Research Article

The Distribution of the Interval between Events of a Cox Process with Shot Noise Intensity

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Applying piecewise deterministic Markov processes theory, the probability generating function of a Cox process, incorporating with shot noise process as the claim intensity, is obtained. We also derive the Laplace transform of the distribution of the shot noise process at claim jump times, using stationary assumption of the shot noise process at any times. Based on this Laplace transform and from the probability generating function of a Cox process with shot noise intensity, we obtain the distribution of the interval of a Cox process with shot noise intensity for insurance claims and its moments, that is, mean and variance.

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1. Introduction

In insurance modeling, the Poisson process has been used as a claim arrival process. Extensive discussion of the Poisson process, from both applied and theoretical viewpoints, can be found in [1–6]. However there has been a significant volume of literature that questions the suitability of the Poisson process in insurance modeling [7, 8]. From a practical point of view, there is no doubt that the insurance industry needs a more suitable claim arrival process than the Poisson process that has deterministic intensity.

As an alternative point process to generate the claim arrivals, we can employ a Cox process or a doubly stochastic Poisson process [9–15]. An important book on Cox processes is the book by Bening and Korolev [16], where the applications in both insurance and finance are discussed. A Cox process provides us with the flexibility to allow the intensity not only to depend on time but also to be a stochastic process. Dassios and Jang [17] demonstrated how a Cox process with shot noise intensity could be used in the pricing of catastrophe reinsurance and derivatives.
It is important to measure the time interval between the claims in insurance. Thus in this paper, we examine the distribution of the interval of a Cox process with shot noise intensity for insurance claims. The result of this paper can be used or easily modified in computer science/telecommunications modeling, electrical engineering, and queueing theory.

We start by defining the quantity of interest; this is a doubly stochastic point process of claim arrivals. Then, we derive the probability generating function of a Cox process with shot noise intensity using piecewise deterministic Markov processes (PDMPs) theory, for which see the appendix. The piecewise deterministic Markov processes theory is a powerful mathematical tool for examining nondiffusion models. For details, we refer the reader to \[17–25\]. In Section 3, we derive the Laplace transform of the distribution of the shot noise process at claim times, using stationary assumption of the shot noise process at any times. Using this Laplace transform within the probability generating function of a Cox process with shot noise intensity, we derive the distribution between events of a Cox process with shot noise intensity. These can be insurance claims for examples. We also derive the first two moments of this distribution. Section 4 contains some concluding remarks.

2. A Cox process and the shot noise process

A Cox process (or a doubly stochastic Poisson process) can be viewed as a two-step randomisation procedure. A process \( \lambda_t \) is used to generate another process \( N_t \) by acting as its intensity. That is, \( N_t \) is a Poisson process conditional on \( \lambda_t \) which itself is a stochastic process (if \( \lambda_t \) is deterministic then \( N_t \) is a Poisson process). Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one adopted by Brémaud \[15\].

**Definition 2.1.** Let \((\Omega,F,P)\) be a probability space with information structure given by \( F = \{\mathcal{I}_t, t \in [0,T]\} \). Let \( N_t \) be a point process adapted to \( F \). Let \( \lambda_t \) be a nonnegative process adapted to \( F \) such that

\[
\int_0^t \lambda_s \, ds < \infty \text{ almost surely (no explosions).} \tag{2.1}
\]

If for all \( 0 \leq t_1 \leq t_2 \) and \( u \in \mathbb{R} \)

\[
E\{e^{iu(N_{t_2} - N_{t_1})} \mid \mathcal{I}_{t_1}\} = \exp \left\{ \left( e^{iu} - 1 \right) \int_{t_1}^{t_2} \lambda_s \, ds \right\} \tag{2.2}
\]

then \( N_t \) is called a \( \mathcal{I}_t \)-doubly stochastic Poisson process with intensity, \( \lambda_t \) where \( \mathcal{I}_t \) is the \( \sigma \)-algebra generated by \( \lambda \) up to time \( t \), that is, \( \mathcal{I}_t = \sigma\{\lambda_s; s \leq t\} \).

