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Parisian ruin with exponential claims

Working paper

Parisian ruin with exponential claims

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Abstract

In this paper, we extend the concept of ruin in risk theory to the Parisian type of ruin. For this to occur, the surplus process must fall below zero and stay negative for a continuous time interval of specified length. Working with a classical surplus process with exponential jump size, we obtain the Laplace transform of the time of ruin and the probability of ruin in the infinite horizon. We also consider a diffusion approximation and use it to obtain similar results for the Brownian motion with drift.

Keywords: ruin, Parisian type of ruin, risk process, Laplace transform, ruin probability.

1 Introduction

We consider a classical surplus process in continuous time $\{X_t\}_{t>0}$

$$X_t = x + ct - \sum_{k=0}^{N_t} Y_k, \tag{1}$$

where $x \geq 0$ is the initial reserve, c is a constant rate of premium payment per time unit, N_t is the number of claims up to time t which has a Poisson distribution with parameter λ, and Y_k, $k = 1, 2, ...$, are claim sizes which are independent and identically distributed non-negative random variables that are also independent of N_t. We also assume $c > \lambda E(Y_1)$ (the *net profit condition*). Define the stopping time

$$T_x = \inf \{t > 0 \mid X_t < 0\}. \tag{2}$$

The event of ruin in infinite time horizon can be expressed as $\{T_x < \infty\}$. The density of T_x and the probability of ruin have been widely studied. See for example [9], [10], [11], [12], [17], [18], [21] and [24].

In this paper, we extend the concept of ruin to the Parisian type of ruin. The idea comes from the Parisian options, the prices of which depend on the excursions of the underlying asset prices above or below a barrier. An example is a *Parisian down-and-out option*, the owner of which loses the option if the underlying asset price S reaches the level l and remains constantly below this

1

level for a time interval longer than d. For details and extensions, see [4], [5], [6], [7], [8], [22] and [23].

Parisian type ruin will occur if the surplus falls below zero and stays below zero for a continuous time interval of length d. In some respects, this might be a more appropriate measure of risk than classical ruin as it gives the office some time to put its finances in order.

In order to introduce the concept of Parisian type of ruin mathematically, we will first define the excursion. Set

$$
\begin{aligned}
g_t^X &= \sup\{s < t \mid \operatorname{sign}(X_s)\operatorname{sign}(X_t) \leq 0\}, & (3)\\
d_t^X &= \inf\{s > t \mid \operatorname{sign}(X_s)\operatorname{sign}(X_t) \leq 0\}, & (4)
\end{aligned}
$$

with the usual convention, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$, where

$$
\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}.
$$

The trajectory between g_t^X and d_t^X is the excursion of process X which straddles time t. Assuming $d > 0$, we now define

$$
\tau_d^X = \inf\{t > 0 \mid \mathbf{1}_{\{X_t < 0\}}(t - g_t^X) \geq d\}. \tag{5}
$$

We can see that τ_d^X is therefore the first time that the length of the excursion of process X below 0 reaches given level d. We also define the events $\{\tau_d^X \leq t\}$ and $\{\tau_d^X < \infty\}$ to be the Parisian type ruin in the finite and infinite horizons. We are interested in the corresponding probabilities

$$
P\left(\tau_d^X \leq t\right)
$$

and

$$
P\left(\tau_d^X < \infty\right).
$$

We will restrict ourselves here to claim sizes that are exponentially distributed as this is a case where explicit results can be obtained. We therefore assume that the claim sizes have density $\alpha e^{-\alpha x}$, where $x > 0$. From the net profit condition above, we also have that $c > \frac{\lambda}{\alpha}$.

In Section 2 we provide results on hitting times that will be used in Section 3 to give the Laplace transform of the stopping time τ_d^X. In Section 4 we derive the Parisian type ruin probability in the infinite horizon. In Section 5 we introduce a diffusion approximation and thus obtain results for the Brownian motions.

