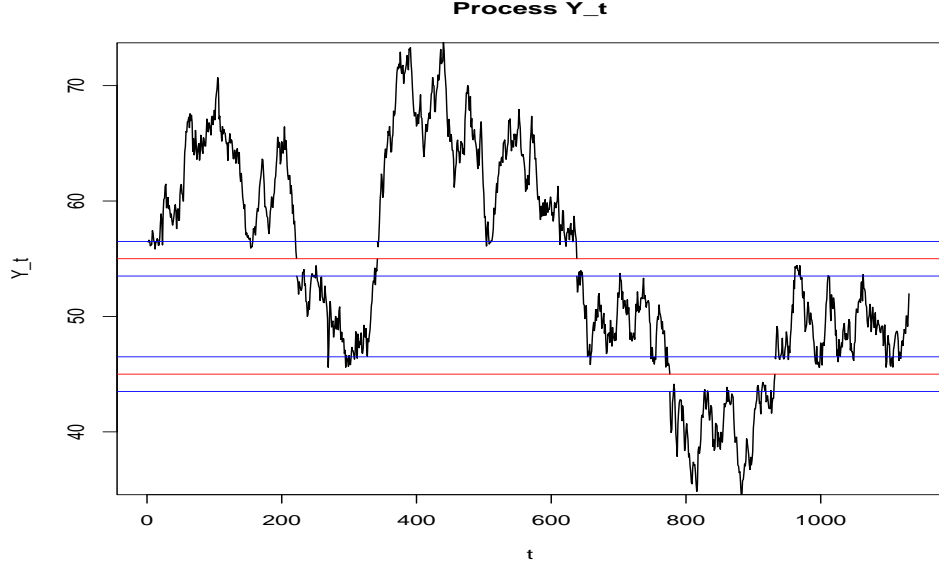


Figure 3: A Sample Path of  $Y_t^{(\epsilon)}$

superscript). For  $Z^Y$ , we have the transition densities (see [4])

$$p_{12}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon + \mu t)^2}{2t}\right\}, \quad (16)$$

$$p_{21}(t) = \exp\left\{\mu\epsilon - \frac{\mu^2 t}{2}\right\} ss_t(l_1 - l_2 - \epsilon, l_1 - l_2), \quad (17)$$

$$p_{24}(t) = \exp\left\{-\mu(l_1 - l_2 - \epsilon) - \frac{\mu^2 t}{2}\right\} ss_t(\epsilon, l_1 - l_2), \quad (18)$$

$$p_{31}(t) = \exp\left\{\mu(l_1 - l_2 - \epsilon) - \frac{\mu^2 t}{2}\right\} ss_t(\epsilon, l_1 - l_2), \quad (19)$$

$$p_{34}(t) = \exp\left\{-\mu\epsilon - \frac{\mu^2 t}{2}\right\} ss_t(l_1 - l_2 - \epsilon, l_1 - l_2), \quad (20)$$

$$p_{43}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(\epsilon - \mu t)^2}{2t}\right\}, \quad (21)$$

where

$$ss_t(x, y) = \sum_{k=-\infty}^{\infty} \frac{(2k+1)y - x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{((2k+1)y - x)^2}{2t}\right\}.$$

Also we know that

$$p_{23}(t) = p_{32}(t) = p_{14}(t) = p_{41}(t) = 0. \quad (22)$$



Clearly, all the arguments above apply to the standard Brownian motion, which is a special case of  $W_t^\mu$  when  $\mu = 0$ .

### 3 Results for the semi-Markov model

In §2 we have introduced the Markov process  $(Z_t^S, V_t^S)$ . Now we apply the same definition to the doubly perturbed Brownian motion  $Y_t$ ; therefore we have  $(Z_t^Y, V_t^Y)$ , where  $Z_t^Y$  is the current state of  $Y_t$ , taking value from state space  $\{1, 2, 3, 4\}$  and  $V_t^Y$  is the time  $Y_t$  has spent in current state.  $V_t^Y$  is also a stochastic process. Now we consider a function of the form

$$f(V_t^Y, Z_t^Y, t) = f_{Z_t^Y}(V_t^Y, t),$$

where  $f_i$ ,  $i = 1, 2, 3, 4$  are functions from  $\mathbb{R}^2$  to  $\mathbb{R}$ . The generator  $\mathcal{A}$  is defined as an operator such that

$$f(V_t^Y, Z_t^Y, t) - \int_0^s \mathcal{A}f(V_s^Y, Z_s^Y, s) ds$$

is a martingale (see [10], chapter 2). Therefore solving

$$\mathcal{A}f = 0$$

subject to certain conditions will provide us with martingales of the form  $f(V_t^Y, Z_t^Y, t)$  to which we can apply the optional stopping theorem to obtain the Laplace transform we are interested in. More precisely, we will have

$$\left\{ \begin{array}{l} \mathcal{A}f_1(u, t) = \frac{\partial f_1(u, t)}{\partial t} + \frac{\partial f_1(u, t)}{\partial u} + \lambda_{12}(u)(f_2(0, t) - f_1(u, t)) \\ \mathcal{A}f_2(u, t) = \frac{\partial f_2(u, t)}{\partial t} + \frac{\partial f_2(u, t)}{\partial u} + \lambda_{21}(u)(f_1(0, t) - f_2(u, t)) + \lambda_{24}(u)(f_4(0, t) - f_2(u, t)) \\ \mathcal{A}f_3(u, t) = \frac{\partial f_3(u, t)}{\partial t} + \frac{\partial f_3(u, t)}{\partial u} + \lambda_{31}(u)(f_1(0, t) - f_3(u, t)) + \lambda_{34}(u)(f_4(0, t) - f_3(u, t)) \\ \mathcal{A}f_4(u, t) = \frac{\partial f_4(u, t)}{\partial t} + \frac{\partial f_4(u, t)}{\partial u} + \lambda_{43}(u)(f_4(0, t) - f_3(u, t)) \end{array} \right. ,$$

Assume  $f_i$  has the form

$$f_i(u, t) = e^{-\beta t} g_i(u).$$

By solving the equation  $\mathcal{A}f = 0$ , i.e.  $\begin{cases} \mathcal{A}f_1 = 0 \\ \mathcal{A}f_2 = 0 \\ \mathcal{A}f_3 = 0 \\ \mathcal{A}f_4 = 0 \end{cases}$  subject to  $\begin{cases} g_1(d_1) = \alpha_1 \\ g_2(d_2) = \alpha_2 \\ g_3(d_2) = \alpha_3 \\ g_4(d_2) = \alpha_4 \end{cases}$

we can get

$$g_i(u) = \alpha_i \exp \left\{ - \int_u^{d_i} \left( \beta + \sum_{j \neq i} \lambda_{ij}(v) \right) dv \right\} \quad (23)$$

$$+ \sum_{j \neq i} g_j(0) \int_u^{d_i} \lambda_{ij}(s) \exp \left\{ - \int_u^s \left( \beta + \sum_{j \neq i} \lambda_{ij}(v) \right) dv \right\} ds.$$

In our case, we are only interested in the excursion outside the corridor. Hence, we set  $d_2$  and  $d_3$  to be  $\infty$ . Also  $\lim_{d_2 \rightarrow \infty} g_2(d_2) = \lim_{d_3 \rightarrow \infty} g_3(d_3) = 0$  gives  $\alpha_2 = \alpha_3 = 0$ . Therefore, we have

$$g_1(0) = \alpha_1 e^{-\beta d_1} \bar{P}_1(d_1) + \left\{ g_1(0) \hat{P}_{21}(\beta) + g_4(0) \hat{P}_{24}(\beta) \right\} \tilde{P}_{12}(\beta), \quad (24)$$

$$g_4(0) = \alpha_4 e^{-\beta d_4} \bar{P}_4(d_4) + \left\{ g_1(0) \hat{P}_{31}(\beta) + g_4(0) \hat{P}_{34}(\beta) \right\} \tilde{P}_{43}(\beta). \quad (25)$$