Equation (2.2) gives us

\[
\Pr\{N_{t_2} - N_{t_1} = k \mid \lambda_s; t_1 \leq s \leq t_2\} = \frac{\exp \left( - \int_{t_1}^{t_2} \lambda_s \, ds \right) \left( \int_{t_1}^{t_2} \lambda_s \, ds \right)^k}{k!}, \tag{2.3}
\]

\[
\Pr\{\tau_2 > t \mid \lambda_s; t_1 \leq s \leq t_2\} = \Pr\{N_{t_2} - N_{t_1} = 0 \mid \lambda_s; t_1 \leq s \leq t_2\} = \exp \left( - \int_{t_1}^{t_2} \lambda_s \, ds \right). \tag{2.4}
\]
where $\tau_k = \inf\{t > 0 : N_t = k\}$. Therefore from (2.4), we can easily find that

$$\Pr(\tau_2 \leq t) = E\left\{\lambda_t \exp\left(-\int_{t_1}^{t_2} \lambda_s ds\right)\right\}.$$  \hspace{1cm} (2.5)

If we consider the process $\Lambda_t = \int_0^t \lambda_s ds$ (the aggregated process), then from (2.3) we can also easily find that

$$E(\theta^{N_t - N_t}) = E\{e^{-(1-\theta)(\Lambda_t - \Lambda_1)}\},$$  \hspace{1cm} (2.6)

where $\theta$ is a constant between 0 and 1. Equation (2.6) suggests that the problem of finding the distribution of $N_t$, the point process, is equivalent to the problem of finding the distribution of $\Lambda_t$, the aggregated process. It means that we just have to find the probability generating function (p.g.f.) of $N_t$ to retrieve the moment generating function (m.g.f.) of $\Lambda_t$ and vice versa.

One of the processes that can be used to measure the impact of primary events is the shot noise process [26–28]. The shot noise process is particularly useful within the claim arrival process as it measures the frequency, magnitude, and time period needed to determine the effect of primary events. As time passes, the shot noise process decreases as more and more claims are settled. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of doubly stochastic Poisson process to measure the number of claims due to primary events, that is, we will use it as a claim intensity function to generate the Cox process. We will adopt the shot noise process used by Cox and Isham [26]:

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i=1}^{M_t} Y_i e^{-\delta (t-S_i)},$$  \hspace{1cm} (2.7)

where

- (i) $\lambda_0$ is initial value of $\lambda_t$;
- (ii) $\{Y_i\}_{i=1,2,...}$ is a sequence of independent and identically distributed random variables with distribution function $G(y)$ ($y > 0$), where $E(Y_i) = \mu_1$;
- (iii) $\{S_i\}_{i=1,2,...}$ is the sequence representing the event times of a Poisson process $M_t$ with constant intensity $\rho$;
- (iv) $\delta$ is rate of exponential decay.

We assume that the Poisson process $M_t$ and the sequences $\{Y_i\}_{i=1,2,...}$ are independent of each other. Figure 1 is the graph illustrating shot noise process. Figure 2 is the graph illustrating a Cox process with shot noise intensity.

The generator of the process $(\Lambda_t, \lambda_t, t)$ acting on a function $f(\Lambda, \lambda, t)$ belonging to its domain is given by

$$Af(\Lambda, \lambda, t) = \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial \Lambda} - \delta \lambda \frac{\partial f}{\partial \lambda} + \rho \left[\int_{0}^{\infty} f(\Lambda, \lambda + y, t) dG(y) - f(\Lambda, \lambda, t)\right].$$  \hspace{1cm} (2.8)
For $f(\Lambda, \lambda, t)$ to belong to the domain of the generator $A$, it is sufficient that $f(\Lambda, \lambda, t)$ is differentiable with respect to $\Lambda, \lambda, t$ for all $\Lambda, \lambda, t$ and that $|\int_0^\infty f(\cdot, \lambda, \cdot) dG(y) - f(\cdot, \lambda, \cdot)| < \infty$.

Let us find a suitable martingale in order to derive the probability generating function (p.g.f.) of $N_t$ at time $t$.

**Theorem 2.2.** Let us assume that $\Lambda_t$ and $\lambda_t$ evolve up to a fixed time $t^*$. Considering constants $k_1$ and $k_2$ are such that $k_1 \geq 0$ and $k_2 \geq -k_1 e^{-\delta t^*}$,

$$
\exp (-k_1 \delta \Lambda_t) \exp \{- (k_1 + k_2 e^{\delta t}) \lambda_t\} \exp \left[ \rho \int_0^t \{ 1 - \hat{g}(k_1 + k_2 e^{\delta s}) \} ds \right]
$$

(2.9)

is a martingale, where $\hat{g}(u) = \int_0^u e^{-uy} dG(y)$ and $t > 0$.