2 Definitions

We consider the X_t with $x = 0$ at first. In this section we are going to introduce a semi-Markov model consisting of two states, the state when the process is above the 0 and the state when it is below. Therefore we define

$$
Z_t^X = \begin{cases} 1, & \text{if } X_t > 0 \\ 2, & \text{if } X_t < 0 \end{cases}.
$$

We can now express the variables defined above in terms of Z_t^X:

$$g_t^X = \sup\{s < t \mid Z_s^X \neq Z_t^X\}, \tag{6}$$

$$d_t^X = \inf\{s > t \mid Z_s^X \neq Z_t^X\}, \tag{7}$$

$$\tau_d^X = \inf\{t > 0 \mid \mathbf{1}_{\{Z_t^X = 2\}}(t - g_t^X) \geq d\}. \tag{8}$$

We then define

$$V_t^X = t - g_t^X,$$

the time Z_t^X has spent in the current state. It is easy to prove that (Z_t^X, V_t^X) is a Markov process. Z_t^X is therefore a semi-Markov process with the state space $\{1, 2\}$, where 1 stands for the state when the stochastic process X is above 0 and 2 corresponds to the state below 0.

Furthermore, we set $U_{i,k}^X$, $i = 1, 2$ and $k = 1, 2, \cdots$ to be the time Z^X spends in state i when it visits i for the kth time. And we have, for each given i and k there exist some t satisfying that

$$U_{i,k}^X = V_{d_t^X}^X = d_t^X - g_t^X.$$

Notice that assuming that the jump size Y_k is exponentially distributed, it is a well-known result that the size of the overshoots are also exponentially distributed with the same parameter. Therefore the excursions above 0 and below 0 are independent. Consequently, we have that $U_{1,k}^X$, $k = 1, 2, \cdots$, are independent and identically distributed, so as for $U_{2,k}^X$, $k = 1, 2, \cdots$, and $U_{i,k}^X$, $i = 1, 2$, $k = 1, 2, \cdots$, are independent with each other. We therefore define the transition density for Z^X:

$$p_{ij}(t) = \lim_{\Delta t \to 0} \frac{P(t < U_{i,k}^X < t + \Delta t)}{\Delta t},$$

$$P_{ij}(t) = P(U_{i,k}^X < t), \quad \bar{P}_{ij}(t) = P(U_{i,k}^X > t).$$

We have

$$P_{ij}(t) = \int_0^t p_{ij}(s)\mathrm{d}s = 1 - \bar{P}_{ij}(t),$$

which is actually the probability that the process will stay in state i for no more than time t. We will see in the later discussion that the condition $c > \frac{\lambda}{\alpha}$ results in $P\left(U_{1,k}^X = \infty\right) > 0$ for all k (we adopt the convention $U_{i,k}^X = \infty$ if the process never leaves state i at its kth excursion); therefore $\int_0^{+\infty} p_{12}(s)\mathrm{d}s < 1$, i.e. with a positive probability, the process will stay in state 1 forever. Hence, in this case $\bar{P}_{12}(t) > \int_t^{+\infty} p_{12}(s)\mathrm{d}s$.

Moreover, in the definition of Z^X, we deliberately ignore the situation when $X_t = 0$. The reason is that

$$\int_0^t \mathbf{1}_{\{X_u = 0\}}\mathrm{d}u = 0.$$

3

We will now show how to get $p_{ij}(t)$. We use $\hat{P}_{ij}(\beta)$ to represent the Laplace transform of $p_{ij}(t)$, i.e.

$$\hat{P}_{ij}(\beta) = \int_0^\infty e^{-\beta t} p_{ij}(t) \mathrm{d}t = E\left(e^{-\beta U_{i,k}^X}\right). \tag{9}$$

