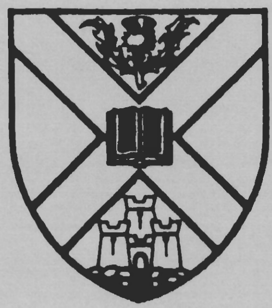

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A DUALITY RELATIONSHIP FOR INTEGER PROGRAMMES*

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Abstract

Two procedures for solving Integer Programmes (IPs) are described. When applied respectively to IPs whose Linear Programming relaxations are duals a correspondence between the two procedures is maintained. This correspondence is proved and shown to result in two reduced models with the same coefficients. One model (the Primal) reduces to a disjunction of inequalities and congruences. The other model (the Dual) reduces to a single equation and a series of homogeneous linear congruences. A numerical example is given.

*A version of this paper was given at the XI International Symposium on Mathematical Programming under the title "The Dual of a Integer Programme". We use a slightly less presumptuous title here.

1. INTRODUCTION

It has been shown by Lee (3) and Williams (5) how integer variables can be eliminated from a system of inequalities using the decision procedure of Presburger (4). By successively eliminating integer variables between inequalities and congruences an Integer Programme (IP) is reduced to a disjunction of inequalities and congruences involving a single variable representing the objective function. This procedure can be regarded as a generalisation of Fourier-Motzkin elimination for Linear Programmes. The version of the procedure described here represents a simplification of that in (3) and (5) which allows a pattern to become apparent.

Another method of solving IPs has been described by Williams (6). This involves successively eliminating constraints reducing the model to a Knapsack Problem together with a series of homogeneous linear congruences.

If the two procedures are applied to models where LP relaxations are duals then the resultant reduced models bear a close relationship to each other involving the same coefficients. This relationship is analogous to the relationship between final tableaus of the Primal and Dual Simplex algorithms when applied to dual models. Hence the original IP models can, in a sense, be regarded as duals.

2. THE STANDARD FORM OF THE PRIMAL MODEL

We will consider a pure IP model in the form:

Minimise $\sum_{j \in J'} c_j x_j$

PI: s.t. $\sum_{j \in J'} a_{ij} x_j \geqslant b_i$ for all $i \in I'$

$x_j \geqslant 0$ and integer for all $j \in J'$

Where $I' = \{1,2,\ldots,m\}$, $J' = \{1,2,\ldots,n\}$

It is convenient to convert this model into the form:

Minimise $\quad x_0$

s.t. $\quad -\sum_{j\in J'} c_j x_j + x_0 \geqslant 0$

P2 : $\quad \sum_{j\in J'} a_{ij}x_j \geqslant b_i$ for all $i \in I'$

$\qquad x_j \geqslant 0$ and integer for all $j \in J'$

This is more compactly expresssd as:

Minimise $\quad x_0$

P3: $\quad$ s.t. $\sum_{j\in J} a_{ij} x_j \geqslant b_i$ for all $i \in I$

$\qquad x_j \geqslant 0$ for all $j \in J'$

$\qquad x_j \equiv 0 \pmod{1}$ for all $j \in J$

where $\quad I = \{0,1,2,\ldots,m\}$

$\qquad J = \{0,1,2,\ldots,n\}$

and $\quad a_{00} = 1$, $a_{0j} = -c_j$ for all $j \in J - \{0\}$

In the course of the elimination procedure described here we will generate a more general class of models than that above, involving more complicated linear congruences. It is therefore more convenient to describe the above model as a special case of this more general form of model.

Minimise $\quad x_0$

$$\text{P:}\quad \text{s.t.} \bigvee_{h_r \in H_r} \begin{cases} \sum_{j\in J} a_{ij} x_j \geqslant b_i + \sum_{r\in R} f_{ir} h_r & \text{for all } i \in I \\ x_j \geqslant 0 & \text{for all } j \in J - \{0\} \\ \sum_{j \in J} d_{kj} x_j \equiv e_k + \sum_{r\in R} g_{kr} h_r \pmod{M_k} & \text{for all } k \in K \end{cases}$$

Where f_{ir} are non-negative coefficients

and the congruences $k \in K$ taken together imply:

$$\sum_{j \in J} a_{ij} x_j \equiv b_i + \sum_{r \in R} f_{ir} h_r \pmod{M} \text{ for all } i \in I \quad (1$$

R, K and H_r are suitable finite index sets taken from Z. We assume all coefficients are integral.