**Proof.** Define $W_t = \delta \Lambda_t + \lambda_t$ and $Z_t = \lambda_t e^{\delta t}$, then the generator of the process $(W_t, Z_t, t)$ acting on a function $f(w, z, t)$ is given by

$$
Af(w, z, t) = \frac{\partial f}{\partial t} + \rho \left[ \int_0^\infty f(w + y, z + ye^{\delta t}, t) dG(y) - f(w, z, t) \right].
$$

(2.10)
and \( f(w, z, t) \) has to satisfy \( A f = 0 \) for \( f(W_t, Z_t, t) \) to be a martingale. We try a solution of the form \( e^{-k_1 w} e^{-k_2 z} h(t) \), where \( h(t) \) is a differentiable function. Then we get the following equation:

\[
h'(t) - \rho [1 - \tilde{g}(k_1 + k_2 e^{\delta t})] h(t) = 0. \tag{2.11}
\]

\( e^{-k_1 w} e^{-k_2 z} h(t) \) belongs to the domain of the generator because of our choice of \( k_1, k_2 \); the function is bounded for all \( t \leq t^* \) and our process evolves up to time \( t^* \) only. Solving (2.11)

\[
h(t) = Ke^{\int_0^t [1 - \tilde{g}(k_1 + k_2 e^{\delta s})] ds}, \tag{2.12}
\]

where \( K \) is an arbitrary constant. Therefore

\[
e^{-k_1 W_t} e^{-k_2 Z_t} e^{\int_0^t [1 - \tilde{g}(k_1 + k_2 e^{\delta s})] ds} \tag{2.13}
\]

is a martingale and hence the result follows. \( \square \)

**Corollary 2.3.** Let \( \nu_1 \geq 0, \ \nu_2 \geq 0, \ \nu \geq 0, \ 0 \leq \theta \leq 1, \) and \( t_1, \ t_2 \) be fixed times. Then

\[
E\{e^{-\nu_1 (N_{t_2} - N_{t_1})} e^{-\nu_2 \lambda_{t_2}} | N_{t_1}, \lambda_{t_1}\} = \exp \left[ - \left\{ \frac{\nu_1}{\delta} + \left( \frac{\nu_2 - \nu_1}{\delta} \right) \right\} \lambda_{t_1} \right] \tag{2.14}
\]

\[
\times \exp \left[ - \rho \int_0^{t_2 - t_1} \left[ 1 - \tilde{g} \left( \frac{\nu_1}{\delta} + \left( \frac{\nu_2 - \nu_1}{\delta} \right) e^{-\delta s} \right) \right] ds \right],
\]

\[
E\{\theta^{(N_{t_2} - N_{t_1})} e^{-\nu \lambda_{t_2}} | N_{t_1}, \lambda_{t_1}\} = \exp \left[ - \left\{ \frac{1 - \theta}{\delta} + \left( \frac{\nu - 1 - \theta}{\delta} \right) \right\} \lambda_{t_1} \right] \tag{2.15}
\]

\[
\times \exp \left[ - \rho \int_0^{t_2 - t_1} \left[ 1 - \tilde{g} \left( \frac{1 - \theta}{\delta} + \left( \frac{\nu - 1 - \theta}{\delta} \right) e^{-\delta s} \right) \right] ds \right].
\]

**Proof.** We set \( k_1 = \nu_1/\delta, \ k_2 = (\nu_2 - \nu_1/\delta)e^{-\delta t_1}, \ t^* \geq t_2 \) in Theorem 2.2 and (2.14) follows immediately. Equation (2.15) follows from (2.14) and (2.6). \( \square \)

Now we can easily derive the probability generating function (p.g.f.) of \( N_t \) and the Laplace transform of \( \lambda_t \) using Corollary 2.3.

