Angelos Dassios and Shanle Wu

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Working paper

Barrier Strategies with Parisian Delay

Angelos Dassios†, Shanle Wu*

†*Department of Statistics, London School of Economics
Houghton Street, London WC2A 2AE
†a.dassios@lse.ac.uk
*s.wu3@lse.ac.uk

Abstract

In this paper, we apply the single barrier strategy to optimize the dividend payment in the situation where there is a time lag $d > 0$ between decision and implementation. Using a Brownian motion with drift as the surplus process, we obtain the optimal barrier b^* which maximises the expected present value of dividends. We also show that the longer the implementation delay, the smaller the optimal barrier will be.

Keywords: Parisian implementation delay, single barrier strategy, surplus process, Brownian motion with drift.

1 Introduction

The dividends problem was first put forward by De Finetti [12]. He considered a discrete-time model and showed that in order to maximize the expectation of the discounted dividends paid to the shareholders of a company, the optimal strategy must be a barrier strategy and the level of the barrier can be determined.

As the continuous counterpart of De Finetti's model, we consider a company with initial surplus $x > 0$. If no dividends are paid, the surplus at time t is

$$S_t = x + \mu t + \sigma W_t,\ t \geq 0, \tag{1}$$

with $\mu > 0$, $\sigma > 0$ and W being a standard Brownian motion starting from 0. Denote the aggregate dividends paid by time t by D_t. The modified surplus at time t is $X_t - D_t$. Without the Parisian implementation delay, whenever the modified surplus reaches the level of the barrier, the "overflow" will be paid as dividends. Let r be the force of interest. Gerber and Shiu [14] have obtained the optimal barrier b^* which maximizes the expected present value of all dividends until ruin, i.e.

$$E\left(\int_0^T e^{-rt}\mathrm{d}D_t\right),$$

where

$$T = \inf\{t \geq 0 \mid X_t - D_t = 0\}$$

is the time of ruin. Another reference for this problem is [16]. The Brownian motion with drift is considered as an approximation of the surplus process. More results on relevant problems can be found in [1], [4] page 168-174, [6], [15], [18], [19], [21] and [22].

In this paper, we introduce the Parisian implementation delay to the dividend paying process. We assume that there is a time lag $d > 0$ between the decision and implementation. During this period, if the modified surplus keeps staying above the barrier, a dividend of size equal to the overflow above the barrier will be paid at the end of the period; otherwise, no dividend will be paid. In this sense, the decision to pay a dividend is reversible. This is motivated by a similar problem solved in [7] where the authors study investment and disinvestment decisions in situations where there is a time lag from the time when the decision is taken to the time when the decision is implemented. Such problems have not been studied very extensively. In addition to [7], there is a similar idea in [13] and also in [2] but there the decision is not reversible. We only consider the case when the initial surplus x is less then the barrier b as is also the case in [14]. A special feature of this constrained strategy is that dividends are not paid continuously, but they are paid as a series of discrete payments of size equal to the amount by which the surplus is above the barrier after the delay.

The Parisian criterion originates from the Parisian options, the prices of which depend on the excursions of the underlying asset prices above or below a barrier. An example is a *Parisian down-and-out option*, the owner of which loses the option if the underlying asset price S reaches the level l and remains constantly below this level for a time interval longer than d . For details and extensions, see [5], [8], [9], [10], [11], [17] and [20].

In Section 2 we give the mathematical definitions and set out the model. In Section 3 we calculate some expectations which will be use in Section 4 to calculate the expected present value of dividend payment. In Section 4 the optimal barrier b^* is obtained. We also discuss the relationship between the optimal barrier and the length of delay.

2 Definitions

In order to introduce the Parisian implementation delay mathematically, we will first define the excursion. Set

$$g_{b,t}^S = \sup\{s \leq t \mid S_s = b\}, \qquad d_{b,t}^S = \inf\{s \geq t \mid S_s = b\} \tag{2}$$

with the usual convention, $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$. The trajectory between $g_{b,t}^S$ and $d_{b,t}^S$ is the excursion of process S either above or below b, which straddles time t. Assuming $d > 0$, we now define