Note that the model P involves a conjunction of a <u>disjunction</u> of contraints in the form of inequalities and congruences. We prefer to keep the model in this Conjunctive Normal Form in constrast to the treatment of (3) and (5). It is important to note however that each inequality indexed by I and each congruence indexed by K is really a disjunction of inequalities or congruences over values of h_r for $h_r \in H_r$

Such a disjunction of inequalities, for a particular i, is a tightening of the simple inequality:

$$\sum_{j \in J} a_{ij} x_j \geqslant b_i$$

It is shown in (7) that a set of linear congruences implies a modulus and residue for any given linear expression. We stipulate in P that the congruences $k \in K$ must be such that the implied modulus for <u>each</u> $i \in I$ is M.

P3 above is clearly a special case of P when:

$R = \emptyset$, $K = j$, $M_k = 1$ for all $k \in K$,

$d_{kj} = 0$ for $k \neq j$, $d_{kk} = 1$, $e_k = 0$ for all $k \in K$

3. THE ELIMINATION OF A VARIABLE FROM THE PRIMAL MODEL

The algorithm described in (3) and (5) proceeds by successively eliminating the variables $x_1, x_2, \ldots, x_n$ from the model. By keeping the model in Conjunctive Normal Form we manage here to simplify this process considerably as well as maintaining a pattern which corresponds to the dual method described in section 5.

Since there is complete flexibility in the order for eliminating the variables we will describe the elimination of a general variable x_p from model P.

STEP 1 We take $a = \operatorname{lcm}_i (a_{ip})$ and substitute y for ax_p throughout the inequalities. In addition we make this substitution in the congruences, multiplying the modulus and terms by a factor if necessary, in order to keep integral coefficients. The congruence $y \equiv 0 (\text{mod } a)$ is also appended. For convenience we will name the new coefficients of y in the inequalities a_{ip} remembering that now $\gcd_i (a_{ip}) = 1$. The implied congruences (1) will now be:

$$a_{ip}y + \sum_{j \in J - \{p\}} a_{ij}x_j \equiv b_i + \sum_{r \in R} f_{ir} h_r \ (\text{mod } M_i) \text{ for all } i \in I \quad (2)$$

STEP 2 By means of the Generalised Chinese Remainder Theorem we can aggregate the congruences to produce at most one congruence involving the variable y together with other congruences independent of y. Full details of the Generalised Chinese Remainder Theorum can be found in, for example Dickson (1), and are therefore omitted here. For simplicity we will write the single congruence involving y as:

$$\alpha_p y + \sum_{j \in J - \{p\}} \alpha_j x_j \equiv \beta \pmod{N} \qquad (3)$$

where β is an expresion involving the constant and h_r terms.

It is shown in (7) that if a series of congruences:

$$\sum_{j \in J} a_{ij} x_j \equiv b_i \pmod{M_i} \text{ for all } i \in I$$

imply a congruence

$$\sum_{j \in J} c_j x_j \equiv d \pmod{M}$$

there exist multipliers λ_i such that:

$$\sum_{i \in I} a_{ij} \lambda_i \equiv c_j \pmod{M} \text{ for all } j \in J$$

$$\sum_{i \in I} b_i \lambda_i \equiv d \pmod{M}$$

$$M = \gcd_i (M_i \lambda_i)$$

Since for congruences (2) $\gcd_i (a_{ip}) = 1$ these congruences can be added in suitable multiples λ_i to imply a congruence (mod M) where y has a coefficient of 1.

Therefore there exist multipliers μ, μ_1, μ_2 etc. which can be applied to congruence (3) and the remaining congruences (not involving y) respectivily in this transformed model P to imply the congruence in which y has a coefficient of 1.

Hence: $\mu \alpha_p \equiv 1 \pmod{M}, \; \mu N \equiv 0 \pmod{M}$

Multiplying (3) through by μ Model P can now be expressed as:

Minimise x_0

s.t.

$$\text{P1} \quad \bigvee_{h_r \in H_r} \begin{array}{l} a_{ip} y + \sum_{j \in J-\{p\}} a_{ij} x_j \geqslant b_i + \sum_{r \in R} f_{ir} h_r \text{ for all } i \in I \\ y + \sum_{j \in J-\{p\}} \alpha_j x_j \equiv \beta \pmod{M} \end{array}$$

& other congruences not involving y

where gcd $(a_{ip}) = 1$ and the terms $\mu \alpha_j$, $\mu \beta$ have, for convenience, been renamed α_j and β respectively.