**Corollary 2.4.** The probability generating function of \( N_t \) is given by

\[
E\{\theta^{(N_{t_2} - N_{t_1})} | \lambda_{t_1}\} = \exp \left[ - \left\{ \frac{1 - \theta}{\delta} \right\} \left( 1 - e^{-\delta (t_2 - t_1)} \right) \right] \lambda_{t_1} \tag{2.16}
\]

\[
\times \exp \left[ - \rho \int_0^{t_2 - t_1} \left[ 1 - \tilde{g} \left( \frac{1 - \theta}{\delta} \right) \left( 1 - e^{-\delta s} \right) \right] ds \right],
\]
the Laplace transform of the distribution of \( \lambda_t \) is given by

\[
E \{ e^{-\nu \lambda_t} \mid \lambda_0 \} = \exp \left( -\nu \lambda_0 \right) \exp \left[ -\rho \int_0^t \left( 1 - \hat{g}(ve^{-\delta s}) \right) ds \right]
\]

(2.17)

and if \( \lambda_t \) is asymptotic (stationary), it is given by

\[
E \{ e^{-\nu \lambda_t} \} = \exp \left[ -\rho \int_0^\infty \left( 1 - \hat{g}(ve^{-\delta s}) \right) ds \right]
\]

(2.18)

which can also be written as

\[
E \{ e^{-\nu \lambda_t} \} = \exp \left\{ -\frac{\rho}{\delta} \int_0^\nu \hat{G}(u) du \right\},
\]

(2.19)

where \(
\hat{G}(u) = (1 - \hat{g}(u))/u.
\)

Proof. If we set \( \nu = 0 \) in (2.15) then (2.16) follows. Equation (2.17) follows if we either set \( \nu \) = 0 in (2.14) or set \( \theta = 1 \) in (2.15). Let \( t \to \infty \) in (2.17) and the result follows immediately. \( \square \)

Theorem 2.2, Corollaries 2.3 and 2.4 can be found in [17, 19], but they have been included here for completeness and for comparison purposes.

If we differentiate (2.17) and (2.19) with respect to \( \nu \) and put \( \nu = 0 \), we can easily obtain the first moments of \( \lambda_t \), that is,

\[
E(\lambda_t \mid \lambda_0) = \frac{\mu_1 \rho}{\delta} + \left( \lambda_0 - \frac{\mu_1 \rho}{\delta} \right) e^{-\delta t},
\]

(2.20)

\[
E(\lambda_t) = \frac{\mu_1 \rho}{\delta}.
\]

(2.21)

The higher moments can be obtained by differentiating them further, that is,

\[
\text{Var} \left( \lambda_t \mid \lambda_0 \right) = \left( 1 - e^{-2\delta t} \right) \frac{\mu_2 \rho}{2\delta},
\]

\[
\text{Var} \left( \lambda_t \right) = \frac{\mu_2 \rho}{2\delta},
\]

(2.22)

where \( \mu_2 = E(Y^2) = \int_0^\infty y^2 dG(y) \).

3. The distribution of the interval between events of a Cox process with
shot noise intensity and its moment

Let us examine the Laplace transform of the distribution of the shot noise intensity at claim times. To do so, let us denote the time of the \( n \)th claim of \( N_t \) by \( \tau_n \) and denote the value of \( \lambda_t \), when \( N_t \) takes the value \( n \) for the first time by \( \lambda_{\tau_n} \). Since a claim occurs at time \( \tau \), this implies that the intensity at claim times, \( \lambda_{\tau} \), should be higher than the intensity at any times.
Therefore the distribution of $\lambda_t$ should not be the same as the distribution of $\lambda_t$, which will be clear from Theorem 3.2.

Let us start with the following lemma in order to obtain the Laplace transform of the distribution of the shot noise intensity at claim times. We assume that the claims and jumps (or primary events) in shot noise intensity do not occur at the same time.

**Lemma 3.1.** Let $N_t$ be a Cox process with shot noise intensity $\lambda_t$. Let $A$ be the generator of the process $\lambda_t$ and suppose that $f(\lambda)$ is a function belonging to its domain and furthermore that it satisfies

$$\lim_{t \to \infty} E\left\{ f(\lambda_t) \exp \left( -\int_0^t \lambda_s \, ds \right) \mid \lambda_0 \right\} = 0. \quad (3.1)$$

If $h(\lambda)$ is such that

$$\lambda \{ h(\lambda) - f(\lambda) \} + Af(\lambda) = 0 \quad (3.2)$$

then

$$E \{ h(\lambda_{\tau_1}) \mid \lambda_0 \} = f(\lambda_0). \quad (3.3)$$

**Proof.** From (3.2)

$$f(\lambda_t) + \int_0^t \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds \quad (3.4)$$

is a martingale and since $\tau_1$ is a stopping time, where $\Pr(\tau_1 \leq s) = \Pr(N_s > 0)$ and $N_s$ is $\lambda_s$-measurable, we have

$$Ef \{ (\lambda_{\tau_1} \mid \lambda_0) \} + E \left[ \int_0^{\tau_1} \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds \mid \lambda_0 \right] = f(\lambda_0). \quad (3.5)$$

