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Decision procedures in formal logic and mathematical programming algorithms

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DECISION PROCEDURES IN FORMAL LOGIC AND MATHEMATICAL PROGRAMMING ALGORITHMS

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ABSTRACT

The relationship between decision procedures in formal logic and algorithms for Linear Programming (LP) and Integer Programming (IP) are discussed. A decision procedure of Langford for densely ordered sets is described and shown to yield the Fourier-Motzkin-elimination method for LP. A decision procedure of Presburger for arithmetic without multiplication is described and shown to yield an algorithm for IP analogous to the Fourier-Motzkin method for LP. Both algorithms are applied to a small numerical example.

1. INTRODUCTION

It is not widely known by Mathematical Programmers that algorithms for Linear (LP) and Integer (IP) Programming arise in the field of Formal Logic. One of the major discoveries in modern logic has been that some mathematical theories are undecidable. By this it is meant that there does not exist an algorithm for deciding the truth or falsity of an arbitrary proposition in such a theory. One such undecidable theory is the arithmetic of the natural numbers, including operations for both addition and multiplication. Godel (7) has shown that this theory is sufficiently "big" that it is impossible to have a general finite computational procedure for deciding whether any statement is true or false. A much more rigorous and full explanation of such results can be obtained from textbooks in logic such as Mendelson (11).

Given that some mathematical theories are undecidable it is of interest to try to find "smaller" theories and see if they are decidable. Two such theories are considered in this paper. The first is the Theory of Densely Ordered Sets. This theory consists of the apparatus of formal logic known as the predicate calculus together with the relation " $<$ ", and a densely ordered set such as the rational numbers. A proof of the decidability of this theory has been given by Langford (10). This proof is constructive and consists of exhibiting a decision procedure for ascertaining the truth or falsity of propositions made in the theory. The decision procedure can very easily be adapted to an algorithm for solving Linear Programming problems. This algorithm turns out to be the Fourier - Motzkin elimination method.

A description of the Fourier - Motzkin method is given in Dantzig (4). The second theory considered in this paper is the arithmetic of the natural numbers with multiplication excluded. This theory has been shown to be decidable by Presburger (15). Again the proof involves exhibiting a decision procedure for ascertaining the truth or falsity of propositions made in the theory. This decision procedure can be adapted to an algorithm for solving Integer Programming problems. As far as the author is aware this does not correspond to any existing algorithm for integer programming. It can, however, be regarded as an extension of the Fourier - Motzkin method to deal with integer variables. In fact the two decision procedures (Langford's and Presburger's) can easily be combined to give a general Mixed Integer Linear Programming algorithm. The extension of the Fourier - Motzkin method to deal with integer programming problems has been attempted before by Bradley (2) and, for the knapsack problems, by Cabot (3). Neither of these two extensions is, however, the same as that given here.

The purpose of this paper is not to promote either of these algorithms as methods for solving real life models. It is rather to increase the awareness of mathematical programmers to some results in logic. Hopefully it will also lead to a greater understanding of the relationship of integer programming to linear programming and why the former is so much more difficult.

A small numerical example is solved first as a linear programming problem (Langford's procedure) and then as an integer programming problem (Presburger's procedure).

2. LOGICAL FORMALISATION OF LP AND IP PROBLEMS

All LP and IP problems can be stated in the form :

(1) Maximise
or minimise $\quad Z = \sum_{j=1}^{n} c_j x_j$

subject to the following constraints

$$\sum_{j=1}^{n} a_{ij} x_j - b_i = 0 \qquad 1 \leq i \leq m_1$$

Constraints R_i, $1 \leq i \leq m$

$$\sum_{j=1}^{n} a_{ij} x_j - b_i \leq 0 \qquad m_1+1 \leq i \leq m_2$$

$$\sum_{j=1}^{n} a_{ij} x_j - b_i \geq 0 \qquad m_2+1 \leq i \leq m$$

Constraints R_{m+j}, $1 \leq j \leq n$

$$x_j \geq 0 \qquad 1 \leq j \leq n$$

It is convenient to treat the objective function by means of the equality constraint:

$$R_0 \qquad \sum_{j=1}^{n} c_j x_j - Z = 0$$

The objective is then to maximise or minimise Z

The problem can then be formally stated as :