Consider the equation

$$-\beta + cv_\beta + \lambda\left(\frac{\alpha}{v_\beta + \alpha} - 1\right) = 0, \tag{10}$$

which has two roots,

$$v_\beta^+ = \frac{\sqrt{(c\alpha + \beta + \lambda)^2 - 4c\alpha\lambda} - (c\alpha - \beta - \lambda)}{2c}, \tag{11}$$

and

$$v_\beta^- = \frac{-\sqrt{(c\alpha + \beta + \lambda)^2 - 4c\alpha\lambda} - (c\alpha - \beta - \lambda)}{2c}. \tag{12}$$

First of all, we want to look at the length of an excursion below 0, i.e. $U_{2,k}^X$, $k = 1, 2, 3, \ldots$. Define the stopping time

$$T_x = \inf\{t > 0, X_t = 0 \mid X_0 = x, x < 0\}.$$

It has been shown in [16] that

$$E\left(\exp\left(-\beta T_x\right)\right) = \exp\left(v_\beta^+ x\right).$$

According to the definitions of the process X and $U_{2,k}^X$ and the argument above, every excursion below 0 starts from an overshoot below 0 with length $|x|$ following the exponential distribution with parameter α and follows by the excursion with length T_x. We have therefore

$$
\begin{aligned}
\hat{P}_{21}(\beta) &= E\left(e^{-\beta U_{2,k}^X}\right) = \int_0^\infty E\left(e^{-\beta T_{-x}}\right) \alpha e^{-\alpha x} \mathrm{d}x \\
&= \int_0^\infty \exp\left(-v_\beta^+ x\right) \alpha e^{-\alpha x} \mathrm{d}x \\
&= \frac{2c\alpha}{\sqrt{(\beta + \lambda + c\alpha)^2 - 4c\lambda\alpha} + (\beta + \lambda + c\alpha)}.
\end{aligned}
$$

Inverting this Laplace transform with respect to β gives the transition density

$$p_{21}(t) = \sqrt{\frac{c\alpha}{\lambda}} e^{-(\lambda + c\alpha)t} t^{-1} I_1\left(2t\sqrt{c\lambda\alpha}\right). \tag{13}$$

The formulae for the inversion can be found in [3].

For the length of an excursion above 0, i.e. $U_{1,k}^X$, $k = 1, 2, 3, \ldots$, we define the stopping time

$$T_0 = \inf\{t > 0, X_t < 0 \mid X_0 = 0\}.$$

4

By results in [15], [17] and [18] and the independence of the time and the size of the overshoot, i.e. T_0 and X_{T_0} we have

$$E\left(e^{-\beta T_0}\right) E\left(\exp\left(v_\beta^- X_{T_0}\right)\right) = 1.$$

And we also know that X_{T_0} follows exponential distribution with parameter α. Therefore

$$
\begin{aligned}
\hat{P}_{12}(\beta) &= E\left(e^{-\beta U_{1,k}^X}\right) = E\left(e^{-\beta T_0}\right) = \frac{1}{E\left(\exp\left(v_\beta^- X_{T_0}\right)\right)} \\
&= \frac{1}{\int_0^\infty \exp\left(-v_\beta^- x\right) \alpha e^{-\alpha x} \mathrm{d}x} \\
&= \frac{2\lambda}{\sqrt{(\beta + \lambda + c\alpha)^2 - 4c\lambda\alpha} + (\beta + \lambda + c\alpha)}.
\end{aligned}
$$

Inverting $\hat{P}_{12}(\beta)$ gives

$$p_{12}(t) = \sqrt{\frac{\lambda}{c\alpha}} e^{-(\lambda + c\alpha)t} t^{-1} I_1\left(2t\sqrt{c\lambda\alpha}\right) \tag{14}$$

(see [3] for the formulae).

3 The Laplace Transform of τ_d^X

In this section we give the Laplace transform of τ_d^X for the cases when $x = 0$ and when $x > 0$ together with the proofs.