Conditioning on the realisation $\lambda_v$, $0 \leq v \leq t$, $\tau_1$ is distributed with density

$$\lambda_{\tau_1} \exp \left( -\int_0^{\tau_1} \lambda_u \, du \right) \quad (3.6)$$
on \((0, t)\) and a mass \(\exp(-\int_0^t \lambda_u \, du)\) at \(t\). Hence,

\[
E\{ f(\lambda_{\tau_1}) \mid \lambda_v, \; 0 \leq v \leq t \} = \int_0^t \left\{ f(\lambda_r) \lambda_r \exp \left( -\int_0^r \lambda_u \, du \right) \right\} \, dr + f(\lambda_t) \exp \left( -\int_0^t \lambda_u \, du \right),
\]

(3.7)

\[
E \left[ \int_0^t \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds \mid \lambda_v, \; 0 \leq v \leq t \right]
= \int_0^t \left\{ \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds \lambda_r \exp \left( -\int_0^r \lambda_u \, du \right) \right\} \, dr
+ \int_0^t \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds \exp \left( -\int_0^t \lambda_u \, du \right).
\]

(3.8)

Changing the order of integration on the first term of this, it becomes

\[
= \int_0^t \lambda_s \exp \left( -\int_0^r \lambda_u \, du \right) \, dr \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds
+ \int_0^t \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds \exp \left( -\int_0^t \lambda_u \, du \right)
= \int_0^t \left\{ \exp \left( -\int_0^s \lambda_u \, du \right) - \exp \left( -\int_0^t \lambda_u \, du \right) \right\} \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds
+ \int_0^t \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds \exp \left( -\int_0^t \lambda_u \, du \right)
= \int_0^t \exp \left( -\int_0^s \lambda_u \, du \right) \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds.
\]

(3.9)

Adding (3.7) and (3.9), we notice that more terms cancel and we get

\[
E \left\{ f(\lambda_{\tau_1}) + \int_0^t \lambda_s [h(\lambda_s) - f(\lambda_s)] \, ds \mid \lambda_v, \; 0 \leq v \leq t \right\}
= \int_0^t \exp \left( -\int_0^s \lambda_u \, du \right) \lambda_s h(\lambda_s) \, ds + f(\lambda_t) \exp \left( -\int_0^t \lambda_u \, du \right)
= E \left\{ h(\lambda_{\tau_1}) 1_{\tau_1 \leq t} \mid \lambda_v, \; 0 \leq v \leq t \right\} + f(\lambda_t) \exp \left( -\int_0^t \lambda_u \, du \right),
\]

(3.10)
and hence

\[
E\left\{ f(\lambda_t) + \int_0^t \lambda_s \{ h(\lambda_s) - f(\lambda_s) \} ds \mid \lambda_0 \right\} = E\left\{ (h(\lambda_t) 1_{|\tau_i| \geq t}) + f(\lambda_t) \exp \left( - \int_0^t \lambda_u du \right) \mid \lambda_0 \right\}.
\] (3.11)

From (3.5), we then have

\[
E\left\{ (h(\lambda_t) 1_{|\tau_i| \geq t}) + f(\lambda_t) \exp \left( - \int_0^t \lambda_u du \right) \mid \lambda_0 \right\} = f(\lambda_0)
\] (3.12)

and setting \( t \to \infty \), we get (3.3).

Assuming that the shot noise process \( \lambda_t \) is stationary, let us derive the Laplace transform of the distribution of the shot noise process at claim times, \( \lambda_{\tau_i} \).

**Theorem 3.2.** If the shot noise process \( \lambda_t \) is stationary, the Laplace transform of the distribution of the shot noise process at claim times is given by

\[
E(e^{-\nu \lambda_{\tau_i}}) = \frac{\bar{G}(u)}{\mu_1} \exp \left\{ - \frac{\rho}{\delta} \int_0^u \bar{G}(v) dv \right\},
\] (3.13)

where \( \bar{G}(u) = (1 - \bar{g}(u))/u \) and \( \bar{g}(u) = \int_0^u e^{-\nu y} dG(y) \).

