Point Processes with Contagion and an Application to Credit Risk

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Abstract

We introduce a new point process, the dynamic contagion process, by generalising the self excited Hawkes process (with exponential decay) by Hawkes (1971) and the Cox process with shot noise intensity by Dassios and Jang (2003). Our process includes both self excited and externally excited jumps, which can be used to model the dynamic contagion impact from endogenous and exogenous factors of the underlying system. We have systematically analysed the theoretical properties of this new process, based on the piecewise deterministic Markov process theory developed by Davis (1984), and the extension of the martingale methodology used by Dassios and Jang (2003). The analytic expressions of the Laplace transform of the intensity process and probability generating function of the point process have been derived. An explicit example of specified jumps with exponential distributions is also given. The object of this study is to produce a general mathematical framework for modelling the dependence structure of arriving events with dynamic contagion, which has the potential to be applicable to a variety of problems in economics, finance and insurance, such as credit risk and catastrophe risk. We provide an application of this process to credit risk, and the simulation algorithm for further industrial implementation and statistical analysis.

Dynamic Contagion Process

Mathematical Definition

The **dynamic contagion process** is a point process N_t, such that,

$$P\left\{N_{t+\Delta t} - N_t = 1 \middle| N_t\right\} = \lambda_t \Delta t + o(\Delta t), \quad P\left\{N_{t+\Delta t} - N_t > 1 \middle| N_t\right\} = o(\Delta t),$$

where Δt is a sufficient small time interval, and the non-negative intensity process λ_t follows the piecewise deterministic dynamics with positive jumps,

$$\lambda_t = a + (\lambda_0 - a)\, e^{-\delta t} + \int_0^t e^{-\delta(t-s)} \mathrm{d}Y_s^{(1)} + \int_0^t e^{-\delta(t-s)} \mathrm{d}Y_s^{(2)},$$

where

- $a \geq 0$ is the reversion level;
- $\lambda_0 > 0$ is the initial value of λ_t;
- $\delta > 0$ is the constant rate of exponential decay;
- $Y_t^{(1)} = \sum_{i=1}^{M_t} Y_i^{(1)}$, $\left\{Y_i^{(1)}\right\}_{i=1,2,\ldots}$ is a sequence of independent identical distributed positive random variables (externally excited jumps) with distribution function $H(y), y > 0$, and M_t is a Poisson process with constant intensity $\rho > 0$;
- $Y_t^{(2)} = \sum_{j=1}^{N_t} Y_j^{(2)}$, $\left\{Y_j^{(2)}\right\}_{j=1,2,\ldots}$ is a sequence of independent identical distributed positive random variables (self excited jumps) with distribution function $G(y), y > 0$;
- The sequences $\left\{Y_i^{(1)}\right\}_{i=1,2,\ldots}$, $\left\{Y_j^{(2)}\right\}_{j=1,2,\ldots}$ and the point process M_t are assumed to be independent of each other.

The infinitesimal generator of the dynamic contagion process (λ_t, N_t, t) acting on a function $f(\lambda, n, t)$ within its domain $\Omega(\mathcal{A})$ is given by

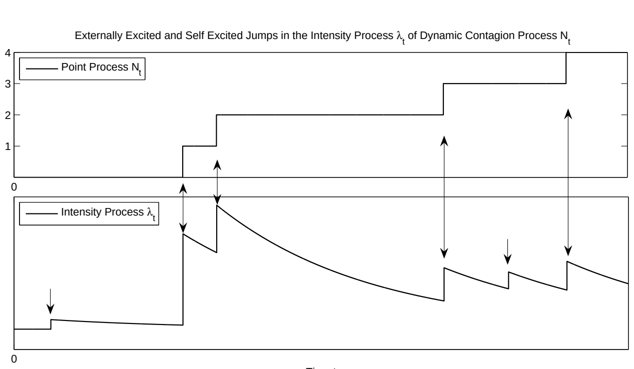
$$\begin{aligned}\mathcal{A}f(\lambda,n,t) &= \frac{\partial f}{\partial t} + \delta(a-\lambda)\frac{\partial f}{\partial \lambda} + \rho\left(\int_0^\infty f(\lambda+y,n,t)\mathrm{d}H(y) - f(\lambda,n,t)\right) \\ &\quad + \lambda\left(\int_0^\infty f(\lambda+y,n+1,t)\mathrm{d}G(y) - f(\lambda,n,t)\right),\end{aligned}$$