Solving (24) and (25) gives

$$g_1(0) \quad (26)$$

$$= \frac{\alpha_1 e^{-\beta d_1} \bar{P}_1(d_1) \left( 1 - \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) \right) + \alpha_4 e^{-\beta d_4} \bar{P}_4(d_4) \hat{P}_{24}(\beta) \tilde{P}_{12}(\beta)}{1 - \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) - \hat{P}_{31}(\beta) \tilde{P}_{43}(\beta) \hat{P}_{24}(\beta) \tilde{P}_{12}(\beta)},$$

$$g_4(0) \quad (27)$$

$$= \frac{\alpha_4 e^{-\beta d_4} \bar{P}_4(d_4) \left( 1 - \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) \right) + \alpha_1 e^{-\beta d_1} \bar{P}_1(d_1) \hat{P}_{31}(\beta) \tilde{P}_{43}(\beta)}{1 - \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) - \hat{P}_{31}(\beta) \tilde{P}_{43}(\beta) \hat{P}_{24}(\beta) \tilde{P}_{12}(\beta)}.$$

where

$$\hat{P}_{ij}(\beta) = \int_0^\infty e^{-\beta s} p_{ij}(s) ds, \quad (28)$$

$$\tilde{P}_{ij}(\beta) = \int_0^{d_i} e^{-\beta s} p_{ij}(s) ds. \quad (29)$$

As a result, we have obtained the martingale

$$M_t = f(V_t^Y, t) = e^{-\beta t} g_{Z_t^Y}(V_t^Y), \quad i = 1, 2, 3, 4. \quad (30)$$

We now can apply the optional stopping theorem to  $M_t$  with the stopping time  $\tau^Y \wedge t$ , where  $\tau^Y$  is the stopping time defined by (6):

$$E(M_{\tau^Y \wedge t}) = E(M_0). \quad (31)$$

The right hand side of (31) is

$$E(M_{\tau^Y \wedge t}) = E(M_{\tau^Y} \mathbf{1}_{\{\tau^Y < t\}}) + E(M_t \mathbf{1}_{\{\tau^Y > t\}}).$$

Furthermore,

$$\begin{aligned} & E(M_{\tau^Y} \mathbf{1}_{\{\tau^Y < t\}}) \\ &= E\left(M_{\tau^Y} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}} \mathbf{1}_{\{\tau_1^Y < t\}}\right) + E\left(M_{\tau^Y} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}} \mathbf{1}_{\{\tau_4^Y < t\}}\right) \\ &= E\left(e^{-\beta\tau^Y} g_1(d_1) \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}} \mathbf{1}_{\{\tau_1^Y < t\}}\right) + E\left(e^{-\beta\tau^Y} g_4(d_4) \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}} \mathbf{1}_{\{\tau_4^Y < t\}}\right) \\ &= \alpha_1 E\left(e^{-\beta\tau^Y} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}} \mathbf{1}_{\{\tau_1^Y < t\}}\right) + \alpha_4 E\left(e^{-\beta\tau^Y} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}} \mathbf{1}_{\{\tau_4^Y < t\}}\right). \end{aligned}$$

We also have

$$E(M_t \mathbf{1}_{\{\tau^Y > t\}}) = e^{-\beta t} E\left(g_{Z_t^Y}(V_t^Y) \mathbf{1}_{\{\tau^Y > t\}}\right),$$

where  $Z_t^Y$  can take values 1, 2, 3 or 4.

When  $Z_t^Y = 1$  or 4, since  $\tau^Y > t$ , we have  $0 \leq V_t^Y < d_1 \wedge d_4$ . According to the definition of  $g_i(\mu)$  in (23), we have  $g_1(V_t^Y)$  and  $g_4(V_t^Y)$  are bounded.

When  $Z_t^Y = 2$  or 3, since  $\lim_{d_2 \rightarrow \infty} g_2(d_2) = \lim_{d_3 \rightarrow \infty} g_3(d_3) = 0$  and looking at (23) with  $d_2$  and  $d_3$  replaced by  $\infty$  we have that  $g_2(V_t^Y)$  and  $g_3(V_t^Y)$  are bounded.

Therefore

$$\lim_{t \rightarrow \infty} E(M_t \mathbf{1}_{\{\tau^Y > t\}}) = 0.$$

Hence we have

$$\lim_{t \rightarrow \infty} E(M_{\tau^Y \wedge t}) = \alpha_1 E\left(e^{-\beta\tau^Y} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}}\right) + \alpha_4 E\left(e^{-\beta\tau^Y} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}}\right). \quad (32)$$

The left hand side of (31) gives

$$\lim_{t \rightarrow \infty} E(M_0) = E(M_0) = \begin{cases} g_1(0), & Y_0^{(\epsilon)} = l_1 + \epsilon \\ g_4(0), & Y_0^{(\epsilon)} = l_2 - \epsilon \end{cases}.$$

By taking the proper  $\alpha_1$  and  $\alpha_4$ , we will have when  $Y_0^{(\epsilon)} = l_1 + \epsilon$

$$\begin{aligned} & E\left(e^{-\beta\tau^Y} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}}\right) \tag{33} \\ &= \frac{e^{-\beta d_1} \bar{P}_{12}(d_1) \left(1 - \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta)\right)}{1 - \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) - \hat{P}_{31}(\beta) \tilde{P}_{43}(\beta) \hat{P}_{24}(\beta) \tilde{P}_{12}(\beta)}, \\ & E\left(e^{-\beta\tau^Y} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}}\right) \tag{34} \\ &= \frac{e^{-\beta d_4} \bar{P}_{43}(d_4) \hat{P}_{24}(\beta) \tilde{P}_{12}(\beta)}{1 - \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) - \hat{P}_{31}(\beta) \tilde{P}_{43}(\beta) \hat{P}_{24}(\beta) \tilde{P}_{12}(\beta)}; \end{aligned}$$

when  $Y_0^{(\epsilon)} = l_2 - \epsilon$

$$\begin{aligned}
& E \left( e^{-\beta\tau^Y} \mathbf{1}_{\{\tau_1^Y < \tau_4^Y\}} \right) & (35) \\
& = \frac{e^{-\beta d_1} \bar{P}_{12}(d_1) \hat{P}_{31}(\beta) \tilde{P}_{43}(\beta)}{1 - \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) - \hat{P}_{31}(\beta) \tilde{P}_{43}(\beta) \hat{P}_{24}(\beta) \tilde{P}_{12}(\beta)}, \\
& E \left( e^{-\beta\tau^Y} \mathbf{1}_{\{\tau_1^Y > \tau_4^Y\}} \right) & (36) \\
& = \frac{e^{-\beta d_4} \bar{P}_{43}(d_4) \left( 1 - \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) \right)}{1 - \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) - \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) + \hat{P}_{21}(\beta) \tilde{P}_{12}(\beta) \hat{P}_{34}(\beta) \tilde{P}_{43}(\beta) - \hat{P}_{31}(\beta) \tilde{P}_{43}(\beta) \hat{P}_{24}(\beta) \tilde{P}_{12}(\beta)}.
\end{aligned}$$

## 4 Main Results

In §2 we have stated that the main difficulty with the Brownian Motion is that its origin point is regular, i.e. the probability that  $W_t^\mu$  will return to the origin at arbitrarily small time is 1. We have therefore introduced the new processes  $Y_t^{(\epsilon)}$  and  $(Z_t^Y, V_t^Y)$  with transition densities for  $Z_t^Y$  defined in (16) to (22).

In order to simplify the expressions, we define

$$\Psi(x) = 2\sqrt{\pi}x\mathcal{N}(\sqrt{2}x) - \sqrt{\pi}x + e^{-x^2},$$

where  $\mathcal{N}(\cdot)$  is the cumulative distribution function for the standard Normal Distribution.