**Proof.** From Lemma 3.1, which implies that if \( f(\lambda) \) and \( h(\lambda) \) are such that

\[
\lambda \{ h(\lambda) - f(\lambda) \} - \delta \lambda f'(\lambda) + \rho \left\{ \int_0^\infty f(\lambda + y) dG(y) - f(\lambda) \right\} = 0
\] (3.14)

and (3.1) is satisfied, we have

\[
E\{ h(\lambda_{\tau_i}) \mid \lambda_{\tau_i} \} = f(\lambda_{\tau_i})
\] (3.15)

by starting the process from \( \tau_i \). Employing \( f(\lambda) = \{ \lambda - \bar{g}(\nu)/(1 - \bar{g}(\nu)) \} e^{-\nu \lambda} \), the function \( f(\lambda) \) clearly satisfies (3.1) and substituting into (3.14), then we have

\[
\lambda \left\{ h(\lambda) - \lambda e^{-\nu \lambda} + \frac{\bar{g}'(\nu)}{1 - \bar{g}(\nu)} e^{-\nu \lambda} \right\} + \delta \nu \lambda \left\{ \lambda - \frac{\bar{g}'(\nu)}{1 - \bar{g}(\nu)} \right\} e^{-\nu \lambda} - \delta \lambda e^{-\nu \lambda} = -\rho \lambda e^{-\nu \lambda} \{ \bar{g}(\nu) - 1 \}.
\] (3.16)

Divide by \( \lambda \) and simplify then we have

\[
h(\lambda) = \lambda e^{-\nu \lambda} (1 - \delta \nu) + \delta e^{-\nu \lambda} - (1 - \delta \nu) \frac{\bar{g}'(\nu)}{1 - \bar{g}(\nu)} e^{-\nu \lambda} + \rho e^{-\nu \lambda} \{ 1 - \bar{g}(\nu) \}.
\] (3.17)
From (3.15), it is given that
\[ E\{h(\tau_{\nu})\} = E[E\{h(\tau_{\nu}) \mid \lambda_{\nu}\}] = E\{f(\lambda_{\nu})\}. \] (3.18)

So put (3.17) into (3.18), then
\[
E\left[\lambda_{\nu} \exp \left(-\nu\lambda_{\nu}\right)(1 - \delta\nu) + \delta \exp \left(-\nu\lambda_{\nu}\right) \frac{\hat{g}(\nu)}{1 - \hat{g}(\nu)} \times \exp \left(-\nu\lambda_{\nu}\right) + \rho \exp \left(-\nu\lambda_{\nu}\right)\{1 - \hat{g}(\nu)\}\right]
= E\left\{\lambda_{\nu} \exp \left(-\nu\lambda_{\nu}\right) - \frac{\hat{g}(\nu)}{1 - \hat{g}(\nu)} \exp \left(-\nu\lambda_{\nu}\right)\right\}. \tag{3.19}
\]

When the process \(\lambda_{\nu}\) is stationary, \(\lambda_{\nu,i}\), and \(\lambda_{\nu}\) have the same distribution whose Laplace transform we denote by \(H(\nu) = E(e^{-\nu\lambda_{\nu}})\). Therefore from (3.19), we have
\[
-(1 - \delta\nu)H'(\nu) - (1 - \delta\nu) \frac{\hat{g}(\nu)}{1 - \hat{g}(\nu)} H(\nu) + [\delta + \rho\{1 - \hat{g}(\nu)\}] H(\nu) = -H'(\nu) - \frac{\hat{g}(\nu)}{1 - \hat{g}(\nu)} H(\nu). \tag{3.20}
\]

Divide both sides of (3.20) by \(\delta\nu\), then we have
\[
H'(\nu) + \frac{\hat{g}(\nu)}{1 - \hat{g}(\nu)} H(\nu) + \left\{\frac{1}{\nu} + \frac{\rho}{\delta} \frac{1 - \hat{g}(\nu)}{\nu}\right\} H(\nu) = 0. \tag{3.21}
\]