(2) Find the maximum (or minimum) Z such that

$$\exists x_1 \exists x_2 \ldots \exists x_n \exists z [R_0 . R_1 . R_2 . \ldots R_{m+n}]$$

A decision procedure involves a method of eliminating the quantifiers " $\exists$ " together with their associated variables x_j, Z. If only the quantifiers $\exists x_j$ and the associated x_j, are eliminated the result is a proposition of the form :

(3) $\quad \exists z\, [T_0, T_1, \ldots T_p]$

where the T_i involve at most the variable z

It is straightforward to find the maximum (minimum) value of z which makes the proposition in brackets true. If no value of z makes it true the original problem (1) was infeasible. If values of z exist making it true but there is no maximum (minimum) the problem (1) was unbounded.

If the X_j variables are integer the elimination process is different than if they are continuous.

3. CONTINUOUS LINEAR PROGRAMMING (LP PROBLEMS)

Suppose the variables X_j, z appearing in the predicates of (2) are taken from a densely ordered set with neither a first nor last element (such a set might be the rational numbers or the real numbers). Relations " $<$ " and " $=$ " can be defined between elements of the set together with axioms for combining such relations with the logical connectives. This theory has been shown to be decidable by Langford (10). A description of his decision procedure is given by Mendelson (11) (pp. 94 - 95).

For the purpose of LP it is more convenient to work with the relation " $\leq$ ". The decision procedure can easily be adapted to involve only the arithmetic relations " $\leq$ " and " $=$ ". To apply the procedure to LP the arithmetic operations of addition and subtraction will also be added to the system.

The variables and constants will be taken from the set of rational numbers. It is also convenient to allow multiplication and division by rational constants. This is not necessary but makes computation easier. The use of the symbol " $\geqslant$ " can be avoided by multiplying by -1 and making the inequalities " $\leqslant$ " i.e.

$$\sum_{j=1}^{n} a_{ij} x_j - b_i \geqslant 0$$

is the same as

$$\sum_{j=1}^{n} (-a_{ij}) x_j + b_i \leqslant 0$$

Problem (3) can then be stated in the form:

(4) Find the maximum (minimum) z such that

$$\exists x_1 \exists x_2 \ldots \exists x_n \exists z \left[S_0 . S_1 . S_2 \ldots S_{m+n} \right]$$

where S_i are all equalities (=) or less - than - or - equal inequalities ($\leqslant$)

For simplicity in the rest of this section the quantifiers in the above expression will be ignored and the variables eliminated from the expression in brackets.

Langford's decision procedure amounts to performing the following steps for each variable in turn :

Step 1. Choose the next variable X_j which has not been eliminated.

Step 2. Consider those (in)equalities S_i in which X_j has a non-zero coefficient.

Step 3. Divide through the (inequalities) by the absolute value of the coefficient of X_j (This is not strictly necessary but is computationally more convenient).

Step 4. Partition the (in)equalities into 3 groups (E), (L) and (G). Group (E) consists of the equalities, group (L) of the inequalities in which Xj has coefficient +1 and group (G) of the inequalities in which Xj has coefficient -1.

Typical (in)equalities from each group are:

(E) $\pm X_j + S = 0$

(L) $X_j + t \leq 0$

(G) $-X_j + u \leq 0$

s, t and u represent expressions involving the variables which have not been eliminated (apart from Xj) together with constants.

Step 5. The following three possibilities must be distinguished.

(a) There exists at least one equality in (E).

(b) All (in)equalities are of the form (L) or all are of the form (G).