Theorem 1 *For X_t with $x = 0$, we have*

$$E\left(e^{-\beta \tau_d^X}\right) = \frac{e^{-\beta d} \bar{P}_{21}(d) \hat{P}_{12}(\beta)}{1 - \hat{P}_{12}(\beta) \tilde{P}_{21}(\beta)}, \tag{15}$$

where

$$
\begin{aligned}
\bar{P}_{21}(d) &= 1 - \int_0^d \sqrt{\frac{c\alpha}{\lambda}} e^{-(\lambda + c\alpha)t} t^{-1} I_1\left(2t\sqrt{c\lambda\alpha}\right) \mathrm{d}t, \tag{16} \\
\tilde{P}_{21}(\beta) &= \int_0^d \sqrt{\frac{c\alpha}{\lambda}} e^{-(\beta + \lambda + c\alpha)t} t^{-1} I_1\left(2t\sqrt{c\lambda\alpha}\right) \mathrm{d}t, \tag{17} \\
\hat{P}_{12}(\beta) &= \frac{2\lambda}{\sqrt{(\beta + \lambda + c\alpha)^2 - 4c\lambda\alpha} + (\beta + \lambda + c\alpha)}, \tag{18}
\end{aligned}
$$

and $I_1(x)$ is the modified Bessel function of the first kind.

Proof: A_k denotes the event that the first time the length of the excursion in state 2, i.e. below 0, reaches d happens during the kth excursion in this state, i.e.

$$\{A_k\} = \left\{\tau_d^X \text{ is achieved in the } k\text{th excursion in state 2}\right\}.$$

So we have

$$E\left(e^{-\beta\tau_d^X}\right) = \sum_{k=1}^{\infty} E\left(e^{-\beta\tau_d^X} \mid A_k\right) P\left(A_k\right). \tag{19}$$

Notice that given A_k, τ_d^X is comprised of k full excursions above 0, $k-1$ below 0 with the length less than d and last one with the length d, i.e.

$$\tau_d^X \mid A_k = \sum_{n=1}^{k-1} \left(U_{1,n}^X + U_{2,n}^X\right) + U_{1,k}^X + d \mid U_{2,1}^X < d, \cdots, U_{2,k-1}^X < d, U_{2,k}^X \geq d.$$

More importantly, $U_{1,n}^X$'s have distribution P_{12}; $U_{2,n}^X$'s have distribution P_{21} and all these variables are independent of each other. As a result,

$$
\begin{aligned}
&E\left(e^{-\beta\tau_d^X} \mid A_k\right)\\
=\ & E\left(e^{-\beta\left(\sum_{n=1}^{k-1}\left(U_{1,n}^X+U_{2,n}^X\right)+U_{1,k}^X+d\right)} \,\Big|\, U_{2,1}^X < d, \cdots, U_{2,k-1}^X < d, U_{2,k}^X \geq d\right)\\
=\ & \sum_{k=1}^{\infty} e^{-\beta d}\left\{\int_0^{+\infty} e^{-\beta u} p_{12}(u)\mathrm{d}u\right\}^k \left\{\int_0^d e^{-\beta u}\frac{p_{21}(u)}{P_{21}(d)}\mathrm{d}u\right\}^{k-1}.
\end{aligned}
$$

Also

$$P(A_k) = P_{21}(d)^{k-1}\bar{P}_{21}(d).$$

We have therefore

$$
\begin{aligned}
&E\left(e^{-\beta\tau_d^X}\right)\\
=\ & \sum_{k=1}^{\infty} E\left(e^{-\beta\tau_d^X}|A_k\right) P\left(A_k\right)\\
=\ & \sum_{k=1}^{\infty} e^{-\beta d}\left\{\int_0^{+\infty} e^{-\beta u} p_{12}(u)\mathrm{d}u\right\}^k \left\{\int_0^d e^{-\beta u}\frac{p_{21}(u)}{P_{21}(d)}\mathrm{d}u\right\}^{k-1} P_{21}(d)^{k-1}\bar{P}_{21}(d)\\
=\ & \frac{e^{-\beta d}\bar{P}_{21}(d)\int_0^{+\infty} e^{-\beta s} p_{12}(s)\mathrm{d}s}{1 - \int_0^{+\infty} e^{-\beta s} p_{12}(s)\mathrm{d}s \int_0^d e^{-\beta s} p_{21}(s)\mathrm{d}s}.
\end{aligned}
$$

$\square$

We should also consider the case when $x > 0$.

