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NARROW-BAND ANALYSIS OF NONSTATIONARY PROCESSES[1]

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The behavior of averaged periodograms and cross-periodograms of a broad class of nonstationary processes is studied. The processes include nonstationary ones that are fractional of any order, as well as asymptotically stationary fractional ones. The cross-periodogram can involve two nonstationary processes of possibly different orders, or a nonstationary and an asymptotically stationary one. The averaging takes place either over the whole frequency band, or over one that degenerates slowly to zero frequency as sample size increases. In some cases it is found to make no asymptotic difference, and in particular we indicate how the behavior of the mean and variance changes across the two-dimensional space of integration orders. The results employ only local-to-zero assumptions on the spectra of the underlying weakly stationary sequences. It is shown how the results can be applied in fractional cointegration with unknown integration orders.

1. Introduction. In the analysis of time series that are believed prone to nonstationarity, the behavior of bilinear and quadratic forms is of prime interest. For univariate time series, Gaussian rules of inference lead to consideration of quadratic forms, and Gaussian methods developed by Whittle (1951) and others in stationary short-range dependent environments were extended to unit root nonstationary ones by Box and Jenkins (1971), with limit theory developed by Dickey and Fuller (1979) and many subsequent authors. In case of multivariate time series, the Gaussian approach covers not only jointly dependent modelling but also linear regression, and in either case bilinear and quadratic forms arise. Again, limit theory for stationary short-range dependent vector processes has been extended to unit roots, activity in this direction fuelled by considerable econometric interest in the possible existence of cointegrated structures, positing the existence of a linear combination of related unit root series which has short-range dependence.

The scope of time series analysis has considerably expanded with the development of methods and theory for stationary and nonstationary long-range dependent or fractional processes. A fractional view of time series regards the stationary short-range dependent and unit root processes as mere points (at $\beta = 0$ and $\beta = 1$, respectively) on the real line of processes indexed by integration order β. For univariate processes, a loose definition of integration order (the article employs a more general one) is "that degree of differencing

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needed to convert a stationary or nonstationary process to one with spectral density that is positive and continuous at zero frequency." Limit theory for Whittle estimates of parametric stationary long-range dependent series has been developed by Fox and Taqqu (1986) and others, while recently cointegration of multiple nonstationary fractional time series has been considered by Chan and Terrin (1995), Jeganathan (1999, 2001), Dolado and Marmol (1998) and others, though this topic is still in its infancy.

Narrow-band frequency domain analysis has been a major focus of the long-range dependence literature. A stationary long-range dependent univariate series is usually thought of as having a spectral pole at zero frequency, with spectral density behaving like $\lambda^{-2\beta}$ nearby, where λ indicates frequency, and $0 < \beta < \frac{1}{2}$. Methods of estimating β based on a band of frequencies around zero that degenerates slowly as sample size increases were considered by Geweke and Porter-Hudak (1983), Künsch (1986, 1987) and Robinson (1994a, b, 1995a, b), the asymptotic theory of the latter author imposing few or no conditions on spectral behaviour away from zero frequency and thereby demonstrating a signal advantage of such 'semiparametric' methods.

The main theoretical concern of Robinson (1994a) was the convergence of the discretely averaged periodogram of a univariate series, over a degenerating band of Fourier frequencies, but one of his applications of this theory was to cointegration of bivariate stationary long-range dependent series $\{y_t, z_t, t = 0, \pm 1, \ldots\}$. It was envisaged that whereas y_t and z_t each has integration order $\beta \in (0, \frac{1}{2})$, there exists an unknown ν such that the unobservable series ζ_t in

$$y_t = \nu z_t + \zeta_t \tag{1.1}$$

has integration order $\alpha < \beta$. The ζ_t by construction thus have the character of regression errors, at least after mean-correction, but there is no prior reason to suppose that they possess the classical property of orthogonality with z_t, $\mathrm{Cov}(\zeta_t, z_t) = 0$. Were y_t, z_t nonstationary, but ζ_t stationary, or "less nonstationary" than y_t, z_t, such that the signal-to-noise ratio $\sum_{t=1}^n \zeta_t^2 / \sum_{t=1}^n z_t^2$ converges stochastically to zero as sample size n tends to infinity, the least squares estimate (LSE) of ν would be consistent, as demonstrated by, for example, Stock (1987), in case y_t, z_t have a unit root but ζ_t is short-range dependent ($\alpha = 0, \beta = 1$). When y_t, z_t are stationary, however, the LSE is generally inconsistent when there is correlation between z_t and ζ_t. However, Robinson (1994a) showed that the narrow-band least squares estimate (NBLSE) of ν, namely the ratio of the real part of the averaged cross-periodogram of y_t, z_t to the averaged periodogram of z_t, averaging across the m lowest Fourier frequencies where $m \to \infty$ but $m/n \to 0$ as $n \to \infty$, is consistent for ν. This is due to the spectrum of z_t dominating that of ζ_t near zero frequency, since $\alpha < \beta$, even though the respective variances (equivalently, the spectra integrated over the whole sampling frequency band) are both finite and positive. Robinson (1994b) discussed optimal choice of m.

Cointegration of stationary long range-dependent series has been of interest in a financial context, for example for the three-dimensional vector of exchange rates between three currencies. However, financial series may also be nonstationary, as is typically believed to be the case with macroeconomic ones, while cointegration has also been of interest in other fields, such as ecology, where nonstationarity can arise, and in general not only are integration orders likely to be unknown, but also we may not even know whether or not the series is stationary. Thus, given its superiority over the LSE in stationary environments, there is interest in analyzing the performance of the NBLSE in nonstationary ones.

Cointegration provides a motivation for the theoretical contribution of the present paper, an examination of the averaged cross-periodogram, and the sample covariance, of a bivariate series, one element of which is nonstationary and the other is either nonstationary or (asymptotically) stationary. We derive and compare leading terms in the asymptotic bias and variance of these statistics, leading to a qualitative classification of behavior depending on integration orders of the time series, for example, whether the integration orders sum to less than one or greater than one is important, while the case when one of them is zero and the other unity (familiar from the unit root cointegration literature) is seen to be quite special. Our modelling of the series is notably general. They are linear filters of short-range dependent series. The filters have desirable commutativity properties and cover standard fractional differencing, and in general produce low frequency stochastic trends. Consequently, it is the low frequency behavior of the short-range dependent innovations that is important, as our results and conditions stress; in the spirit of Robinson (1994a, b) our conditions entail only mild restrictions at zero frequency and have little implication for higher frequencies.

Our results clarify the extent to which the (cross-) periodogram averaged over all Fourier frequencies, equivalently the sample (co-) variance, is approximated by the average over only frequencies near zero, possibly an asymptotically negligible proportion of the sampling frequencies. Intuitively, this is due to a dominance of low frequency contributions. When the limit distribution of the sample (co-) variance can be characterized by means of invariance principles for nonstationary fractional series, of Marinucci and Robinson (2000), we may thence simply deduce limit distributional behavior of the averaged (cross-) periodogram. When applied to cointegration, we can then characterize the limit distributions of both the LSE and NBLSE. These distributions, and rates of convergence, reflect integration orders. Over some range of these, the LSE and NBLSE have the same limit distribution and convergence rate, but over another they do not, the NBLSE suffering from less bias and consequently even converging faster.

The following section defines the basic averaged (cross-) periodogram statistic and its implementations of particular interest. Section 3 demonstrates an approach to modelling nonstationary and asymptotically stationary sequences, with derivation of useful properties. Sections 4 and 5 cover, respectively asymptotics for the mean and variance of the averaged (cross-) periodogram under this type of model. Section 6 applies the results to the LSE and NBLSE for cointegrated nonstationary series. Sections 7–9 give proofs of results of Sections 3–5, respectively.

2. The averaged cross-periodogram. For a sequence ζ_t, $t = 1, \dots, n$, we define the discrete Fourier transform

$$w_\zeta(\lambda) = \frac{1}{(2\pi n)^{1/2}} \sum_t \zeta_t e^{it\lambda}, \tag{2.1}$$

where $\sum_t$ will always denote $\sum_{t=1}^n$; with also a sequence ξ_t, $t = 1, \dots, n$, we define the (cross-) periodogram

$$I_{\zeta\xi}(\lambda) = w_\zeta(\lambda) w_\xi(-\lambda).$$

Denoting by $\lambda_j = 2\pi j/n$, for integer j, the Fourier frequencies, and by $1(\cdot)$ the indicator function, we define the averaged (cross-) periodogram,

$$\widehat{F}_{\zeta\xi}(l,m) = \frac{2\pi}{n}\left[2\Re e\left\{\sum_{j=l}^{m} I_{\zeta\xi}(\lambda_j)\right\} - I_{\zeta\xi}(0)1(l=0) - I_{\zeta\xi}(\pi)1\left(m = \frac{n}{2}\right)\right] \tag{2.2}$$

for integers l, m such that $0 \le l \le m \le n/2$, noting that $I_{\zeta\xi}$ has period 2π, that $\Re e\{I_{\zeta\xi}(\lambda)\}$ is symmetric about $\lambda = 0$ and $\lambda = \pi$, and that $I_{\zeta\xi}(\pi)$ is real-valued. We have for all such m,

$$\widehat{F}_{\zeta\xi}(1, m) = \widehat{F}_{\zeta\xi}(0, m) - \bar{\zeta}\bar{\xi}, \tag{2.3}$$

with the notation $\bar{a} = n^{-1} \sum_t a_t$, so that omission of zero frequency entails a sample mean correction. We shall always consider only $l = 0$ or $l = 1$, though properties for other fixed (as $n \to \infty$) values of l are the same as those for $l = 1$. On the other hand, the final term in (2.2) can make a non-zero contribution only when $m = n/2$, for which n must be even. Defining $\tilde{n} = [n/2]$, where $[\cdot]$ denotes the integer part, the orthogonality of the complex exponential implies that, irrespective of whether n is even or odd,

$$\widehat{F}_{\zeta\xi}(0, \tilde{n}) = \frac{2\pi}{n} \sum_{j=1}^{n} I_{\zeta\xi}(\lambda_j) = \frac{1}{n} \sum_t \zeta_t \xi_t, \tag{2.4}$$

the sample second (cross-) moment, so that from (2.3), $F_{\zeta\xi}(1, \tilde{n})$ is the corresponding statistic based on deviations from sample means.

The real part operator in (2.2) is redundant when $m = \tilde{n}$, but not in other cases of interest. We shall sometimes generalize $m = \tilde{n}$ to

$$m \le \tilde{n}; \qquad m \to \infty \quad \text{as } n \to \infty, \tag{2.5}$$

but more often contradict $m = \tilde{n}$ by

$$m < \tilde{n}; \qquad \frac{1}{m} + \frac{m}{n} \to 0 \quad \text{as } n \to \infty, \tag{2.6}$$

so that $\widehat{F}_{\zeta\xi}$ is based on a degenerating band of frequencies.

Under (2.6), $\widehat{F}_{\zeta\xi}$ has principally been of interest in connection with estimating the (cross-) spectral density of covariance stationary processes. As a matter of notation, if ζ_t, ξ_t, $t = 0, \pm 1, \ldots$, are jointly covariance stationary with a (cross-) spectral density $f_{\xi\zeta}(\lambda)$, the latter satisfies

$$\text{(2.7)} \quad \operatorname{Cov}(\zeta_0, \xi_j) = E(\zeta_0 - E\zeta_0)(\xi_j - E\xi_0) = \int_{\Pi} f_{\zeta\xi}(\lambda) e^{ij\lambda} d\lambda, \quad j = 0, \pm 1, \ldots,$$

where $\Pi = [-\pi, \pi]$. Under regularity conditions and (2.6), $\pi n \widehat{F}_{\zeta\zeta}(1, m)/m$ consistently estimates $f_{\zeta\zeta}(0)$ [see Brillinger (1975)]. When the latter is infinite (so ζ_t has long-range dependence), Robinson (1994a, b) studied asymptotic properties of $\widehat{F}_{\zeta\zeta}(1, m)$, with multivariate generalization given by Lobato (1997). We are concerned, however, with $\widehat{F}_{\zeta\xi}(l, m)$ when neither ζ_t nor ξ_t is stationary, though one of them can be asymptotically stationary; the following section describes such processes and their properties. An identity readily deduced from (2.2),

$$\text{(2.8)} \qquad \widehat{F}_{\zeta\xi}(l, m) = \widehat{F}_{\zeta\xi}(l, \tilde{n}) - \widehat{F}_{\zeta\xi}(m + 1, \tilde{n}), \qquad m < \tilde{n},$$

is important in our context because the second term on the right is sometimes asymptotically dominated by the first; this is not the case when ζ_t, ξ_t are both asymptotically stationary.

Relative to the literature on quadratic forms of stationary long-range dependent processes, following Fox and Taqqu (1985), $\widehat{F}_{\zeta\xi}(l, \tilde{n})$ cover very specialized quadratic forms and we can envisage how $\widehat{F}_{\zeta\xi}(l, m)$, for general m, can likewise be generalized. On the other hand the possible bilinear aspect, with allowance for nonstationary ζ_t, ξ_t, or a mixture of asymptotically stationary and nonstationary processes, represents in itself a considerable theoretical development, not only when $m < \tilde{n}$ [where indeed the forms considered in the stationary literature do not even quite cover $\widehat{F}_{\zeta\xi}(0, m)$, say] but even when $m = \tilde{n}$. As it is, our simple forms can be used to approximate ones with a factor $\sigma(\lambda_j)$ in the summand of (2.2), where $\sigma(\lambda)$ is nonzero and sufficiently well behaved at $\lambda = 0$, while the allowance for poles and zeros in $\sigma(\lambda)$ would affect the character of the results more interestingly, as would tapering, but require a considerably more lengthy discussion. Our possibly bivariate setting means that results for the averaged periodogram matrix are immediately covered for vector series with possibly different integration orders. Note also that while the stationary quadratic form literature focusses directly on limit distributional properties, our leading concern is with comparison of $\widehat{F}_{\zeta\xi}(l, m)$ satisfying (2.5) and (2.6) through their first and second moments. These comparisons vary considerably with α and β, and to the extent that $\widehat{F}_{\zeta\xi}(l, m)$ approximates the "time domain" statistics $\widehat{F}_{\zeta\xi}(l, \tilde{n})$ [see (2.3), (2.4)], functional limit theory for vector nonstationary fractional processes of Marinucci and Robinson (2000) can be used to characterize limit distributional theory, as mentioned in Section 6.

3. Nonstationary sequences. We first define classes of weight sequences which will generate classes of nonstationary, including asymptotically stationary, processes.

DEFINITION 3.1. $\Phi(\alpha)$ is the class of sequences $\{\phi_t^{(\alpha)}, t = 0, 1, \dots\}$ such that

$$\phi_t^{(0)} = 1(t = 0), \tag{3.1}$$

and for $\alpha > 0$, as $t \to \infty$,

$$\phi_t^{(\alpha)} \sim \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.2}$$

$$\left|\phi_t^{(\alpha)} - \phi_{t+1}^{(\alpha)}\right| = O\left(\frac{|\phi_t^{(\alpha)}|}{t}\right), \tag{3.3}$$

where "$\sim$" means that the ratio of left- and right-hand sides tends to 1, and $\Gamma(\cdot)$ is the Gamma function.

There is no loss of generality in the scale restrictions implicit in (3.1) and (3.2). It is possible to extend the definition, and subsequent results of the article, to cover $\alpha < 0$, but we have focused on $\alpha \geq 0$ here due to space limitations and because this covers the cases of greatest practical interest. When $0 < \alpha < 1$, (3.2), (3.3) define $\{\phi_t^{(\alpha)}\}$ as quasi-monotonically convergent to zero and of pure bounded variation in the sense of Yong [(1974), pages 2, 4]. In particular, (3.2) and (3.3) are satisfied by $\phi_t^{(\alpha)} = t^{\alpha-1}/\Gamma(\alpha)$, but only (3.2) by $\phi_t^{(\alpha)} = t^{\alpha-1}/\Gamma(\alpha) + t^{\beta-1}1$ (t even), for $\alpha - 1 < \beta < \alpha$ (though it would be possible to show that the results of following sections hold also for the latter type of sequence).

For our purposes the class $\Phi(\alpha)$ is motivated principally by the sequence $\phi_t^{(\alpha)} = \Delta_t^{(\alpha)}$, where

$$\Delta_t^{(\alpha)} = \frac{\Gamma(t+\alpha)}{\Gamma(\alpha)\Gamma(t+1)}, \qquad t \geq 0, \tag{3.4}$$

with the conventions $\Gamma(0) = \infty$, $\Gamma(0)/\Gamma(0) = 1$, given by the formal expansion

$$\Delta^{-\alpha} = \sum_{t=0}^{\infty} \Delta_t^{(\alpha)} L^t, \tag{3.5}$$

where L is the lag operator and $\Delta = 1 - L$ is the difference operator. Using Stirling's formula, we have $\{\Delta_t^{(\alpha)}\} \in \Phi(\alpha)$, for all $\alpha \geq 0$. For integer α, Δ^α is familiar from Box and Jenkins' (1971) "ARIMA" modelling of nonstationary series. In particular,

$$\Delta_t^{(1)} = 1, \qquad t \geq 0, \tag{3.6}$$

is used to generate "unit root" series in their framework. The somewhat special nature of (3.4) relative to (3.2) and (3.3), even when α is fixed at 1, is notable in view of the vast econometric literature focussing on (3.6). In fact, some of our work involving $\alpha = 1$ (see Theorem 4.3) requires some strengthening of (3.3) [see (4.15) and (4.18)], but still greater generality than (3.6) is afforded. When α is nonintegral, Δ^{α} is the fractional difference operator arising in modelling of "FARIMA" series. A cosinusoidal modification of Definition 3.1 would enable study of stationary or nonstationary cyclic or seasonal behavior.