<u>STEP 3</u> We partition the set of inequalities I in three subsets:

$$Z_p = \{i : a_{ip} = 0\}$$
$$L_p = \{i : a_{ip} > 0\}$$
$$G_p = \{i : a_{ip} < 0\}$$

Three cases are distinguished in order to eliminate y.

(i) $Z_p = I$, (ii) $L_p = \phi$ or $G_p = \phi$, (iii) $L_p \neq \phi$, $G_p \neq \phi$

In case (i) we remove the congruence (3) involving y.

In case (ii) we remove congruence (3) and those inequalities for which $a_{ip} \neq 0$.

Case (iii) is the non-trivial case.

Let $A = a'_{ip} a_{ip} = \underset{i \in L_p \cup G_p}{\mathrm{lcm}} (a_{ip})$ for all $i \in L_p \cup G_p$.

We then take each $i \in L_p$ together with each $i \in G_p$.

Let e.g. $i_1 \in L_p$, $i_2 \in G_p$. The corresponding inequalities can be written:

$$a'_{i_2 p}\Big(-b_{i_2} + \sum_{j \in J-\{p\}} a_{i_2 j} x_j - \sum_{r \in R} f_{i_2 r} h_r\Big) \geqslant Ay \geqslant a'_{i_1 p}\Big(b_{i_1} - \sum_{j \in J-\{p\}} a_{i_1 p} x_j + \sum_{r \in R} f_{i_1 r} h_r\Big) \quad (4)$$

and the congruence involving y:

$$Ay \equiv A\Big(-\sum_{j \in J-\{p\}} \alpha_j x_j + \beta\Big) \pmod{AM} \quad (5)$$

substituting Z for A_y gives :

$$a'_{i_2 p}\Big(-b_{i_2} + \sum_{j \in J-\{p\}} a_{i_2 j} x_j - \sum_{r \in R} f_{i_2 r} h_r\Big) \geqslant Z \geqslant a'_{i_1 p}\Big(b_i - \sum_{j \in J-\{p\}} a_{i_1 p} x_j + \sum_{r \in R} f_{i_1 r} h_r\Big) \quad (6)$$

$$Z \equiv A\Big(-\sum_{j \in J-\{p\}} \alpha_j x_j + \beta\Big) \pmod{AM} \quad (7)$$

$$Z \equiv 0 \pmod{A} \quad (8)$$

Congruence (8) is clearly redundant being implied by (7).

(6) and (7) can then be written:

$$a'_{i_2 p}\Big(-b_{i_2} + \sum_{j\in J-\{p\}} a_{i_2 j}x_j - \sum_{r\in R} f_{i_2 r}h_r\Big) + A\Big(\sum_{j\in J-\{p\}}\alpha_j x_j - \beta\Big)$$

$$\geq AMl$$

$$\geq a'_{i_1 p}\Big(b_{i_1} - \sum_{j\in J-\{p\}} a_{i_1 j}x_j + \sum_{r\in R} f_{i_1 r}h_r\Big) + A\Big(\sum_{j\in J-\{p\}}\alpha_j x_j - \beta\Big) \qquad (9$$

for some integer l.

Following the discussion in (3), (5) or the discussion of Presburger's precedure in Kreiseland Krivine (2), (9) can be written:

$$\bigvee_{\substack{h_{|R|+1}\in\{0,1,\dots AM-1\}\\ h_{|R|+2}\in\{0,1,\dots AM-1\}}} \begin{aligned} &\sum_{j\in J-\{p\}}(a'_{i_1 p}a_{i_1 j} + a'_{i_2 p}a_{i_2 j})x_j \geq (a'_{i_1 p}b_{i_1} + a'_{i_2 p}b_{i_2}) \\ &\qquad + \sum_{r\in R}(a'_{i_1 p}f_{i_1 r} + a'_{i_2 p}f_{i_2 r})h_r \\ &\qquad + h_{|R|+1} + h_{|R|+2} \\ &a'_{i_1 p}\sum_{j\in J-\{p\}} a_{i_1 j}x_j \equiv a'_{i_1 p}b_{i_1} + a'_{i_1 p}\sum_{r\in R} f_{i_1 r}h_r - h_{|R|+1} \pmod{AM} \\ &a'_{i_2 p}\sum_{j\in J-\{p\}} a_{i_2 j}x_j \equiv a'_{i_2 p}b_{i_2} + a'_{i_2 p}\sum_{r\in R} f_{i_2 r}h_r - h_{|R|+2} \pmod{AM} \end{aligned}$$