(c) There do not exist any equalities in (E) and not all the inequalities are of the same form.

In case (a) one of the equalities is used to "substitute" for Xj in the other (in)equalities. This substitution can be effected by adding or subtracting the equality from the other (in)equalities to eliminate Xj.

In case (b) all (in)equalities in which Xj occurs are ignored.

In case (c) each inequality in (L) is added to each inequality in (G) to eliminate Xj i.e. all possible pairs of inequalities, one from (L) and one from (G) are combined.

Step 6. when all the variables x_j have been eliminated by the above steps only the variable z remains.

The problem is now in the form :

(5) Find the maximum (minimum) z such that

$$[T_0, T_1, \ldots T_p]$$

where the T_i involve at most the variable z

Three cases must be distinguished :

(a) Some T_i are of the form

$k \leq 0$ where k is negative

(b) No T_i involve z with a positive coefficient and case (a) does not apply.

(c) There exist T_i involving z with a positive coefficient.

If case (a) occurs the original problem (1) was infeasible

If case (b) occurs the original problem (1) was unbounded

If case (c) occurs the original problem (1) has a solution. The maximum value of the objective function is given by the maximum value of z satisfying (5). Values of the variables x_j giving rise to this maximum value of z can be obtained by "backtracking" in the following way: Trace those original (in)equalities S_i in (4) which resulted in the strictest of the (in)equalities T_i involving z i.e. that which determined the maximum z. These original (in)equalities can be all treated as equations and then solved to give an optimal solution to the problem.

The above procedure was used by Fourier (6) for solving LP problems. It was also considered by Motzkin (12).

Dantzig (4) describes the method as the Fourier - Motzkin method. A detailed description of the application of the method is given by Duffin (5). Computational refinements can be made by selecting the variables Xj to be eliminated in an order which minimises the number of (in)equalities resulting from Stage 5. Kohler (9) has greatly improved the computational efficiency of the method by excluding obviously redundant inequalities during the course of solution. In effect he exploits the result that only vertex solutions need be considered. The possibility of applying the method to the dual problem has been considered by Abadie (1). In spite of these variations and refinements the method is not computationally efficient for practical problems. This is because of the large build up in inequalities which can occur at Step 5. Kohler gives some computational results on this. Nevertheless the method is of interest. Duffin uses it to prove duality. For small problems solved by hand calculation the method is easier and quicker than the simplex algorithm.

4. INTEGER PROGRAMMING (IP) PROBLEMS.

Suppose the variables Xj, appearing in the predicates R_i of (2) are taken from the set of natural numbers. The relation " = " and the function " + " are used. In fact the theory amounts to arithmetic without multiplication. Presburger (13) gives a decision procedure for this theory. He achieves this by using the relations " < " and " $\equiv (\mathrm{mod}\ k)$ " where k is a positive integer. " = " can be defined using these relations. A full description of the method is given by Hilbert and Bernays (8).

No extra difficulty is involved in the procedure by allowing negative integers and subtraction. Also it is convenient to deal with the relations " $\leq$ " and " = " instead of " $<$ " since most mathematical programming problems are conventionally stated in this form. It is now posible to state the original (integer) problem (1) in the form (4) where the (in)equalities S_i are all either equalities or "less - than - or - equal to" inequalities.

The steps of Presburger's decision procedure applied to IP problems will be gone through in an analogous fashion to those for Langford's. Before doing this, however, a sub-section will be devoted to highlighting the difference in the two procedures.

4.1 Eliminating integer variables between inequalities

Many of the steps of Presburger's procedure are the same as those for Langford's. A major complication arises, however, in steps 3 and 4. It is no longer possible, in general, to divide through (in)equalities by the absolute value of the coefficient of Xj. For example consider the inequalities

(i) $$2x - 3y - 5 \leq 0$$

(ii) $$-3x + 2y - 1 \leq 0$$

It is not permissable to convert (i)&(ii) to the inequalities (iii and (iv).