$$\tau_0^S = \inf\{t \geq 0 \mid S_t = b\}, \tag{3}$$

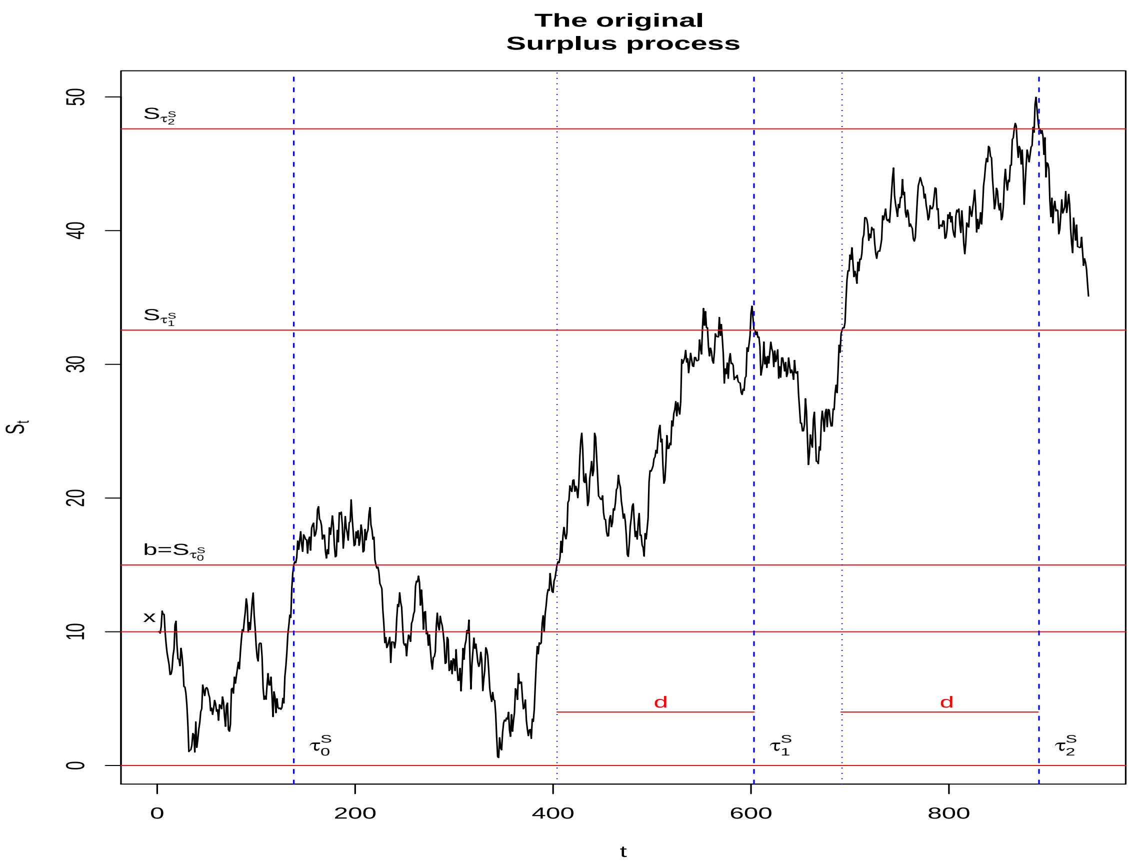

Figure 1: A Sample Path of the original surplus process S

the first time S hits barrier b and a series of stopping times

$$\tau_i^S = \inf\left\{ t > \tau_{i-1}^S \;\middle|\; \mathbf{1}_{\left\{S_t > S_{\tau_{i-1}^S}\right\}} \left(t - g^S_{S_{\tau_{i-1}^S}, t} \right) \geq d \right\}, \; i = 1, 2, \cdots . \tag{4}$$

τ_i^S is the first time after τ_{i-1}^S that the length of excursion above $S_{\tau_{i-1}^S}$ reaches d (see Figure 1).

The modified surplus process is

$$Y_t = S_t \mathbf{1}_{\{0 \leq t \leq \tau_0^S\}} + \sum_{i=0}^{\infty} \left(S_t - S_{\tau_i^S} + b \right) \mathbf{1}_{\{\tau_i^S \leq t \leq \tau_{i+1}^S\}} \tag{5}$$

(see Figure 2).

Define the first time of ruin for Y to be

$$T = \inf\left\{ t \geq 0 \mid Y_t \leq 0 \right\}. \tag{6}$$

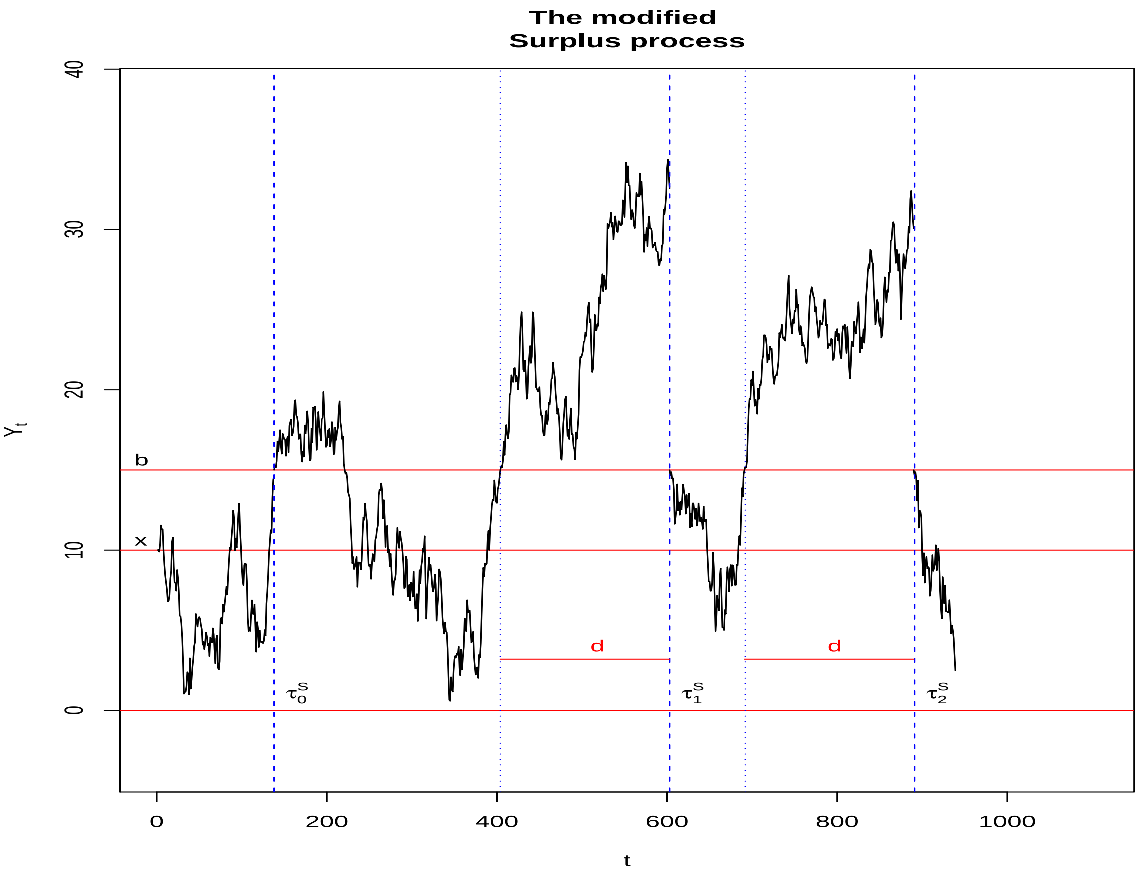

Figure 2: A Sample Path of the modified surplus process Y

As is the case with barrier strategy, ruin is certain for the modified process, i.e.

$$P(T < \infty) = 1.$$

For any $t < T$, we have the aggregate dividends paid by time t

$$\begin{aligned} Z_t &= X_t - Y_t \\ &= 0\mathbf{1}_{\{0 \le t < \tau_1^S\}} + \sum_{i=1}^{\infty} \left(S_{\tau_i^S} - b\right) \mathbf{1}_{\{\tau_i^S \le t \le \tau_{i+1}^S\}} \\ &= \sum_{i=1}^{n} \left(S_{\tau_i^S} - S_{\tau_{i-1}^S}\right) \mathbf{1}_{\{t \le \tau_1^S\}}, \end{aligned}$$

where n is the unique integer which satisfies $\tau_n^S \le t < \tau_{n+1}^S$. We are interested in the present value of the total dividend payment before ruin of Y defined by

$$V(x,b) = \sum_{i=1}^{\infty} e^{-r\tau_i^S} \left(S_{\tau_i^S} - S_{\tau_{i-1}^S}\right) \mathbf{1}_{\left\{\inf_{0 \le t \le \tau_i^S} \{Y_t\} > 0\right\}}. \tag{7}$$

We would like to maximize its expectation $E(V(x,b))$.