where $\Omega(\mathcal{A})$ is the domain for the generator $\mathcal{A}$ such that $f(\lambda,n,t)$ is differentiable with respect to λ, t for all λ, n and t, and

$$\left|\int_0^\infty f(\lambda+y,n,t)\mathrm{d}H(y) - f(\lambda,n,t)\right| < \infty, \quad \left|\int_0^\infty f(\lambda+y,n+1,t)\mathrm{d}G(y) - f(\lambda,n,t)\right| < \infty.$$

Graphic Illustration

To give an intuitive picture of this new process, we present the figure below for illustrating how the externally excited jumps $Y^{(1)}$ (marked by single arrow $\downarrow$) and self excited jumps $Y^{(2)}$ (marked by double arrow $\updownarrow$) in the intensity process λ_t interact with its dynamic contagion point process N_t.



In this more general framework of the dynamic contagion process, the classic Cox process with shot noise intensity, introduced by Dassios and Jang (2003) for pricing catastrophe reinsurance and derivatives, can be recovered, by setting reversion level $a = 0$ and eliminating the self excited jumps $Y^{(2)}$; The Hawkes process (with the exponential decay), used by Errais, Giesecke and Goldberg (2009) for modelling the portfolio credit risk in a top-down approach framework, can be recovered, by setting the intensity $\rho = 0$ of the externally excited jumps $Y^{(1)}$.

Laplace Transform of the Intensity Process λ_t

The conditional Laplace transform λ_T given λ_0 at time $t = 0$, under the condition $\delta > \mu_{1_G}$, is given by

$$\mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right] = \exp\left(-\int_{\mathcal{G}_{v,1}^{-1}(T)}^{v} \frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1}\mathrm{d}u\right) \times \exp\left(-\mathcal{G}_{v,1}^{-1}(T)\lambda_0\right),$$

where

$$\mu_{1_G} =: \int_0^\infty y\mathrm{d}G(y), \qquad \mathcal{G}_{v,1}(L) =: \int_L^v \frac{\mathrm{d}u}{\delta u + \hat{g}(u) - 1}.$$

The Laplace transform of the asymptotic distribution of λ_T under the condition $\delta > \mu_{1_G}$ is given by

$$\mathbb{E}\left[e^{-v\lambda_T}\right] = \exp\left(-\int_0^v \frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1}\mathrm{d}u\right).$$

If λ_0 follows the distribution above, then the process λ_T is also stationary.

Example: Jumps with Exponential Distributions

If both the externally excited and self excited jumps follow exponential distributions, i.e. $Y^{(1)} \sim \mathrm{Exp}(\alpha)$ and $Y^{(2)} \sim \mathrm{Exp}(\beta)$, then, under the stationarity condition $\delta\beta > 1$, the stationary distribution of λ_T is given by

$$\begin{cases} a + \tilde{\Gamma}_1 + \tilde{\Gamma}_2 & \text{for } \alpha \geq \beta \\ a + \tilde{\Gamma}_3 + \tilde{B} & \text{for } \alpha < \beta \text{ and } \alpha \neq \beta - \frac{1}{\delta} \\ a + \tilde{\Gamma}_4 + \tilde{P} & \text{for } \alpha = \beta - \frac{1}{\delta} \end{cases},$$

where independent random variables

$$\tilde{\Gamma}_1 \sim \mathrm{Gamma}\left(\frac{1}{\delta}\left(a + \frac{\rho}{\delta(\alpha-\beta)+1}\right), \frac{\delta\beta-1}{\delta}\right); \tilde{\Gamma}_2 \sim \mathrm{Gamma}\left(\frac{\rho(\alpha-\beta)}{\delta(\alpha-\beta)+1}, \alpha\right);$$