Practical interest in $\Phi(\alpha)$ will further be strengthened by means of the following lemma. In the sequel we write ϕ_t in place of $\phi_t^{(\alpha)}$, dropping the superscript; the dependence on α will be indicated by the statement $\{\phi_t\} \in \Phi(\alpha)$.

LEMMA 3.1. *Let* $\{\phi_t\} \in \Phi(\alpha)$, $\{\psi_t\} \in \Phi(\beta)$, $\alpha, \beta \geq 0$. *Then*

$$\chi_t \stackrel{\text{def}}{=} \sum_{j=0}^{t} \phi_j \psi_{t-j} \in \Phi(\alpha + \beta). \tag{3.7}$$

The next lemma [see also Kokoszka and Taqqu (1996), Lemma 3.1], describing properties of the complex partial sum,

$$S_{uv}(\lambda, \alpha) = \sum_{t=u}^{v} \phi_t e^{it\lambda}, \qquad \{\phi_t\} \in \Phi(\alpha),$$

for λ real, will be of considerable use in the sequel. Throughout the article, C denotes a generic positive constant.

LEMMA 3.2. *Let* $\{\phi_t\} \in \Phi(\alpha)$. *Then for* $0 \leq u < v$, $0 \leq |\lambda| \leq \pi$,

$$S_{uv}(\lambda, 0) = 1(u = 0), \tag{3.8}$$

$$|S_{uv}(\lambda, \alpha)| \leq C \min\left(v^{\alpha}, \frac{(u+1)^{\alpha-1}}{|\lambda|}, \frac{1}{|\lambda|^{\alpha}}\right), \qquad 0 < \alpha \leq 1, \tag{3.9}$$

$$|S_{uv}(\lambda, \alpha)| \leq C \min\left(v^{\alpha}, \frac{v^{\alpha-1}}{|\lambda|}\right), \qquad \alpha > 1. \tag{3.10}$$

Also, for $0 < \alpha < 1$, *as* $\lambda \to 0^+$,

$$\Re e\{S_{0\infty}(\lambda, \alpha)\} \sim \cos\frac{\alpha\pi}{2}\lambda^{-\alpha}, \qquad \Im m\{S_{0\infty}(\lambda, \alpha)\} \sim \sin\frac{\alpha\pi}{2}\lambda^{-\alpha}. \tag{3.11}$$

Short range dependent processes are given as follows.

DEFINITION 3.2. I is the class of zero-mean scalar covariance stationary sequences $\{\eta_t, t = 0, \pm 1, \ldots\}$ having spectral density $f_{\eta\eta}(\lambda)$ [cf. (2.7)] that is positive and continuous at $\lambda = 0$.

The zero-mean restriction is costless in our discussion of $\widehat{F}_{\zeta\xi}(l, m)$ when $l = 1$. Robinson and Marinucci (2000) study the averaged periodogram in case of additive time trends, though they obtain only upper bounds rather than our precise limits in Sections 4 and 5, and under stronger conditions on the stochastic component. We generate long-range dependent processes as follows.

DEFINITION 3.3. For $\alpha \geq 0$, $I(\alpha)$ is the class of processes $\{\zeta_t, t = 0, \pm 1, \ldots\}$ such that for $\{\eta_t\} \in I$ and $\{\phi_t\} \in \Phi(\alpha)$,

$$\zeta_t = \sum_{s=-\infty}^{t} \phi_{t-s}\{\eta_s 1(s \geq 1)\}. \tag{3.12}$$

LEMMA 3.3. *Let $\{\zeta_t\} \in I(\alpha)$ and let*

$$\xi_t = \sum_{s=-\infty}^{t} \psi_{t-s}\{\zeta_s 1(s \geq 1)\}, \tag{3.13}$$

where $\{\psi_t\} \in \Phi(\beta)$. Then $\{\xi_t\} \in I(\alpha + \beta)$.

We can thus view processes in $I(\alpha)$ as having possibly been passed through a succession of Φ-filters, whether by nature or the statistician, including the difference filter given in (3.4), (3.5).

Notice that Definition 3.3 implies $\zeta_t = 0$, $t \leq 0$, as a consequence of ζ_t being $(\eta_1, \ldots, \eta_t)$-measurable, which is itself motivated by the fact that, for $\{\phi_t\} \in \Phi(\alpha)$, the untruncated process

$$\rho_t = \sum_{s=-\infty}^{t} \phi_{t-s}\eta_s \tag{3.14}$$

is not well defined in the mean square sense when $\alpha \geq \frac{1}{2}$. However, for $\alpha < \frac{1}{2}$, ρ_t is, unlike ζ_t, covariance stationary, for example when $\alpha = 0$, we have $\zeta_t = \eta_t 1(t \geq 1)$. We have preferred to give a single definition for all $\alpha \geq 0$; for $\alpha < \frac{1}{2}$, ζ_t is "asymptotically covariance stationary" in a sense indicated in the following lemma [see also Parzen (1963), Dahlhaus (1997)] which also describes second order properties in the "purely" nonstationary case $\alpha \geq \frac{1}{2}$. Define

$$\phi(\lambda) = \sum_{s=0}^{\infty} \phi_s e^{is\lambda}, \qquad \phi_t(\lambda) = \sum_{s=0}^{t-1} \phi_s e^{is\lambda}. \tag{3.15}$$

LEMMA 3.4. *Let $\{\phi_t\} \in \Phi(\alpha)$, $\{\eta_t\} \in I$.*

(i) *Let $0 \leq \alpha < \frac{1}{2}$. Then $\{\rho_t\}$ is covariance stationary with spectral density $f_{\rho\rho}(\lambda) = |\phi(\lambda)|^2 f_{\eta\eta}(\lambda)$, satisfying*

$$f_{\rho\rho}(\lambda) \sim f_{\eta\eta}(0)\lambda^{-2\alpha} \quad \text{as } \lambda \to 0^+. \tag{3.16}$$

The "time varying spectral density" of ζ_t, $f_{\zeta\zeta}^{(t)}(\lambda) = |\phi_t(\lambda)|^2 f_{\eta\eta}(\lambda)$ satisfies

$$\lim_{\lambda + (\lambda t)^{-1} \to 0+} \left\{ f_{\zeta\zeta}^{(t)}(\lambda) \Big/ f_{\rho\rho}(\lambda) \right\} = 1 \tag{3.17}$$

and in addition we have, uniformly in $j \geq 0$,

$$\mathrm{Cov}(\zeta_t, \zeta_{t+j}) - \mathrm{Cov}(\rho_0, \rho_j) = O(t^{\alpha-(1/2)}). \tag{3.18}$$

(ii) *Let* $\alpha = \frac{1}{2}$. *Then for all* $j \geq 0$, *as* $t \to \infty$,

$$\frac{\mathrm{Cov}(\zeta_t, \zeta_{t+j})}{\log t} \to 2f_{\eta\eta}(0), \tag{3.19}$$

where the convergence is uniform in $j = o(\log t)$.

(iii) *Let* $\alpha > \frac{1}{2}$. *Then for all* $j \geq 0$, *as* $t \to \infty$,

$$t^{1-2\alpha}\mathrm{Cov}(\zeta_t, \zeta_{t+j}) \to \frac{2\pi f_{\eta\eta}(0)}{\Gamma(\alpha)^2(2\alpha - 1)}, \tag{3.20}$$

where the convergence in uniform in $j = o(t^{2\alpha-1})$ *for* $\frac{1}{2} < \alpha < 1$, $j = o(t/\log t)$ *for* $\alpha = 1$, *and* $j = o(t)$ *for* $\alpha > 1$.

Note that (3.17) holds despite $f^{(t)}_{\zeta\zeta}(\lambda)$ having no pole at $\lambda = 0$ for finite t even when $\alpha > 0$, unlike $f_{\rho\rho}(\lambda)$. By comparison (3.18) is a weak result, but a time domain version of (3.17) would require stronger conditions, in effect on $f_{\eta\eta}(\lambda)$ for all λ, an approximation for $\mathrm{Cov}(\rho_t, \rho_{t+j})$ as $j \to \infty$ can be influenced by a pole in $f_{\eta\eta}(\lambda)$ for some $\lambda \neq 0$, for example. Lemma 3.4 foreshadows the main results of the paper in its reliance on only mild, local-to-zero, conditions on $f_{\eta\eta}(\lambda)$.

4. The mean of the averaged periodogram. We consider the statistic $\widehat{F}_{\zeta\xi}(l, m)$ in (2.2), where $\{\zeta_t\} \in \Phi(\alpha)$, $\{\xi_t\} \in \Phi(\beta)$ and

$$0 \leq \alpha \leq \beta, \qquad \beta \geq \tfrac{1}{2}. \tag{4.1}$$

Thus only ζ_t can be asymptotically stationary. Strictly speaking, the case where both are asymptotically stationary in our sense has not been covered in the literature, but in view of Lemma 3.4 it is predictable that the results will be too similar to the stationary cases covered by Robinson (1994a, b), Lobato (1997) to be worth reporting. Of course when $\alpha \geq \frac{1}{2}$, our results for (4.1) include the case where $\zeta_t \equiv \xi_t$, the same nonstationary process. There is no loss of generality in the requirement $\alpha \leq \beta$.

We introduce the following definition.

DEFINITION 4.1. I_2 is the class of jointly covariance stationary bivariate processes $\{\eta_t, \theta_t, t = 0, \pm 1, \ldots\}$ such that $\{\eta_t\} \in I$, $\{\theta_t\} \in I$ and $f_{\eta\theta}(\lambda)$ is continuous at $\lambda = 0$.

With ζ_t generated by (3.12) we take

$$\xi_t = \sum_{s=-\infty}^{t} \psi_{t-s}\{\theta_s 1(s \geq 1)\}, \tag{4.2}$$

where $\{\psi_t\} \in \Phi(\beta)$.

DEFINITION 4.2. $I(\alpha, \beta)$ is the class of bivariate processes $\{\zeta_t, \xi_t, t = 0, \pm 1, \ldots\}$ such that (3.12) and (4.2) hold with $\{\eta_t, \theta_t\} \in I_2$.

Depending on the values of α and β, $E\{\widehat{F}_{\zeta\xi}(0, m)\}$ may or may not differ negligibly from $E\{\widehat{F}_{\zeta\xi}(1, m)\}$, and so in view of (2.3) we first estimate $E(\bar{\zeta}\bar{\xi})$ and, more generally, the covariance structure of discrete Fourier transforms $w_\zeta(\lambda_j)$, $w_\xi(\lambda_k)$ at fixed j, k, to extend results of Künsch (1986), Hurvich and Beltrao (1993), Hurvich and Ray (1995), Robinson (1995a). Denote by the superscripts R and I the real and imaginary part, respectively.

LEMMA 4.1. *Let $\{\zeta_t, \xi_t\} \in I(\alpha, \beta)$. Then for $(A, B) = (R, R), (R, I), (I, R), (I, I)$,*

$$\lim_{n\to\infty} n^{-\alpha-\beta} E\left\{w_\zeta^A(\lambda_j) w_\xi^B(\lambda_k)\right\} = f_{\eta\theta}(0) \int_0^1 U_j^A(z;\alpha) U_k^B(z;\beta)\, dz, \tag{4.3}$$

where $U_j^A(z;\alpha)$ and $U_j^B(z;\alpha)$ are, respectively, the real and imaginary parts of

$$U_j(z;\alpha) = \frac{1(\alpha > 0)}{\Gamma(\alpha)} \int_0^{1-z} y^{\alpha-1} e^{2\pi i j(y+z)}\, dy + 1(\alpha = 0) e^{2\pi i j z}. \tag{4.4}$$

Thus,

$$\lim_{n\to\infty} n^{1-\alpha-\beta} E(\bar{\zeta}\bar{\xi}) = \frac{2\pi f_{\eta\theta}(0)}{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha+\beta+1)}. \tag{4.5}$$

For finite m, Lemma 4.1 can be applied to calculate the limit $E\{\widehat{F}_{\zeta\xi}(l, m)\}$. Under (2.5) or (2.6) the behavior of $E\{\widehat{F}_{\zeta\xi}(l, m)\}$ varies significantly across the following five mutually exhaustive subsets of (4.1):

$$\alpha \ge 0, \qquad \beta \ge \tfrac{1}{2}, \qquad \alpha + \beta < 1, \tag{4.6}$$

$$\alpha > 0, \qquad \beta \ge \tfrac{1}{2}, \qquad \alpha + \beta = 1, \tag{4.7}$$

$$\alpha = 0, \qquad \beta = 1, \tag{4.8}$$

$$\alpha = 0, \qquad \beta > 1, \tag{4.9}$$

$$\alpha > 0, \qquad \beta > \tfrac{1}{2}, \qquad \alpha + \beta > 1. \tag{4.10}$$

In (4.6) and (4.7) ζ_t is asymptotically stationary and β is small enough that the combined memory $\alpha+\beta$ of ζ_t and ξ_t is less than one in (4.6), while in (4.7) it equals one but the familiar $I(0)/I(1)$ case (4.8) of the econometric literature is excluded. In (4.9) and (4.10) it exceeds one. In (4.10), $\beta > \frac{1}{2}$ is actually implied by $\alpha + \beta > 1$, in view of (4.1).

Consider first case (4.6). Define

$$\psi(\lambda) = \sum_{t=0}^{\infty} \psi_t e^{it\lambda},$$

which [like $\phi(\lambda)$] is infinite at $\lambda = 0$ but is well defined for $\lambda \ne 0$, mod(2π), from Lemma 3.2.

THEOREM 4.1. *Let* $\{\zeta_t, \xi_t\} \in I(\alpha, \beta)$ *under* (4.6). *Then for* $l = 0, 1$,

$$\lim_{n\to\infty} E\{\widehat{F}_{\zeta\xi}(l, \tilde{n})\} = \int_{\Pi} \phi(\lambda)\psi(-\lambda)f_{\eta\theta}(\lambda)d\lambda, \tag{4.11}$$

where the right side is finite, *and under* (2.6),

$$\lim_{n\to\infty} \lambda_m^{\alpha+\beta-1} E\{\widehat{F}_{\zeta\xi}(l, m)\} = 2f_{\eta\theta}(0)\frac{\cos(\alpha-\beta)\frac{\pi}{2}}{1-\alpha-\beta}. \tag{4.12}$$

Neither (4.11) nor (4.12) is affected by mean correction. Most interestingly, the results are identical to those which may be obtained if both ζ_t and ξ_t are stationary or asymptotically stationary, so $\alpha, \beta < \frac{1}{2}$, which automatically implies $\alpha + \beta < 1$; thus sufficiently small memory in ζ_t can compensate for the nonstationarity in ξ_t, though for given $\alpha + \beta$ (4.6) has the potential for a larger $\alpha-\beta$ and consequently smaller $\cos(\alpha-\beta)\pi/2$ factor in (4.12) than when $0 < \alpha, \beta < \frac{1}{2}$. The latter factor is positive, and so the limit (4.12) shares the sign of $f_{\eta\theta}(0)$ (which is real-valued by the continuity assumption and oddness of the quadrature spectrum).

THEOREM 4.2. *Let* $\{\zeta_t, \xi_t\} \in I(\alpha, \beta)$ *under* (4.7). *Then for* $l = 0, 1$,

$$\lim_{n\to\infty} \frac{1}{\log n} E\{\widehat{F}_{\zeta\xi}(l, \tilde{n})\} = 2f_{\eta\theta}(0)\sin\alpha\pi = 2f_{\eta\theta}(0)\sin\beta\pi, \tag{4.13}$$

and under (2.5),

$$\lim_{n\to\infty} \frac{1}{\log m} E\{\widehat{F}_{\zeta\xi}(l, m)\} = 2f_{\eta\theta}(0)\sin\alpha\pi = 2f_{\eta\theta}(0)\sin\beta\pi. \tag{4.14}$$

The degeneration condition (2.6) now leaves little difference between the expectations of the broad- and narrow-band statistics, in fact for $m \sim n^a$, $0 < a < 1$, they have the same convergence rates. Note that just as Theorem 4.1 covered the case $\beta = \frac{1}{2}$, the border of the nonstationary region, so Theorem 4.2 covers $\alpha = \beta = \frac{1}{2}$.