3 Some results for τ_i^S and $S_{\tau_i^S}$

In this section we aim to calculate

$$E\left(e^{-r\tau_0^S} \mathbf{1}_{\left\{\inf_{0 \le t \le \tau_0^S} \{Y_t\} > 0\right\}}\right),$$

$$E\left(e^{-r\left(\tau_1^S - \tau_0^S\right)} \mathbf{1}_{\left\{\inf_{\tau_0^S \le t \le \tau_1^S} \{Y_t\} > 0\right\}}\right),$$

and

$$E\left(e^{-r\left(\tau_1^S - \tau_0^S\right)} \left(S_{\tau_1^S} - S_{\tau_0^S}\right) \mathbf{1}_{\left\{\inf_{\tau_0^S \le t \le \tau_1^S} \{Y_t\} > 0\right\}}\right),$$

which will be used to obtain the optimal barrier in next section.

Now set

$$W_t^{\mu} = \mu t + \sigma W_t,$$

with $\mu > 0$, $\sigma > 0$ and W being a standard Brownian motion starting from 0. Define

$$\tau_0^* = \inf\left\{t \ge 0 \mid W_t^{\mu} = b - x\right\},$$

$$\tau_d^{W^{\mu}} = \inf\left\{t > 0 \mid \mathbf{1}_{\{W_t^{\mu} > 0\}} \left(t - g_{0,t}^{W^{\mu}}\right) \ge d\right\}.$$

According to the definitions in (3) and (4), $\tau_1^S - \tau_0^S$ is the first time the process, started from b, reaches an excursion above b with length d and therefore has the same law as $\tau_d^{W^{\mu}}$. Together with the fact that S is translation invariant,

$S_{\tau_1^S} - S_{\tau_0^S}$ has the same law as $W^{\mu}_{\tau_d^{W^\mu}}$. Furthermore, the event $\{Y_t > 0\}$ for $0 \le t \le \tau_0^S$ is equivalent to the event $\{W_t^\mu > -x\}$; and the event $\{Y_t > 0\}$ for $\tau_0^S \le t \le \tau_1^S$ is equivalent to the event $\left\{W^\mu_{t-\tau_0^S} > -b\right\}$. Consequently, we have that

$$E\left(e^{-r\tau_0^S}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_0^S}\{Y_t\}>0\right\}}\right) = E\left(e^{-r\tau_0^*}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_0^*}\{W_t^\mu\}>-x\right\}}\right), \quad (8)$$

$$E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le \tau_1^S}\{Y_t\}>0\right\}}\right) = E\left(e^{-r\tau_d^{W^\mu}}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_d^{W^\mu}}\{W_t^\mu\}>-b\right\}}\right), \quad (9)$$

$$E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\left(S_{\tau_1^S}-S_{\tau_0^S}\right)\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le \tau_1^S}\{Y_t\}>-b\right\}}\right) = E\left(e^{-r\tau_d^{W^\mu}}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_d^{W^\mu}}\{W_t^\mu\}>0\right\}}W^\mu_{\tau_d^{W^\mu}}\right). \quad (10)$$

The expectation presented by (8) is a well known result for the first exit time of Brownian motions with drift (see [3]):

$$\begin{aligned} E\left(e^{-r\tau_0^S}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_0^S}\{Y_t\}>0\right\}}\right) &= E\left(e^{-r\tau_0^*}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_0^*}\{W_t^\mu\}>-x\right\}}\right) \\ &= \exp\left\{\frac{\mu(b-x)}{\sigma^2}\right\}\frac{\sinh\left(\frac{x}{\sigma}\sqrt{2r+\frac{\mu^2}{\sigma^2}}\right)}{\sinh\left(\frac{b}{\sigma}\sqrt{2r+\frac{\mu^2}{\sigma^2}}\right)}. \end{aligned} \quad (11)$$

For (9) and (10), we need to use the same technique as that in [8], [9] and [10]. First of all, in order to avoid the problems caused by the peculiar properties of Brownian motions sample paths, we introduce the perturbed Brownian motion $X^{(\epsilon)}$, where $\epsilon > 0$ as follows. Define a sequence of stopping times

$$\begin{aligned} \delta_0 &= 0, \\ \sigma_n &= \inf\left\{t > \delta_n \,|\, W_t^\mu = -\epsilon\right\}, \\ \delta_{n+1} &= \inf\left\{t > \sigma_n \,|\, W_t^\mu = 0\right\}, \end{aligned}$$

where $n = 0, 1, \cdots$. Now define

$$X_t^{(\epsilon)} = \begin{cases} W_t^\mu + \epsilon, & \text{if} \quad \delta_n \le t < \sigma_n \\ W_t^\mu, & \text{if} \quad \sigma_n \le t < \delta_{n+1} \end{cases}, \text{ (see Figure 3).}$$

By introducing the jumps to the original Brownian motion, we get this new process $X^{(\epsilon)}$ which has a very clear structure of excursions above and below 0, i.e. the excursions above and below 0 alternate with the length of each excursion greater than 0. In the Appendix we prove that the expectation of the variables defined based on $X^{(\epsilon)}$ converge to those based on W^μ as ϵ goes to 0. As a result,