**Theorem 1** *For a Brownian Motion  $W_t^\mu$ ,  $\tau_1^{W^\mu}$ ,  $\tau_4^{W^\mu}$ ,  $\tau^{W^\mu}$  defined as in (2), (5) and (6) with  $S_t = W_t^\mu$ , we have the following Laplace transforms:*

when  $W_0^\mu = l_1$ ,

$$E \left( e^{-\beta\tau^{W^\mu}} \mathbf{1}_{\{\tau_1^{W^\mu} < \tau_4^{W^\mu}\}} \right) = \frac{G_1(d_1, d_4, \mu)}{G(d_1, d_4, \mu)}; \quad (37)$$

$$E \left( e^{-\beta\tau^{W^\mu}} \mathbf{1}_{\{\tau_1^{W^\mu} > \tau_4^{W^\mu}\}} \right) = \frac{G_2(d_4, d_1, -\mu)}{G(d_1, d_4, \mu)}; \quad (38)$$

$$E \left( e^{-\beta\tau^{W^\mu}} \right) = \frac{G_1(d_1, d_4, \mu) + G_2(d_4, d_1, -\mu)}{G(d_1, d_4, \mu)}; \quad (39)$$

when  $W_0^\mu = l_2$ ,

$$E \left( e^{-\beta\tau^{W^\mu}} \mathbf{1}_{\{\tau_1^{W^\mu} < \tau_4^{W^\mu}\}} \right) = \frac{G_2(d_1, d_4, \mu)}{G(d_1, d_4, \mu)}; \quad (40)$$

$$E \left( e^{-\beta\tau^{W^\mu}} \mathbf{1}_{\{\tau_1^{W^\mu} > \tau_4^{W^\mu}\}} \right) = \frac{G_1(d_4, d_1, -\mu)}{G(d_1, d_4, \mu)}; \quad (41)$$

$$E \left( e^{-\beta\tau^{W^\mu}} \right) = \frac{G_1(d_4, d_1, -\mu) + G_2(d_1, d_4, \mu)}{G(d_1, d_4, \mu)}; \quad (42)$$

where

$$\begin{aligned}
G_1(x, y, z) &= e^{-2(l_1-l_2)\sqrt{2\beta+z^2}-\beta x} \left\{ \sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi xy}{2}} \right\} \\
&+ \frac{\left(1 - e^{-2(l_1-l_2)\sqrt{2\beta+z^2}}\right) e^{-\beta x}}{2\sqrt{2\beta+z^2}} \left\{ \Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi x}{2}} \right\} \\
&\left\{ \sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\frac{(2\beta+z^2)y}{2}}\right) + \sqrt{(2\beta+z^2)y} \right\}, \tag{43}
\end{aligned}$$

$$G_2(x, y, z) = e^{-(l_1-l_2)(\sqrt{2\beta+z^2}-z)-\beta x} \left\{ \sqrt{y}\Psi\left(|z|\sqrt{\frac{x}{2}}\right) + z\sqrt{\frac{\pi xy}{2}} \right\}, \tag{44}$$

$$\begin{aligned}
G(x, y, z) &= e^{-2(l_1-l_2)\sqrt{2\beta+z^2}} \left\{ \sqrt{y}\Psi\left(\sqrt{\frac{(2\beta+z^2)x}{2}}\right) + \sqrt{x}\Psi\left(\sqrt{\frac{(2\beta+z^2)y}{2}}\right) \right\} \\
&+ \frac{\left(1 - e^{-2(l_1-l_2)\sqrt{2\beta+z^2}}\right)}{2\sqrt{2\beta+z^2}} \left\{ \Psi\left(\sqrt{\frac{(2\beta+z^2)x}{2}}\right) + \sqrt{\frac{(2\beta+z^2)\pi x}{2}} \right\} \\
&\left\{ \sqrt{\frac{2}{\pi}}\Psi\left(\sqrt{\frac{(2\beta+z^2)y}{2}}\right) + \sqrt{(2\beta+z^2)y} \right\}. \tag{45}
\end{aligned}$$

**Proof:** We apply the transition densities in (16) to (22) to the results in (33) to (36) and take the limit as  $\epsilon \rightarrow 0$ . According to the definition of  $Y^{(\epsilon)}$ , we know that

$$Y_t^{(\epsilon)} \xrightarrow{a.s.} W_t^\mu, \quad \text{for all } t.$$

As we saw in [9], since  $Y_t^{(\epsilon)} \xrightarrow{a.s.} W_t^\mu$ , for all  $t$ , by taking the limit  $\epsilon \rightarrow 0$ , the quantities defined based on  $Y_t^{(\epsilon)}$  converge to those based on Brownian motion with drift. Therefore we will get the results shown by (37), (38), (40) and (41). We can therefore get (39) and (42) by

$$E\left(e^{-\beta\tau^{W^\mu}}\right) = E\left(e^{-\beta\tau^{W^\mu}} \mathbf{1}_{\{\tau_1^{W^\mu} < \tau_4^{W^\mu}\}}\right) + E\left(e^{-\beta\tau^{W^\mu}} \mathbf{1}_{\{\tau_1^{W^\mu} > \tau_4^{W^\mu}\}}\right).$$

□

**Corollary 1.1** For a standard Brownian Motion ( $\mu = 0$ ), we have when  $W_0 = l_1$ ,

$$E\left(e^{-\beta\tau^W} \mathbf{1}_{\{\tau_1^W < \tau_4^W\}}\right) = \frac{G_1(d_1, d_4, 0)}{G(d_1, d_4, 0)}; \tag{46}$$

$$E\left(e^{-\beta\tau^W} \mathbf{1}_{\{\tau_1^W > \tau_4^W\}}\right) = \frac{G_2(d_4, d_1, 0)}{G(d_1, d_4, 0)}; \tag{47}$$

$$E\left(e^{-\beta\tau^W}\right) = \frac{G_1(d_1, d_4, 0) + G_2(d_4, d_1, 0)}{G(d_1, d_4, 0)}; \tag{48}$$

when  $W_0 = l_2$ ,

$$E\left(e^{-\beta\tau^W} \mathbf{1}_{\{\tau_1^W < \tau_4^W\}}\right) = \frac{G_2(d_1, d_4, 0)}{G(d_1, d_4, 0)}; \quad (49)$$

$$E\left(e^{-\beta\tau^W} \mathbf{1}_{\{\tau_1^W > \tau_4^W\}}\right) = \frac{G_1(d_4, d_1, 0)}{G(d_1, d_4, 0)}; \quad (50)$$

$$E\left(e^{-\beta\tau^W}\right) = \frac{G_1(d_4, d_1, 0) + G_2(d_1, d_4, 0)}{G(d_1, d_4, 0)}; \quad (51)$$

where

$$G_1(x, y, 0) = e^{-2(l_1-l_2)\sqrt{2\beta}-\beta x}\sqrt{y} + \frac{(1 - e^{-2(l_1-l_2)\sqrt{2\beta}})e^{-\beta x}}{2\sqrt{2\beta}} \left\{ \sqrt{\frac{2}{\pi}}\Psi(\sqrt{\beta y}) + \sqrt{2\beta y} \right\}, \quad (52)$$

$$G_2(x, y, 0) = e^{-(l_1-l_2)\sqrt{2\beta}-\beta x}\sqrt{y}, \quad (53)$$

$$G(x, y, 0) = e^{-2(l_1-l_2)\sqrt{2\beta}} \left\{ \sqrt{y}\Psi(\sqrt{\beta x}) + \sqrt{x}\Psi(\sqrt{\beta y}) \right\} + \frac{(1 - e^{-2(l_1-l_2)\sqrt{2\beta}})}{2\sqrt{2\beta}} \left\{ \Psi(\sqrt{\beta x}) + \sqrt{\beta\pi x} \right\} \left\{ \sqrt{\frac{2}{\pi}}\Psi(\sqrt{\beta y}) + \sqrt{2\beta y} \right\}. \quad (54)$$

**Remark 1:** By taking the limit  $l_1 - l_2 \rightarrow 0$ , we can get the result for the single barrier two-sided excursion case as in [9].