Though Theorem 4.2 does not cover (4.8), putting $\alpha = 0$ or $\beta = 1$ annihilates the limits (4.13) and (4.14), suggesting a faster rate of convergence under (4.8). This is indeed the outcome, implying that the $I(0, 1)$ case (4.8), which looms large in the econometric literature within an autoregressive framework, is also rather special within the fractional domain. These results do require a strengthening of the condition on $\{\zeta_t, \xi_t\}$. Define the function

$$h_{\eta\theta}(\lambda) = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}(\omega_{-|j|} - \omega_{|j|+1})\cos j\lambda, \qquad \lambda \in \Pi,$$

where

$$\omega_j = \sum_{l=|j|}^{\infty} \gamma_{l\,\mathrm{sign}(j)}, \qquad \gamma_j = \mathrm{Cov}(\eta_0, \theta_j), \qquad j = 0, \pm 1, \ldots,$$

with the convention that sign(0) is negative.

THEOREM 4.3. *Let* $\{\zeta_t, \xi_t\} \in I(0,1)$, *so* (4.8) *holds.*

(i) *If also* $h_{\eta\theta}(\lambda)$ *is integrable on* Π *and*

$$\sum_{j=0}^{\infty} |\psi_j - \psi_{j+1}| < \infty, \tag{4.15}$$

then

$$\lim_{n\to\infty} E\{\widehat{F}_{\zeta\xi}(0,\tilde{n})\} = \sum_{j=0}^{\infty} \psi_j \gamma_{-j}, \tag{4.16}$$

$$\lim_{n\to\infty} E\{\widehat{F}_{\zeta\xi}(1,\tilde{n})\} = \tfrac{1}{2}(\omega_0 - \omega_1) + \sum_{j=0}^{\infty} (\psi_j - 1)\gamma_{-j}. \tag{4.17}$$

(ii) *If also* $h_{\eta\theta}(\lambda)$ *is continuous at* $\lambda = 0$, (2.6) *holds and*

$$\sum_{j=0}^{\infty} |\psi_j - 1| < \infty, \tag{4.18}$$

then

$$\lim_{n\to\infty} E\{\widehat{F}_{\zeta\xi}(0,m)\} = \pi f_{\eta\theta}(0), \tag{4.19}$$

$$\lim_{n\to\infty} \frac{n}{m} E\{\widehat{F}_{\zeta\xi}(1,m)\} = 2\pi h_{\eta\theta}(0) + 4\pi f_{\eta\theta}(0) \sum_{j=0}^{\infty} (\psi_j - 1). \tag{4.20}$$

It is sufficient for the conditions on $h_{\eta\theta}(\lambda)$ that $\sum |j\gamma_j| < \infty$, which is implied if $f_{\eta\theta}(\lambda)$ is differentiable with derivative satisfying a Lipschitz condition of degree greater than $\frac{1}{2}$ [see Zygmund (1977), page 240] but a global smoothness condition is not implied, though by the Riemann–Lebesgue lemma $\omega_{-|j|} - \omega_{|j|+1} \to 0$ as $|j| \to \infty$. Note that if $\gamma_j \equiv -\gamma_j$ (as is true if $\eta_t \equiv \theta_t$, for example), we have $h_{\eta\theta}(\lambda) \equiv f_{\eta\theta}(\lambda)$, so the additional conditions are vacuous. The mean-corrected narrow-band statistic $\widehat{F}_{\zeta\xi}(1,m)$ [but not $\widehat{F}_{\zeta\xi}(0,m)$] has expectation of smaller order than that of either full band statistic. Sensitivity is found, except in (4.19), to the precise values of the sequence $\{\psi_t\}$, rather than simply their asymptotic value (in this case, 1). In the usual case $\psi_t \equiv 1$, stressed in the econometric literature, (4.16), (4.17) are already known though seemingly only under more global frequency domain conditions. Condition (4.15) is only slightly stronger than (3.3) since we have $\alpha = 1$ in Definition 3.1, while (4.18) is stronger than (4.15), by the triangle inequality. Note that (4.19) can be interpreted as a limit of (4.12) with $l = 1$, on putting $\alpha = 0$ and then letting β tend to 1.

THEOREM 4.4. *Let* $\{\zeta_t, \xi_t\} \in I(0, \beta)$, $\beta > 1$, *so* (4.9) *holds*.

(i) *If also* $|\omega_{-1}| < \infty$,

$$\lim_{n\to\infty} n^{1-\beta} E\{\widehat{F}_{\zeta\xi}(0, \tilde{n})\} = 0, \tag{4.21}$$

$$\lim_{n\to\infty} n^{1-\beta} E\{\widehat{F}_{\zeta\xi}(1, \tilde{n})\} = -\frac{2\pi f_{\eta\theta}(0)}{\Gamma(\beta+2)}. \tag{4.22}$$

(ii)

$$\lim_{n\to\infty} E\{\widehat{F}_{\zeta\xi}(0, \tilde{n})\} = \sum_{j=0}^{\infty} \psi_j \gamma_{-j} \tag{4.23}$$

if the right side is finite.

Part (i) of the theorem shows that $E\{n^{-1}\sum_t \zeta_t\xi_t\}$ is of smaller order than $E(\bar{\zeta}\bar{\xi})$, while the former is shown in part (ii) to be finite as long as the γ_{-j} decay fast enough, as is the case for any $\beta > 1$ if $\{\zeta_t, \xi_t\}$ is an "ARMA" process. Mean-correction now affects the order of magnitude of the expectation of full-band statistics. The present case (4.9) is somewhat anomalous, the discontinuity at $\alpha = 0$ in Definition 3.1 taking effect, and by way of contrast with Theorems 4.1–4.3 it can be inferred that the $\widehat{F}(l, m)$ can actually have larger expectation for $m < \tilde{n}$; we have been unable to obtain an attractive result in this case.

The other way to achieve $\alpha + \beta > 1$ is to allow $\alpha > 0$, and now the choice of m makes no difference.

THEOREM 4.5. *Let* $\{\zeta_t, \xi_t\} \in I(\alpha, \beta)$ *under* (4.10). *Then under* (2.5),

$$\lim_{n\to\infty} n^{1-\alpha-\beta} E\{\widehat{F}_{\zeta\xi}(0, m)\} = \frac{2\pi f_{\eta\theta}(0)}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta)(\alpha+\beta-1)}, \tag{4.24}$$

$$\lim_{n\to\infty} n^{1-\alpha-\beta} E\{\widehat{F}_{\zeta\xi}(1, m)\} = \frac{A(\alpha, \beta) 2\pi f_{\eta\theta}(0)}{\Gamma(\alpha)\Gamma(\beta)}, \tag{4.25}$$

where

$$A(\alpha, \beta) = \frac{\alpha\beta(\alpha+\beta-1) - \alpha(\alpha-1) - \beta(\beta-1)}{\alpha\beta(\alpha+\beta-1)(\alpha+\beta)(\alpha+\beta+1)},$$

and thus

$$\lim_{n\to\infty} n^{1-\alpha-\beta} E\{\widehat{F}_{\zeta\xi}(m+1, \tilde{n})\} = 0. \tag{4.26}$$

The distinctive feature of Theorem 4.5 is that $E\{\widehat{F}_{\zeta\xi}(l, \tilde{n})\}$ is dominated by an arbitrarily slowly increasing number of low frequency components. As in some of our earlier results, the rate of convergence is improved if η_t and θ_t are fully incoherent at zero frequency, not necessarily at all frequencies. Note that only (2.5) is imposed, so that we also cover the case where m increases as fast as n.

5. The variance of the averaged periodogram. Unlike in the case of the mean, we can give a single theorem to describe the variance of $\widehat{F}_{\zeta\xi}(l, m)$ when

$$0 \le \alpha \le \beta, \qquad \beta > \tfrac{1}{2}, \tag{5.1}$$

though different proofs are needed over different portions of this region. Thus we now omit the borderline case $\alpha = 0, \beta = \frac{1}{2}$, which seems too special to include in view of the particular treatment it requires.

We need to extend some earlier definitions.

DEFINITION 5.1. I_3 is the class of jointly fourth-order stationary bivariate processes $\{\eta_t, \theta_t, t = 0, \pm 1, \ldots\}$, such that $\{\eta_t, \theta_t\} \in I_2$ and the cumulant spectral density $f_{\eta\theta\eta\theta}(\lambda, \mu, \omega)$ given by

$$\mathrm{Cum}\{\eta_s, \theta_t, \eta_u, \theta_v\} = \int_\Pi \int_\Pi \int_\Pi f_{\eta\theta\eta\theta}(\lambda, \mu, \omega) e^{i(t-s)\lambda + i(u-s)\mu + i(v-s)\omega}\, d\lambda\, d\mu\, d\omega$$

is continuous at $\lambda = \mu = \omega = 0$ and satisfies

$$\sup_{\mu, \omega \in \Pi} \int_\Pi |f_{\eta\theta\eta\theta}(\lambda, \mu, \omega)|^2\, d\lambda < \infty. \tag{5.2}$$

DEFINITION 5.2. I_4 is the class of jointly fourth-order stationary bivariate processes $\{\eta_t, \theta_t, t = 0, \pm 1, \ldots\}$ such that $\{\eta_t, \theta_t\} \in I_3$ and $f_{\eta\eta}(\lambda), f_{\theta\theta}(\lambda)$ are square integrable.

DEFINITION 5.3. For $j = 3, 4$, $I_j(\alpha, \beta)$ is the class of bivariate processes $\{\zeta_t, \xi_t, t = 0, \pm 1, \ldots\}$ such that (3.12) and (4.2) hold for $\{\eta_t, \theta_t\} \in I_j$.

We introduce, for $\alpha, \beta, \gamma, \delta > 0$,

$$p(x, y; \alpha, \beta) = \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y z^{\alpha-1}(z + x)^{\beta-1}\, dz, \qquad 0 \le y \le 1 - x,$$

$$P(\alpha, \beta, \gamma, \delta) = 2 \int_0^1 \int_0^{1-x} p(x, y; \alpha, \beta) p(x, y; \gamma, \delta) dy\, dx,$$

$$q(x; \alpha, \beta) = \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta + 1)} \int_0^x (x - y)^{\alpha-1}(1 - y)^\beta\, dy,$$

$$Q(\alpha, \beta, \gamma, \delta) = \int_0^1 q(x; \alpha, \beta) q(x; \gamma, \delta) dx,$$

and for $\beta > \frac{1}{2}$,

$$P(0, \beta, \beta, 0) = 0, \qquad P(0, 0, \beta, \beta) = \frac{(2\pi)^2}{\Gamma(\beta)^2 2\beta(2\beta - 1)},$$

$$Q(0, \beta, \beta, 0) = \frac{(2\pi)^2}{\Gamma(2\beta + 2)},$$

$$Q(0, 0, \beta, \beta) = \frac{(2\pi)^2}{\Gamma(\beta)\Gamma(\beta + 1)} \int_0^1 \int_0^x (x - y)^{\beta - 1}(1 - y)^\beta \, dy \, dx.$$

Also, define

$$R(\alpha, \beta, \gamma, \delta) = \frac{(2\pi)^2}{\Gamma(\alpha + 1)\Gamma(\beta + 1)(\alpha + \beta + 1)\Gamma(\gamma + 1)\Gamma(\delta + 1)(\gamma + \delta + 1)},$$

$$S(\alpha, \beta, \gamma, \delta) = P(\alpha, \beta, \gamma, \delta) - 2Q(\alpha, \beta, \gamma, \delta) + R(\alpha, \beta, \gamma, \delta).$$

THEOREM 5.1. *Let* $\{\zeta_t, \xi_t\} \in I_3(\alpha, \beta)$ *for* $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$ *and* $\{\zeta_t, \xi_t\} \in I_4(\alpha, \beta)$ *for* $0 \le \alpha \le \frac{1}{2}$, $\beta > \frac{1}{2}$. *Then under* (2.6),

$$\lim_{n\to\infty} n^{2(1-\alpha-\beta)} \operatorname{Var}\{\widehat{F}_{\zeta\xi}(0, m)\} = f^2_{\eta\theta}(0)P(\alpha, \beta, \beta, \alpha) + f_{\eta\eta}(0)f_{\theta\theta}(0)P(\alpha, \alpha, \beta, \beta), \tag{5.3}$$

$$\lim_{n\to\infty} n^{2(1-\alpha-\beta)} \operatorname{Var}\{\widehat{F}_{\zeta\xi}(1, m)\} = f^2_{\eta\theta}(0)S(\alpha, \beta, \beta, \alpha) + f_{\eta\eta}(0)f_{\theta\theta}(0)S(\alpha, \alpha, \beta, \beta), \tag{5.4}$$

$$\lim_{n\to\infty} n^{2(1-\alpha-\beta)} \operatorname{Var}\{\widehat{F}_{\zeta\xi}(m + 1, \tilde{n})\} = 0. \tag{5.5}$$

As (5.3)–(5.5) indicate, throughout the region (5.1) $\operatorname{Var}\{\widehat{F}_{\zeta\xi}(l, m)\}$ is asymptotically dominated by the contribution from an arbitrarily slowly increasing number of low frequencies. The variance is generally increased when $f_{\eta\theta}(0) \neq 0$, though this does not affect the rate of convergence, or divergence. The square integrability requirement on $f_{\eta\eta}$ and $f_{\theta\theta}$ (and thence on $f_{\eta\theta}$) when $\alpha \le \frac{1}{2}$ seems unavoidable and is, for example, essential for sample autocovariances of stationary sequences to be $n^{1/2}$-consistent [see Hannan (1976)]. The fourth cumulant requirement seems mild by the standards of such conditions in the literature; (5.2) is milder than boundedness of $f_{\eta\theta\eta\theta}$, but stronger than square integrability. We suspect that it could be further relaxed, but the proof would further lengthen the paper and our current condition is automatically satisfied when η_t, θ_t are Gaussian. In any case the absence from the limiting variances (5.3) and (5.4) of any fourth cumulant contribution is fortunate, and also distinctive from the stationary situation.

6. Cointegration application. We define observable sequences $\{y_t, z_t, t = 0, 1, \ldots\}$ such that (1.1) holds, or equivalently

$$y_t = \zeta_t + \nu\xi_t, \qquad z_t = \xi_t, \tag{6.1}$$

where ν is unknown and $\{\zeta_t, \xi_t\} \in I(\alpha, \beta)$ under (4.1) with

$$\alpha < \beta. \tag{6.2}$$

From (6.1), y_t and z_t have a common, nonstationary, component ξ_t, while y_t has an additional component ζ_t that can be nonstationary or asymptotically stationary. It is readily possible to apply the results of the preceding sections to a model with additional components in y_t and z_t, with smaller memory parameters, and to a model with vector observables of arbitrary dimension, but we keep the setting as simple as possible to conserve on notation. We deduce (1.1) from (6.1) and as discussed in Section 1 consider estimating ν by

$$\hat{\nu}_l = \widehat{F}_{yz}(l, \tilde{n})/\widehat{F}_{zz}(l, \tilde{n}), \qquad l = 0, 1,$$

and also by

$$\tilde{\nu}_l = \widehat{F}_{yz}(l, m)/\widehat{F}_{zz}(l, m), \qquad l = 0, 1,\ m < \tilde{n},$$

so that $\hat{\nu}_l$ is the LSE with ($l = 1$) or without ($l = 0$) intercept, and under (2.6) $\tilde{\nu}_l$ is the NBLSE, likewise mean-corrected or not. When (2.5) holds with $m \sim cn$, $0 < c < 1$, then $\tilde{\nu}_l$ is based on a nondegenerate band of frequencies, following the idea of Hannan (1963). Phillips (1991) considered a spectral form of estimate in cointegration with $\alpha = 0$ or $\beta = 1$, though his proofs concerned weighted autocovariance estimates rather than averaged periodogram ones, and in a nonstationary environment these are not necessarily close asymptotically.

Our main interest is in comparison of $\hat{\nu}_l, \tilde{\nu}_l$ across l, m in terms of bias and convergence rates but we can also attempt to characterize limit distributions. It follows from Theorems 4.5 and 5.1 that $n^{1-2\beta}\widehat{F}_{\xi\xi}(l, \tilde{n})$ and, when $\alpha + \beta > 1$, $n^{1-\alpha-\beta}\widehat{F}_{\zeta\xi}(l, \tilde{n})$, have mean and variance which both have finite but nonzero limits, motivating, though not implying, the following assumption which is unprimitive but eases the exposition.