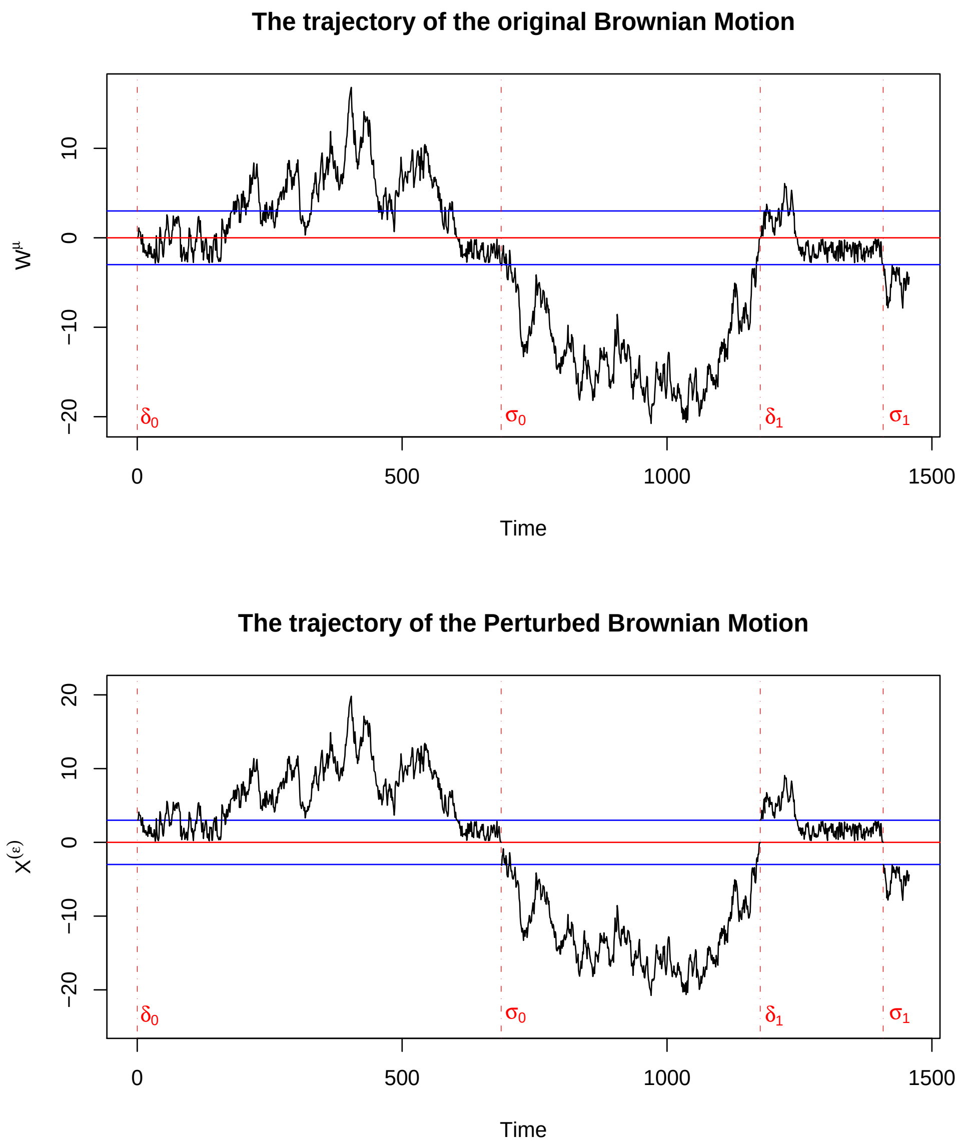

Figure 3: A Sample Path of the Perturbed Brownian motion $X^{(\epsilon)}$

we can obtain the results for W^μ by carrying out the calculations for $X^{(\epsilon)}$ and taking the limit $\epsilon \to 0$. Hence we will focus on studying the excursions of $X^{(\epsilon)}$.

For $X^{(\epsilon)}$, similarly, we can define

$$g_{0,t}^X = \sup\left\{s \le t \mid X_s^{(\epsilon)} = 0\right\}, \quad d_{0,t}^X = \inf\left\{s \ge t \mid X_s^{(\epsilon)} = 0\right\}, \tag{12}$$

$$\tau_d^X = \inf\left\{t > 0 \mid \mathbf{1}_{\left\{X^{(\epsilon)}>0\right\}}\left(t - g_{0,t}^X\right) \ge d\right\}. \tag{13}$$

Furthermore, we set U_k^X, $k = 1, 2, \cdots$ to be the length of the kth excursion of $X^{(\epsilon)}$ above 0 and V_k^X, $k = 1, 2, \cdots$ to be the length of the kth excursion of $X^{(\epsilon)}$ below 0 before $X^{(\epsilon)}$ ever falls below $-b$. Notice that U_k^X $k = 1, 2, \cdots$ are i.i.d, so are V_k^X $k = 1, 2, \cdots$ and U_k^X and V_k^X are independent. We therefore define the densities for U_k^X and V_k^X $k = 1, 2, \cdots$:

$$p_1(t) = \lim_{\Delta t \to 0} \frac{P\left(t < U_k^X < t + \Delta t\right)}{\Delta t}, \quad p_2(t) = \lim_{\Delta t \to 0} \frac{P\left(t < V_k^X < t + \Delta t\right)}{\Delta t};$$

$$P_1(t) = P\left(U_k^X < t\right), \quad P_2(t) = P\left(V_k^X < t\right);$$

$$\bar{P}_1(t) = P\left(U_k^X > t\right), \quad \bar{P}_2(t) = P\left(V_k^X > t\right).$$

We have

$$P_i(t) = \int_0^t p_i(s)\mathrm{d}s = 1 - \bar{P}_i(t),$$

which is actually the probability that the process will stay above (or below) 0 for no more than time t. More precisely, according to the definition of $X^{(\epsilon)}$, we actually have:

$$p_1(s) = \frac{\epsilon}{\sigma\sqrt{2\pi s^3}} \exp\left\{-\frac{(\epsilon + \mu s)^2}{2\sigma^2 s}\right\}, \tag{14}$$

$$p_2(s) = \exp\left\{\frac{\mu\epsilon}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right\} ss_t\left(\frac{b-\epsilon}{\sigma}, \frac{b}{\sigma}\right), \tag{15}$$

where

$$ss_t(x, y) = \sum_{k=-\infty}^{\infty} \frac{(2k+1)y - x}{\sqrt{2\pi t^3}} \exp\left\{-\frac{((2k+1)y - x)^2}{2t}\right\}.$$

In fact, $p_1(s)$ is the density of the first time W^μ which starts from ϵ hits 0; and $p_2(s)$ is the density of the first time W^μ which starts from $-\epsilon$ exit the corridor $(-b, 0)$ from 0.