(iii) $$x - \frac{3}{2}y - \frac{5}{2} \leq 0$$

(iv) $$-x + \frac{2}{3}y - \frac{1}{3} \leq 0$$

since non integral expressions might then arise.

In order to eliminate the integer variable X between (i) and (ii) it is convenient to multiply (i) by 3 and (ii) by 2, to give the inequalities (v) and (vi).

(v) $\quad 6x - 9y - 15 \leq 0$

(vi) $\quad -6x + 4y - 2 \leq 0$

Taken together these two inequalities can be expressed as

(vii) $\quad 4y - 2 \leq 6x \leq 9y + 15$

If x is taken from a densely ordered set of numbers (such as the rationals) no difficulty arises in eliminating x. The expression (vii) is equivalent to the expression (viii)

(viii) $\quad 4y - 2 \leq 9y + 15$

This clearly gives (ix)

(ix) $\quad -5y \leq 17$

Obviously this inequality can be obtained more easily by adding (iii) and (iv) together as in the Fourier - Motzkin method.

For the case considered here, however, x is an integer variable. The import of (vii) is not simply that a rational number lies between the left-hand-side and right-hand-side expressions. It is that a multiple of 6 lies between the two expressions. We wish to be able to say :

(x)

$$\begin{array}{l} \vdots \\ (4y - 2 \leq -6 \leq 9y + 15) \\ \vee\ (4y - 2 \leq 0 \leq 9y + 15) \\ \vee\ (4y - 2 \leq 6 \leq 9y + 15) \\ \vee\ (4y - 2 \leq 12 \leq 9y + 15) \\ \vdots \end{array}$$

etc.

If upper and lower (such as 0) bounds are known for x then (x) only involves a finite number of possibilities and the elimination of x would be straightforward. If no such bounds are known then (x) would present an infinite disjunction of inequalities.

By introducing the relation " $\equiv$ (mod 2) " it is possible to express (x) in a finite manner as (xi)

(xi)

$$4y - 2 \leq 9 \text{[illegible]} \,.\, 3y + 5 \equiv 0 \pmod{2}$$

$$\vee\ 4y - 2 \leq 9y - \text{[illegible]} \,.\, 3y + 5 - 1 \equiv \text{[illegible]}$$

In (x) and (xi) $\vee$ is the connective " or "

. is the connective " and "

The above statement can be written rather more compactly but is easier to understand if written in the above form.

This use of the relation " $\equiv$ (mod k) " is the crux of Presburger's procedure as it allows a finite computational procedure to be applied to any integer programming problem (even if there are an infinite number of integer points in the feasible region).

Once the relation " $\equiv$ (mod k) " has been introduced into a problem by eliminating a variable it may be necessary to take account of it in eliminating subsequent variables. The general procedure for eliminating a variable xj from a problem must therefore suppose that the variable is involved in such congruence relations as well as in (in)equality relations. The general method of eliminating a variable between all the possible kinds of relations is given in the next sub-section.

A detailed description of the procedure is also given in Hilbert and Bernays (8) but using only the relations " $<$ " and " $\equiv$ (mod k) ".

4.2 Presburger's procedure applied to IP problems

The original problem will be expressed in terms of the (in)equalities S_0, S_1, ... S_{m+n} of (4) where Xj and z will be considered as integer variables. The steps of the procedure will be given analogously to those for Langford's procedure but will be labelled 1^1, 2^1 etc.

Step 1^1 Choose the next variable Xj which has not been eliminated.

Step 2^1 Consider those (in)equalities and congruences in which Xj has a non-zero coefficient,

Step 3^1 Omit

Step 4^1 Partition the (in)equalities and congruences into 4 groups (E), (L), (G) and (M). Group (E) consists of the equalities, group (L) of the inequalities in which Xj has a positive coefficient, group (G) of the inequalities in which Xj has a negative coefficient and group (M) of the congruences.