ASSUMPTION 6.1. *For $l = 0, 1$, there exist random variables $\Phi_l(\beta), \Psi_l(\alpha, \beta)$ such that $\Phi_l(\beta) \neq 0$ almost surely and*

$$n^{1-2\beta}\widehat{F}_{\xi\xi}(l, \tilde{n}) \xrightarrow{d} \Phi_l(\beta), \qquad \beta > \tfrac{1}{2}, \tag{6.3}$$

$$n^{1-\alpha-\beta}\widehat{F}_{\zeta\xi}(l, \tilde{n}) \xrightarrow{d} \Psi_l(\alpha, \beta), \qquad \alpha + \beta \geq 1. \tag{6.4}$$

We can deduce (6.3) and (6.4) from the continuous mapping theorem if there exist jointly dependent processes $U(r;\alpha)$, $V(r;\beta)$, $0 \leq r \leq 1$, such that

$$\left\{n^{1/2-\alpha}\zeta_{[nr]}, n^{1/2-\beta}\xi_{[nr]}\right\} \Rightarrow \left\{U(r;\alpha), V(r;\beta)\right\} \quad \text{as } n \to \infty,\ 0 \leq r \leq 1, \tag{6.5}$$

where "$\Rightarrow$" denotes a suitable notion of weak convergence [see Billingsley (1968), pages 30, 111–123]. Then $\Phi_0(\beta) = \int_0^1 V(r;\beta)^2\,dr$, $\Phi_1(\beta) = \Phi_0(\beta) - \{\int_0^1 V(r;\beta)\,dr\}^2$, $\Psi_0(\alpha,\beta) = \int_0^1 U(r;\alpha)V(r;\beta)\,dr$, $\Psi_1(\alpha,\beta) - \int_0^1 U(r;\alpha)V(r;\beta)\,dr$. Sufficient conditions for (6.5) given by Marinucci and Robinson (2000) [which develops earlier work of Akonom and Gourieroux (1987), Silveira (1991)], are that ϕ_t, ψ_t are given by $\Delta_t(\alpha), \Delta_t(\beta)$, while $(\eta_t, \theta_t)' = \sum_{j=-\infty}^{\infty} A_j \varepsilon_{r-j}$, the A_j being 2×2 matrices such that $\sum_{j=0}^{\infty}\sum_{|k|=j+1}^{\infty} \|A_k\|^2 < \infty$ where $\|\cdot\|$ is Euclidean norm, the ε_t being independent and identically distributed with zero mean and finite qth moment for $q > \max(2, 2/(2\alpha-1), 2/(2\beta-1))$, while $\sum_{j=-\infty}^{\infty} A_j$ and the covariance matrix of ε_t have full rank. These conditions are implied by Gaussian "FARIMA" (ζ_t, ξ_t), such that (η_t, θ_t) is a stationary and invertible "ARMA" sequence ,while on the other hand implying that $\{\zeta_t, \xi_t\} \in I_4(\alpha, \beta)$. Then for $\alpha, \beta > \frac{1}{2}$ we have (6.5) with U, V being "Type II fractional Brownian motion" [see Marinucci and Robinson (1999)],

$$(U(r;\alpha), V(r;\beta)) = \int_0^r \{(r-s)^{\alpha-1}\, dB_1(s), (r-s)^{\beta-1}\, dB_2(s)\}, \tag{6.6}$$

where $B(r) = \{B_1(r), B_2(r)\}'$ is 2×1 Brownian motion with $EB(r) = 0$ and

$$E\{B(r_1)B(r_2)'\} = 2\pi \min(r_1, r_2) \begin{bmatrix} f_{\eta\eta}(0) & f_{\eta\theta}(0) \\ f_{\eta\theta}(0) & f_{\theta\theta}(0) \end{bmatrix}.$$

When $\alpha \le \frac{1}{2}$, V is given as in (6.6) under a simplified version of the conditions. We cannot so characterize $\Psi_l(\alpha, \beta)$ when $\alpha + \beta > 1$ but $0 < \alpha \le \frac{1}{2}$ since on the one hand the continuous mapping theorem does not apply, while on the other ζ_t cannot be approximated by a semimartingale. The latter property holds when $\alpha = 0, \beta = 1$ [case (4.8)] where, when $\psi_t \equiv 1$,

$$\Psi_0(0,1) = \int_0^1 B_2(1)\, dB_1(r) + \omega_0, \qquad \Psi_1(0,1) = \Psi_0(0,1) - B_1(1)B_2(1) - \pi f_{\eta\theta}(0),$$

ω_0 representing the limiting expectation of $\widehat{F}_{\zeta\xi}(0, \tilde{n})$ from (4.16), and $\frac{1}{2}(\omega_0 - \omega_1) = \omega_0 - \pi f_{\eta\theta}(0)$ that of $\widehat{F}_{\zeta\xi}(1, \tilde{n})$, from (4.17).

PROPOSITION 6.1. *Let* $\{\zeta_t, \xi_t\} \in I_4(\alpha, \beta)$ *under* (4.6) *and let* (6.1), (6.2) *and* (6.3) *of Assumption* 6.1 *hold. Then as* $n \to \infty$,

$$n^{2\beta-1}(\hat{\nu}_l - \nu) \to_d \frac{\int_\Pi \phi(\lambda)\psi(\lambda) f_{\eta\theta}(\lambda)\, d\lambda}{\Phi_l(\beta)}, \qquad l = 0, 1,$$

and under (2.6),

$$n^{\beta-\alpha} m^{\alpha+\beta-1}(\tilde{\nu}_l - \nu) \to_d \frac{2(2\pi)^{1-\alpha-\beta} f_{\eta\theta}(0) \dfrac{\cos(\beta-\alpha)(\pi/2)}{1-\alpha-\beta}}{\Phi_l(\beta)}, \qquad l = 0, 1.$$

PROOF. Write

$$\hat{a}_l = \widehat{F}_{\zeta\xi}(l, \tilde{n}), \hat{b}_l = \widehat{F}_{\xi\xi}(l, \tilde{n}), \tilde{a}_l = \widehat{F}_{\zeta\xi}(l, m), \tilde{b}_l = \widehat{F}_{\xi\xi}(l, m).$$

Thus $\hat{\nu}_l - \nu = \hat{a}_l/\hat{b}_l, \tilde{\nu}_l - \nu = \tilde{a}_l/\tilde{b}_l$. Now $\tilde{b}_l = \hat{b}_l - \{\hat{b}_l - \tilde{b}_l - E(\hat{b}_l - \tilde{b}_l)\} - E(\hat{b}_l - \tilde{b}_l)$. The term in braces is $o_p(n^{2\beta-1})$ from (5.5) of Theorem 5.1, while from (4.24) and (4.25) of Theorem 4.5, $E(\hat{b}_l - \tilde{b}_l) = o(n^{2\beta-1})$. Thus from

Assumption 6.1 we have $n^{1-2\beta}\tilde{b}_l$, $n^{1-2\beta}\hat{b}_l \to_d \Phi_l(\beta)$. Next, from Theorem 5.1, $\hat{a}_l = E\hat{a}_l + O_p(n^{\alpha+\beta-1})$ and $\tilde{a}_l = E\tilde{a}_l + O_p(n^{\alpha+\beta-1})$, so that $\lambda_m^{\alpha+\beta-1}\tilde{a}_l = \lambda_m^{\alpha+\beta-1}E\tilde{a}_l + O_p(m^{\alpha+\beta-1})$. The proof is then routinely completed by means of Theorem 4.1. □

PROPOSITION 6.2. *Let* $\{\zeta_t, \xi_t\} \in I_4(\alpha, \beta)$ *under* (4.7) *and let* (6.1), (6.2) *and* (6.3) *of Assumption* 6.1 *hold. Then as* $n \to \infty$,

$$\frac{n^{2\beta-1}}{\log n}(\hat{\nu}_l - \nu) \xrightarrow{d} \frac{2f_{\eta\theta}(0)\sin\beta\pi}{\Phi_l(\beta)}, \qquad l = 0, 1,$$

and under (2.5),

$$\frac{n^{2\beta-1}}{\log m}(\tilde{\nu}_l - \nu) \xrightarrow{d} \frac{2f_{\eta\theta}(0)\sin\beta\pi}{\Phi_l(\beta)}, \qquad l = 0, 1.$$

PROOF. From Theorem 5.1, $\hat{a}_l - E\hat{a}_l$, $\tilde{a}_l - E\tilde{a}_l$ are $O_p(1)$, so that $\hat{a}_l/\log n$, $\tilde{a}_l/\log m \to_p 2f_{\eta\theta}(0)\sin\beta\pi$ by Theorem 4.2, and the remaining proof follows from that of Proposition 6.1. □

PROPOSITION 6.3. *Let* $\{\zeta_t, \xi_t\} \in I_4(0, 1)$ *and let* (6.1), *Assumption* 6.1 *and the additional assumptions of Theorem* 4.3 *hold. Then as* $n \to \infty$,

$$n(\hat{\nu}_l - \nu) \xrightarrow{d} \frac{\Psi_l(0, 1)}{\Phi_l(1)}, \qquad l = 0, 1,$$

and under (2.6),

$$n(\tilde{\nu}_l - \nu) \xrightarrow{d} \frac{\Psi_l(0, 1) - \frac{1}{2}(\omega_0 - \omega_1) - \sum_{j=0}^{\infty}(\psi_j - 1)\gamma_{-j}}{\Phi_l(1)}, \quad l = 0, 1. \tag{6.7}$$

PROOF. We have $\hat{a}_l \to_d \Psi_l(0, 1)$ by Assumption 6.1. Write $\tilde{a}_l = \{\hat{a}_l - E\hat{a}_l\} - \{\hat{a}_l - \tilde{a}_l - E(\hat{a}_l - \tilde{a}_l)\} + E\tilde{a}_l$. For $l = 1$, the last two terms are respectively $o_p(1)$ by Theorem 5.1, and $O(m/n)$ by (4.20) of Theorem 4.3, whereas by Assumption 6.1 and (4.17) of Theorem 4.3, $\hat{a}_1 - E\hat{a}_1$ converges in distribution to the numerator on the right of (6.7). For $l = 0$, the only difference is that $E\tilde{a}_0 \to \pi f_{\eta\theta}(0)$ from (4.19), and since $E\hat{a}_0 \to \sum_{j=0}^{\infty}\psi_j\gamma_{-j}$ we get the same correction term in the numerator as when $l = 1$. The proof is again completed by that of Proposition 6.1. □

PROPOSITION 6.4. *Let* $\{\zeta_t, \xi_t\} \in I_4(0, \beta)$, *for* $\beta > 1$, *and let* (6.1) *and Assumption* 6.1 *hold. Then as* $n \to \infty$, *for* $l = 0, 1$,

$$n^{\beta}(\hat{\nu}_l - \nu) \xrightarrow{d} \frac{\Psi_l(0, \beta)}{\Phi_l(\beta)}.$$

The proof is routine.

PROPOSITION 6.5. *Let* $\{\zeta_t, \xi_t\} \in I_4(\alpha, \beta)$, *for* $\alpha > 0$, $\alpha + \beta > 1$, *and let* (6.1), (6.2) *and Assumption* 6.1 *hold. Then as* $n \to \infty$, *for* $l = 0, 1$,

$$n^{\beta-\alpha}(\hat{\nu}_l - \nu) \xrightarrow{d} \frac{\Psi_l(\alpha, \beta)}{\Phi_l(\beta)}, \tag{6.8}$$

and under (2.5),

$$n^{\beta-\alpha}(\tilde{\nu}_l - \hat{\nu}_l) \xrightarrow{p} 0, \tag{6.9}$$

and thus

$$n^{\beta-\alpha}(\tilde{\nu}_l - \nu) \xrightarrow{d} \frac{\Psi_l(\alpha, \beta)}{\Phi_l(\beta)}. \tag{6.10}$$

PROOF. The proof of (6.8) is routine, and (6.10) will follow from (6.8) and (6.9). To prove (6.9), write $\tilde{\nu}_l - \hat{\nu}_l = (\tilde{a}_l - \hat{a}_l)/\hat{b}_l + \tilde{a}_l(\tilde{b}_l^{-1} - \hat{b}_l^{-1})$. Now $\tilde{a}_l - \hat{a}_l = o_p(n^{\alpha+\beta-1})$ and $\tilde{a}_l = O_p(n^{\alpha+\beta-1})$ by Theorem 4.5, while $\tilde{b}_l^{-1} - \hat{b}_l^{-1} = (\hat{b}_l\tilde{b}_l)^{-1}(\hat{b}_l - \tilde{b}_l) = o_p(n^{1-2\beta})$ by the proof of Proposition 6.1 and Assumption 6.1. □

Proposition 6.4 has convergence rate compatible with those of Propositions 6.3 and 6.5, but only deals with the full-band statistics $\hat{\nu}_l$, in view of a remark following Theorem 4.4. Proposition 6.5 shows that when $\alpha > 0$ and the combined memory $\alpha + \beta$ of the observables and cointegrating error exceeds that of the usual case $\alpha = 0$, $\beta = 1$, $\tilde{\nu}_l$ has the same convergence rate and limit distribution as $\hat{\nu}_l$, so that nothing asymptotically is lost by neglecting high frequencies, even all those outside a band around zero that decays arbitrarily slower than n^{-1}. In Propositions 6.1–6.3, $\tilde{\nu}_l$ is found to have the capacity to beat $\hat{\nu}_l$ when it is less affected by the "bias" due to correlation between ζ_t and ξ_t in (6.1). In Proposition 6.3, when $\alpha = 0$, $\beta = 1$, rates of convergence are identical but $\tilde{\nu}_1$ eliminates the "second-order bias" [see Phillips (1991)] namely the expectation of $\Psi_1(0, 1)$; more particularly, the "second order bias" of $\tilde{\nu}_1$ is only $O(m/n^2)$, which is of smaller order than $1/n$ under (2.6). Monte Carlo simulations [see Robinson and Marinucci (1997)] demonstrate the consequent superiority of $\tilde{\nu}_1$ in smallish samples. (Note that $\tilde{\nu}_0$ does not share this desirable property of $\tilde{\nu}_1$.) In Proposition 6.2, $\alpha > 0$, $\beta < 1$ but again $\alpha + \beta = 1$, and here the comparison depends on m. If m increases at the same rate as n, as permitted by (2.5), so $\log m \sim \log n$, then $\hat{\nu}_l$ and $\tilde{\nu}_l$ have the same convergence rate and limit distribution. On the other hand if (2.6) holds there are essentially two possibilities of interest. If $m \sim cn^d$, for $c > 0, 0 < d < 1$, then $\tilde{\nu}_l$ has the same convergence rate as $\hat{\nu}_l$ but it is numerically shrunk towards ν. If $\log m = o(\log n)$, for example if $m = \log\log n$, then $\tilde{\nu}_l$ converges faster than $\hat{\nu}_l$. This latter phenomenon is more dramatically evident in Proposition 6.1, where, with $\alpha + \beta < 1$, $\tilde{\nu}_l$'s bias-reducing qualities really come to the fore; the more slowly m increases the better.

A different definition of nonstationary fractional processes (which would lead to different limit distributional forms for our estimates) entails integer differences having stationary long memory, or negative memory with invertibility. Chan and Terrin (1995) [see also Sowell (1990)] nest this kind of behavior in a vector autoregression (AR) and study the LSE of the AR coefficients, while Marinucci (2000) studies estimates similar to those in our paper, replacing averaged periodograms by weighted sums of sample autocovariances. Jeganathan (1999, 2001) also employs this definition of fractional nonstationarity, considering cointegration in a first order AR model driven by a simple parametric fractional stationary process, considering also the possibility that the AR coefficient is less than one in absolute value. He establishes asymptotic properties of maximum likelihood estimates based on a general but known distributional form for the innovations, the estimates of the cointegrating coefficient ν having a mixed normal asymptotic distribution, leading to a standard, χ^2, null limit distribution for Wald statistics for testing hypotheses on ν, analogous to results of Phillips (1991) in case $\alpha = 0$, $\beta = 1$ is known. The convergence rates of his estimates of ν correspond in our setting to $n^{\beta-\alpha}$ for $\beta-\alpha > \frac{1}{2}$, $(n/\log n)^{1/2}$ for $\beta-\alpha = \frac{1}{2}$ and $n^{1/2}$ for $0 < \beta-\alpha < \frac{1}{2}$. We believe such rates are optimal over our broader (α, β) space, and $\hat{\nu}_l$ and $\tilde{\nu}_l$ achieve them when $\alpha+\beta > 1$ and $\beta-\alpha \geq \frac{1}{2}$, or when $\alpha = 0$, $\beta = 1$ (see Propositions 6.3–6.5) but not otherwise (see Propositions 6.1 and 6.2). In fact, as Theorem 4.1 hints, the $n^{\beta-\alpha}$ rate, for any α, β such that $0 \leq \alpha < \beta$, may be achievable by the NBLSE with m fixed as $n \to \infty$, for example $\tilde{\nu}_1$ with $m = 1$; when $\alpha+\beta \leq 1$ such that $(\alpha, \beta) \neq (0, 1)$ this estimate converges faster than our estimates which assume $m \to \infty$, essentially because even less bias is incurred. However, such an estimate is likely to be unstable with an unusually dispersed limit distribution, for example in case $m = 1$ and random walk Gaussian z_t, its denominator is proportional to a χ_2^2 variate. Alternatively, the limitation of convergence rates in Propositions 6.1 and 6.2 due to coherence between ζ_t and z_t raises the possibility of achieving the optimal rates by a form of bias-correction. However this would require estimating the constant numerators in the limit distributions, which in turn would necessitate computing estimates of α, β and of other nuisance parameters/functions, while theoretical justification would require further assumptions and considerable extra proof.This kind of effort seems better directed to achieving and justifying estimates of ν which not only achieve optimal rates but also the desirable mixed normal limit distributional behavior for parametric or semiparametric forms of our vector nonstationary processes. Such estimates require sufficiently good preliminary estimates of ν, for which our present estimates suffice, but we also believe these are of interest in themselves, the LSE for its computational simplicity and familiarity, and the NBLSE for its bias-reducing property and illustration of the dominating importance of low frequencies in cointegration analysis.