Now in order to calculate (9) we first need to calculate

$$E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_d^X}\left\{X_t^{(\epsilon)}\right\} > -b\right\}}\right).$$

Let A_i denote the event that the first time the length of the excursion above zero reaches d happens during the ith excursion above zero. Since excursions above and below alternate, given event A_i, before the ruin of Y τ_d^X is comprised of $i-1$ full excursions below zero, none of which crosses level $-b$ and $i-1$ full excursions above zero with the length less than d and the last one with the length d, i.e.

$$\tau_d^X \mathbf{1}_{\left\{\inf_{0\le t\le \tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}}\Bigg| A_i = \sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)+d\Bigg| U_k^X<d, k=1,2,\cdots i-1, U_i^X\ge d. \tag{16}$$

We have therefore

$$\begin{aligned}
& E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}}\right)\\
= & \sum_{i=1}^{\infty} E\left(\exp\left\{-r\sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)-rd\right\}\Bigg| U_k^X<d, k=1,2,\cdots i-1, U_i^X\ge d\right)P\left(A_i\right)
\end{aligned}$$

Since U_k^X, $k=1,2,\cdots$ are i.i.d and so are V_k^X, $k=1,2,\cdots$ and U_k^X and V_k^X are independent, we have that

$$\begin{aligned}
& E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}}\right)\\
= & e^{-rd}\sum_{i=1}^{\infty} E\left(e^{-rU_1^X}\Big| U_1^X<d\right)^{i-1} E\left(e^{-rV_1^X}\right)^{i-1} P\left(U_1^X<d\right)^{i-1} P\left(U_i^X\ge d\right)\\
= & e^{-rd}\sum_{i=1}^{\infty}\left(\int_0^d e^{-rs}\frac{p_1(s)}{P\left(U_1^X<d\right)}\mathrm{d}s\right)^{i-1}\left(\int_0^\infty e^{-rs}p_2(s)\mathrm{d}s\right)^{i-1} P\left(U_1^X<d\right)^{i-1} P\left(U_i^X\ge d\right)\\
= & \frac{e^{-rd}P\left(U_i^X\ge d\right)}{1-\int_0^d e^{-rs}p_1(s)\mathrm{d}s\int_0^\infty e^{-rs}p_2(s)\mathrm{d}s}.
\end{aligned}$$

We can then calculate

$$P\left(U_i^X\ge d\right)=\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d}+\frac{\epsilon}{\sigma\sqrt{d}}\right)-e^{-2\frac{\mu\epsilon}{\sigma^2}}\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d}-\frac{\epsilon}{\sigma\sqrt{d}}\right),$$

$$\begin{aligned}
\int_0^d e^{-rs}p_1(s)\mathrm{d}s = & \exp\left\{-\frac{\left(\mu+\sqrt{2r\sigma^2+\mu^2}\right)\epsilon}{\sigma^2}\right\}\mathscr{N}\left(\sqrt{\left(2r+\frac{\mu^2}{\sigma^2}\right)d}-\frac{\epsilon}{\sigma\sqrt{d}}\right)\\
& +\exp\left\{-\frac{\left(\mu-\sqrt{2r\sigma^2+\mu^2}\right)\epsilon}{\sigma^2}\right\}\mathscr{N}\left(-\sqrt{\left(2r+\frac{\mu^2}{\sigma^2}\right)d}-\frac{\epsilon}{\sigma\sqrt{d}}\right),
\end{aligned}$$

$$\int_0^\infty e^{-rs} p_2(s)\mathrm{d}s = \exp\left\{\frac{\mu\epsilon}{\sigma^2}\right\} \frac{\sinh\left(\frac{(b-\epsilon)}{\sigma}\sqrt{2r+\frac{\mu^2}{\sigma^2}}\right)}{\sinh\left(\frac{b}{\sigma}\sqrt{2r+\frac{\mu^2}{\sigma^2}}\right)}.$$

By taking the limit $\epsilon \to 0$ we have that

$$\begin{aligned}
& E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\right) \qquad (17)\\
= \; & E\left(e^{-r\tau_d^{W^\mu}}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^{W^\mu}}\{W_t^\mu\}>-b\right\}}\right)\\
= \; & \frac{e^{-rd}\left[2\frac{\mu}{\sigma}\mathcal{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right)+\sqrt{\frac{2}{\pi d}}\exp\left(-\frac{\mu^2 d}{2\sigma^2}\right)\right]}{\frac{2\sqrt{2r+\frac{\mu^2}{\sigma^2}}}{\exp\left(2\frac{b}{\sigma}\sqrt{2r+\frac{\mu^2}{\sigma^2}}\right)-1}+2\sqrt{2r+\frac{\mu^2}{\sigma^2}}\mathcal{N}\left(\sqrt{\left(2r+\frac{\mu^2}{\sigma^2}\right)d}\right)+\sqrt{\frac{2}{\pi d}}\exp\left\{-\frac{(2r\sigma^2+\mu^2)d}{2\sigma^2}\right\}}
\end{aligned}$$

(we prove in the Appendix that the convergence is valid when taking the limit).

For (11), we calculate $E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}}X_{\tau_d^X}^{(\epsilon)}\right)$ and take the limit $\epsilon \to 0$. A_i is defined as above. According to (16), we have

$$\begin{aligned}
& E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}}X_{\tau_d^X}^{(\epsilon)}\middle|A_i\right)\\
= \; & E\left(\exp\left\{-r\sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)-rd\right\}X^{(\epsilon)}_{\sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)+d}\middle|U_k^X<d, k=1,2,\cdots i-1, U_i^X\ge d\right).
\end{aligned}$$

By definition, we know that

$$X^{(\epsilon)}_{\sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)}=\epsilon.$$

Therefore we have

$$\begin{aligned}
& E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>0\right\}}X_{\tau_d^X}^{(\epsilon)}\middle|A_i\right)\\
= \; & E\left(\exp\left\{-r\sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)-rd\right\}X^{(\epsilon)}_{\sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)+d}\middle|X^{(\epsilon)}_{\sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)}=\epsilon,\right.\\
& \left.U_k^X<d, k=1,2,\cdots i-1, U_i^X\ge d\right).
\end{aligned}$$