($|R|$ is used to represent the highest member of the index set R)

For the purposes of the algorithm it is only necessary to include one of the congruences above and the corresponding one of the terms $h_{|R|+1}$ $h_{|R|+2}$. In order, however, to demonstrate the correspondence with the transomations of the Dual model we retain both congruences and terms $h_{|R|+1}$, $h_{|R|+2}$.

It is possible to simplify the above disjunction by noting that the congruences can only hold for certain values of $h_{|R|+1}$ and $h_{|R|+2}$.
Since all terms apart from $h_{|R|+1}$ in the first of the two congruences above are multiples of a'_{i_1p} we need only consider values of $h_{|R|+1}$ which are multiples of a'_{i_1p}. Similarly we need only consider values of $h_{|R|+2}$ which are multiples of a'_{i_2p}. The fact that the congruences guarantee congruence (2) for each $i \in I$ allows us to further restrict $h_{|R|+1}$ to multiples of $a'_{i_1p}M$ and $h_{|R|+2}$ to multiples of $a'_{i_2p}M$.

For convenience we rename $h_{|R|+1}$ as $a'_{i_2p}Mh_{|R|+1}$ and $h_{|R|+2}$ as $a'_{i_2p}M\,h_{|R|+2}$.
The above disjunction can then be written as:

$$\sum_{j\in J-\{p\}}(a'_{i_1p}a_{i_1j}+a'_{i_2p}a_{i_2j})x_j \geqslant (a'_{i_1p}b_{i_1}+a'_{i_2p}b_{i_2}) + \sum_{r\in R}(a'_{i_1p}f_{i_1r}+a'_{i_2p}f_{i_2r})h_r + a'_{i_1p}M\,h_{|R|+1}+a'_{i_2p}M\,h_{|R|+2}$$

$$\bigvee_{\substack{h_{|R|+1}\in\{0,1,\dots,a_{i_2}-1\}\\ h_{|R|+2}\in\{0,1,\dots,a_{i_1}-1\}}}\sum_{j\in J-\{p\}}a_{i_1p}x_j \equiv b_{i_1}+\sum_{r\in R}f_{i_1r}h_r - M\,h_{|R|+1} \pmod{a_{i_1p}M}$$

$$\sum_{j\in J-\{p\}}a_{i_2p}x_j \equiv b_{i_2}+\sum_{r\in R}f_{i_2r}h_r - M\,h_{|R|+2} \pmod{a_{i_2p}M}$$

The two congruences generated clearly imply that the inequality generated can also be regarded as a congruence modulus AM.

The above procedure is carried out between each pair from Lp and Gp to generate an inequality and pair of congruences in each case. We also include the inequality $y \geqslant 0$ as a special case of Lp.

4. THE STANDARD FORM OF THE DUAL MODEL

We will consider a pure IP model in the form:

Maximise $\sum_{i \in I'} b_i y_i$

D1 :

s.t. $\sum_{i \in I'} a_{ij} y_i \leq c_j$ for all $j \in J'$

$y_i \geqslant 0$ and integer for all $i \in I'$

I' and J' are index sets as defined in section 2.

Clearly we have chosen D1 as an IP whose linear programming relaxation is the dual of the linear programming relaxation of P1.

Corresponding to the conversions in section 2 we can convert D1 to the form:

Maximise $\sum_{i \in I'} b_i y_i$

D2 : s.t. $\sum_{i \in I'} a_{ij} y_i - c_j y_0 \leq 0$ for all $j \in J'$

$y_0 = 1$

$y_i \geqslant 0$ and integer for all $i \in I$

or more compactly as:

$$\text{D3:}\quad \begin{array}{ll} \text{Maximise} & \sum_{i \in I} b_i y_i \\ \text{s.t.} & -c_j y_0 + \sum_{i \in I} a_{ij} y_i + z_j = 0 \quad \text{for all } j \in J \\ & y_0 = 1 \\ & y_i \equiv 0 \pmod{1} \quad \text{for all } i \in I \\ & z_j \equiv 0 \pmod{1} \quad \text{for all } j \in J \end{array}$$

The sets I, J and the other new coefficients are as defined in section 1.