7. Proofs for Section 3.

PROOF OF LEMMA 3.1. The proof when $\alpha = 0$ and/or $\beta = 0$ is trivial so assume $\alpha > 0$, $\beta > 0$. By integral approximation we have

$$\chi_t \sim \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx \sim \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}$$

to verify (3.2). For $0 < r < t$ we may write

$$\chi_t - \chi_{t+1} = \sum_{s=0}^{r} \phi_s(\psi_{t-s} - \psi_{t+1-s}) - \phi_{r+1}\psi_{t-r} + \sum_{s=0}^{t-r-1} (\phi_{t-s} - \phi_{t+1-s})\psi_s.$$

Taking $r = [t/2]$, all three terms are easily seen to be $O(t^{\alpha+\beta-2}) = O(|\chi_t|/t)$, to verify (3.3). □

PROOF OF LEMMA 3.2. The proof of (3.8) is trivial so consider (3.9) and (3.10) with $\alpha > 0$. Drop the argument α from $S_{uv}(\lambda, \alpha)$ and omit the trivially easy case $v = u + 1$. Obviously $|S_{uv}(\lambda)| \le Cv^{\gamma}$. For $\alpha \in (0, 1)$ write, for $u < s < v$,

$$S_{uv}(\lambda) = \sum_{t=u}^{s-1} \phi_t e^{it\lambda} + \sum_{t=s}^{v-1}(\phi_t - \phi_{t+1}) \sum_{r=s}^{t} e^{ir\lambda} + \phi_v \sum_{t=s}^{v} e^{it\lambda}$$

by summation-by-parts. Thus because

$$\left| \sum_{r=s}^{t} e^{ir\lambda} \right| \le \frac{C(t-s)}{1 + (t-s)|\lambda|}, \qquad |\lambda| \le \pi, \tag{7.1}$$

[see, e.g., Zygmund (1977), page 51], (3.2) and (3.3) imply that $|S_{uv}(\lambda)| \le C(s^{\alpha} + s^{\alpha-1}/|\lambda|)$. For $c \in (0, \pi)$ we may choose $s = [c/|\lambda|]$ when $c/v < |\lambda| < c/(u+1)$, which gives the bound $C/|\lambda|^{\alpha}$ for such λ. On the other hand we also have

$$S_{uv}(\lambda) = \sum_{t=u}^{v-1}(\phi_t - \phi_{t+1}) \sum_{s=u}^{t} e^{is\lambda} + \phi_v \sum_{t=u}^{v} e^{it\lambda} \tag{7.2}$$

to deduce $|S_{uv}(\lambda)| \le C(u+1)^{\alpha-1}/|\lambda|$ for $0 < \alpha < 1$ from (3.2), (3.3), (7.1). Since $v^{\alpha} \le C/|\lambda|^{\alpha}$ for $0 < |\lambda| \le c/v$ and $(u+1)^{\alpha-1}/|\lambda| \le C/|\lambda|^{\alpha}$ for $c/(u+1) \le |\lambda| \le \pi$ the bound $C/|\lambda|^{\alpha}$ holds for all $\lambda \in (0, \pi]$ when $0 < \alpha < 1$, to complete the proof of (3.9). For $\alpha > 1$, (7.2) gives instead $|S_{uv}(\lambda)| \le Cv^{\alpha-1}/|\lambda|$ to complete the proof of (3.10). Finally (3.11) follows directly from Theorem III-11 of Yong (1974) and a reflection formula for the Gamma function. □

PROOF OF LEMMA 3.3. We have $\xi_t = 0$, $t \le 0$, and for $t \ge 1$,

$$\xi_t = \sum_{s=1}^{t} \psi_{t-s} \sum_{r=1}^{s} \phi_{s-r}\eta_r = \sum_{s=1}^{t} \chi_{t-s}\eta_s,$$

where χ_t is given in (3.7). □

PROOF OF LEMMA 3.4. (i) The first statement is standard while (3.16) follows from the stated formula for $f_{\rho\rho}$, (3.11) and $\{\eta_t\} \in I$. For $\alpha > 0$ write

$\bar{\phi}_t(\lambda) = \sum_{s=t}^{\infty} \phi_s e^{is\lambda}$. From (3.10) we have $|\phi_t(\lambda)| \le C|\lambda|^{-\alpha}$ and $|\bar{\phi}_t(\lambda)| \le Ct^{\alpha-1}/|\lambda|$, so

$$\begin{aligned} \big| |\phi(\lambda)|^2 - |\phi_t(\lambda)|^2 \big| &\le \big| \phi_t(\lambda)\bar{\phi}_t(-\lambda) + \phi_t(-\lambda)\bar{\phi}_t(\lambda) \big| + |\bar{\phi}_t(\lambda)|^2 \\ &\le C\big(t^{\alpha-1}|\lambda|^{-\alpha-1} + t^{2\alpha-2}\lambda^{-2}\big) \\ &\le C|\lambda|^{-2\alpha}\big((t|\lambda|)^{\alpha-1} + (t|\lambda|)^{2(\alpha-1)}\big), \end{aligned}$$

whence (3.17) follows by reference to (3.16). To prove (3.18), note that, for $j \ge 0$,

$$\operatorname{Cov}(\zeta_t, \zeta_{t+j}) = \int_\Pi \phi_t(\lambda)\phi_{t+j}(-\lambda) f_{\eta\eta}(\lambda) e^{ij\lambda}\, d\lambda, \tag{7.3}$$

so its deviation from $\operatorname{Cov}(\rho_0, \rho_j)$ is bounded by

$$\int_\Pi \big\{ |\phi(\lambda)\bar{\phi}_{t+j}(-\lambda)| + |\bar{\phi}_t(\lambda)\phi_{t+j}(-\lambda)| \big\} f_{\eta\eta}(\lambda)\, d\lambda. \tag{7.4}$$

Fix $\delta > 0$. Because $\{\eta_t\} \in I$ we can choose $\varepsilon > 0$ such that

$$\sup_{|\lambda|<\varepsilon} |f_{\eta\eta}(\lambda) - f_{\eta\eta}(0)| < \delta. \tag{7.5}$$

Also, we have

$$\frac{1}{2\pi} \int_\Pi \left| \sum_{s=u}^{v} \phi_s e^{is\lambda} \right|^2 d\lambda = \sum_{s=u}^{v} \phi_s^2. \tag{7.6}$$

Thus by the Schwarz inequality the contribution to (7.4) from the integral over $(-\varepsilon, \varepsilon)$ is bounded by

$$2\{f_{\eta\eta}(0) + \delta\} \left\{ \sum_{s=0}^{\infty} \phi_s^2 \sum_{s=t}^{\infty} \phi_s^2 \right\}^{1/2} = O(t^{\alpha-(1/2)})$$

as $t \to \infty$, while the contribution from $[-\pi, -\varepsilon] \cup [\varepsilon, \pi]$ is bounded by

$$\frac{C}{\varepsilon^2} t^{\alpha-1} \int_\Pi f_{\eta\eta}(\lambda)\, d\lambda = O(t^{\alpha-1}),$$

using (3.9).

(ii) and (iii). The difference between (7.3) and $2\pi f_{\eta\eta}(0) \sum_{s=0}^{t-1} \phi_s^2$ is

$$f_{\eta\eta}(0) \int_\Pi \big\{ \phi_t(\lambda)\phi_{t+j}(-\lambda)e^{ij\lambda} - |\phi_t(\lambda)|^2 \big\} d\lambda \tag{7.7}$$

$$+ \int' \big\{ f_{\eta\eta}(\lambda) - f_{\eta\eta}(0) \big\} \phi_t(\lambda)\phi_{t+j}(-\lambda)e^{ij\lambda}\, d\lambda \tag{7.8}$$

$$+ \int'' \big\{ f_{\eta\eta}(\lambda) - f_{\eta\eta}(0) \big\} \phi_t(\lambda)\phi_{t+j}(-\lambda)e^{ij\lambda}\, d\lambda, \tag{7.9}$$

where, here and subsequently,

$$\int' = \int_{|\lambda|<\varepsilon}, \qquad \int'' = \int_{|\lambda|\ge\varepsilon}.$$

Now (7.7) is zero for $j = 0$, and for $j \geq 1$ it is bounded by

$$\begin{aligned} C\left|\sum_{s=0}^{t-1} \phi_s(\phi_{s+j} - \phi_s)\right| &\leq C\sum_{s=0}^{t-1} |\phi_s| \sum_{r=0}^{j-1} |\phi_{s+r+1} - \phi_{s+r}| \\ &\leq C\sum_{s=0}^{t-1} |\phi_s| \sum_{r=0}^{j-1} |\phi_{s+r}|/(s+r), \end{aligned}$$

using (3.3). The last expression is, uniformly, $O(j\sum_1^t s^{\alpha-3}) = O(j)$ for $\frac{1}{2} \leq \alpha < 1$, $O(j\log t)$ for $\alpha = 1$, and $O(j\sum_1^t s^{\alpha-2}(s+j)^{\alpha-1}) = O(jt^{2\alpha-2})$ for $\alpha > 1$. Because (7.8) is $O(\delta\sum_0^{t+j}\phi_s^2)$, which is uniformly $O(\delta\log t)$ for $\alpha = \frac{1}{2}$ and $O(\delta t^{2\alpha-1})$ for $\alpha > \frac{1}{2}$, while (7.9) is, from (3.9), $O((t+j)^{2\alpha-2}) = O(t^{2\alpha-2})$ uniformly, the proof may then be routinely completed, noting that $\sum_{s=0}^t \phi_s^2 \sim (\log t)/\pi$ for $\alpha = \frac{1}{2}$ and $\sum_{s=0}^t \phi_s^2 \sim t^{2\alpha-1}/\Gamma^2(\alpha)(2\alpha-1)$ for $\alpha > \frac{1}{2}$. $\square$

8. Proofs for Section 4.

Proof of Lemma 4.1. Though (4.3) is of independent interest it is not in this generality of much importance to the sequel, while a full proof would require introduction of notation which would not find subsequent use. We thus give the proof only of (4.5), which is equivalent to (4.3) with $A = B = R$, $j = k = 0$, the full proof of (4.3) being only notationally more complex. We first provide some basic derivations which will be useful also in subsequent proofs. In view of (2.1), (3.12), (3.13), (3.15), (4.2), we can write

$$w_\zeta(\lambda) = \frac{1}{(2\pi n)^{1/2}}\sum_t \phi_{n-t+1}(\lambda)\eta_t e^{it\lambda}, \qquad w_\xi(\lambda) = \frac{1}{(2\pi n)^{1/2}}\sum_t \psi_{n-t+1}(\lambda)\theta_t e^{it\lambda},$$

where $\psi_t(\lambda) = \sum_{s=0}^{t-1}\psi_s e^{is\lambda}$. From (2.7)

$$EI_{\zeta\xi}(\lambda) = \frac{1}{2\pi n}\int_\Pi \chi_n(\lambda,\mu)f(\mu)\,d\mu, \tag{8.1}$$

where for brevity we write $f(\mu) = f_{\eta\theta}(\mu)$, and $\chi_n(\lambda,\mu) = \phi_n(\lambda,-\mu)\psi_n(-\lambda,\mu)$, in which, for example,

$$\begin{aligned} \phi_n(\lambda,\mu) &= \sum_t e^{it(\lambda+\mu)}\phi_t(-\mu) \\ &= \sum_t \phi_{n-t+1}(\lambda)e^{it(\lambda+\mu)} = \sum_t \phi_{n-t}e^{i(n-t)\lambda}D_t(\lambda+\mu), \end{aligned} \tag{8.2}$$

the final equality following by summation-by-parts with

$$D_t(\lambda) = \sum_{s=1}^t e^{is\lambda},$$

the Dirichlet kernel, and all three representations in (8.2) finding use in the sequel. From (7.1),

$$|\phi_n(\lambda,\mu)| \leq \frac{Cn^{\alpha+1}}{1+n|\lambda+\mu|}, |\psi_n(\lambda,\mu)| \leq \frac{Cn^{\beta+1}}{1+n|\lambda+\mu|}, \quad 0 \leq |\lambda+\mu| \leq \pi. \tag{8.3}$$

Fix $\delta > 0$, then choose $\varepsilon \in (0, \pi)$ such that

$$\sup_{|\lambda|<\varepsilon} |f(\lambda) - f(0)| < \delta. \tag{8.4}$$

We deduce from (8.1) that

$$E(\bar{\zeta}\bar{\xi}) = \frac{2\pi}{n} EI_{\zeta\xi}(0) = \frac{1}{n^2}\int_\Pi \chi_n(0, \mu) f(\mu)\, d\mu,$$

which can be written, for $\varepsilon \in (0, \pi)$, as

$$\frac{f(0)}{n^2}\int_\Pi \chi_n(0, \mu)\, d\mu + \frac{1}{n^2}\int' \chi_n(0, \mu)\bar{f}(\mu)\, d\mu + \frac{1}{n^2}\int'' \chi_n(0, \mu)\bar{f}(\mu)\, d\mu, \tag{8.5}$$

writing $\bar{f}(\mu) = f(\mu) - f(0)$. Since

$$\int_\Pi D_s(\lambda) D_t(-\lambda)\, d\lambda = 2\pi \min(s, t), \tag{8.6}$$

the first component of (8.5) is, from (8.2),

$$\frac{2\pi f(0)}{n^2}\sum_t \sum_s \phi_{n-s}\psi_{n-t} \min(s, t) = \frac{2\pi f(0)}{n^2}\sum_t \sum_{r=0}^{n-t} \phi_r \sum_{s=0}^{n-t} \psi_s.$$

For $\alpha = 0$, $\alpha > 0$ this is, respectively,

$$\frac{2\pi f(0)}{n^2}\sum_s s\psi_{n-s} \sim \frac{2\pi f(0) n^{\beta-1}}{\Gamma(\beta)}\int_0^1 x(1-x)^{\beta-1}\, dx \sim \frac{2\pi f(0) n^{\beta-1}}{\Gamma(\beta+2)},$$

$$\begin{aligned} &\frac{2\pi f(0) n^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \int_0^{1-x} y^{\alpha-1} dy \int_0^{1-x} z^{\beta-1}\, dz\, dx(1 + o(1)) \\ &\quad \sim \frac{2\pi f(0) n^{\alpha+\beta-1}}{\Gamma(\alpha+1)\Gamma(\beta+1)(\alpha+\beta+1)}, \end{aligned}$$

as $n \to \infty$. The second term in (8.5) is bounded by

$$\begin{aligned} &\frac{C\delta}{n^2}\int' \Big\{\sum_s |\phi_{n-s}|\, |D_s(\mu)|\Big\}\Big\{\sum_t |\psi_{n-t}|\, |D_t(\mu)|\Big\} d\mu \\ &\qquad \le \frac{C\delta}{n^2}\sum_s |\phi_{n-s}| \sum_t |\psi_{n-t}| \max_{1\le t\le n}\int_\Pi |D_t(\mu)|^2\, d\mu \le C\delta n^{\alpha+\beta-1}, \end{aligned} \tag{8.7}$$

using (8.6). Because δ is arbitrary the second term of (8.5) can be neglected. The final term of (8.5) is bounded by [cf. (8.7)]

$$\begin{aligned} \frac{C}{n^2}\sum_t |\phi_{n-t}| \sum_t |\psi_{n-t}| \int_\varepsilon^\pi \mu^{-2}|\bar{f}(\mu)|\, d\mu &\le \frac{Cn^{\alpha+\beta-2}}{\varepsilon^2}\big[\{\mathrm{Var}(\eta_t)\mathrm{Var}(\theta_t)\}^{1/2} + |f(0)|\big] \\ &= O(n^{\alpha+\beta-2}). \qquad \square \end{aligned}$$

PROOF OF THEOREM 4.1. Abbreviate $\widehat{F}_{\zeta\xi}$ to $\widehat{F}$. We first prove (4.12), where for any $\varepsilon > 0$ we can choose n such that $2\lambda_m < \varepsilon$. Take $l = 1$. From (8.1), $E\{\widehat{F}(1, m)\}$ is the real part of

$$\frac{2}{n^2}\sum_{j=1}^{m}\int' \chi_n(\lambda_j, \mu) f(\mu)\, d\mu + \frac{2}{n^2}\sum_{j=1}^{m}\int'' \chi_n(\lambda_j, \mu) f(\mu)\, d\mu. \tag{8.8}$$

From (8.3), the second term is bounded in modulus by

$$Cmn^{\alpha+\beta-2}\int_{\Pi} |f(\lambda)|\, d\lambda = O(mn^{\alpha+\beta-2}) = o\biggl(\Bigl(\frac{n}{m}\Bigr)^{\alpha+\beta-1}\biggr). \tag{8.9}$$

The difference between the first term of (8.8) and

$$\frac{2}{n^2} f(0)\sum_{j=1}^{m}\int' \chi_n(\lambda_j, \mu)\, d\mu \tag{8.10}$$

is bounded by

$$\frac{2\delta}{n^2}\sum_{j=1}^{m}\int_{\Pi} |\chi_n(\lambda_j, \mu)|\, d\mu \le \frac{C\delta}{n^2}\sum_{j=1}^{m}\biggl\{\sum_t |\phi_t(\lambda_j)|^2 \sum_t |\psi_t(\lambda_j)|^2\biggr\}^{1/2}, \tag{8.11}$$

using the Schwarz inequality and, for example, from (8.2),

$$\int_{\Pi} |\phi_n(\lambda, -\mu)|^2\, d\mu = 2\pi \sum_t |\phi_t(\lambda)|^2. \tag{8.12}$$

From (3.9), the factor in braces in (8.11) is $O(n^2|\lambda_j|^{-2(\alpha+\beta)})$, so that (8.11) is bounded by

$$C\delta n^{\alpha+\beta-1}\sum_{j=1}^{m} j^{-\alpha-\beta} \le C\delta\Bigl(\frac{n}{m}\Bigr)^{\alpha+\beta-1}, \tag{8.13}$$

and can thus be neglected because δ is arbitrary.