**Remark 2:** If we only want to consider the excursion above a barrier, we can let  $l_2 \rightarrow -\infty$ . Similarly, for the one below a barrier, we can let  $l_1 \rightarrow +\infty$ . These results have been shown in [9].

**Corollary 1.2** For a Brownian Motion  $W_t^\mu$ ,  $\tau^{W^\mu}$  defined as in (6) with  $S_t = W_t^\mu$ , we have the following Laplace transforms:

when  $W_0^\mu = x_0$ ,  $x_0 > l_1$ ,

$$\begin{aligned} & E\left(e^{-\beta\tau^{W^\mu}}\right) \\ &= \left\{ e^{-(\mu+\sqrt{2\beta+\mu^2})(x_0-l_1)} \mathcal{N}\left(\sqrt{(2\beta+\mu^2)d_1} - \frac{x_0-l_1}{\sqrt{d_1}}\right) \right. \\ & \quad \left. + e^{-(\mu-\sqrt{2\beta+\mu^2})(x_0-l_1)} \mathcal{N}\left(-\sqrt{(2\beta+\mu^2)d_1} - \frac{x_0-l_1}{\sqrt{d_1}}\right) \right\} \frac{G_1(d_1, d_4, \mu) + G_2(d_4, d_1, -\mu)}{G(d_1, d_4, \mu)} \\ & \quad + e^{-\beta d_1} \left\{ 1 - e^{-(\mu+|\mu|)(x_0-l_1)} \mathcal{N}\left(|\mu|\sqrt{d_1} - \frac{x_0-l_1}{\sqrt{d_1}}\right) \right. \\ & \quad \left. - e^{-(\mu-|\mu|)(x_0-l_1)} \mathcal{N}\left(-|\mu|\sqrt{d_1} - \frac{x_0-l_1}{\sqrt{d_1}}\right) \right\}; \end{aligned} \quad (55)$$

when  $W_0^\mu = x_0$ ,  $l_2 \leq x_0 \leq l_1$ ,

$$\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) \tag{56} \\
&= \frac{e^{(l_1-x_0)\mu} \left\{ e^{\sqrt{2\beta+\mu^2}(x_0-l_2)} - e^{-\sqrt{2\beta+\mu^2}(x_0-l_2)} \right\} \{G_1(d_1, d_4, \mu) + G_2(d_4, d_1, -\mu)\}}{\left\{ e^{\sqrt{2\beta+\mu^2}(l_1-l_2)} - e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)} \right\} G(d_1, d_2, \mu)} \\
&+ \frac{e^{(l_2-x_0)\mu} \left\{ e^{\sqrt{2\beta+\mu^2}(l_1-x_0)} - e^{-\sqrt{2\beta+\mu^2}(l_1-x_0)} \right\} \{G_2(d_1, d_4, \mu) + G_1(d_4, d_1, -\mu)\}}{\left\{ e^{\sqrt{2\beta+\mu^2}(l_1-l_2)} - e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)} \right\} G(d_1, d_2, \mu)};
\end{aligned}$$

when  $W_0^\mu = x_0$ ,  $x_0 < l_2$ ,

$$\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) \tag{57} \\
&= \left\{ e^{(\mu-\sqrt{2\beta+\mu^2})(l_2-x)} \mathcal{N}\left(\sqrt{(2\beta+\mu^2)d_4} - \frac{l_2-x}{\sqrt{d_4}}\right) \right. \\
&+ e^{(\mu+\sqrt{2\beta+\mu^2})(l_2-x)} \mathcal{N}\left(-\sqrt{(2\beta+\mu^2)d_4} - \frac{l_2-x}{\sqrt{d_4}}\right) \left. \right\} \frac{G_1(d_4, d_1, -\mu) + G_2(d_1, d_4, \mu)}{G(d_1, d_4, \mu)} \\
&+ e^{-\beta d_4} \left\{ 1 - e^{(\mu-|\mu|)(l_2-x)} \mathcal{N}\left(|\mu|\sqrt{d_4} - \frac{l_2-x}{\sqrt{d_4}}\right) \right. \\
&\left. - e^{(\mu+|\mu|)(l_2-x)} \mathcal{N}\left(-|\mu|\sqrt{d_4} - \frac{l_2-x}{\sqrt{d_4}}\right) \right\}. \tag{58}
\end{aligned}$$

**Proof:** We will first prove the case when  $x_0 > l_1$ . Define  $T = \inf\{t \mid W_t^\mu = l_1\}$ , i.e the first time  $W_t^\mu$  hits  $l_1$ . By definition, we have  $\tau^{W^\mu} = d_1$ , if  $T \geq d_1$ ;  $\tau^{W^\mu} = T + \tau^{\tilde{W}^\mu}$ , if  $T < d_1$ , where  $\tilde{W}^\mu$  here stands for a Brownian motion with drift started from  $l_1$ . As a result

$$\begin{aligned}
& E\left(e^{-\beta\tau^{W^\mu}}\right) \\
&= E\left(e^{-\beta\tau^{W^\mu}} \mathbf{1}_{\{T \geq d_1\}}\right) + E\left(e^{-\beta\tau^{W^\mu}} \mathbf{1}_{\{T < d_1\}}\right) \\
&= e^{-\beta d_1} P(T \geq d_1) + E\left(e^{-\beta T} \mathbf{1}_{\{T < d_1\}}\right) E\left(e^{-\beta\tau^{\tilde{W}^\mu}}\right)
\end{aligned}$$

$E\left(e^{-\beta\tau^{\tilde{W}^\mu}}\right)$  has been calculated in Theorem 1 (see (39)). The density for  $T$  is given in [4] as

$$p_{x_0} = \frac{|l_1 - x_0|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(l_1 - x_0 - \mu t)^2}{2t}\right\}.$$

We can therefore calculate

$$\begin{aligned}
P(T \geq d_1) &= 1 - e^{-(\mu+|\mu|)(x_0-l_1)} \mathcal{N}\left(|\mu|\sqrt{d_1} - \frac{x_0-l_1}{\sqrt{d_1}}\right) \\
&\quad - e^{-(\mu-|\mu|)(x_0-l_1)} \mathcal{N}\left(-|\mu|\sqrt{d_1} - \frac{x_0-l_1}{\sqrt{d_1}}\right),
\end{aligned}$$

$$\begin{aligned}
E\left(e^{-\beta T} \mathbf{1}_{\{T < d_1\}}\right) &= e^{-(\mu + \sqrt{2\beta + \mu^2})(x_0 - l_1)} \mathcal{N}\left(\sqrt{(2\beta + \mu^2)d_1} - \frac{x_0 - l_1}{\sqrt{d_1}}\right) \\
&\quad + e^{-(\mu - \sqrt{2\beta + \mu^2})(x_0 - l_1)} \mathcal{N}\left(-\sqrt{(2\beta + \mu^2)d_1} - \frac{x_0 - l_1}{\sqrt{d_1}}\right).
\end{aligned}$$

We therefore get the result in (55). For the case when  $x_0 < l_2$ , we can apply the same argument.