In the course of the procedure we will generate more congruences and we will therefore consider the more general form of model:

$$\text{D:}\quad \begin{array}{ll} \text{Maximise} & \frac{1}{M} \sum_{i \in I} b_i y_i \\ \text{s.t.} & \sum_{i \in I} a_{i0} y_i = M \\ & \sum_{i \in I} a_{ij} y_i + M z_j = 0 \quad \text{for all } j \in J' \\ & \sum_{i \in I} f_{ir} y_i \equiv 0 \pmod{M} \quad \text{for all } r \in R \end{array}$$

All coefficients are integral and f_{ir} are non-negative.

5. THE ELIMINATION OF A CONSTRAINT FROM THE DUAL MODEL

The algorithm discribed in (6) proceeds by successively eliminating the homogeneous "balance" constraints from D. There is complete flexibility over the order in which we eliminate these constraints. Therefore we will describe the elimination of a general constraint $p \in J$. In order to mirror the treatment in section 3 we will number the steps.

STEP 1 We divide constraint p through by the greatest common divisor of the coefficients a_{ip} . For convenience we rename the new coefficients a_{ip} .

Step 2 of Section 3 has no counterpart and is therefore omitted.

STEP 3 Following the partitioning of I into subsets in section 3 we distinguish three cases:

(i) $Z_p = I$, (ii) $L_p = \emptyset$ or $G_p = \emptyset$, (iii) $L_p \neq \emptyset$, $G_p \neq 0$.

In case (i) the constraint is already empty and can be removed.

In case (ii) we remove all variables y_i for which $a_{ip} \neq 0$.

For case (iii) we introduce new variables $V_{\ell m}$ to satisfy the conditions:

$$\sum_m V_{im} = a_{ip} y_i \text{ for all } i \in L_p \qquad (10)$$

$$\sum_\ell V_{\ell j} = a_{jp} y_j \text{ for all } j \in G_p \qquad (11)$$

$$\sum_m V_{|L_p|+1, m} = M z_p \qquad (12)$$

where l and m are indexed over sets of cardinalities $|L_p|+1$ and $|G_p|$ respectively.

Substituting (10), (11) and (12) for the y_i into D, constraint p disappears.

In order to illustrate the general form of the transformed model we, take $i_1 \in L_p$ and $i_2 \in G_p$ and consider the coefficients of the new variable $v_{i_1 i_2}$.

The objective coefficient will be:

$$\frac{1}{M}\left(\frac{b_{i_1}}{a_{i_1 p}} + \frac{b_{i_2}}{a_{i_2 p}}\right)$$

The coefficient in a general constraint $j \in J$ will be:

$$\left(\frac{a_{i_1 j}}{a_{i_1 p}} + \frac{a_{i_2 j}}{a_{i_2 p}}\right)$$

the coefficient in a general congruence $r \in R$ will be:

$$\left(\frac{f_{i_1 r}}{a_{i_1 p}} + \frac{f_{i_2 r}}{a_{i_2 p}}\right)$$

It is also necessary to impose extra congruences to guarantee the integrality of y_i after the substitution. These are:

$$\sum_m v_{im} \equiv 0 \ (\mathrm{mod}\ a_{ip}) \ \text{for all } i \in L_p \qquad (12)$$

$$\sum_l v_{lj} \equiv 0 \ (\mathrm{mod}\ a_{jp}) \ \text{for all } j \in G_p \qquad (13)$$

$$\sum_m v_{|L_p|+1,m} \equiv 0 \ (\mathrm{mod}\ M)$$

In order to produce integral coefficients throughout the transformed model we multiply the objective, constraints and congruences through by

$$A = a'_{ip}\, a_{ip} = \underset{i \in L_p \cup G_p}{\mathrm{lcm}} (a_{ip})$$

and replace this coefficient M throughout by AM. This gives the transformed coefficients as :

Objective: $\frac{1}{AM}\left(a'_{i_1 p} b_{i_1} + a'_{i_2 p} b_{i_2}\right)$

Constraints: $\left(a'_{i_1 p} a_{i_1 p} + a'_{i_2 p} a_{i_2 j}\right)$ for all $j \in J$

Original Congruences: $\left(a'_{i_1 p} f_{i_1 r} + a'_{i_2 p} f_{i_2 r}\right)$ for all $r \in R$
(Modulus AM)

New Congruences: $a'_{i_1 p} M$
(Modulus AM)

$a'_{i_2 p} M$

The coefficients appearing above for the new variable $V_{i_1 i_2}$ are clearly the same as those appearing in the new inequality of the Primal model once the corresponding variable has been eliminated.