Typical (in)equalities and congruences from each group are :

(E) $\pm p_1 X_j + s = 0$

(L) $p_2 X_j + t \leq 0$

(G) $-p_3 X_j + u \leq 0$

(M) $\pm p_4 X_j + v \equiv 0 \pmod{k}$

s, t, u and v represent expressions involving the variables which have not been eliminated (apart from Xj) together with constants. P_1. P_2, P_3 and P_4 are positive integers.

Step 5^1 The following three possibilities must be distinguished.

(a) There exists at least one equality in (E)

(b) All (in)equalities are of the form (L) or all are of the form (G) and there are no equalities. As a special case of this there may be no inequalities or equalities (only congruences).

(c) There do not exist any equalities in (E) and not all the inequalities are of the same form.

In case (a) one of the equalities is used to substitute for Xj in the other (in)equalities and the congruences. To do this it may be necessary to multiply (in)equalities and congruences by integer quantities in order to give Xj the same coefficient in each e.g. suppose the following equality (i) is being used to substitute in the inequality (ii) and the congruence (iii)

(i) $$p_1 x_j + s = 0$$

(ii) $$p_2 x_j + t \leq 0$$

(iii) $$p_4 x_j + v \equiv 0 \pmod{k}$$

Let the least common multiple (LCM) (or any multiple) of P_1 and P_2 be $a\,(= q_1 p_1 = q_2 p_2)$

(i) and (ii) become

(i)[1] $$a x_j + q_1 s = 0$$

and

(ii)[1] $$a x_j + q_2 t \leq 0$$

Eliminating Xj between (i) and (ii) gives (iv)

(iv) $$-q_1 s + q_2 t \leq 0$$

Let the LCM of P_2 and $P4$ be $b\,(= q_3 p_1 = q_4 p_4)$

(i) and (iii) become

(i)[11] $$b x_j + q_3 s = 0$$

and

(iii)^1 $\qquad b x_j + q_4 v \equiv 0 \pmod{q_4 k}$.

Eliminating Xj between (i)^{11} and (iii)^1 gives (v)

(v) $\qquad -q_3 s + q_4 v \equiv 0 \pmod{q_4 k}$

In case (b) all (in)equalities in which Xj occurs are removed. If Xj occurs in any congruence it must be eliminated between each pair. Suppose Xj occurs in the following pair of congruences

(vi) $\qquad p_4 x_j + v \equiv 0 \pmod{k}$

(vii) $\qquad p_5 x_j + w \equiv 0 \pmod{l}$

Let the least common multiple (or any multiple) of P_4 and P_5 be a $(= q_4 p_4 = q_5 p_5)$

(vi) and (vii) then become (viii) and (ix)

(viii) $\qquad a x_j + q_4 v \equiv 0 \pmod{q_4 k}$

(ix) $\qquad a x_j + q_5 w \equiv 0 \pmod{q_5 l}$

Eliminating Xj between these congruences gives (x)

(x) $\qquad q_4 v - q_5 w \equiv 0 \pmod{g}$

where g is the greatest common divisor (GCD) of $q_4 k$ and $q_5 l$.

Frequently g will turn out to be 1 showing the congruence (x) is vacuous and may be ignored.

Should Xj only occur in one congruence (vi) X_j is eliminated from it to give the congruence (xi)

(xi) $\qquad v \equiv 0 \pmod{g}$

where g is the GCD of p_4 and k.

In case (c) it is necessary to take each triplet of an inequality from (L), an inequality from (G) and an congruence from (M) in turn and eliminate Xj between them. A typical triplet is given below

(A)

$$p_2 x_j - t \le 0$$

$$-p_2 x_j + u \le 0$$

$$p_2 x_j + v \equiv 0 \pmod{\hat{k}}$$

It may happen that there are no congruences. In this case the elimination is simplified, ($\hat{k}$ can then be taken as 1.) Such special cases are considered after deriving the general elimination.