$$\tilde{\Gamma}_3 \sim \mathrm{Gamma}\left(\frac{a+\rho}{\delta}, \frac{\delta\beta-1}{\delta}\right); \tilde{B} \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N_1} X_i^{(1)}, N_1 \sim \mathrm{NegBin}\left(\frac{\rho}{\delta}\frac{\beta-\alpha}{\gamma_1-\gamma_2}, \frac{\gamma_2}{\gamma_1}\right), X_i^{(1)} \sim \mathrm{Exp}(\gamma_1);$$

$$\tilde{\Gamma}_4 \sim \mathrm{Gamma}\left(\frac{a+\rho}{\delta}, \alpha\right); \tilde{P} \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N_2} X_i^{(2)}, N_2 \sim \mathrm{Poisson}\left(\frac{\rho}{\delta^2\alpha}\right), X_i^{(2)} \sim \mathrm{Exp}(\alpha);$$

and $\gamma_1 = \max\left\{\alpha, \frac{\delta\beta-1}{\delta}\right\}$, $\gamma_2 = \min\left\{\alpha, \frac{\delta\beta-1}{\delta}\right\}$. $\tilde{B}$ follows a compound negative binomial distribution with underlying exponential jumps; $\tilde{P}$ follows a compound Poisson distribution with underlying exponential jumps.

Particularly, for the non-self-excited case, λ_T follows a shifted Gamma distribution,

$$\lambda_T \stackrel{\mathcal{D}}{=} a + \tilde{\Gamma}_5,$$

where

$$\tilde{\Gamma}_5 \sim \mathrm{Gamma}\left(\frac{\rho}{\delta}, \alpha\right),$$

which recovers the result by Dassios and Jang (2003); For the Hawkes process, λ_T follows a shifted Gamma distribution,

$$\lambda_T \stackrel{\mathcal{D}}{=} a + \tilde{\Gamma}_6,$$

where

$$\tilde{\Gamma}_6 \sim \mathrm{Gamma}\left(\frac{a}{\delta}, \frac{\delta\beta-1}{\delta}\right).$$

Probability Generating Function of the Point Process N_t

The conditional probability generating function of N_T given λ_0 and $N_0 = 0$ at time $t = 0$, under the condition $\delta > \mu_{1_G}$, is given by

$$\mathbb{E}\left[\theta^{N_T}\middle|\lambda_0\right] = \exp\left(-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{a\delta u + \rho[1-\hat{h}(u)]}{1 - \delta u - \theta\hat{g}(u)}\mathrm{d}u\right) \times \exp\left(-\mathcal{G}_{0,\theta}^{-1}(T)\lambda_0\right),$$

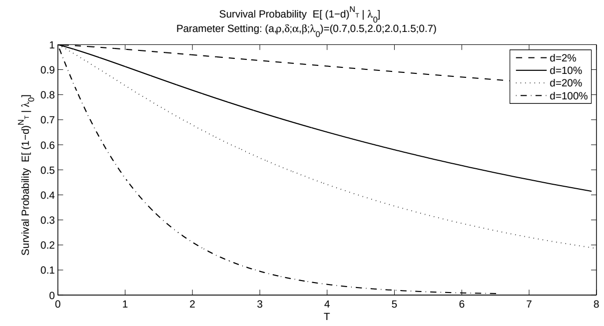
where

$$\mathcal{G}_{0,\theta}(L) =: \int_0^L \frac{\mathrm{d}u}{1 - \delta u - \theta\hat{g}(u)} \qquad 0 \leq \theta < 1.$$