5. A NUMERICAL EXAMPLE

The Primal Model

Minimise x_0

s.t
$$\begin{array}{rl}
x_0 - 18x_1 + 3x_2 & \geq 0 \\
4x_1 + x_2 & \geq 5 \\
9x_1 - 2x_2 & \geq -1 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
x_1 & \equiv 0 \pmod{1} \\
x_2 & \equiv 0 \pmod{1}
\end{array}$$

Eliminating x_1 (and simplifying the resulting congruences) we obtain:

$$\bigvee_{\substack{h_1 \in H_1 \\ h_2 \in H_2 \\ h_3 \in H_3}} \left\{ \begin{array}{rl}
2x_0 + 15x_2 & \geq 45 + 2h_1 + 9h_2 \\
2x_0 - 2x_2 & \geq -4 + 2h_1 + 4h_3 \\
2x_0 + 6x_2 & \geq 0 + 2h_1 + 30h_4 \\
x_2 & \geq 0 \\
x_2 & \equiv 5 + 9h_2 + 4h_3 \pmod{36} \\
x_0 & \equiv 1 + h_1 + h_2 \pmod{2} \\
x_0 & \equiv 3 + h_1 + 6h_3 \pmod{9}
\end{array} \right.$$

where $H_1 = \{0, 1, \ldots, 17\}$, $H_2 = \{0, 1, \ldots, 3\}$, $H_3 = \{0, 1, \ldots, 8\}$

Note that the congruences imply that each of the new inequalities can also be regarded as congruences modulus 36. This can be seen by applying multipliers of 15, 18, 20 respectively to the congruences to imply that the first inequality can be treated as a congruence modulus 36. For the ~~second~~ inequality the appropriate multipliers are 34,18,20. For the third inequality the appropriate multipliers are 6,18,20.

Eliminating x_2 (and simplifying the resulting congruences) we obtain:-

$$\bigvee \begin{cases} 34x_0 \geqslant 30 + 34h_1 + 18h_2 + 60h_3 + 72h_4 + 540h_5 \\ 40x_0 \geqslant -60 + 40h_1 + 60h_3 + 540h_5 + 180h_6 \\ 30x_0 \geqslant -60 + 30h_1 + 60h_3 - 540h_5 + 30h_7 \end{cases}$$

$$x_0 \equiv 1065 + h_1 + 117h_2 + 60h_3 + 1008h_4 + 90h_6 \pmod{1080}$$

$$0 \equiv 5 + 9h_2 + 6h_3 + 33h_7 \pmod{36}$$

$$0 \equiv h_2 + 2h_3 + h_4 + 2h_6 \pmod{3}$$

$$0 \equiv 1 + h_3 + h_5 + h_6 \pmod{2}$$

$h_1 \in H_1$
$h_2 \in H_2$
$h_3 \in H_3$
$h_4 \in H_4$
$h_5 \in H_5$
$h_6 \in H_6$
$h_7 \in H_7$

where $H_4 \in \{0, 1, \ldots, 14\}$, $H_5 \in \{0, 1\}$, $H_6 \{0, 1, \ldots, 5\}$ $H_7 \in \{0, 1, \ldots, 35\}$

Note that the congruences imply that the inequalities can each be regarded as a congruence modulus 1080. The congruence multipliers for the first inequality are 34, 0, 360, 540. For the second inequality they are 40, 0, 720, 540. For the third inequality they are 30, 1050, 0, 540.

The solution to the above model (minimising x_6) is:

$x_0 = 3$, $h_1 = h_2 = h_3 = h_5 = 0$, $h_4 = 1$, $h_6 = 1$, $h_7 = 5$.