When  $l_2 \leq x_0 \leq l_1$ , we define  $\tilde{T} = \inf(t \mid W_t^\mu \notin (l_2, l_1))$ . By definition, we have  $\tau^{W^\mu} = T + \tau^{\tilde{W}^\mu}$ , if  $W_T^\mu = l_1$ ;  $\tau^{W^\mu} = T + \tau^{\underline{W}^\mu}$ , if  $W_T^\mu = l_2$ , where  $\underline{W}^\mu$  stands for a Brownian motion with drift started from  $l_2$ . Consequently,

$$\begin{aligned}
&E\left(e^{-\beta \tau^{W^\mu}}\right) \\
&= E\left(e^{-\beta T} e^{-\beta \tau^{\tilde{W}^\mu}} \mathbf{1}_{\{T=l_1\}}\right) + E\left(e^{-\beta T} e^{-\beta \tau^{\underline{W}^\mu}} \mathbf{1}_{\{T=l_2\}}\right) \\
&= E\left(e^{-\beta T} \mathbf{1}_{\{T=l_1\}}\right) E\left(e^{-\beta \tau^{\tilde{W}^\mu}}\right) + E\left(e^{-\beta T} \mathbf{1}_{\{T=l_2\}}\right) E\left(e^{-\beta \tau^{\underline{W}^\mu}}\right)
\end{aligned}$$

$E\left(e^{-\beta \tau^{\tilde{W}^\mu}}\right)$  and  $E\left(e^{-\beta \tau^{\underline{W}^\mu}}\right)$  have been obtained by Theorem 1, (39) and (42). According to [4], we have

$$\begin{aligned}
E\left(e^{-\beta T} \mathbf{1}_{\{T=l_1\}}\right) &= \frac{e^{(l_1-x_0)\mu} \left\{ e^{\sqrt{2\beta+\mu^2}(x_0-l_2)} - e^{-\sqrt{2\beta+\mu^2}(x_0-l_2)} \right\}}{e^{\sqrt{2\beta+\mu^2}(l_1-l_2)} - e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}}, \\
E\left(e^{-\beta T} \mathbf{1}_{\{T=l_2\}}\right) &= \frac{e^{(l_2-x_0)\mu} \left\{ e^{\sqrt{2\beta+\mu^2}(l_1-x_0)} - e^{-\sqrt{2\beta+\mu^2}(l_1-x_0)} \right\}}{e^{\sqrt{2\beta+\mu^2}(l_1-l_2)} - e^{-\sqrt{2\beta+\mu^2}(l_1-l_2)}}.
\end{aligned}$$

We have therefore obtained (56).  $\square$

**Theorem 2** *The probability that  $W_t^\mu$  with  $W_0^\mu = x_0$ ,  $l_2 \leq x_0 \leq l_1$ , achieves an excursion above  $l_1$  with length as least  $d_1$  before it achieves an excursion below  $l_2$  with length at least  $d_4$  is*

$$\begin{aligned}
P\left(\tau_1^{W^\mu} < \tau_4^{W^\mu}\right) &= \frac{e^{(l_1-x_0)\mu} \left\{ e^{|\mu|(x_0-l_2)} - e^{-|\mu|(x_0-l_2)} \right\} F_1(d_1, d_4, \mu)}{\left\{ e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)} \right\} F(d_1, d_4, \mu)} \\
&\quad + \frac{e^{(l_2-x_0)\mu} \left\{ e^{|\mu|(l_1-x_0)} - e^{-|\mu|(l_1-x_0)} \right\} F_2(d_1, d_4, \mu)}{\left\{ e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)} \right\} F(d_1, d_4, \mu)},
\end{aligned} \tag{59}$$

$$\begin{aligned}
P\left(\tau_1^{W^\mu} > \tau_4^{W^\mu}\right) &= \frac{e^{(l_1-x_0)\mu} \left\{ e^{|\mu|(x_0-l_2)} - e^{-|\mu|(x_0-l_2)} \right\} F_2(d_4, d_1, -\mu)}{\left\{ e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)} \right\} F(d_1, d_4, \mu)} \\
&\quad + \frac{e^{(l_2-x_0)\mu} \left\{ e^{|\mu|(l_1-x_0)} - e^{-|\mu|(l_1-x_0)} \right\} F_1(d_4, d_1, -\mu)}{\left\{ e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)} \right\} F(d_1, d_4, \mu)};
\end{aligned} \tag{60}$$



where

$$F_1(x, y, z) = e^{-2(l_1-l_2)|z|} \left\{ \sqrt{y} \Psi \left( |z| \sqrt{\frac{x}{2}} \right) + z \sqrt{\frac{\pi xy}{2}} \right\} \quad (61)$$

$$+ \frac{(1 - e^{-2(l_1-l_2)|z|})}{2|z|} \left\{ \Psi \left( |z| \sqrt{\frac{x}{2}} \right) + z \sqrt{\frac{\pi x}{2}} \right\} \left\{ \sqrt{\frac{2}{\pi}} \Psi \left( |z| \sqrt{\frac{y}{2}} \right) + |z| \sqrt{y} \right\},$$

$$F_2(x, y, z) = e^{-(l_1-l_2)(|z|-z)} \left\{ \sqrt{y} \Psi \left( |z| \sqrt{\frac{x}{2}} \right) + z \sqrt{\frac{\pi xy}{2}} \right\}, \quad (62)$$

$$F(x, y, z) = e^{-2(l_1-l_2)|z|} \left\{ \sqrt{y} \Psi \left( |z| \sqrt{\frac{x}{2}} \right) + \sqrt{x} \Psi \left( |z| \sqrt{\frac{y}{2}} \right) \right\} \quad (63)$$

$$+ \frac{(1 - e^{-2(l_1-l_2)|z|})}{2|z|} \left\{ \Psi \left( |z| \sqrt{\frac{x}{2}} \right) + |z| \sqrt{\frac{\pi x}{2}} \right\} \left\{ \sqrt{\frac{2}{\pi}} \Psi \left( |z| \sqrt{\frac{y}{2}} \right) + |z| \sqrt{y} \right\}.$$

**Proof:** From Theorem 1 and (56) in Corollary 1.2, we actually know that, when  $W_0^\mu = x_0$ ,  $l_2 \leq x_0 \leq l_1$ ,

$$E \left( e^{-\beta \tau^{W^\mu}} \mathbf{1}_{\{\tau_1^{W^\mu} < \tau_4^{W^\mu}\}} \right) = \frac{e^{(l_1-x_0)\mu} \{e^{|\mu|(x_0-l_2)} - e^{-|\mu|(x_0-l_2)}\} G_1(d_1, d_4, \mu)}{\{e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)}\} G(d_1, d_4, \mu)} \quad (64)$$

$$+ \frac{e^{(l_2-x_0)\mu} \{e^{|\mu|(l_1-x_0)} - e^{-|\mu|(l_1-x_0)}\} G_2(d_1, d_4, \mu)}{\{e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)}\} G(d_1, d_4, \mu)},$$

$$E \left( e^{-\beta \tau^{W^\mu}} \mathbf{1}_{\{\tau_1^{W^\mu} > \tau_4^{W^\mu}\}} \right) = \frac{e^{(l_1-x_0)\mu} \{e^{|\mu|(x_0-l_2)} - e^{-|\mu|(x_0-l_2)}\} G_2(d_4, d_1, -\mu)}{\{e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)}\} G(d_1, d_4, \mu)} \quad (65)$$

$$+ \frac{e^{(l_2-x_0)\mu} \{e^{|\mu|(l_1-x_0)} - e^{-|\mu|(l_1-x_0)}\} G_1(d_4, d_1, -\mu)}{\{e^{|\mu|(l_1-l_2)} - e^{-|\mu|(l_1-l_2)}\} G(d_1, d_4, \mu)}.$$

Setting  $\beta = 0$  in (64) and (65) yields the results.  $\square$

Theorem 2 leads to the following remarkable result.

**Corollary 2.1** *The probability that a standard Brownian motion  $W_t$  with  $W_0 = x_0$ ,  $l_2 \leq x_0 \leq l_1$ , we have*

$$P(\tau_1^W < \tau_4^W) = \frac{\sqrt{d_4} + (x_0 - l_2) \sqrt{\frac{2}{\pi}}}{\sqrt{d_1} + \sqrt{d_4} + (l_1 - l_2) \sqrt{\frac{2}{\pi}}}, \quad (66)$$

$$P(\tau_1^W > \tau_4^W) = \frac{\sqrt{d_1} + (l_1 - x_0) \sqrt{\frac{2}{\pi}}}{\sqrt{d_1} + \sqrt{d_4} + (l_1 - l_2) \sqrt{\frac{2}{\pi}}}. \quad (67)$$

**Remark:** When we take  $l_1 \rightarrow 0$ ,  $l_2 \rightarrow 0$ ,  $x_0 \rightarrow 0$ , we can get the results for the one barrier case as in [9].

We will now extend Corollary 1.2 to obtain the joint distribution of  $W_t$  and  $\tau^W$  at an exponential time. This is an application of (56) and Girsanov's theorem.