An Application in Credit Risk

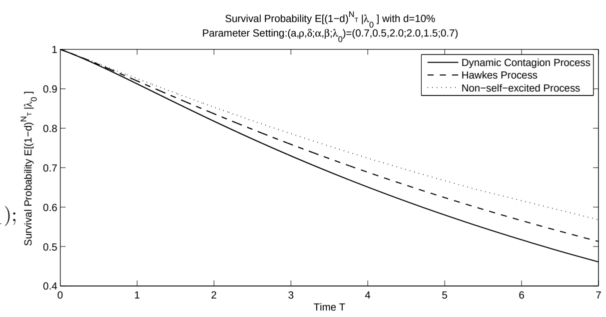
Our motivation of applying the dynamic contagion process to model the credit risk is a combination of Duffie and Singleton (1999) and Lando (1998). Duffie and Singleton (1999) introduced the affine processes to model the default intensity. Lando (1998), the extension of Jarrow, Lando and Turnbull (1997), used the state of credit ratings as an indicator of the likelihood of default, and modelled the underlying credit rating migration driven by a probability transition matrix with Cox processes in a finite-state Markov process framework. However, we go beyond this and model the bad events that can possibly lead to credit default, and the number and the intensity of these events are modelled by the dynamic contagion process.

The point process N_t is to model the number of bad events released from the underlying company. It is driven by a series of bad events $Y^{(2)}$ from itself and the common bad events $Y^{(1)}$ widely in the whole market via its intensity process λ_t. The impact of each event decays exponentially with constant rate δ. We assume each jump, or bad event, can result to default with a constant probability d, $0 < d \leq 1$, which measures and quantifies the resistance level. Therefore, the survival probability conditional on the (initial) current intensity λ_0 at time T is $P_s(T) = \mathbb{E}\left[(1-d)^{N_T}\middle|\lambda_0\right]$, which can be calculated simply by letting $\theta = 1 - d$ in the conditional probability generating function. By setting the parameters $(a, \rho, \delta; \alpha, \beta; \lambda_0) = (0.7, 0.5, 2.0; 2.0, 1.5; 0.7)$, the term structure of the survival probabilities $p_s(T)$ based on $d = 2\%$, 10%, 20% and 100% are shown in the figure below.



By contrast with Lando (1998), we possibly could consider different values of d correspond to different credit ratings, by assuming these bad events are all related to the company's credit ratings.

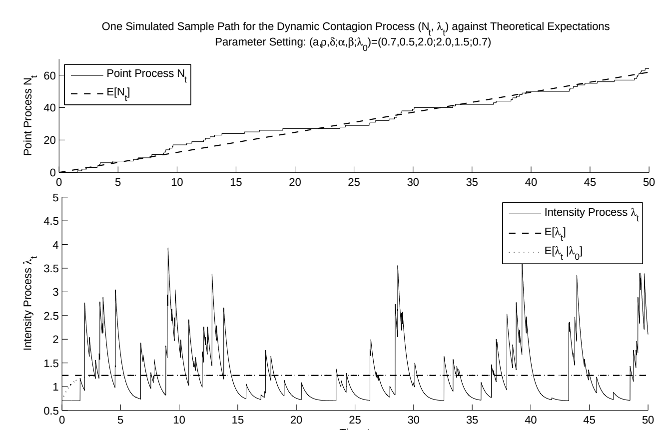
We also provide a comparison for the survival probabilities based on three main processes discussed in this paper: dynamic contagion process, Hawkes process (by setting $\rho = 0$) and non-self-exciting process (by setting $\beta = \infty$), with the same parameter setting and fixed $d = 10\%$. The results are shown in the figure below.



We can see that, the dynamic contagion process, as the most general case of the three processes, generates the lowest survival probability, and the differences between the other two processes explain the impact from the endogenous and exogenous factors respectively. This process is capable to capture more aspects of the risk, which is particularly useful for modelling the risks during the economic downturn involving more clustering bad economic events.

Monte Carlo Simulation

To make easier for further industrial applications and statistical analysis, alternatively, we derive the simulation algorithm for one sample path of the general dynamic contagion process (N_t, λ_t), which applies to any distribution assumption for jump sizes, $H(y)$ and $G(y)$ for externally and self excited jumps, respectively. Here, we use the same parameter setting under the exponential distribution assumption for the jump sizes, and for instance one simulated sample path (N_t, λ_t) with $T = 50$ is provided in the figure below. For comparison, the theoretical expectations $\mathbb{E}[\lambda_t]$, $\mathbb{E}[\lambda_t|\lambda_0]$ and $\mathbb{E}[N_t]$ are also plotted.



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