This yields the optimal solution to the original model:

$x_0 = 3$, $x_1 = 1$, $x_2 = 5$.

The Dual Model

Maximise $5y_1 - y_2$

s.t.
$$y_0 = 1$$
$$-18y_0 + 4y_1 + 9y_2 + y_3 = 0$$
$$3y_0 + y_1 - 2y_2 + y_4 = 0$$

$$y_0, y_1, y_2, y_3, y_4 \geqslant 0$$ and integral.

Eliminating the first homogeneous constraint we obtain:

Maximise $\frac{1}{36}(45v_1 - 4v_2)$

s.t.

$$
\begin{aligned}
2v_1 + 2v_2 + 2v_3 & = 36 \\
15v_1 - 2v_2 + 6v_3 + 36y_4 & = 0 \\
v_1 + v_2 + v_3 & \equiv 0 \pmod{18} \\
v_1 & \equiv 0 \pmod{4} \\
v_2 & \equiv 0 \pmod{9}
\end{aligned}
$$

Eliminating the second homogeneous constraint we obtain:

Maximise $\frac{1}{1080}(30w_1 - 60w_2 - 60w_3)$

s.t.

$$
\begin{aligned}
34w_1 - 60w_2 - 60w_3 & = 1080 \\
17w_1 + 20w_2 + 15w_3 & \equiv 0 \pmod{540} \\
2w_1 & \equiv 0 \pmod{120} \\
15w_1 + 15w_2 + 15w_3 & \equiv 0 \pmod{270} \\
w_1 & \equiv 0 \pmod{15} \\
w_1 + w_2 + w_3 & \equiv 0 \pmod{2} \\
w_2 & \equiv 0 \pmod{6} \\
w_3 & \equiv 0 \pmod{36}
\end{aligned}
$$

These congruences can be simplified but are left in this form in order to demonstrate the correspondence with the Primal Model. If the congruences are all expressed modulus 1080 we have the same coefficients as in the Primal Model.

The optimal solution to this model is:

$w_1 = 0$, $w_2 = 0$, $w_3 = 36$, Objective = -2 leading to the optimal solution to the original model:

$y_o = 1, \ y_1 = 0, \ y_2 = 2, \ y_3 = 0, \ y_4 = 1,$ Objective = -2.

6. THE CORRESPONDENCE BETWEEN PRIMAL AND DUAL MODELS

Given a Primal model in the form P it is clearly possible to construct a Dual model D. Whatsmore if we eliminate variables from P and the corresponding constraints from D as described in sections 3 and 4 this correspondence is shown to be maintained.

Unfortunately given a Dual model D the Primal model P is not defined uniquely. For example the (infeasible) Dual model:

Maximise $\frac{1}{30} \cdot 96y$

s.t.

$$24y = 30$$
$$y \equiv 0 \quad \text{(mod 6)}$$
$$y \equiv 0 \quad \text{(mod 15)}$$

has alternative Primal models.

Minimise x

s.t.

$$24x \geqslant 96 + 5h_1 + 2h_2$$
$$\bigvee_{\substack{h_1 \in \{0,1,2,3,4\} \\ h_2 \in \{0,1\}}} \begin{array}{lll} 2x \equiv 6 + h_1 & & (\text{mod } 6) \\ 7x \equiv 33 & + h_2 & (\text{mod } 15) \end{array}$$

OR

Minimise x

s.t.

$$\bigvee_{\substack{h_1 \in \{0,1,2,3,4\} \\ h_2 \in \{0,1\}}} \begin{array}{lll} 24x \geq 96 + 5h_1 + 2h_2 & & \\ 6x \equiv 6 + h_1 & & (\text{mod } 6) \\ 12x \equiv 3 + h_2 & & (\text{mod } 15) \end{array}$$

since in both models the congruences together imply that the inequality can be treated as a congruence modulus 30.

7. FURTHER OBSERVATIONS

In a subsequent paper it is hoped to investigate the relationship between the solutions of the Primal and Dual models. This relationship is clearly not straightforward given the ambiguity in the Primal Model corresponding to the particular Dual.

Another area for further investigation is the form of the value function determined by the solution to the Primal model. The elimination of variables is obviously unchanged for any right-hand-side coefficients. Therefore the optimal value of the objective function can be given in terms of these right-hand-side coefficients using a disjunction of inequalities and congruences.

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