Let the LCM of p_2 [illegible] be p ([illegible])

Denote $q_1 t$ by t_1, $q_2 u$ by [illegible]

and $q_3 \hat{k}$ by k

If Xj is eliminated from the above triplet of inequalities and congruences (B) is obtained.

(B)

$$u_1 + t_1 \le 0 \cdot u_1 + v_1 \equiv 0 \text{ [illegible]}$$

$\vee\ u_1$ [illegible]

$\vee\ u_1$ [illegible] $\le 0) \cdot u_1 +$ [illegible]

⋮

$\vee\ u_1$ [illegible] $(\hat{k}-1)$ [illegible] $v + (\hat{k}$ [illegible]

(to save brackets the connective " · " is regarded as more binding that the connective " V " in the above expressions).

Special cases

(i) No congruence involves Xj ((M) is empty). In this case it is possible to regard $\mathcal{R}$ as 1, $P_4 = 1$ and V (= V_1) as 0 in (A).

p becomes the LCM of P_2, P_3 and $\mathcal{R}_1 = p$

(B) simplifies to (C)

$$
\text{(C)}\qquad
\begin{aligned}
& u_1 + t_1 \leq 0 , \quad u \equiv 0 \pmod{P_3} \\
\vee\ & u_1 + t_1 + q_3 \leq 0 , \quad u + 1 \equiv 0 \pmod{P_3} \\
\vee\ & u_1 + t_1 + 2q_3 \leq 0 , \quad u + 2 \equiv 0 \pmod{P_3} \\
& \vdots \\
\vee\ & u_1 + t_1 + (p_3 - 1)q_3 \leq 0 , \quad u + (p_3 - 1) \equiv 0 \pmod{p_3}
\end{aligned}
$$

(C) can be written in another form (D) which will involve a smaller disjunction if $p_2 < p_3$

$$
\text{(D)}\qquad
\begin{aligned}
& u_1 + t_1 \leq 0 , \quad t \equiv 0 \pmod{p_2} \\
\vee\ & u_1 + t_1 + q_2 \leq 0 , \quad t + 1 \equiv 0 \pmod{p_2} \\
\vee\ & u_1 + t_1 + 2q_2 \leq 0 , \quad t + 2 \equiv 0 \pmod{p_2} \\
& \vdots \\
\vee\ & u_1 + t_1 + (p_2 - 1)q_2 \leq 0 , \quad t + (p_2 - 1) \equiv 0 \pmod{p_2}
\end{aligned}
$$

(ii) In case (i) if $P_3 = 1$ (C) can be further simplified to (E)

(E) $\quad p_2 u + t \leq 0$

(iii) In case (i) if $P_2 = 1$ (D) can be further simplified to (F)

(F) $\quad u + p_3 t \leq 0$

(iv) Cases (ii) and (iii) combine if $P_2 = P_3 = 1$ to give (G)

(G) $\quad u + t \leq 0$

i.e. the two inequalities

$$
\begin{aligned}
X_j + t &\leq 0 \\
\text{and} \quad - X_j + u &\leq 0
\end{aligned}
$$

are simply added together as in Langford's procedure (the Fourier - motzkin method).

It is worth pointing out that for a unimodular matrix the congruence relation " $\equiv$ (mod ρ) " will never enter the calculations and the coefficients F_2 and F_3 will always be 1. Hence all eliminations of a variable will be identical in the LP and IP cases.

Step 6 When all the variables X_j have been eliminated by the above steps only the variable z remains. The problem will now be in a form involving simple (in)equalities and congruences containing at the most the variable z . These simple (in)equalities and congruences will be connected by the connectives " $\vee$ " and " $\cdot$ ". It will be possible to state this form of the problem in a number of ways. For simplicity it will be stated in disjunctive form as :

(5) Find the maximum (minimum) such that

$$(T_{10} \cdot T_{11} \cdots T_{1p1}) \vee (T_{20} \cdot T_{21} \cdots T_{2p2}) \vee \cdots \vee (T_{Q0} \cdot T_{Q1} \cdots T_{QPQ})$$

Three cases must be distinguished.