Theorem 2 *For X_t, with $X_0 = x > 0$ we have*

$$E\left(e^{-\beta\tau_d^X}\right) = \frac{\alpha + v_{\bar{\beta}}^{-}}{\alpha} e^{-\beta d + v_{\beta}^{-}x} \frac{\bar{P}_{21}(d)}{1 - \hat{P}_{12}(\beta)\tilde{P}_{21}(\beta)}. \tag{20}$$

6

Proof: When $x > 0$, we need to find the Laplace transform for $U_{1,1}^X$, which has different distribution from $U_{1,k}^X$, $k = 2, 3, 4, ...$

Applying the optional sampling theorem to the martingale $e^{-\beta t + v_\beta^- X_t}$ (it is easy to check that $e^{-\beta t + v_\beta^- X_t}$ is a martingale), we have

$$E\left(e^{-\beta U_{1,1}^X + v_\beta^- X_{U_{1,1}^X}} \mid X_0 = x\right) = e^{v_\beta^- x}.$$

Since the distribution of the overshoot of this process, i.e. $-X_{U_{1,1}^X}$, is still an exponential distribution with parameter α and it is independent of the time of overshoot, i.e. $U_{1,1}^X$, we have

$$E\left(e^{-\beta U_{1,1}^X + v_\beta^- X_{U_{1,1}^X}} \mid X_0 = x\right) = \frac{\alpha}{\alpha + v_\beta^-} E\left(e^{-\beta U_{1,1}^X} \mid X_0 = x\right).$$

Therefore

$$E\left(e^{-\beta U_{1,1}^X} \mid X_0 = x\right) = \frac{\alpha + v_\beta^-}{\alpha} e^{v_\beta^- x}. \tag{21}$$

As a result,

$$E\left(e^{-\beta \tau_d^X}\right)$$
$$= E\left(e^{-\beta \tau_d^X} \mathbf{1}_{\{U_{2,1}^X \geq d\}}\right) + E\left(e^{-\beta \tau_d^X} \mathbf{1}_{\{U_{2,1}^X < d\}}\right)$$
$$= e^{-\beta d} E\left(e^{-\beta U_{1,1}^X} \mathbf{1}_{\{U_{2,1}^X \geq d\}}\right) + E\left(e^{-\beta(U_{1,1}^X + U_{2,1}^X)} \mathbf{1}_{\{U_{2,1}^X < d\}}\right) E\left(e^{-\beta \tau_2^{\tilde{X}}}\right),$$

where $\tilde{X}$ is the same process with $X_0 = 0$. $E\left(e^{-\beta \tau_2^{\tilde{X}}}\right)$ has been calculated in Theorem 1. Since $U_{1,1}^X$ and $U_{2,1}^X$ are independent, we have

$$E\left(e^{-\beta \tau_d^X}\right)$$
$$= e^{-\beta d} E\left(e^{-\beta U_{1,1}^X}\right) P\left(U_{2,1}^X \geq d\right) + E\left(e^{-\beta U_{1,1}^X}\right) E\left(e^{-\beta U_{2,1}^X} \mathbf{1}_{\{U_{2,1}^X < d\}}\right) E\left(e^{-\beta \tau_2^{\tilde{X}}}\right)$$
$$= \frac{\alpha + v_\beta^-}{\theta} e^{-\beta d + v_\beta^- x} \int_d^\infty p_{21}(t) dt + \frac{\alpha + v_\beta^-}{\alpha} e^{v_\beta^- x} \int_0^d e^{-\beta t} p_{21}(t) dt E\left(e^{-\beta \tau_d^{\tilde{X}}}\right)$$
$$= \frac{\alpha + v_\beta^-}{\alpha} e^{-\beta d + v_\beta^- x} \frac{\bar{P}_{21}(d)}{1 - \hat{P}_{12}(\beta) \tilde{P}_{21}(\beta)}.$$

$\square$

By taking $\beta = 0$, we can obtain the probability that τ_d^X will ever be achieved.