Applying the strong Markov property and transition invariance of $X^{(\epsilon)}$ gives

$$\begin{aligned}
& E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>0\right\}}X_{\tau_d^X}^{(\epsilon)}\middle|A_i\right) \\
= & E\left(\exp\left\{-r\sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)-rd\right\}X_d^{(\epsilon)}\middle|U_k^X<d,k=1,2,\cdots i-1,U_i^X\ge d\right) \\
= & E\left(\exp\left\{-r\sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)-rd\right\}\middle|U_k^X<d,k=1,2,\cdots i-1\right)E\left(X_d^{(\epsilon)}\middle|U_i^X\ge d\right) \\
= & E\left(\exp\left\{-r\sum_{k=1}^{i-1}\left(U_k^X+V_k^X\right)-rd\right\}\middle|U_k^X<d,k=1,2,\cdots i-1\right)E\left(X_d^{(\epsilon)}\middle|U_1^X\ge d\right) \\
= & E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>0\right\}}\middle|A_i\right)E\left(X_d^{(\epsilon)}\middle|U_1^X\ge d\right).
\end{aligned}$$

And therefore

$$\begin{aligned}
& E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>0\right\}}X_{\tau_d^X}^{(\epsilon)}\right) \\
= & \sum_{i=1}^{\infty}E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>0\right\}}\middle|A_i\right)E\left(X_d^{(\epsilon)}\middle|U_1^X\ge d\right)P\left(A_i\right) \\
= & E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>0\right\}}\right)E\left(X_d^{(\epsilon)}\middle|U_1^X\ge d\right).
\end{aligned}$$

$E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>0\right\}}\right)$ has been obtained above. We will now focus on $E\left(X_d^{(\epsilon)}\middle|U_1^X\ge d\right)$. First of all according to the definition we have

$$U_1^X=\inf\left\{t\ge 0 \;\middle|\; X^{(\epsilon)}\le 0\right\}.$$

For $X^{(\epsilon)}$ we have that

$$X_{U_1^X\wedge d}^{(\epsilon)}-\mu\left(U_1^X\wedge d\right)$$

is a martingale. $U_1^X\wedge d$ is a bounded stopping time. Hence

$$\begin{aligned}
\epsilon &= E\left(X_{U_1^X\wedge d}^{(\epsilon)}-\mu\left(U_1^X\wedge d\right)\right) \\
&= E\left(X_{U_1^X\wedge d}^{(\epsilon)}\right)-\mu E\left(U_1^X\wedge d\right) \\
&= E\left(X_{U_1^X}^{(\epsilon)}\;\middle|\;U_1^X<d\right)P\left(U_1^X<d\right)+E\left(X_d^{(\epsilon)}\;\middle|\;U_1^X>d\right)P\left(U_1^X>d\right)-\mu E\left(U_1^X\wedge d\right) \\
&= E\left(X_d^{(\epsilon)}\;\middle|\;U_1^X>d\right)P\left(U_1^X>d\right)-\mu E\left(U_1^X\wedge d\right).
\end{aligned}$$

As a result,

$$E\left(X_d^{(\epsilon)}\,\Big|\,U_1^X > d\right) = \frac{\epsilon + \mu E\left(U_1^X \wedge d\right)}{P\left(U_1^X > d\right)},$$

where

$$E\left(U_1^X \wedge d\right) = E\left(U_1^X \mathbf{1}_{\{U_1^X < d\}}\right) + dP\left(U_1^X > d\right),$$

$$P\left(U_1^X > d\right) = 1 - \int_0^d p_1(t)\mathrm{d}t = \mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d} + \frac{\epsilon}{\sigma\sqrt{d}}\right) - e^{-2\frac{\mu\epsilon}{\sigma^2}}\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d} - \frac{\epsilon}{\sigma\sqrt{d}}\right),$$

$$E\left(U_1^X \mathbf{1}_{\{U_1^X < d\}}\right) = \int_0^d t p_1(t)\mathrm{d}t = \frac{\epsilon}{\mu}\left[e^{-\frac{2\epsilon\mu}{\sigma^2}}\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d} - \frac{\epsilon}{\sigma\sqrt{d}}\right) - \mathscr{N}\left(-\frac{\mu}{\sigma}\sqrt{d} - \frac{\epsilon}{\sigma\sqrt{d}}\right)\right].$$

Therefore

$$E\left(X_d^{(\epsilon)}\,\Big|\,U_1^X > d\right) = \frac{\epsilon\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d} + \frac{\epsilon}{\sigma\sqrt{d}}\right) + \epsilon e^{-\frac{2\epsilon\mu}{\sigma^2}}\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d} - \frac{\epsilon}{\sigma\sqrt{d}}\right)}{\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d} + \frac{\epsilon}{\sigma\sqrt{d}}\right) - e^{-\frac{2\epsilon\mu}{\sigma^2}}\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d} - \frac{\epsilon}{\sigma\sqrt{d}}\right)} + \mu d. \tag{18}$$

We have therefore obtained

$$E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_d^X}\left\{X_t^{(\epsilon)}\right\}>0\right\}} X_{\tau_d^X}^{(\epsilon)}\right).$$

Taking the limit $\epsilon \to 0$ gives

$$\begin{aligned} & E\left(e^{-r\left(\tau_1^S - \tau_0^S\right)}\left(S_{\tau_1^S} - S_{\tau_0^S}\right)\mathbf{1}_{\left\{\inf_{\tau_0^S \le t\le \tau_1^S}\{Y_t\}>-b\right\}}\right) \qquad (19)\\ = \; & E\left(e^{-r\tau_d^{W^\mu}}\mathbf{1}_{\left\{\inf_{0\le t\le \tau_d^{W^\mu}}\{W_t^\mu\}>0\right\}} W^\mu_{\tau_d^{W^\mu}}\right) \\ = \; & \frac{e^{-rd}\left\{\frac{\mu d}{\sigma}\left[2\frac{\mu}{\sigma}\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right) + \sqrt{\frac{2}{\pi d}}\exp\left(-\frac{\mu^2 d}{2\sigma^2}\right)\right] + 2\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right)\right\}}{\frac{2\sqrt{2r+\frac{\mu^2}{\sigma^2}}}{\exp\left(2\frac{b}{\sigma}\sqrt{2r+\frac{\mu^2}{\sigma^2}}\right)-1} + 2\sqrt{2r+\frac{\mu^2}{\sigma^2}}\mathscr{N}\left(\sqrt{\left(2r+\frac{\mu^2}{\sigma^2}\right)d}\right) + \sqrt{\frac{2}{\pi d}}\exp\left\{-\frac{(2r\sigma^2+\mu^2)d}{2\sigma^2}\right\}}. \end{aligned}$$

Notice that here we assume $\mu > 0$ as this is the usual assumption in practice. The results for $\mu < 0$ can also be calculated using the same method.