(a) No Z can be found to satisfy any of the above conjunctions.

(b) Some Z can be found satisfying some of the above conjunctions and there is no maximum (minimum) such Z.

(c) A finite maximum (minimum) Z can be found satisfying some of the conjunctions.

If case (a) occurs the original problem was infeasible
If case (b) occurs the original problem was unbounded
If case (c) occurs the original problem has a solution.
The maximum (minimum) value of the objective function is given by the maximum (minimum) integer Z satisfying (5)'

To obtain the values of X_j which give rise to this maximum (minimum) value a "backtracking" procedure can again be employed. This is not as straightforward as in the LP case since "binding" inequalities will not necessarily become equations. The easiest procedure consists of examining those (in)equalities and congruences prior to the elimination of the last X_j variable which combined to give rise to the final (in)equality determining the maximum (minimum) value of Z. These (in)equalities and congruences will determine the value of the last X_j eliminated. Proceeding in this manner the values of the X_j can be determined in the reverse order to which they were eliminated.

The numerical example given later should clarify the method.

5. A NUMERICAL EXAMPLE

The following problem will be considered first as a continuous LP problem and then as an IP problem.

$$\text{Minimise} \quad z = x_1 + x_2 + x_3$$

$$(\alpha_1) \qquad \begin{aligned} \text{subject to} \quad x_1 + 4x_2 + x_3 &\geq b_1 \\ -x_1 + x_2 &\leq b_2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Since no extra difficulty is involved in solving the problem for general right-hand-sides this will be done.

It is convenient to substitute for x_3 and consider the problem in the following form with all constraints "$\leq$"

$$\text{minimise} \quad z$$

$$(\alpha_2) \qquad \begin{array}{lrl} \text{subject to} & -3x_2 - z \leq -b_1 & \text{R1} \\ & -x_1 + x_2 \leq b_2 & \text{R2} \\ & x_1 + x_2 - z \leq 0 & \text{R3} \\ & -x_1 \leq 0 & \text{R4} \\ & -x_2 \leq 0 & \text{R5} \end{array}$$

5.1 The continuous LP problem

Eliminating X_1 the following inequalities are produced

$$(\alpha_3) \qquad \begin{array}{rl} -3x_2 - z \leq -b_1 & \text{R1} \\ 2x_2 - z \leq b_2 & \text{R2, R3} \\ x_2 - z \leq 0 & \text{R3, R4} \\ -x_2 \leq 0 & \text{R5} \end{array}$$

Eliminating X_2 the following inequalities are produced (after dividing through the inequalities R1 by 3 and R2, R3 by 2).

(α_4)

$-\frac{5}{[illegible]}z \le -\frac{[illegible]}{[illegible]} + [illegible]b$	R1, R2, R3
$-\frac{4}{3}z \le -\frac{b_1}{3}$	R1, R3, R4
$-\frac{1}{2}z \le \frac{b_2}{2}$	R2, R3, R5
$-z \le 0$	R3, R4, R5

These inequalities can be rewritten as

(α_5)

$z \ge \frac{2}{5}b_1 - [illegible]b_2$	R1, R2, R3
$z \ge \frac{b_1}{4}$	R1, R3, R4
$z \ge -b_2$	R2, R3, R5
$z \ge 0$	R3, R4, R5

To give a specific problem the values of b_1 and b_2 will be taken as $b_1 = 3$, $b_2 = 0$

This given the minimum value of z as $6/5$ obtained from R1, R2, R3.