Corollary 2.1 *For X_t with $X_0 = x > 0$, we have*

$$P\left(\tau_d^X < \infty\right) = \frac{\lambda}{c\alpha} e^{(\frac{\lambda}{c} - \alpha)x} \frac{c\alpha \bar{P}_{21}(d)}{c\alpha - \lambda P_{21}(d)}. \tag{22}$$

Remark: From (22) we can see that the Parisian ruin probability actually equals to the ruin probability multiplied by a constant. In fact, the Parisian type ruin probability can also be calculated in the following way:

$$P\left(\tau_d^X < \infty\right) = P\left(T_x < \infty\right) \int_0^\infty P\left(\tau_d^{\bar{X}} < \infty \mid X_{T_x} = -y\right) \alpha e^{-\alpha y} \mathrm{d}y, \quad (23)$$

where T_x has the same definition as in (2) and $\bar{X}$ is the risk process with $X_0 = X_{T_x}$. Therefore $P\left(T_x < \infty\right)$ is the ruin probability which has been well studied,

$$P\left(T_x < \infty\right) = \frac{\lambda}{c\alpha} e^{\left(\frac{\lambda}{c} - \alpha\right)x}. \quad (24)$$

By using the same method in Theorem 1, we can calculate that

$$\int_0^\infty E\left(e^{-\beta \tau_d^{\bar{X}}} \mid X_0 = -y\right) \alpha e^{-\alpha y} \mathrm{d}y = \frac{e^{-\beta d} \bar{P}_{21}(d)}{1 - \hat{P}_{12}(\beta)\tilde{P}_{21}(d)}. \quad (25)$$

By taking $\beta = 0$ in (25) we have

$$\int_0^\infty P\left(\tau_d^{\bar{X}} < \infty \mid X_{T_x} = -y\right) \alpha e^{-\alpha y} \mathrm{d}y = \frac{c\alpha \bar{P}_{21}(d_2)}{c\alpha - \lambda P_{21}(d_2)}. \quad (26)$$

Substituting (24) and (26) in (23) gives the same result as in corollary 2.1.

4 A diffusion approximation

Set

$$c = \mu + \frac{\sigma^2 \alpha}{2}, \quad \lambda = \frac{\sigma^2 \alpha^2}{2,}$$

with $\mu > 0$ and let $\alpha \to +\infty$. The process $X_t - \mu t - x$ converges weakly in $D\left[0, \infty\right)$ to a standard Brownian motion W_t with $W_0 = 0$ and hence X_t converges to a Brownian motion with drift

$$W_t^\mu = x + \mu t + \sigma W_t.$$

See for example [2], pp 117-118 and also [21], pp 159-160. Moreover, the events

$$\left\{\tau_d^X \leq t\right\}$$

and

$$\left\{\sup_{0 \leq s \leq t} \left\{\mathbf{1}_{\{X_s < 0\}}\left(s - g_s^X\right)\right\} \geq d\right\}$$

are identical and since

$$\sup_{0 \leq s \leq t} \left\{\mathbf{1}_{\{X_s < 0\}}\left(s - g_s^X\right)\right\}$$

is a continuous functional of X_t on $D[0, \infty)$ a.e., we can conclude that

$$\lim_{\alpha \to \infty} P\left(\tau_d^X \le t\right) = P\left(\tau_d^{W^\mu} \le t\right) \qquad (27)$$

for all t; and therefore

$$\lim_{\alpha \to \infty} E\left(e^{-\beta \tau_d^X} \mid X_0 = x, x > 0\right) = E\left(e^{-\beta \tau_d^{W^\mu}} \mid W_0^\mu = x, x > 0\right). \qquad (28)$$