4 The optimal barrier

In this section, we show that there exists an unique barrier b^* which maximizes the expectation as long as $x < b^*$. Set $V(b)$ to be the discounted value of the total dividends payment at the first time S hits barrier b. We can then express $E(V(x,b))$ in terms of $E(V(b))$ as follows:

$$\begin{aligned} & E(V(x,b)) \\ = \; & E\left(e^{-r\tau_0^S}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_0^S}\{Y_t\}>0\right\}}\sum_{i=1}^{\infty}e^{-r\left(\tau_i^S-\tau_0^S\right)}\left(S_{\tau_i^S}-S_{\tau_{i-1}^S}\right)\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_i^S}\{Y_t\}>0\right\}}\right). \end{aligned}$$

By the strong Markov property of S, we have that

$$\begin{aligned} & E(V(x,b)) \\ = \; & E\left(e^{-r\tau_0^S}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_0^S}\{Y_t\}>0\right\}}\right)E\left(\sum_{i=1}^{\infty}e^{-r\left(\tau_i^S-\tau_0^S\right)}\left(S_{\tau_i^S}-S_{\tau_{i-1}^S}\right)\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_i^S}\{Y_t\}>0\right\}}\right) \\ = \; & E\left(e^{-r\tau_0^S}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_0^S}\{Y_t\}>0\right\}}\right)E(V(b)). \end{aligned}$$

We have obtained $E\left(e^{-r\tau_0^S}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_0^S}\{Y_t\}>0\right\}}\right)$ in (11). For $E(V(b))$ we have

$$\begin{aligned} & E(V(b)) \\ = \; & E\left(\sum_{i=1}^{\infty}e^{-r\left(\tau_i^S-\tau_0^S\right)}\left(S_{\tau_i^S}-S_{\tau_{i-1}^S}\right)\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_i^S}\{Y_t\}>0\right\}}\right) \\ = \; & E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\left(S_{\tau_1^S}-S_{\tau_0^S}\right)\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\right) \\ & +E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\sum_{i=2}^{\infty}e^{-r\left(\tau_i^S-\tau_1^S\right)}\left(S_{\tau_i^S}-S_{\tau_{i-1}^S}\right)\mathbf{1}_{\left\{\inf_{\tau_1^S\le t\le\tau_i^S}\{Y_t\}>0\right\}}\right). \end{aligned}$$

Applying the strong Markov property again, we have that

$$\begin{aligned} & E(V(b)) \\ = \; & E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\left(S_{\tau_1^S}-S_{\tau_0^S}\right)\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\right) \\ & +E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\right)E\left(\sum_{i=2}^{\infty}e^{-r\left(\tau_i^S-\tau_1^S\right)}\left(S_{\tau_i^S}-S_{\tau_{i-1}^S}\right)\mathbf{1}_{\left\{\inf_{\tau_1^S\le t\le\tau_i^S}\{Y_t\}>0\right\}}\right). \end{aligned}$$

Since S is translation invariant, it follows that

$$\begin{aligned} E(V(b)) \; = \; & E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\left(S_{\tau_1^S}-S_{\tau_0^S}\right)\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\right) \\ & +E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\right)E(V(b)), \end{aligned}$$

and therefore

$$E\left(V(b)\right)=\frac{E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\left(S_{\tau_1^S}-S_{\tau_0^S}\right)\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\right)}{1-E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\right)}.$$

As a result, we have

$$E\left(V(x,b)\right)=E\left(e^{-r\tau_0^S}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_0^S}\{Y_t\}>0\right\}}\right)\frac{E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\left(S_{\tau_1^S}-S_{\tau_0^S}\right)\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\right)}{1-E\left(e^{-r\left(\tau_1^S-\tau_0^S\right)}\mathbf{1}_{\left\{\inf_{\tau_0^S\le t\le\tau_1^S}\{Y_t\}>0\right\}}\right)}. \tag{20}$$

Substituting (11), (17) and (19) into (20) gives

$$E\left(V(x,b)\right)=\frac{C_1\exp\left\{\frac{\mu}{\sigma^2}b\right\}}{C_2\exp(\alpha b)+C_3\exp(-\alpha b)}, \tag{21}$$

where

$$\begin{aligned}
\alpha &= \frac{1}{\sigma}\sqrt{2r+\frac{\mu^2}{\sigma^2}},\\
C_1 &= 2\exp\left\{-rd-\frac{\mu x}{\sigma^2}\right\}\sinh(\alpha x)\left\{\frac{\mu d}{\sigma}\left[2\frac{\mu}{\sigma}\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right)+\sqrt{\frac{2}{\pi d}}\exp\left(-\frac{\mu^2 d}{2\sigma^2}\right)\right]+2\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right)\right\},\\
C_2 &= 2\alpha\sigma\mathscr{N}\left(\alpha\sigma\sqrt{d}\right)-2\frac{\mu}{\sigma}e^{-rd}\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right),\\
C_3 &= 2\alpha\sigma\mathscr{N}\left(-\alpha\sigma\sqrt{d}\right)+2\frac{\mu}{\sigma}e^{-rd}\mathscr{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right).
\end{aligned}$$

In order to maximize this expectation, we need to solve

$$\frac{\mathrm{d}}{\mathrm{d}b}E\left(V(x,b)\right)=0,$$

which gives

$$b^*=\frac{1}{2\alpha}\ln\frac{\left(\alpha\sigma^2+\mu\right)C_3}{\left(\alpha\sigma^2-\mu\right)C_2}. \tag{22}$$

Furthermore at b^*

$$\frac{\mathrm{d}^2}{\mathrm{d}b^2}E\left(V(x,b)\right)<0.$$

Therefore $E\left(V(x,b)\right)$ is maximized at b^*.