The difference between (8.10) and

$$\frac{2f(0)}{n^2}\sum_{j=1}^{m}\int_{\Pi} \chi_n(\lambda_j, \mu)\, d\mu \tag{8.14}$$

is $O(n^{\alpha+\beta-2}m) = o((n/m)^{\alpha+\beta-1})$ using (8.3) again, so it remains to estimate the real part of (8.14), which is $4\pi f(0)$ times

$$\frac{1}{n^2}\sum_{j=1}^{m}\sum_t \chi_t(\lambda_j) = \frac{1}{n}\sum_{j=1}^{m} \phi(\lambda_j)\psi(-\lambda_j) \tag{8.15}$$

$$- \frac{1}{n^2}\sum_{j=1}^{m}\biggl\{\phi(\lambda_j)\sum_t \bar{\psi}_t(-\lambda_j) + \sum_t \bar{\phi}_t(\lambda_j)\psi(-\lambda_j)\biggr\} \tag{8.16}$$

$$+ \frac{1}{n^2}\sum_{j=1}^{m}\sum_t \bar{\phi}_t(\lambda_j)\bar{\psi}_t(-\lambda_j), \tag{8.17}$$

where $\chi_t(\lambda) = \phi_t(\lambda)\psi_t(-\lambda)$ and $\bar{\psi}_t(\lambda) = \sum_{s=t}^{\infty} \psi_s e^{is\lambda}$. For $\alpha = 0$, (8.17) is zero. For $\alpha > 0$, applying (3.9), we bound (8.17) by

$$\frac{C}{n^2}\sum_{j=1}^{m}\left(\sum_{t=1}^{[1/2\lambda_j]} \lambda_j^{-\alpha-\beta} + \lambda_j^{-2}\sum_{t=[1/2\lambda_j]}^{n} t^{\alpha+\beta-2}\right) \le \frac{C}{n^2}\sum_{j=1}^{m}\lambda_j^{-1-\alpha-\beta}$$
$$\le Cn^{\alpha+\beta-1} = o\left(\left(\frac{n}{m}\right)^{\alpha+\beta-1}\right).$$

Likewise, for $\alpha > 0$, (8.16) is bounded by

$$\frac{C}{n^2}\sum_{j=1}^{m}\left(\lambda_j^{-\alpha-1}\sum_t t^{\beta-1} + \lambda_j^{-\beta-1}\sum_t t^{\alpha-1}\right) \le Cn^{\alpha+\beta-1}\sum_{j=1}^{m}(j^{-\alpha-1} + j^{-\beta-1})$$
$$= o\left(\left(\frac{n}{m}\right)^{\alpha+\beta-1}\right),$$

whereas for $\alpha = 0$, (8.16) is bounded by $Cn^{-2}\sum_{j=1}^{m}\lambda_j^{-1}\sum_t t^{\beta-1} \le Cn^{\beta-1}\log m = o((n/m)^{\beta-1})$. Finally, the right side of (8.15) has, from (3.11), real part

$$\begin{aligned}&\frac{1}{n}\left(\cos\frac{\alpha\pi}{2}\cos\frac{\beta\pi}{2} + \sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}\right)\sum_{j=1}^{m}\lambda_j^{-\alpha-\beta}(1+o(1))\\ &\qquad \sim \frac{\cos(\alpha-\beta)\frac{\pi}{2}}{2\pi(1-\alpha-\beta)}\cdot\lambda_m^{1-\alpha-\beta}\end{aligned} \tag{8.18}$$

as $n \to \infty$, to complete the proof of (4.12) with $l = 1$. The proof for $l = 0$ follows from (2.3) and Lemma 4.1, due to $\alpha + \beta < 1$.

To prove (4.11) with $l = 0$, we can deduce from (8.1) that

$$E\{\widehat{F}(0,\tilde{n})\} = \frac{1}{n}\sum_t\int_\Pi \chi_t(\mu) f(\mu)\,d\mu, \tag{8.19}$$

which differs from (4.11) by

$$\frac{1}{n}\int_\Pi\left\{\phi(\mu)\sum_t\bar{\psi}_t(-\mu) + \sum_t\bar{\phi}_t(\mu)\psi(-\mu)\right\}f(\mu)\,d\mu \tag{8.20}$$

$$+\frac{1}{n}\int_\Pi\sum_t\bar{\phi}_t(\mu)\bar{\psi}_t(-\mu)f(\mu)\,d\mu. \tag{8.21}$$

From (3.9) and (8.4), we can bound (8.21) by

$$C\int'|\mu|^{-\alpha-\beta}\,d\mu + \frac{C}{n\varepsilon^2}\sum_t t^{\alpha+\beta-2}\int_\Pi|f(\mu)|\,d\mu \le C\left(\varepsilon^{1-\alpha-\beta} + \frac{1}{n\varepsilon^2}\right) = o(1),$$

with the same bound resulting for (8.20). Finiteness of (4.11) follows similarly, by bounding it by $C(\varepsilon^{1-\alpha-\beta} + \varepsilon^{-2})$. Thus (4.11) is proved with $l = 0$, and thence with $l = 1$ by Lemma 4.1. □

PROOF OF THEOREM 4.2. Given (4.13) and (4.14) for $l = 0$, they hold also for $l = 1$ due to Lemma 4.1 and $\alpha + \beta = 1$, so we can ignore l. The proof of (4.14) closely follows that of (4.12). In place of (8.9) we have the bound $O(m/n) = o(\log m)$, while the right side of (8.13) is $O(\delta \log m) = o(\log m)$. The argument for replacing (8.14) by (8.10) holds, as does that for neglecting (8.15)–(8.17), while (8.18) is $(\sin \alpha\pi/2\pi) \log m(1 + o(1))$. To prove (4.13), we can write (8.19) as

$$\text{(8.22)} \qquad \frac{1}{n} \sum_t \left\{ f(0) \int_\Pi \chi_t(\mu)\, d\mu + \int' \chi_t(\mu) \bar{f}(\mu)\, d\mu + \int'' \chi_t(\mu) \bar{f}(\mu)\, d\mu \right\}.$$

The contribution from the first term in the braces is

$$\text{(8.23)} \qquad \frac{2\pi f(0)}{n} \sum_t \sum_{s=0}^{t-1} \phi_s \psi_s \sim \frac{2\pi f(0)}{\Gamma(\alpha)\Gamma(1-\alpha)} \left(1 + \sum_{s=1}^{n} s^{-1}\right) \sim 2 \sin \alpha\pi f(0) \log n.$$

That from the remaining terms can be bounded, respectively, by

$$\begin{aligned} &\frac{\delta}{n} \sum_t \int_{-n^{-1}}^{n^{-1}} |\chi_t(\mu)|\, d\mu + \frac{\delta}{n} \sum_t \int_{n^{-1} \le |\mu| \le \varepsilon} |\chi_t(\mu)|\, d\mu \\ &\qquad \le \frac{C\delta}{n^2} \sum_t t + C\delta \int_{n^{-1}}^{\varepsilon} \mu^{-1}\, d\mu \le \delta C(1 + \log n), \end{aligned}$$

and by $(C/\varepsilon^2) \int_\Pi |f(\mu)|\, d\mu < C$, using (3.9). □

PROOF OF THEOREM 4.3. First note that (4.17) and (4.19) follow from Lemma 4.1 and (4.16) and (4.18), respectively, since (4.5) is $\pi f(0)$. To prove (4.16) note first that $\omega_0 = \sum_{j=-\infty}^{0} \gamma_j$, $\omega_1 = \sum_{j=1}^{\infty} \gamma_j$ are both finite, because $2\pi f(0) = \omega_0 + \omega_1$ and $\int_\Pi h(\lambda)\, d\lambda = \omega_0 - \omega_1$ both are, writing $h = h_{\eta\theta}$. Direct calculation gives

$$\text{(8.24)} \qquad \begin{aligned} E\{\widehat{F}(0, \tilde{n})\} &= \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) \psi_j \gamma_{-j} \\ &= \sum_{j=0}^{\infty} \psi_j \gamma_{-j} - \sum_{j=n}^{\infty} \psi_j \gamma_{-j} - \frac{1}{n} \sum_{j=0}^{n-1} j \psi_j \gamma_{-j}. \end{aligned}$$

By summation-by-parts, the second term is bounded by

$$\text{(8.25)} \qquad \sum_{j=n}^{\infty} |\psi_j - \psi_{j+1}| \left| \sum_{l=n}^{j} \gamma_{-l} \right| \le \sum_{j=n}^{\infty} |\psi_j - \psi_{j+1}| \big(|\omega_{-n}| + |\omega_{-j-1}|\big) \to 0$$

as $n \to \infty$, whereas the final term is bounded by

$$\text{(8.26)} \qquad \frac{1}{n} \sum_{j=0}^{n-1} \big(j|\psi_j - \psi_{j+1}| + |\psi_{j+1}|\big) |\omega_{-j-1}| + \left(\frac{n-1}{n}\right) |\psi_{n-1}|\, |\omega_{-n}|$$

which tends to 0 for similar reasons. Thus (4.16) is proved.

It is convenient to first prove (4.20) when $\psi_t \equiv 1$, and then estimate the "error." Write $\tilde{\omega}_l = \sum_{k\le l} \gamma_k$, whence

$$E\{I_{\zeta\xi}(\lambda_j)\} = \frac{1}{2\pi n}\sum_s\sum_t(\tilde{\omega}_{t-s} - \tilde{\omega}_{-s})e^{i(s-t)\lambda_j}$$

$$(8.27)\qquad = \frac{1}{2\pi}\sum_1^{n-1}\left(1-\frac{l}{n}\right)(\tilde{\omega}_l + \tilde{\omega}_{-l})e^{-il\lambda_j} + \frac{\tilde{\omega}_0}{2\pi}$$

because

$$(8.28)\qquad D_n(\lambda_j) = n, \qquad j = 0, \mathrm{mod}(n),$$

$$(8.29)\qquad = 0, \qquad \text{otherwise}.$$

For $l \le 0$ we have $\tilde{\omega}_l = \omega_l$, whereas for $l \ge 0$ we have $\tilde{\omega}_l = \omega_0 + \omega_1 - \omega_{l+1}$, so (8.27) has real part

$$\frac{1}{2\pi}\sum_1^{n-1}\left(1-\frac{l}{n}\right)(\omega_0+\omega_1+\omega_{-l}-\omega_{l+1})\cos l\lambda_j + \frac{\omega_0}{2\pi}$$

$$(8.30)\qquad = \frac{1}{4\pi}\sum_{1-n}^{n-1}\left(1-\frac{|l|}{n}\right)(\omega_{-|l|}-\omega_{|l|+1})\cos l\lambda_j,$$

in view of (8.29) and

$$\sum_{l=0}^{n-1} l\cos l\lambda_j = -\frac{n}{2}, \qquad 1 \le j \le n-1.$$

Now (8.30) is the Cesaro sum, to $n-1$ terms, of the Fourier series of $h(\lambda_j)/2$. Equivalently,

$$(8.31)\qquad E\left\{\frac{n}{m}\widehat{F}(1,m)\right\} = \frac{1}{nm}\sum_{j=1}^m\int_\Pi |D_n(\lambda-\lambda_j)|^2 h(\lambda)\,d\lambda.$$

Fix $\delta > 0$. There exists $\varepsilon > 0$ such that $|h(\lambda) - h(0)| < \delta$ for $0 < |\lambda| \le \varepsilon$. Let n be large enough that $2\lambda_m < \varepsilon$. The difference between the right-hand side of (8.31) and $2\pi h(0)$ is bounded by

$$\frac{1}{n}\left\{\delta\max_{1\le j\le m}\int' |D_n(\lambda-\lambda_j)|^2\,d\lambda + \sup_{(\varepsilon/2)<|\lambda|<\pi}|D_n(\lambda)|^2\left(\int_\Pi |h(\lambda)|\,d\lambda + 2\pi|h(0)|\right)\right\}$$

which is $O(\delta + n^{-1})$ using (8.6). Because δ is arbitrary, the proof of (4.20) when $\psi_t \equiv 1$ is complete.

The difference between $E\{\widehat{F}(1,m)\}$ and the same thing with $\psi_t \equiv 1$, is, from (8.2), the real part of

$$(8.32)\qquad \frac{2}{n^2}\sum_{j=1}^m\int_\Pi R_n(\lambda_j,\mu)f(\mu)\,d\mu,$$

where

$$R_n(\lambda,\mu) = D_n(\lambda-\mu)\sum_t(\psi_{n-t}-1)e^{it\lambda}D_t(\mu-\lambda).$$

Using (8.6), we may write (8.32) as

$$\frac{4\pi f(0)}{n^2}\sum_t t(\psi_{n-t}-1)\sum_{j=1}^m e^{it\lambda_j} \tag{8.33}$$

$$+\frac{2}{n^2}\sum_{j=1}^m\int' R_n(\lambda_j,\mu)\bar f(\mu)\,d\mu+\frac{2}{n^2}\sum_{j=1}^m\int'' R_n(\lambda_j,\mu)\bar f(\mu)\,d\mu. \tag{8.34}$$

To consider (8.33), we have

$$\frac{1}{n^2}\left|\sum_{t=1}^{n-r}t(\psi_{n-t}-1)\sum_{j=1}^m e^{it\lambda_j}\right|\le\frac{m}{n}\sum_{t=r}^\infty|\psi_t-1|=o\left(\frac{m}{n}\right)$$

as $r\to\infty$. On the other hand the contribution from $t>n-r$ to (8.33) is

$$4\pi f(0)\left\{\frac{1}{n}\sum_{t=0}^{r-1}(\psi_t-1)\sum_{j=1}^m e^{-it\lambda_j}-\frac{1}{n^2}\sum_{t=0}^{r-1}t(\psi_t-1)\sum_{j=1}^m e^{-it\lambda_j}\right\}.$$

We can bound the second term by $Crm/n^2=o(m/n)$, taking $r=o(n)$, whereas, using the inequality $|\cos x-1|\le x^2$, the first term has real part differing from

$$\frac{m}{n}\sum_{t=0}^{r-1}(\psi_t-1)=\frac{m}{n}\sum_{t=0}^{\infty}(\psi_t-1)(1+o(1)) \tag{8.35}$$

by something bounded by

$$\frac{C}{n}\sum_{t=0}^{r-1}|\psi_t-1|\sum_{j=1}^m(t\lambda_j)^2\le\frac{Cm^3r^2}{n^3}\sum_{t=0}^\infty|\psi_t-1|=o\left(\frac{m}{n}\right),$$

since we can at the same time choose $r=o(n/m)$. Since (8.35) delivers the correction term in (4.20), it remains to show that the contribution from (8.34) is $o(m/n)$. Using (8.6) and the Schwarz inequality, its first term is bounded by

$$\begin{aligned}&\frac{\delta}{n^2}\sum_{j=1}^m\left\{\int_\Pi|D_n(\lambda_j-\mu)|^2\,d\mu\int_\Pi\left|\sum_t(\psi_{n-t}-1)e^{it\lambda_j}D_t(\mu-\lambda_j)\right|^2d\mu\right\}^{1/2}\\ &\le\frac{C\delta}{n^{3/2}}\sum_{j=1}^m\left\{\sum_s\sum_t(\psi_{n-s}-1)(\psi_{n-t}-1)e^{i(s-t)\lambda_j}\min(s,t)\right\}^{1/2}\\ &\le\frac{C\delta m}{n}\sum_{t=0}^\infty|\psi_t-1|^2=O\left(\frac{\delta m}{n}\right),\end{aligned}$$

whereas, with $\varepsilon > 2\lambda_m$, its second term is bounded by

$$\frac{Cm}{n^2}\sum_{t=0}^{\infty}|\psi_t - 1|\left\{\int_{\Pi}|f(\mu)|\,d\mu + |f(0)|\right\} = o\left(\frac{m}{n}\right)$$

to complete the proof of (4.20). □

PROOF OF THEOREM 4.4. From (8.24)–(8.26),

$$\left|E\{\widehat{F}(0,\tilde{n})\}\right| \le \frac{C}{r}\sum_{j=1}^{n-1} j^{\beta-1}|\omega_{-j-1}| = o(n^{\beta-1})$$

by the Toeplitz lemma, to prove (4.21) and then, by Lemma 4.1, (4.22). To prove (4.23), consider (8.24) again: the second term on the right is clearly $o(1)$ while the last one can be written $n^{-1}\sum_{j=1}^{n-1}\sum_{l=j}^{\infty}\psi_l\gamma_{-l} - \sum_{l=n}^{\infty}\psi_l\gamma_{-l} \to 0$. □