**Theorem 3** For a standard Brownian Motion  $W_t$  with  $W_0 = x_0$ ,  $l_2 \leq x_0 \leq l_1$  and  $\tau^W$  defined as in (4) with  $S_t = W_t$ , we have the following result:

For the case  $x \geq l_1$ ,

$$P\left(W_{\tilde{T}} \in dx, \tau^W < \tilde{T}\right) = a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4) + a_1(x_0) h(x - l_1, d_1); \quad (68)$$

For the case  $l_2 \leq x < l_1$ ,

$$P\left(W_{\tilde{T}} \in dx, \tau^W < \tilde{T}\right) = a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4); \quad (69)$$

For the case  $x < l_2$ ,

$$P\left(W_{\tilde{T}} \in dx, \tau^W < \tilde{T}\right) = a_1(x_0) f(x - l_1, d_1) + a_2(x_0) f(x - l_2, d_4) + a_2(x_0) h(x - l_2, d_4); \quad (70)$$

where  $\tilde{T}$  is a random variable with an exponential distribution of parameter  $\gamma$  that is independent of  $W_t$  and

$$f(x, y) = \frac{e^{-\sqrt{2\gamma}|x|}}{\sqrt{2\gamma}} - e^{\gamma y - \sqrt{2\gamma}|x|} \sqrt{2\pi y} \mathcal{N}\left(-\sqrt{2\gamma y}\right), \quad (71)$$

$$h(x, y) = \sqrt{2\pi y} e^{\gamma y} \left\{ e^{-\sqrt{2\gamma}|x|} \mathcal{N}\left(\frac{|x|}{\sqrt{y}} - \sqrt{2\gamma y}\right) - e^{\sqrt{2\gamma}|x|} \mathcal{N}\left(-\frac{|x|}{\sqrt{y}} - \sqrt{2\gamma y}\right) \right\}, \quad (72)$$

$$a_1(x_0) = \frac{\gamma \left\{ e^{\sqrt{2\gamma}(x_0 - l_2)} - e^{-\sqrt{2\gamma}(x_0 - l_2)} \right\} b_1(d_1, d_4) + \gamma \left\{ e^{\sqrt{2\gamma}(l_1 - x_0)} - e^{-\sqrt{2\gamma}(l_1 - x_0)} \right\} b_2(d_1, d_4)}{G \left\{ e^{\sqrt{2\gamma}(l_1 - l_2)} - e^{-\sqrt{2\gamma}(l_1 - l_2)} \right\}}, \quad (73)$$

$$a_2(x_0) = \frac{\gamma \left\{ e^{\sqrt{2\gamma}(x_0 - l_2)} - e^{-\sqrt{2\gamma}(x_0 - l_2)} \right\} b_2(d_4, d_1) + \gamma \left\{ e^{\sqrt{2\gamma}(l_1 - x_0)} - e^{-\sqrt{2\gamma}(l_1 - x_0)} \right\} b_1(d_4, d_1)}{G \left\{ e^{\sqrt{2\gamma}(l_1 - l_2)} - e^{-\sqrt{2\gamma}(l_1 - l_2)} \right\}}, \quad (74)$$

$$b_1(x, y) = e^{-2(l_1 - l_2)\sqrt{2\gamma} - \gamma x} \sqrt{y} + \frac{1 - e^{-2\gamma\sqrt{2\gamma}}}{2\sqrt{2\gamma}} e^{-\gamma x} \left\{ \sqrt{\frac{2}{\pi}} \Psi(\sqrt{\gamma y}) + \sqrt{2\gamma y} \right\}, \quad (75)$$

$$b_2(x, y) = e^{-(l_1 - l_2)\sqrt{2\gamma} - \gamma x} \sqrt{y}, \quad (76)$$

$$G = e^{-2(l_1 - l_2)\sqrt{2\gamma}} \left\{ \sqrt{d_4} \Psi(\sqrt{\gamma d_1}) + \sqrt{d_1} \Psi(\sqrt{\gamma d_4}) \right\} + \frac{(1 - e^{-2(l_1 - l_2)\sqrt{2\gamma}})}{2\sqrt{2\gamma}} \left\{ \Psi(\sqrt{\gamma d_1}) + \sqrt{\gamma \pi d_1} \right\} \left\{ \sqrt{\frac{2}{\pi}} \Psi(\sqrt{\gamma d_4}) + \sqrt{2\gamma d_4} \right\}. \quad (77)$$

**Proof:** see appendix.  $\square$

## 5 Pricing double barrier Parisian Options

We want to price a double barrier Parisian call option with the current price of its underlying asset to be  $x$ ,  $L_1 < x < L_2$ , the owner of which will obtain the right to exercise it when either the length of the excursion above the barrier  $L_1$  reaches  $d_1$ , or the length of the excursion below the barrier  $L_2$  reaches  $d_2$  before  $T$ . Its price formula is given by

$$P_{min-in-call} = e^{-rT} E_Q \left( (S_T - K)^+ \mathbf{1}_{\{\tau^S < T\}} \right),$$

where  $S$  is the underlying stock price,  $Q$  denotes the risk neutral measure. The subscript *min-in-call* means it is a call option which will be triggered when the minimum of two stopping times,  $\tau_1^S$  and  $\tau_4^S$ , is less than  $T$ , i.e.  $\tau^S < T$ . We assume  $S$  is a geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = x,$$

where  $L_1 < x < L_2$ ,  $r$  is the risk free rate,  $W_t$  with  $W_0 = 0$  is a standard Brownian motion under  $Q$ . Set

$$m = \frac{1}{\sigma} \left( r - \frac{1}{2} \sigma^2 \right), \quad b = \frac{1}{\sigma} \ln \left( \frac{K}{x} \right), \quad B_t = mt + W_t,$$

$$l_1 = \frac{1}{\sigma} \ln \left( \frac{L_1}{x} \right), \quad l_2 = \frac{1}{\sigma} \ln \left( \frac{L_2}{x} \right).$$

We have

$$S_t = x \exp \left\{ \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\} = x \exp \{ \sigma (mt + W_t) \} = x e^{\sigma B_t}.$$

By applying Girsanov's Theorem, we have

$$P_{min-in-call} = e^{-(r+\frac{1}{2}m^2)T} E_P \left[ (x e^{\sigma B_T} - K)^+ e^{mB_T} \mathbf{1}_{\{\tau^B < T\}} \right],$$

where  $P$  is a new measure, under which  $B_t$  is a standard Brownian motion with  $B_0 = 0$ , and  $\tau^B$  is the stopping time defined with the respect to barrier  $l_1, l_2$ . And we define

$$P_{min-in-call}^* = e^{(r+\frac{1}{2}m^2)T} P_{min-in-call}.$$

We are going to show that we can obtain the Laplace transform of  $P_{min-in-call}^*$  w.r.t  $T$ , denoted by  $\mathcal{L}_T$ .

Firstly, assuming  $\tilde{T}$  is a random variable with an exponential distribution

with parameter  $\gamma$  that is independent of  $W_t$ , we have

$$\begin{aligned}
& E_P \left[ (xe^{\sigma B_{\tilde{T}}} - K)^+ e^{mB_{\tilde{T}}} \mathbf{1}_{\{\tau^B < \tilde{T}\}} \right] \\
&= \int_b^\infty (xe^{\sigma y} - K) e^{my} P \left( B_{\tilde{T}} \in dy, \tau^B < \tilde{T} \right) \\
&= \int_0^\infty \gamma e^{-\gamma T} \int_b^\infty (xe^{\sigma y} - K) e^{my} P \left( B_T \in dy, \tau^B < T \right) dT \\
&= \gamma \int_0^\infty e^{-\gamma T} E_P \left[ (xe^{\sigma B_T} - K)^+ e^{mB_T} \mathbf{1}_{\{\tau^B < T\}} \right] dT \\
&= \gamma \mathcal{L}_T
\end{aligned}$$

Hence we have

$$\mathcal{L}_T = \frac{1}{\gamma} \int_b^\infty (xe^{\sigma y} - K) e^{my} P \left( B_{\tilde{T}} \in dy, \tau^B < \tilde{T} \right).$$

By using the results in Theorem 3, this Laplace transform can be calculated explicitly.