Setting $d=0$ gives the result for the special case when there is no implementation delay

$$b^*=\frac{\sigma}{\sqrt{2r+\frac{\mu^2}{\sigma^2}}}\ln\frac{\sqrt{2r\sigma+\mu^2}+\mu\sigma}{\sqrt{2r\sigma+\mu^2}-\mu\sigma}. \tag{23}$$

This is the result obtained in [14].

For the case with implementation delay, b^* is a function of d and we can actually calculate that

$$\frac{\mathrm{d}b^*}{\mathrm{d}d} = -4\alpha\sigma r e^{-rd}\left\{\frac{1}{\sqrt{2\pi d}}e^{-\frac{\mu^2 d}{2\sigma^2}} + \frac{\mu}{\sigma}\mathcal{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right)\right\} < 0,$$

i.e. b^* is a decreasing function of d. Furthermore, when $d \to \infty$, $b^* \to -\infty$. We therefore need to put a constrain on d since the barrier have to be greater than 0. Solving

$$b^* > 0$$

gives

$$d < \bar{d}, \tag{24}$$

where $\bar{d}$ is the unique solution of

$$\alpha\sigma^2 + \mu - 2\alpha\sigma^2\mathcal{N}\left(\alpha\sigma\sqrt{d}\right) + \mu e^{-rd}\mathcal{N}\left(\frac{\mu}{\sigma}\sqrt{d}\right) = 0.$$

5 Appendix

We show in this section that we can take limits $\epsilon \to 0$ as we did earlier. First of all, we consider two processes W^μ and $\underline{W}^\mu = W^\mu - \epsilon$. According to the definitions, $X^{(\epsilon)}$ satisfies

$$\lim_{\epsilon \to 0} X_t^{(\epsilon)} = W_t^\mu,\ a.s. \text{ for all } t,$$

$$\underline{W}_t^\mu \le X_t^{(\epsilon)} \le W_t^\mu \quad \text{for all } t,$$

and $g_{0,t}^X$ always lies between $g_{0,t}^{\underline{W}^\mu}$ and $g_{0,t}^{W^\mu}$. Since

$$\lim_{\epsilon \to 0} g_{0,t}^{\underline{W}^\mu} = \lim_{\epsilon \to 0} g_{\epsilon,t}^{W^\mu} = g_{0,t}^{W^\mu},$$

we have that

$$\lim_{\epsilon \to 0} g_{0,t}^X = g_{0,t}^{W^\mu},\ a.s.$$

and therefore

$$\lim_{\epsilon \to 0} \mathbf{1}_{\left\{X_t^{(\epsilon)} > 0\right\}}\left(t - g_{0,t}^X\right) = \mathbf{1}_{\{W_t^\mu > 0\}}\left(t - g_{0,t}^{W^\mu}\right)\ a.s.$$

From the definition of τ_d^S we have that

$$\begin{aligned}\left\{\tau_d^{W^\mu} < t\right\} &= \left\{\sup_{0 \le s \le t}\left\{\mathbf{1}_{\{W_s^\mu > 0\}}\left(s - g_{0,s}^{W^\mu}\right)\right\} \ge d\right\} \\ &= \lim_{\epsilon \to 0}\left\{\sup_{0 \le s \le t}\left\{\mathbf{1}_{\left\{X_s^{(\epsilon)} > 0\right\}}\left(t - g_{0,s}^X\right)\right\} \ge d\right\} = \lim_{\epsilon \to 0}\left\{\tau_d^X < t\right\}.\end{aligned}$$

Consequently,

$$\lim_{\epsilon\to 0}\tau_d^X = \tau_d^{W^\mu} \ a.s. \text{ and } \lim_{\epsilon\to 0} X^{(\epsilon)}_{\tau_d^X} = W^\mu_{\tau_d^{W^\mu}} \ a.s.$$

Therefore for any given non-negative constants r,

$$\lim_{\epsilon\to 0} e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}} = e^{-r\tau_d^{W^\mu}}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^{W^\mu}}\{W_t^\mu\}>-b\right\}} \quad a.s.$$

$$\lim_{\epsilon\to 0} e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}} X^{(\epsilon)}_{\tau_d^X} = e^{-r\tau_d^{W^\mu}}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^{W^\mu}}\{W_t^\mu\}>-b\right\}} W^\mu_{\tau_d^{W^\mu}} \quad a.s.$$

Since $\tau_d^X \ge 0$, we also have,

$$\left| e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}} \right| < 1,$$

and

$$\left| e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}} X^{(\epsilon)}_{\tau_d^X} \right| < X^{(\epsilon)}_{\tau_d^X},$$

where we have shown in (18) that

$$E\left(\left|X^{(\epsilon)}_{\tau_d^X}\right|\right) = E\left(X^{(\epsilon)}_{\tau_d^X}\right) = E\left(X_d^{(\epsilon)}\middle| U_1^X > d\right) < \infty.$$

We can then applying the Dominated Convergence Theorem which gives

$$\begin{aligned} E\left(e^{-r\tau_d^{W^\mu}}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^{W^\mu}}\{W_t^\mu\}>-b\right\}}\right) &= E\left(\lim_{\epsilon\to 0} e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}}\right) \\ &= \lim_{\epsilon\to 0} E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}}\right). \end{aligned}$$

$$\begin{aligned} E\left(e^{-r\tau_d^{W^\mu}}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^{W^\mu}}\{W_t^\mu\}>-b\right\}} W^\mu_{\tau_d^{W^\mu}}\right) &= E\left(\lim_{\epsilon\to 0} e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}} X^{(\epsilon)}_{\tau_d^X}\right) \\ &= \lim_{\epsilon\to 0} E\left(e^{-r\tau_d^X}\mathbf{1}_{\left\{\inf_{0\le t\le\tau_d^X}\left\{X_t^{(\epsilon)}\right\}>-b\right\}} X^{(\epsilon)}_{\tau_d^X}\right). \end{aligned}$$

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