Since the inequality R1, R2, R3 in (α_4) which gives rise to the minimum value of z arises from inequalities R1 and R2, R3 in (α_3) these latter inequalities can be solved as equations to give X_2 i.e.

$$X_2 = 3/5$$

R2, R3 arises from R2 and R3 in (α). Solving these inequalities as equations gives $X_1 = 3/5$

Finally since $z = x_1 + x_2 + x_3$

$$x_3 = 0$$

5.2 The IP Problem

Eliminating x_1 between R2 and R3 gives

$$2x_2 - z \leq b_2$$

Eliminating x_1 between R3 and R4 gives

$$x_2 - z \leq 0$$

The Resultant problem after eliminating x_1 is then

$-3x_2 - z \leq -b_1$	R1
$2x_2 - z \leq b_2$	R2, R3
$x_2 - z \leq 0$	R3, R4
$-x_2 \leq 0$	R5

The elimination of x_1 clearly involves special cases and is no different from Fourier - Motzkin elimination.

Eliminating x_2 between R1 and R2, R3 gives

$$-5z \text{ [illegible] } \equiv 0 \pmod 2$$

$$\vee \quad -5z \leq -2b_1 + 3b_2 \text{ [illegible] } \equiv 1 \pmod 2$$

Eliminating x_2 between R1 and R3, R4 gives

$$-4z \leq -b_1$$

Eliminating x_2 between R2, R3 and R5 gives

[illegible]

Eliminating x_2 between R3, R4 and R5 gives

The inequalities and congruences in the problem can now be written as :

$$(\beta_1)\quad 4z \geq [illegible] \left(5z \geq 2 [illegible] \equiv 0 \pmod 2 \;\vee\; 5z \geq [illegible] 3. [illegible] \right)$$

Taking the specific values for b_1 and b_2 of $b_1 = 3$ and $b_2 = 0$ the above expression gives (after eliminating obvious redundancies)

$$(\beta_2)\quad 4z \geq 3. \left(5z \geq [illegible] z \equiv 0 \pmod 2 \;\vee\; 5z \geq 7. [illegible] \right)$$

The minimum value of Z satisfying this expression is 2.

This arises from the expression obtained after eliminating X_2 between R1 and R2, R3 in (β_1)

These inequalities give

$$-3X_2 \leq -1$$

$$\text{and} \qquad 2X_2 \leq 2$$

$$\text{i.e.} \qquad X_2 = 1$$

Considering R2, and R3 of (α_2) gives

$$-X_1 \leq -1$$

$$X_1 \leq 1$$

$$\text{i.e.} \qquad X_1 = 1$$

The optimal integer solution is therefore

$$X_1 = X_2 = 1 \qquad X_3 = 0$$

$$\text{giving} \quad Z = 2$$

6. COMMENTS ON THE ALGORITHMS

The computational difficulties in solving an integer programming problem by the above algorithm will clearly depend very critically on the size of coefficients in the problem. For large coefficients the number of inequalities and congruences generated in (B) of Section 4·2 could be enormous. It is conceivable, however, that the method could prove practical for restricted classes of IP problem.

The algorithm can be used as a method of generating all solutions to an IP problem. It can also be used to obtain the solution in terms of the objective or right-hand-side coefficients in an analogous fashion to the way the Fourier - Motzkin method does this for LP problems, Kohler describes how this may be done by Fourier - Motzkin elimination.

To logicians the interest in both algorithms lies in their demonstration of the decidability of the respective formal theories. To demonstrate decidability it is only necessary to show the finiteness of the procedure. Such theoretical finiteness gives no indication of the amount of computation which may result in practice.

It is worth noting, however, that many IP algorithms do not themselves demonstrate decidability. They sometimes demand that each integer variable be given finite upper and lower bounds. The lattice of integer points then obviously becomes finite. The lattice of integer points satisfying the constraints of the numerical example (α_1) in Section 6 is infinite. Many IP algorithms would be incapable of solving this problem unless extra constraints were added.

Finally the IP algorithm described here might suggest a possible branching strategy that could be exploited to solve IP problems. In general when a variable is eliminated a disjunction of statements is obtained. Each statement in this disjunction could form a possible branching direction.

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