As a result, by taking the limit $\alpha \to +\infty$ in (20) and applying the approximation for the modified Bessel function of the first kind (see [13])

$$\lim_{z \to \infty} I_1(z) = \frac{e^z}{\sqrt{2\pi z}}, \qquad (29)$$

we have

$$E\left(e^{-\beta \tau_d^{W^\mu}} \mid W_0^\mu = x, x > 0\right) = \frac{e^{-\beta d} e^{-\frac{\sqrt{\mu^2 + 2\beta\sigma^2} + \mu}{\sigma^2} x} \int_d^\infty \frac{1}{2\sqrt{2\pi t^3}} e^{-\frac{\mu^2}{2\sigma^2} t} dt}{\sqrt{\frac{\mu^2}{\sigma^2} + 2\beta} + \int_d^\infty \frac{1}{2\sqrt{2\pi t^3}} e^{-\beta t} e^{-\frac{\mu^2}{2\sigma^2} t} dt}. \qquad (30)$$

Calculating the integrals in (30) gives

$$E\left(e^{-\beta \tau_d^{W^\mu}} \mid W_0^\mu = x, x > 0\right)$$

$$= \frac{e^{-\beta d} e^{-\frac{\sqrt{\mu^2 + 2\beta\sigma^2} + \mu}{\sigma^2} x} \left\{ \frac{1}{\sqrt{2\pi d}} e^{-\frac{\mu^2}{2\sigma^2} d} - \frac{\mu}{\sigma} \mathcal{N}\left(-\frac{\mu}{\sigma}\sqrt{d}\right) \right\}}{\sqrt{\frac{\mu^2}{\sigma^2} + 2\beta} + \frac{1}{\sqrt{2\pi d}} e^{-\left(\beta + \frac{\mu^2}{2\sigma^2}\right) d} - \sqrt{2\beta + \frac{\mu^2}{\sigma^2}} \mathcal{N}\left(-\sqrt{\left(2\beta + \frac{\mu^2}{\sigma^2}\right) d}\right)} \qquad (31)$$

The same result with $x = 0$ and $\sigma = 1$ has been obtained in [4], [5] and [23] using different approaches. It is an important result for pricing the Parisian options.

Letting $\beta = 0$ in (30) and (31), we have the Parisian type ruin probability for a Brownian motion with positive drift,

$$P\left(\tau_d^{W^\mu} < \infty \mid W_0^\mu = x, x > 0\right) = \frac{e^{-\frac{2\mu}{\sigma^2} x} \int_d^\infty \frac{1}{2\sqrt{2\pi t^3}} e^{-\frac{\mu^2}{2\sigma^2} t} dt}{\frac{\mu}{\sigma} + \int_d^\infty \frac{1}{2\sqrt{2\pi t^3}} e^{-\frac{\mu^2}{2\sigma^2} t} dt} \qquad (32)$$

$$= \frac{e^{-\frac{2\mu}{\sigma^2} x} \left\{ \frac{1}{\sqrt{2\pi d}} e^{-\frac{\mu^2}{2\sigma^2} d} - \frac{\mu}{\sigma} \mathcal{N}\left(-\frac{\mu}{\sigma}\sqrt{d}\right) \right\}}{\frac{1}{\sqrt{2\pi d}} e^{-\frac{\mu^2}{2\sigma^2} d} + \frac{\mu}{\sigma} \mathcal{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right)} \qquad (33)$$

The same result with $x = 0$ and $\sigma = 1$ can also be found in [5] where the idea of Parisian ruin probability was first introduced.

Remark: It is tempting to derive the Parisian ruin probability by taking the limit as $\alpha \to \infty$ in (22). However, the argument used to get (27) does not generalise in the case of an infinite horizon so we can not argue directly from (22). See [1] pp 196,199, [2] pp 119, [19], [20] and [21] pp 165-166 for more details. A simple way to proceed is via (30) or (31) as we did.

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