PROOF OF THEOREM 4.5. To prove (4.24) with $m = \tilde{n}$ we use (8.19) and (8.22). The left side of (8.23) is

$$\frac{2\pi f(0)}{n\Gamma(\alpha)\Gamma(\beta)}\sum_t\sum_{s=0}^{t-1}s^{\alpha+\beta-2}(1+o(1)) \sim \frac{2\pi f(0)n^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta)(\alpha+\beta-1)}.$$

By the Schwarz inequality, the contribution from the second term in braces in (8.22) is bounded by

$$C\delta n^{\beta-3/2}\sum_t\left\{\int_{\Pi}|\phi_t(\lambda)|^2\,d\lambda\right\}^{1/2}, \tag{8.36}$$

since $\int_{\Pi}|\psi_t(\lambda)|^2\,d\lambda = 2\pi\sum_{s=1}^{t}\psi_s^2 \le Cn^{2\beta-1}$, because $\beta > \frac{1}{2}$. For $\alpha > \frac{1}{2}$, (8.36) is thus clearly $O(\delta n^{\alpha+\beta-1})$, while the same bound holds for $\alpha < \frac{1}{2}$ because the integral in (8.36) is bounded by

$$Ct^{2\alpha}\int_0^{n^{-1}}d\lambda + C\int_{n^{-1}}^{\pi}\lambda^{-2\alpha}\,d\lambda \le Cn^{2\alpha-1},$$

using (3.9). Finally, the contribution from the final term in braces in (8.22) is bounded by

$$\frac{C}{n}\int_{\varepsilon}^{\pi}\sum_t|\phi_t(\mu)\psi_t(\mu)|\{|f(\mu)| + |f(0)|\}\,d\mu,$$

which is $O(n^{\alpha+\beta-1})$ for $\alpha > 1$ and $O(n^{\beta-1})$ for $\alpha \le 1$, on applying (3.10). Thus (4.24) is proved for $m = \tilde{n}$, whence (4.25) follows by incorporating Lemma 4.1. Now with $m < \tilde{n}$, we show that the contribution from the second term on the right of (2.8) is negligible. We can bound $E\{\widehat{F}(m+1,\tilde{n})\}$ by

$$\frac{C}{n^2}\sum_j{}''\int'|\chi_n(\lambda_j,\mu)|\,|f(\mu)|\,d\mu + \frac{C}{n^2}\sum_j{}''\int''|\chi_n(\lambda_j,\mu)|\,|f(\mu)|\,d\mu, \tag{8.37}$$

where $\sum_j''$ denotes $\sum_{j=m+1}^{\tilde{n}}$. Applying the Schwarz inequality and (8.4), (8.12), the first term is bounded by

$$\frac{C}{n^2}\sum_j{}''\left\{\sum_t|\phi_t(\lambda_j)|^2\sum_t|\psi_t(\lambda_j)|^2\right\}^{1/2}. \tag{8.38}$$

On applying (3.9) and (3.10) we find that (8.38) is $O(n^{\alpha+\beta-1}\sum_{j=m}^{\infty}j^{-2}) = O(n^{\alpha+\beta-1}m^{-1})$ when $\alpha > 1$ and $O(n^{\alpha+\beta-1}\sum_{j=m}^{\infty}j^{-\alpha-\min(\beta,1)}) = o(n^{\alpha+\beta-1}\times m^{-\min(\alpha,\alpha+\beta-1)})$ when $\alpha \le 1$, so both are $o(n^{\alpha+\beta-1})$ from (2.5). Finally the second term of (8.37) is bounded by

$$\frac{C}{n^2}\int''\left\{\sum_{j=1}^{n}|\phi_n(\lambda_j,\mu)|^2\sum_{j=1}^{n}|\psi_n(\lambda_j,-\mu)|^2\right\}^{1/2}|f(\mu)|\,d\mu. \tag{8.39}$$

From (8.2), (8.28) and (8.29) we have, for example,

$$\sum_{j=1}^{n}|\psi_n(\lambda_j,-\mu)|^2 = n\sum_t|\psi_t(\mu)|^2. \tag{8.40}$$

Thus the term in braces in (8.39) is $n^2\sum_t|\phi_t(\mu)|^2\sum_t|\psi_t(\mu)|^2$, so from (3.9) and (3.10) we deduce that (8.39) is $O(n^{\alpha+\beta-2})$ when $\alpha > 1$ and $O(n^{\beta-1})$ when $\alpha \le 1$. □

9. Proofs for Section 5.

PROOF OF THEOREM 5.1. We first consider $\widehat{F}(0,\tilde{n})$, which has variance

$$\frac{1}{n^2}\sum_s\sum_t\sum_1^s\phi_{s-q}\sum_1^s\psi_{s-r}\sum_1^t\phi_{t-u}\sum_1^t\psi_{t-v}\{\gamma_{r-u}\gamma_{v-q}+\gamma_{u-q}^{(\eta)}\gamma_{v-r}^{(\theta)}+\kappa_{qruv}\}, \tag{9.1}$$

where $\gamma_j^{(\eta)} = \mathrm{Cov}(\eta_0,\eta_j)$, $\gamma_j^{(\theta)} = \mathrm{Cov}(\theta_0,\theta_j)$, $\kappa_{qruv} = \mathrm{Cum}\{\eta_q,\theta_r,\eta_u,\theta_v\}$. The contribution of the first term in braces to (9.1) can be written

$$\frac{1}{n^2}\sum_s\sum_t a_{st}a_{ts} = \frac{1}{n^2}\sum_s\sum_t(b_{st}+c_{st}+d_{st})(b_{ts}+c_{ts}+d_{ts}),$$

where

$$\begin{aligned} a_{st} &= \int_\Pi \chi_{st}(\mu)f(\mu)\,d\mu, \qquad \chi_{st}(\mu) = \phi_s(\mu)\psi_t(-\mu)e^{i(t-s)\mu},\\ b_{st} &= f(0)\int_\Pi \chi_{st}(\mu)\,d\mu, \qquad c_{st} = \int'\bar{f}(\mu)\chi_{st}(\mu)\,d\mu,\\ d_{st} &= \int''\bar{f}(\mu)\chi_{st}(\mu)\,d\mu. \end{aligned} \tag{9.2}$$

We shall show that

$$\frac{1}{n^2}\sum_s\sum_t a_{st}a_{ts} = \frac{1}{n^2}\sum_s\sum_t b_{st}b_{ts} + o(1) \sim P(\alpha,\beta,\beta,\alpha)n^{2(\alpha+\beta-1)}. \tag{9.3}$$

For $\alpha > 0$, the last relation follows from Definition 3.1, integral approximation and

$$\int_{\Pi} \chi_{st}(\mu)\, d\mu = 2\pi \sum_{j(s,t)}{}' \phi_j \psi_{j+t-s},$$

where $\sum'_{j(s,t)} = \sum_{j=\max(0,\, s-t)}^{s-1}$, whereas for $\alpha = 0$,

$$\frac{1}{n^2} \sum_s \sum_t b_{st} b_{ts} = \frac{4\pi^2}{n^2} f(0)^2 \sum_s \sum_t \psi_{s-t} \psi_{t-s} = O(n^{-1}) = o(n^{2(\beta-1)}).$$

To prove the first relation in (9.3), we first consider the case $\alpha > \frac{1}{2}$ and note that by elementary inequalities it suffices to show that

$$\begin{aligned} &\sum_s \sum_t |b_{st}|^2 = O(n^{2(\alpha+\beta)}), \\ &\sum_s \sum_t |c_{st}|^2 = o(n^{2(\alpha+\beta)}), \\ &\sum_s \sum_t |d_{st}|^2 = o(n^{2(\alpha+\beta)}). \end{aligned} \tag{9.4}$$

By the Schwarz inequality especially and

$$\sum_s \sum_t |b_{st}|^2 \le C n^2 \sum_0^n \phi_j^2 \sum_0^n \psi_j^2 \le C n^{2(\alpha+\beta)},$$

and clearly $\sum_s \sum_t |c_{st}|^2$ has the same bound times δ^2, where δ is arbitrary, to prove the first two components of (9.4). The last component of (9.4) follows from the bound [due to (3.9), (3.10)],

$$|\chi_{st}(\mu)| \le C n^{\max(\alpha-1,\, 0)+\max(\beta-1,\, 0)} |\mu|^{-\min(\alpha,\, 1)-\min(\beta,\, 1)},$$

for $0 < |\mu| \le \pi$, since then $\sum_s \sum_t |d_{st}|^2$ is $O(n^2)$ for $\alpha, \beta \le 1$, $O(n^{2\beta})$ for $\alpha \le 1$, $\beta > 1$, and $O(n^{2(\alpha+\beta-1)})$ for $\alpha, \beta > 1$, to complete the proof of (9.4) in case $\alpha > \frac{1}{2}$.

Now consider the case $\alpha \le \frac{1}{2}$, which is more delicate. Writing

$$G_n(\lambda, \mu, \omega) = \sum_t \phi_t(\lambda) \psi_t(\mu) e^{it\omega},$$

we have

$$\begin{aligned} \left| \sum_s \sum_t c_{st} c_{ts} \right| &\le \delta^2 \int_{\Pi} \int_{\Pi} |G_n(\mu, -\lambda, \lambda - \mu)|^2\, d\mu\, d\lambda \\ &\le 4\pi^2 \delta^2 \sum_s \sum_t \sum_{j(s,\,t)}{}' \phi_j \phi_{j+t-s} \sum_{j(s,t)}{}' \psi_j \psi_{j+t-s} \end{aligned} \tag{9.5}$$

$$\le C\delta^2 \left(n + \sum_{s \ne t} \sum |s-t|^{\alpha-1} \right) \sum_0^n \psi_j^2 \sum_0^n \psi_j^2 = O(\delta^2 n^{2(\alpha+\beta)}) \tag{9.6}$$

for $0 < \alpha \le \frac{1}{2}$, while for $\alpha = 0$ we easily get the bound $O(j^2 n^{2\beta})$. Next, recall that $\bar{\phi}_t(\lambda) = \phi(\lambda) - \phi_t(\lambda)$, and correspondingly introduce

$$\overline{G}_n(\lambda, \mu, \omega) = \phi(\lambda) \sum_t \psi_t(\mu) e^{it\omega} - G_n(\lambda, \mu, \omega).$$

Thus $\sum_s \sum_t d_{st} d_{ts}$ is

$$\begin{aligned} &\int'' \int'' \{\phi(\mu)\psi_n(-\mu, \lambda) - \overline{G}_n(\mu, -\lambda, \lambda - \mu)\} \\ &\quad \times \{\phi(\lambda)\psi_n(-\lambda, \mu) - \overline{G}_n(\lambda, -\mu, \mu - \lambda)\} \bar{f}(\mu)\bar{f}(\lambda)\, d\mu\, d\lambda \\ &\quad \le C \int'' \left\{ \int_\Pi (|\psi_n(-\lambda, \mu)|^2 + |\overline{G}_n(\lambda, -\mu, \mu - \lambda)|^2)\, d\mu \right\} |\bar{f}(\lambda)|^2\, d\lambda \\ &\quad \le C \int'' \left\{ \sum_t |\psi_t(\lambda)|^2 + \left(\sum_t |\bar{\phi}_t(\lambda)| \right)^2 \sum_0^n \psi_j^2 \right\} |\bar{f}(\lambda)|^2\, d\lambda \\ &\quad \le C\big(n^{2\beta - 1} + n^{2(\alpha+\beta)-1}\big) = o(n^{2(\alpha+\beta)}) \end{aligned}$$

for $\alpha \ge 0$. It is then straightforward to show, by similar means, that the remaining components of $n^{-2} \sum_s \sum_t (a_{st} a_{ts} - b_{st} b_{ts})$ are negligible when $0 \le \alpha \le \frac{1}{2}$. This concludes the proof of (9.3).

The contribution from the second term in braces can be handled in almost the same way; the only notable difference is that it is nonnegligible when $\alpha = 0$, but this is easily seen.

We write the contribution to (9.1) from the final, fourth-cumulant, term as

$$\frac{1}{n^2} \int_\Pi \int_\Pi \int_\Pi H_n(\lambda, \mu, \omega) f(\lambda, \mu, \omega)\, d\lambda\, d\mu\, d\omega, \tag{9.7}$$

where

$$H_n(\mu, \lambda, \omega) = G_n(\lambda + \mu + \omega, -\lambda, -\mu - \omega) G_n(-\mu, -\omega, \mu + \omega).$$

To extend the approach used previously, we can write (9.7) as the sum of terms

$$\frac{f(0,0,0)}{n^2} \int_\Pi \int_\Pi \int_\Pi H_n(\mu, \lambda, \omega)\, d\lambda\, d\mu\, d\omega \tag{9.8}$$

$$+ \int_\Pi \int_\Pi \int_\Pi H_n(\lambda, \mu, \omega) \{f(\lambda, \mu, \omega) - f(0,0,0)\}\, d\lambda\, d\mu\, d\omega. \tag{9.9}$$

It is readily verified that (9.8) is $\delta\pi^2 f(0,0,0)/n^2$ times something bounded by

$$\sum_s \sum_t \sum_{j(s,t)}{}' \phi_j \psi_j \phi_{j+t-s} \psi_{j+t-s} \le \sum_{j=0}^n |\phi_j \psi_j| \sum_s \sum_t |\phi_{j+t-s} \psi_{j+t-s}|. \tag{9.10}$$

For $\alpha + \beta < 1$, the sum over s, t is $O(n)$, uniformly in j, so that (9.10) is $O(n)$ also. For $\alpha + \beta = 1$, the sum over s, t is $O(n \log n)$ and (9.10) is $O(n(\log n)^2)$. For $\alpha + \beta > 1$, (9.10) is clearly $O(n^{2(\alpha+\beta)-1})$. It follows that (9.8) is $o(n^{2(\alpha+\beta-1)})$.

Now consider (9.9). For any $\delta > 0$, we can choose ε such that

$$\sup_{|\lambda|<\varepsilon,\, |\mu|<\varepsilon,\, |\lambda|<\varepsilon} |f(\lambda, \mu, \omega) - f(0,0,0)| < \delta. \tag{9.11}$$

Then with $\int'$ having the same meaning as before

$$\int'\int'\int' H_n(\lambda, \mu, \omega)\Big\{f(\lambda, \mu, \omega) - f(0,0,0)\Big\}\, d\lambda\, d\mu\, d\omega \tag{9.12}$$

is bounded by

$$\begin{aligned}\frac{\delta}{n^2}\Bigg\{&\int_\Pi\int_\Pi\int_\Pi |G_n(\lambda+\mu+\omega, -\lambda, -\mu-\omega)|^2\, d\lambda\, d\mu\, d\omega \\ &\times \int_\Pi\int_\Pi\int_\Pi |G_n(-\mu, -\omega, \mu+\omega)|^2\, d\lambda\, d\mu\, d\omega\Bigg\}^{1/2}.\end{aligned} \tag{9.13}$$

Both triple integrals are easily shown to be 2π times $(9.5)/\delta^2$, which is $O(n^{2(\alpha+\beta)})$ [see (9.6)]. By arbitrariness of δ it follows that (9.13) is $o(n^{2(\alpha+\beta-1)})$, so that (9.12) can be neglected. The difference between (9.9) and (9.12) is bounded by

$$\sum_{j=1}^{3} \iiint_{U_j} |H_n(\lambda, \mu, \omega)|\, |f(\lambda, \mu, \omega) - f(0,0,0)|\, d\lambda\, d\mu\, d\omega, \tag{9.14}$$

where $U_1 = \{\lambda\colon \varepsilon \le |\lambda| \le \pi\} \times V_1$, $V_1 = \{\mu\colon \mu \in \Pi\} \times \{\omega\colon \omega \in \Pi\}, U_2 = \{\lambda\colon \lambda \in \Pi\} \times V_2$, $V_2 = \{\mu\colon \varepsilon \le |\mu| \le \pi\} \times \{\omega\colon \omega \in \Pi\}, U_3 = \{\lambda\colon \lambda \in \Pi\} \times V_3$, $V_3 = \{\mu\colon \mu \in \Pi\} \times \{\omega\colon \varepsilon \le |\omega| \le \pi\}$. Then (9.14) is bounded by

$$\left\{\sup_{\mu,\, \omega\in\Pi} \int_\Pi |f(\lambda, \mu, \omega)|^2\, d\lambda + 2\pi f^2(0,0,0)\right\}^{1/2} \tag{9.15}$$

$$\begin{aligned}\times \frac{1}{n^2}\sum_{j=1}^{3}\Bigg\{&\iiint_{U_j} |G_n(\lambda+\mu+\omega, -\lambda, -\mu-\omega)|^2\, d\omega\, d\mu\, d\lambda \\ &\times \iint_{V_j} |G_n(-\mu, -\omega, \mu+\omega)|^2\, d\omega\, d\mu\Bigg\}^{1/2}.\end{aligned} \tag{9.16}$$

Since (9.15) is finite it suffices to show that each of the summands in (9.16) is $o(n^{2(\alpha+\beta)})$. This is achieved by using the fact, already established, that one of the factors in braces in each summand in (9.14) is $O(n^{2(\alpha+\beta)})$, and showing that the other is $o(n^{2(\alpha+\beta)})$. The latter factors are the first one for $j = 1$, and the second one for $j = 2, 3$. The proofs are too similar to those concerning G_n previously to warrant inclusion; we would only note that for the U_j we effectively only integrate over one of ω, μ as before and that $|\lambda| \ge \varepsilon$ on U_1, while $|\mu| > \varepsilon$ on V_2 and $|\omega| \ge \varepsilon$ on V_3.