Solving (3.21), subject to
\[ H(0) = 1 \tag{3.22} \]
then the Laplace transform of a distribution of the shot noise process at claim times is given by
\[
H(\nu) = K \left(\frac{1 - \hat{g}(\nu)}{\nu}\right) \exp \left\{-\frac{\rho}{\delta} \int_{0}^{\nu} \tilde{G}(u) du\right\}, \tag{3.23}
\]
where \(K\) is a constant. Therefore from (3.22), \(K = 1/\mu_{1}\) and
\[
H(\nu) = \frac{1}{\mu_{1}} \frac{1 - \hat{g}(\nu)}{\nu} \cdot \exp \left\{-\frac{\rho}{\delta} \int_{0}^{\nu} \tilde{G}(u) du\right\} = \frac{\hat{G}(\nu)}{\mu_{1}} \cdot \exp \left\{-\frac{\rho}{\delta} \int_{0}^{\nu} \tilde{G}(u) du\right\}. \tag{3.24}
\]

Equation (3.24) provides us with an interesting result. The distribution defined by the Laplace transform (3.24) (and (3.13)) is the same as the distribution of two random variables;
one having the stationary distribution of $\lambda_t$ (see Corollary 2.4) and the other having density $G(y)/\mu_1$, where $G(y) = 1 - G(y)$. Comparing it with the distribution of the shot noise process, $\lambda_t$ at any times, we can easily find that

$$\frac{\hat{G}(v)}{\mu_1} \exp \left\{ - \frac{\rho}{\delta} \int_0^v \hat{G}(u) du \right\} > \exp \left\{ - \frac{\rho}{\delta} \int_0^v \hat{G}(u) du \right\}. \quad (3.25)$$

It is therefore the case that $\lambda_t$ is stochastically larger than $\lambda_t$. In other words, the intensity at claim times is higher than the intensity at any times.

Now let us derive the distribution of the interval of a Cox process with shot noise intensity for insurance claims using Theorem 3.2.

**Corollary 3.3.** Assume that 0 is the time at which a claim of $N_t$ has occurred and the stationary of $\lambda_t$ has been achieved. Then the tail of the distribution of the interval of a Cox process with shot noise intensity is given by

$$\Pr(\tau > t) = \frac{\hat{G}(1/\delta - (1/\delta)e^{-\delta t})}{\mu_1} \exp \left\{ - \frac{\rho}{\delta} \int_0^t \tilde{G} \left( \frac{1}{\delta} - \frac{1}{\delta} e^{-\delta s} \right) ds \right\}. \quad (3.26)$$

**Proof.** From (2.16), the probability generating function of $N_t$ is given by

$$E(\theta N_t | \lambda_0) = \exp \left\{ - \frac{1 - \theta}{\delta} (1 - e^{-\delta t}) \lambda_0 \right\} \exp \left[ - \rho \int_0^t \tilde{G} \left( \frac{1 - \theta}{\delta} (1 - e^{-\delta s}) \right) ds \right]. \quad (3.27)$$

Set $\theta = 0$ in (3.27) and take expectation, then the tail of the distribution of $\tau$ is given by

$$\Pr(\tau > t) = \exp \left[ - \rho \int_0^t \left( 1 - \frac{1}{\delta} e^{\delta s} \right) ds \right] E \left[ \exp \left\{ - \frac{1 - e^{-\delta t}}{\delta} \lambda_0 \right\} \right]. \quad (3.28)$$

Substitute (3.13) into (3.28), then the result follows immediately as 0 is the time at which a claim has occurred and $\lambda_t$ is stationary.

**Corollary 3.4.** The expectation and variance of the interval between claims are given by

$$E(\tau) = \int_0^\infty \Pr(\tau > t) dt = \frac{\delta}{\mu_1 \rho}, \quad (3.29)$$

$$\text{Var}(\tau) = 2 \int_0^\infty \left[ \frac{u \hat{G}(1/\delta - (1/\delta)e^{-\delta u})}{\mu_1} \exp \left\{ - \frac{\rho}{\delta} \int_0^u \hat{G} \left( \frac{1}{\delta} - \frac{1}{\delta} e^{-\delta s} \right) ds \right\} du - \left( \frac{\delta}{\mu_1 \rho} \right)^2 \right]. \quad (3.30)$$
Proof. Integrate (3.26), then (3.29) follows. (3.30) is obtained from

$E(\tau^2) = \int_0^{\infty} t^2 f(t)dt = 2\int_0^{\infty} \left[ \frac{\hat{G}(1/\delta - (1/\delta)e^{-\delta u})}{\mu_1} \exp\left\{ -\frac{\rho}{\delta}\int_0^u \hat{G}\left(\frac{1}{\delta} - \frac{1}{\delta}e^{-\delta s}\right)ds\right\} \right] du.$

(3.31)

An interesting result we can find from (3.29) and (2.21) is that the expected interval between claims is the inverse of the expected number of claims, where the number of claims follows a Cox process with shot noise intensity, which is also the case for a Poisson process.