When  $b \geq l_1$ , i.e.  $K \geq L_1$ , we have

$$\mathcal{L}_T = \frac{x}{\gamma} F_1(\sigma + m) - \frac{K}{\gamma} F_1(m),$$

where

$$\begin{aligned}
F_1(x) &= a_1(0) \left\{ \frac{1}{\sqrt{2\gamma}} - e^{\gamma d_1} \sqrt{2\pi d_1} \mathcal{N} \left( -\sqrt{2\gamma} d_1 \right) \right\} \frac{e^{\sqrt{2\gamma} l_1 + (x - \sqrt{2\gamma}) b}}{\sqrt{2\gamma} - x} \\
&+ a_2(0) \left\{ \frac{1}{\sqrt{2\gamma}} - e^{\gamma d_4} \sqrt{2\pi d_4} \mathcal{N} \left( -\sqrt{2\gamma} d_4 \right) \right\} \frac{e^{\sqrt{2\gamma} l_2 + (x - \sqrt{2\gamma}) b}}{\sqrt{2\gamma} - x} \\
&+ a_1(0) \sqrt{2\pi d_1} e^{\gamma d_1} \left\{ \frac{2xe^{x l_1 - r d_1 + \frac{d_1 x^2}{2}} \mathcal{N} \left( x \sqrt{d_1} - \frac{b - l_1}{\sqrt{d_1}} \right)}{2\gamma - x^2} \right. \\
&\left. + \frac{e^{\sqrt{2\gamma} l_1 + (x - \sqrt{2\gamma}) b} \mathcal{N} \left( \frac{b - l_1}{\sqrt{d_1}} - \sqrt{2\gamma} d_1 \right)}{\sqrt{2\gamma} - x} + \frac{e^{-\sqrt{2\gamma} l_1 + (x + \sqrt{2\gamma}) b} \mathcal{N} \left( -\frac{b - l_1}{\sqrt{d_1}} - \sqrt{2\gamma} d_1 \right)}{\sqrt{2\gamma} + x} \right\};
\end{aligned}$$

when  $l_2 < b < l_1$ , i.e.  $L_2 < K < L_1$ , we have

$$\mathcal{L}_T = \frac{x}{\gamma} F_2(\sigma + m) - \frac{K}{\gamma} F_2(m),$$

where

$$\begin{aligned}
F_2(x) &= \frac{2a_1(0)e^{l_1x}}{2\gamma - x^2} \left\{ 1 + x\sqrt{2\pi d_1}e^{\frac{d_1x^2}{2}} \mathcal{N}\left(x\sqrt{d_1}\right) \right\} \\
&\quad - a_1(0) \left\{ \frac{1}{\sqrt{2\gamma}} - e^{\gamma d_1} \sqrt{2\pi d_1} \mathcal{N}\left(-\sqrt{2\gamma d_1}\right) \right\} \frac{e^{-\sqrt{2\gamma}l_1 + (x+\sqrt{2\gamma})b}}{\sqrt{2\gamma} - x} \\
&\quad + a_2(0) \left\{ \frac{1}{\sqrt{2\gamma}} - e^{\gamma d_4} \sqrt{2\pi d_4} \mathcal{N}\left(-\sqrt{2\gamma d_4}\right) \right\} \frac{e^{\sqrt{2\gamma}l_2 + (x-\sqrt{2\gamma})b}}{\sqrt{2\gamma} - x};
\end{aligned}$$

when  $b \leq l_2$ , i.e.  $K \leq L_2$ , we have

$$\mathcal{L}_T = \frac{x}{\gamma} F_3(\sigma + m) - \frac{K}{\gamma} F_3(m),$$

where

$$\begin{aligned}
F_2(x) &= \frac{2a_1(0)e^{l_1x}}{2\gamma - x^2} \left\{ 1 + x\sqrt{2\pi d_1}e^{\frac{d_1x^2}{2}} \mathcal{N}\left(x\sqrt{d_1}\right) \right\} \\
&\quad - a_1(0) \left\{ \frac{1}{\sqrt{2\gamma}} - e^{\gamma d_1} \sqrt{2\pi d_1} \mathcal{N}\left(-\sqrt{2\gamma d_1}\right) \right\} \frac{e^{-\sqrt{2\gamma}l_1 + (x+\sqrt{2\gamma})b}}{\sqrt{2\gamma} - x} \\
&\quad + \frac{2a_2(0)e^{l_2x}}{2\gamma - x^2} \left\{ 1 - 2\sqrt{\pi d_4 \gamma} e^{\frac{d_4x^2}{2}} \mathcal{N}\left(x\sqrt{d_4}\right) \right\} \\
&\quad - a_2(0) \left\{ \frac{1}{\sqrt{2\gamma}} - e^{\gamma d_4} \sqrt{2\pi d_4} \mathcal{N}\left(-\sqrt{2\gamma d_4}\right) \right\} \frac{e^{-\sqrt{2\gamma}l_2 + (x+\sqrt{2\gamma})b}}{\sqrt{2\gamma} - x} \\
&\quad + a_2(0) \sqrt{2\pi d_4} e^{\gamma d_4} \left\{ \frac{2\sqrt{2\gamma} e^{xl_2 - rd_4 + \frac{d_4x^2}{2}} \mathcal{N}\left(x\sqrt{d_4} - \frac{b-l_2}{\sqrt{d_4}}\right)}{2\gamma - x^2} \right. \\
&\quad \left. - \frac{e^{\sqrt{2\gamma}l_2 + (x-\sqrt{2\gamma})b} \mathcal{N}\left(\frac{b-l_2}{\sqrt{d_4}} - \sqrt{2\gamma d_4}\right)}{\sqrt{2\gamma} - x} - \frac{e^{-\sqrt{2\gamma}l_2 + (x+\sqrt{2\gamma})b} \mathcal{N}\left(-\frac{b-l_2}{\sqrt{d_4}} - \sqrt{2\gamma d_4}\right)}{\sqrt{2\gamma} + x} \right\}.
\end{aligned}$$

**Remark:** The price can be calculated by numerical inversion of the Laplace transform.

So far, we have shown how to obtain the Laplace transform of

$$P_{min-call-in}^* = e^{(r+\frac{1}{2}m^2)T} P_{min-call-in}.$$

For

$$P_{min-call-out} = e^{-rT} E_Q \left( (S_T - K)^+ \mathbf{1}_{\{\tau^S > T\}} \right),$$

we can get the result from the relationship that

$$P_{min-call-out} = e^{-rT} E_Q \left\{ (S_T - K)^+ \right\} - P_{min-call-in}.$$

Furthermore, if we set

$$\tilde{\tau}_L^Y = \tau_{1,L}^Y \vee \tau_{2,L}^Y,$$

we can define another type of Parisian options by  $\tilde{\tau}_L^Y$ :

$$P_{max-call-in} = e^{-rT} E_Q \left( (S_T - K)^+ \mathbf{1}_{\{\tilde{\tau}_L^S < T\}} \right).$$

In order to get its pricing formula, we should use the following relationship:

$$\mathbf{1}_{\{\tilde{\tau}_L^S < T\}} = \mathbf{1}_{\{\tau_{1,L}^S < T\}} + \mathbf{1}_{\{\tau_{2,L}^S < T\}} - \mathbf{1}_{\{\tau_L^S < T\}}.$$

We have therefore

$$P_{max-call-in} = P_{up-in-call} + P_{down-in-call} - P_{min-call-in}.$$

Similarly, from

$$P_{max-call-out} = e^{-rT} E_Q \{ (S_T - K)^+ \} - P_{max-call-in},$$

we can work out  $P_{max-call-out}$ .

## 6 Appendix: Proof of Theorem 3

Let  $T$  be the final time. According to the definition of  $\Psi(x)$ , we have

$$\Psi(x) = 2\sqrt{\pi}x\mathcal{N}(\sqrt{2}x) - \sqrt{\pi}x + e^{-x^2} = \sqrt{\pi}x - \sqrt{\pi}x\text{Erfc}(x) + e^{-x^2}.$$

It is not difficult to show that

$$E \left( e^{-\beta\tau^{W^\mu}} \right) = E \left( \int_0^\infty \beta e^{-\beta T} \mathbf{1}_{\{\tau^{W^\mu} < T\}} dT \right).$$

By Girsanov's theorem, this is equal to

$$\int_0^\infty \beta e^{-(\beta + \frac{1}{2}\mu^2)T - \mu x_0} E \left( e^{\mu W_T} \mathbf{1}_{\{\tau^W < T\}} \right) dT.$$

Setting  $\gamma = \beta + \frac{1}{2}\mu^2$  gives

$$\begin{aligned} E \left( e^{-\beta\tau^{W^\mu}} \right) &= \int_0^\infty (\gamma - \frac{1}{2}\mu^2) e^{-\gamma T - \mu x_0} E \left( e^{\mu W_T} \mathbf{1}_{\{\tau^W < T\}} \right) dT \\ &= \frac{\gamma - \frac{1}{2}\mu^2}{\gamma} e^{-\mu x_0} E \left( e^{\mu W_{\tilde{T}}} \mathbf{1}_{\{\tau^W < \tilde{T}\}} \right), \end{aligned}$$

where  $\tilde{T}$  is a random variable with an exponential distribution of parameter  $\gamma$  that is independent of  $W_t$ . Therefore we have

$$E \left( e^{\mu W_{\tilde{T}}} \mathbf{1}_{\{\tau^W < \tilde{T}\}} \right) = \frac{\gamma e^{\mu x_0}}{\gamma - \frac{1}{2}\mu^2} E \left( e^{-\beta\tau^{W^\mu}} \right)$$

In order to inverse the above moment generating function, we just need to inverse the following expressions:

$$\begin{aligned}\frac{\mu}{\gamma - \frac{\mu^2}{2}} &= \int_0^\infty e^{\mu x} e^{-\sqrt{2\gamma}x} dx - \int_{-\infty}^0 e^{\mu x} e^{\sqrt{2\gamma}x} dx, \\ \frac{1}{\gamma - \frac{\mu^2}{2}} &= \int_0^\infty e^{\mu x} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}x} dx + \int_{-\infty}^0 e^{\mu x} \frac{1}{\sqrt{2\gamma}} e^{\sqrt{2\gamma}x} dx, \\ e^{\frac{d_1}{2}\mu^2} &= \int_{-\infty}^\infty e^{\mu x} \frac{1}{\sqrt{2\pi d_1}} \exp\left\{-\frac{x^2}{2d_1}\right\} dx, \\ 1 - \sqrt{\frac{d_i}{2}} \pi \mu e^{\frac{d_i}{2}\mu^2} \operatorname{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right) &= \int_{-\infty}^0 e^{\mu x} \frac{-x}{d_i} e^{-\frac{x^2}{2d_i}} dx.\end{aligned}$$

The inversion of  $\frac{\mu e^{\frac{d_1}{2}\mu^2}}{\gamma - \frac{\mu^2}{2}}$  is

$$\begin{aligned}&\int_0^\infty e^{-\sqrt{2\gamma}y} \frac{1}{\sqrt{2\pi d_1}} e^{-\frac{(x-y)^2}{2d_1}} dy - \int_{-\infty}^0 e^{\sqrt{2\gamma}y} \frac{1}{\sqrt{2\pi d_1}} e^{-\frac{(x-y)^2}{2d_1}} dy \\ &= e^{\gamma d_1} \left\{ e^{-\sqrt{2\gamma}x} \mathcal{N}\left(\frac{x}{\sqrt{d_1}} - \sqrt{2\gamma}d_1\right) - e^{\sqrt{2\gamma}x} \mathcal{N}\left(-\frac{x}{\sqrt{d_1}} - \sqrt{2\gamma}d_1\right) \right\}.\end{aligned}$$

The inversion of  $\frac{1 - \sqrt{\frac{d_i}{2}} \pi \mu e^{\frac{d_i}{2}\mu^2} \operatorname{Erfc}\left(\sqrt{\frac{d_i}{2}}\mu\right)}{\gamma - \frac{\mu^2}{2}}$  is given below.

For  $x \geq 0$ ,

$$\int_{-\infty}^0 \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}(x-y)} dy = \frac{e^{-\sqrt{2\gamma}x}}{\sqrt{2\gamma}} - e^{\gamma d_i - \sqrt{2\gamma}x} \sqrt{2\pi d_i} \mathcal{N}\left(-\sqrt{2\gamma}d_i\right);$$

For  $x < 0$ ,

$$\begin{aligned}&\int_{-\infty}^x \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}(x-y)} dy + \int_x^0 \frac{-y}{d_i} e^{-\frac{y^2}{2d_i}} \frac{1}{\sqrt{2\gamma}} e^{\sqrt{2\gamma}(x-y)} dy \\ &= \frac{e^{\sqrt{2\gamma}x}}{\sqrt{2\gamma}} - e^{\gamma d_i - \sqrt{2\gamma}x} \sqrt{2\pi d_i} \mathcal{N}\left(\frac{x}{\sqrt{d_i}} - \sqrt{2\gamma}d_i\right) \\ &\quad + e^{\gamma d_i + \sqrt{2\gamma}x} \sqrt{2\pi d_i} \left\{ \mathcal{N}\left(\sqrt{2\gamma}d_i\right) - \mathcal{N}\left(\frac{x}{\sqrt{d_i}} + \sqrt{2\gamma}d_i\right) \right\}.\end{aligned}$$

Consequently, we can get Theorem 3.

## References

- [1] Akahori, J. Some Formulae for a new type of path-dependent option. *Ann. Appl. Prob.* **5**, 383-388, 1995.

- [2] Azema, J. and Yor, M. Etude d'une martingale remarquable. *Seminaire de probabilities XXIII (Lecture Notes in Mathematics 1372)*. Springer, Berlin, 1989.
- [3] Beteman, H. *Table of Integral Transforms, Volume I*. McGraw-Hill book company, INC, 1954.
- [4] Borodin, A. and Salminen, P. *Handbook of Brownian Motion - Facts and Formulae*. Birkhauser, 1996.
- [5] Bertoin, Jean. *Levy Processes*. Cambridge University Press, 1996.
- [6] Chesney, M., Jeanblanc-Picque, M. and Yor, M. Brownian excursions and Parisian barrier options. *Adv. Appl. Prob.*, **29**, 165-184, 1997.
- [7] Chung, K.L. Excursions in Brownian motion. *Ark. math.* **14**, 155-177, 1976.
- [8] Dassios, A. The distribution of the quantiles of a Brownian motion with drift. *Ann. Appl. Prob.* **5**, 389-398, 1995.
- [9] Dassios, A. Wu, S. Two-sided Parisian option with singl barrier. Working paper L.S.E., 2008.
- [10] Davis, M.H.A. *Markov Models and Optimization*. Chapman and Hall, 1994.
- [11] Grandel, J. *Aspect of Risk Theory*. Springer, 1991.
- [12] Hartley, P. Pricing parisian options by Laplace inversion. *Decisions in Economics and Finance*, 2002.
- [13] Labart, C. and Lelong, J. Pricing parisian options. Technical report, ENPC, <http://cermics.enpc.fr/reports/CERMICS-2005/CERMICS-2005-294.pdf>, December 2005.
- [14] Pechtl, A. Some applications of occupation times of Brownian motion with drift in mathematical finance, *Journal of Applied Mathematics and Decision Sciences*, **3**, 63-73, 1999.
- [15] Schröder, M. Brownian excursions and Parisian barrier options: a note. *J. Appl. Prob.* **40(4)**, 855-864, 2003.