4. Conclusion

We started with deriving the probability generating function of a Cox process with shot noise intensity, employing piecewise deterministic Markov processes theory. It was necessary to obtain the distribution of the shot noise process at claim times as it is not the same as the distribution of the shot noise process at any times. Assuming that the shot noise process is stationary, we derived the distribution of the interval of a Cox process with shot noise intensity for insurance claims and its moments from its probability generating function. The result of this paper can be used or easily modified in computer science/telecommunications modeling, electrical engineering, and queueing theory as an alternative counting process to a Poisson process.

Appendix

This appendix explains the basic definition of a piecewise deterministic Markov process (PDMP) that is adopted from [20]. A detailed discussion can also be found in [18, 24].

PDMP is a Markov process $X_t$ with two components $(\xi_t, \zeta_t)$, where $\xi_t$ takes values in a discrete set $K$ and given $\xi_t = n \in K$, $\zeta_t$ takes values in an open set $M_n \subset \mathbb{R}^{d(n)}$ for some function $d : K \rightarrow N$. The state space of $X_t$ is equal to $E = \{(n, z) : n \in K, z \in M_n\}$. We further assume that for every point $x = (n, z) \in E$, there is a unique, deterministic integral curve $\phi_n(t, z) \subset M_n$, determined by a differential operator $\chi_n$ on $\mathbb{R}^{d(n)}$, such that $z = \phi_n(t, z)$. If for some $t_0 \in \mathbb{R}^+$, $X_{t_0} = (n_0, z_0) \in E$, then $\xi_t$, where $t \geq t_0$ follows $\phi_n(t, z_0)$ until either $t = T_0$, some random time with hazard rate of function $\rho$ or until $\xi_t = \partial M_n$, the boundary of $M_n$. In both cases, the process $X_t$ jumps, according to a Markov transition measure $Q$ on $E$, to a point $(n_i, z_i) \in E$. $\xi_t$ again follows the deterministic path $\phi_n$, till a random time $T_i$ (independent of $T_0$) or till $\xi_i = \partial M_{n_i}$, and so forth. The jump times $T_i$ are assumed to satisfy the following condition:

$$\forall t > 0, \quad E\left(\sum_i I(T_i \leq t)\right) < \infty.$$  

(A.1)

The stochastic calculus that will enable us to analyse various models rests on the notion of (extended) generator $A$ of $X_t$. Let $\Gamma$ denotes the set of boundary points of $E, \Gamma = \{(n, z) : n \in K, z \in \partial M_n\}$, and let $A$ be an operator acting on measurable functions $f : E \cup \Gamma \rightarrow \mathbb{R}$ satisfying the following.

(i) The function $t \rightarrow f(n, \phi_n(t, z))$ is absolutely continuous for $t \in [0, t(n, z)]$ for all $(n, z) \in E$. 


In some cases, it is important to have time \( t \) as an explicit component of the PDMP. In those cases \( A \) can be decomposed as \( \partial/\partial t + A_t \), where \( A_t \) is given by (A.2) with possibly time-dependent coefficients.

An application of Dynkin’s formula provides us with the following important result (martingales will always be with respect to the natural filtration \( \sigma\{X_s : s \leq t\} \)).

(a) If for all \( t \), \( f(\cdot,t) \) belongs to the domain of \( A_t \) and \( (\partial/\partial t)f(x,t) + A_t f(x,t) = 0 \), then \( f(X_t,t) \) is a martingale.

(b) If \( f \) belongs to the domain of \( A \) and \( Af(x) = 0 \), then \( f(X_t) \) is a martingale.

The generator of the process \( X_t \) acting on a function \( f(X_t) \) belonging to its domain as described above is also given by

\[
Af(X_t) = \lim_{h \to 0} \frac{E[f(X_{t+h}) | X_t = x] - f(X_t)}{h}. \tag{A.3}
\]

In other words, \( Af(X_t) \) is the expected increment of the process \( X_t \) between \( t \) and \( t + h \), given the history of \( X_t \) at time \( t \). From this interpretation the following inversion formula is plausible, that is,

\[
E[f(X_{t+h}) | X_t = x] - f(X_t) = \int_0^h E[Af(X_{s})] \, ds \tag{A.4}
\]

which is Dynkin’s formula.

**References**