By elementary inequalities and (2.8), (5.3) for $m < \tilde{n}$ will follow from the above proof and (5.5), so we prove the latter. $\mathrm{Var}\{\widehat{F}(m+1, \tilde{n})\}$ is bounded by the real part of

$$\begin{aligned}(9.17)\qquad &\frac{1}{4\pi n^4}\sum_j{}''\sum_k{}''\sum_q\sum_r\sum_s\sum_t \phi_{n-q+1}(\lambda_j)\psi_{n-r+1}(-\lambda_j)\phi_{n-s+1}(-\lambda_k)\\ &\quad\times\psi_{n-t+1}(\lambda_k)e^{i(q-r)\lambda_j-i(s-t)\lambda_k}\{\gamma_{t-q}\gamma_{r-s}+\gamma^{(\eta)}_{s-q}\gamma^{(\theta)}_{t-r}+\kappa_{qrst}\}.\end{aligned}$$

The contribution from the first term in braces may be written as $(4\pi^2 n^4)^{-1}$ times

$$\int_\Pi\int_\Pi V_n(\lambda,\mu)f(\lambda)f(\mu)\,d\lambda\,d\mu, \tag{9.18}$$

where

$$V_n(\lambda,\mu)=\sum_j{}''\sum_k{}''\phi_n(\lambda_j,-\lambda)\psi_n(-\lambda_j,\mu)\phi_n(-\lambda_k,-\mu)\psi_n(\lambda_k,\lambda).$$

We subdivide the integral (9.18) into components $\int'\int'$, $\int''\int''$, $\int'\int''$ and $\int''\int'$. First

$$\begin{aligned}(9.19)\qquad &\left|\int'\int' V_n(\lambda,\mu)f(\lambda)f(\mu)\,d\lambda\,d\mu\right|\\ &\quad\le C\int_\Pi\int_\Pi\left|\sum_j{}''\phi_n(\lambda_j,-\lambda)\psi_n(-\lambda_j,\mu)\right|^2 d\lambda\,d\mu.\end{aligned}$$

The double integral is evaluated as

$$4\pi^2\sum_s\sum_t\left|\sum_j{}''\phi_s(\lambda_j)\psi_t(-\lambda_j)e^{i(t-s)\lambda_j}\right|^2. \tag{9.20}$$

If $\alpha+\beta>1$ and $\alpha>0$, this is bounded by

$$\begin{aligned}&C\sum_s s^{2\max(\alpha-1,0)}\sum_t t^{2\max(\beta-1,0)}n^{2\min(\alpha,1)+2\min(\beta,1)}\left(\sum_j{}'' j^{-\min(\alpha,1)-\min(\beta,1)}\right)^2\\ &\quad\le Cn^{2(\alpha+\beta+1)}m^{-\min(\alpha+\beta-1,\alpha,1)}=o(n^{2(\alpha+\beta+1)})\end{aligned}$$

as desired. For $\alpha=0$, $\beta>1$ (9.20) is, from (8.28), (8.29),

$$4\pi^2 n\sum_j{}''\sum_t|\psi_t(\lambda_j)|^2\le Cn^{2(\beta+1)}\sum_j{}'' j^{-2}=o(n^{2(\alpha+\beta+1)}).$$

For $\alpha+\beta\le 1$ we write (9.20) as

$$\begin{aligned}(9.21)\qquad &\sum_s\sum_t\left|\sum_j{}''\{\phi(\lambda_j)-\bar\phi_s(\lambda_j)\}\psi_t(-\lambda_j)e^{i(t-s)\lambda_j}\right|^2\\ &\quad=\sum_j{}''\sum_k{}''\phi(\lambda_j)\phi(-\lambda_k)\sum_t\psi_t(-\lambda_j)\psi_t(\lambda_k)e^{it(\lambda_j-\lambda_k)}D_n(\lambda_k-\lambda_j)\\ (9.22)\qquad &\quad-\sum_j{}''\sum_k{}''\phi(\lambda_j)\sum_t\psi_t(-\lambda_j)\psi_t(\lambda_k)e^{it(\lambda_j-\lambda_k)}\sum_s\bar\phi_s(-\lambda_k)e^{is(\lambda_k-\lambda_j)}\end{aligned}$$

$$-\sum_j{}''\sum_k{}''\phi(-\lambda_k)\sum_t\psi_t(-\lambda_j)\psi_t(\lambda_k)e^{it(\lambda_j-\lambda_k)}\sum_s\bar{\phi}_s(\lambda_j)e^{is(\lambda_k-\lambda_j)} \tag{9.23}$$

$$+\sum_j{}''\sum_k{}''\sum_s\bar{\phi}_s(\lambda_j)\bar{\phi}_s(-\lambda_k)e^{is(\lambda_k-\lambda_j)}\sum_t\psi_t(-\lambda_j)\psi_t(\lambda_k)e^{it(\lambda_j-\lambda_k)}. \tag{9.24}$$

Because of (8.28) and (8.29), (9.21) is bounded by $n^{2(1+\alpha+\beta)}$ times

$$C\sum_j{}'' j^{-2(\alpha+\beta)} \leq Cm^{1-2(\alpha+\beta)} = o(1),$$

since $\beta > \frac{1}{2}$. On the other hand, (9.22) and (9.23) are bounded by $n^{2(1+\alpha+\beta)}$ times

$$C\sum_j{}'' j^{-\alpha-2\beta-1} + C\sum_j{}'' j^{-\alpha-\beta}\sum_{k>j}k^{-1-\beta}$$
$$\leq Cm^{-\alpha-2\beta} + C\sum_j{}'' j^{-\alpha-2\beta} \leq Cm^{1-\alpha-2\beta} = o(1),$$

and (9.24) is bounded by $n^{2(\alpha+\beta+1)}$ times $O((\sum_j'' j^{-1-\beta})^2) = o(1)$. Thus (9.19) is $o(n^{2(\alpha+\beta+1)})$. The component $\int''\int''$ requires careful treatment. It is bounded by

$$\int''\int''\left|\sum_j{}''\phi_n(\lambda_j,-\lambda)\psi_n(-\lambda_j,\mu)\right|^2|f(\lambda)f(\mu)|\,d\lambda\,d\mu \tag{9.25}$$

$$\begin{aligned}&\leq \int''\sum_{j=1}^n|\phi_n(\lambda_j,-\lambda)|^2|f(\lambda)|\,d\lambda\int''\sum_{k=1}^n|\psi_n(-\lambda_k,\mu)|^2|f(\mu)|\,d\mu\\ &\leq Cn^2\int''\sum_s|\phi_s(\lambda)|^2|f(\lambda)|\,d\lambda\int''\sum_t|\psi_t(\mu)|^2|f(\mu)|\,d\mu\end{aligned} \tag{9.26}$$

from (8.40). Clearly (9.26) is $O(n^{\max(2\alpha,2)+\max(2\beta,2)})$, which is $o(n^{2(\alpha+\beta+1)})$ unless $\alpha+\beta\leq 1$ or $\alpha = 0$. The latter possibilities imply $\alpha < \frac{1}{2}$, when we write (9.25) as

$$\int''\int''\left|\sum_j{}''\left[\sum_s e^{is(\lambda_j-\lambda)}\{\phi(\lambda)-\bar{\phi}_s(\lambda)\}\right]\psi_n(-\lambda_j,\mu)\right|^2|f(\lambda)f(\mu)|\,d\lambda\,d\mu. \tag{9.27}$$

When $\alpha = 0$ the contribution from $\bar{\phi}_s(\lambda)$ is zero, while for $0 < \alpha < \frac{1}{2}$ and $\frac{1}{2} < \beta < 1$ it is bounded by

$$C\int''\int''\sum_{j=1}^n\left|\sum_s e^{is(\lambda_j-\lambda)}\bar{\phi}_s(\lambda)\right|^2\sum_{k=1}^n|\psi_n(-\lambda_k,\mu)|^2|f(\lambda)f(\mu)|\,d\lambda\,d\mu$$
$$\leq Cn^3\int''\int''\sum_s|\bar{\phi}_s(\lambda)|^2\sum_t|\psi_t(\mu)|^2|f(\lambda)f(\mu)|\,d\lambda\,d\mu$$
$$\leq Cn^{2\alpha+3} = o(n^{2(\alpha+\beta+1)})$$

using (8.28), (8.29), (8.40), (3.9). For $\alpha = 0$, $\beta \geq 1$ we bound (9.25) by

$$C\left\{\int_\Pi \int_\Pi \left| \sum_j{}'' D_n(\lambda_j - \lambda)\psi_n(-\lambda_j, \mu)\right|^2 d\lambda\, d\mu\right\}^{1/2} \tag{9.28}$$

$$\times \left\{\int'' \sum_{j=1}^n |D_n(\lambda_j - \lambda)|^2 |f(\lambda)|^2\, d\lambda \int'' \sum_{j=1}^n |\psi_n(-\lambda_j, \mu)|^2 |f(\mu)|^2\, d\mu\right\}^{1/2}. \tag{9.29}$$

Now

$$\int_\Pi D_n(\lambda_j - \lambda) D_n(\lambda - \lambda_k) d\lambda = 2\pi D_n(\lambda_j - \lambda_k), \tag{9.30}$$

which is $2\pi n$ for $j = k$ and 0 otherwise, so (9.28) is bounded by

$$C\left\{n \sum_j{}'' \sum_t |\psi_t(\lambda)|^2\right\}^{1/2} \leq \left\{Cn\left(\sum_j{}'' \lambda_j^{-2}\right) \sum_t t^{2(\beta-1)}\right\}^{1/2} = o(n^{\beta+1}).$$

On the other hand because (8.40) is $O(n^{2\beta})$ for $|\mu| > \varepsilon$ and, for all λ,

$$\sum_{j=1}^n |D_n(\lambda_j - \lambda)|^2 = n^2, \tag{9.31}$$

it follows that (9.29) is $O(n^{\beta+1})$. Thus (9.25) is $o(n^{2(\beta+1)})$ for $\alpha = 0$, $\beta \geq 1$. It remains to consider $\alpha = 0$, $\beta < 1$. From (6.2) we write

$$\psi_n(-\lambda_j, \mu) = \psi(-\mu) D_n(\mu - \lambda_j) - \sum_t e^{it(\mu - \lambda_j)} \bar{\psi}_t(-\mu) \tag{9.32}$$

for $|\mu| > \varepsilon$. Now, using (9.30), (9.31),

$$\begin{aligned}
&\int'' \int'' \left|\psi(-\mu) \sum_j{}'' D_n(\lambda_j - \lambda) D_n(\mu - \lambda_j)\right|^2 |f(\lambda) f(\mu)|\, d\lambda\, d\mu \\
&\quad \leq C \int_\Pi \left\{\int_\Pi \left|\sum_j{}'' D_n(\lambda_j - \lambda) D_n(\mu - \lambda_j)\right|^2 d\lambda\right\} |f(\mu)|^2\, d\mu \\
&\quad \leq Cn^2 \int_\Pi |f(\mu)|^2\, d\mu.
\end{aligned}$$

We bound the contribution to (9.25) from the second term on the right of (9.32) by

$$\left\{\int'' \int'' \left|\sum_j{}'' D_n(\lambda_j - \lambda) \sum_t e^{it(\mu - \lambda_j)} \bar{\psi}_t(-\mu)\right|^2 d\lambda\, d\mu\right\}^{1/2} \tag{9.33}$$

$$\leq \left\{\int'' \int'' \left|\sum_j{}'' D_n(\lambda_j - \lambda) \sum_t e^{i(\mu - \lambda_j)} \bar{\psi}_t(-\mu)\right|^2 |f(\lambda) f(\mu)|^2\right\}^{1/2}. \tag{9.34}$$

Now (9.34) is bounded by

$$C\left\{\iint \sum_{j=1}^{n} |D_n(\lambda_j-\lambda)|^2 \sum_{j=1}^{n} \left|\sum_t e^{it(\mu-\lambda_j)}\bar{\psi}_t(-\mu)\right|^2 |f(\lambda)f(\mu)|^2\, d\lambda\, d\mu\right\}^{1/2}$$

$$\leq C\left(n^3 \int'' \sum_t |\bar{\psi}(-\mu)|^2 |f(\mu)|^2\, d\mu\right)^{1/2}$$

$$\leq C\left(n^3 \sum_t t^{2(\beta-1)}\right)^{1/2} \leq C n^{\beta+1},$$

from (8.28), (8.29), (9.30), (9.31) and (3.9). To deal with (9.33) we employ (9.32) and (8.2) to write

$$\sum_t e^{it(\mu-\lambda_j)}\bar{\psi}_t(-\mu) = \psi(-\mu)D_n(\mu-\lambda_j) - \sum_t \psi_{n-t+1}(-\lambda_j)e^{it(\mu-\lambda_j)}.$$

The contribution to (9.33) from the first term on the right is bounded by

$$\left\{\int_\Pi\int_\Pi \left|\sum_j'' D_n(\lambda_j-\lambda)D_n(\mu-\lambda_j)\right|^2 d\lambda\, d\mu\right\}^{1/2} = O(n^{3/2})$$

from (9.30), while the contribution from the second term is bounded by

$$C\left\{n \sum_j{}'' \sum_t |\psi_{n-t+1}(\lambda_j)|^2\right\}^{1/2} \leq C n^{1+\beta}\left(\sum_j{}'' j^{-2\beta}\right)^{1/2} = o(n^{\beta+1}).$$

It follows that for $\alpha = 0$, $\beta < 1$, (9.25) is $O(n^3) + (O(n^{3/2}) + o(n^{\beta+1}))O(n^{\beta+1}) = o(n^{2(\beta+1)})$. Thus we can neglect the component $\int''\int''$ of (9.18), as we can also $\int'\int''$ and $\int''\int'$ by straightforwardly combining proofs given so far.

Next the contribution of κ_{qrst} to (9.17) is

$$\frac{1}{4\pi^2 n^4}\int_\Pi\int_\Pi\int_\Pi \left\{\sum_j{}'' \phi_n(\lambda_j, -\lambda-\mu-\omega)\psi_n(-\lambda_j, \lambda)\right\} \times \left\{\sum_j{}'' \phi_n(-\lambda_k, \mu)\psi_n(\lambda_k, \omega)\right\} f(\lambda, \mu, \omega)\, d\lambda\, d\mu\, d\omega. \tag{9.35}$$

The contribution of $\int'\int'\int'$ is bounded by

$$\frac{C}{n^4}\left\{\int_\Pi\int_\Pi\int_\Pi \left|\sum_j{}'' \phi_n(\lambda_j, -\lambda-\mu-\omega)\psi_n(-\lambda_j, \lambda)\right|^2 d\lambda\, d\mu\, d\omega \times 2\pi \int_\Pi\int_\Pi \left|\sum_j{}'' \phi_n(-\lambda_k, \mu)\psi_n(\lambda_k, \mu)\right|^2 d\mu\, d\omega\right\}^{1/2}. \tag{9.36}$$

Both factors in braces are bounded by the right side of (9.19), noting that in the first factor we may substitute for $\lambda+\mu+\omega$ and use periodicity of period 2π.

Thus (9.36) $= o(n^{2(\alpha+\beta-1)})$. We omit the proof for the remainder of (9.35) as it is so similar to earlier proofs. This completes the proof for $\widehat{F}(0, m)$.

For $\widehat{F}(1, m)$ we note that

$$\operatorname{Var}\{\widehat{F}(1, m)\} = \operatorname{Var}\{\widehat{F}(0, m)\} - 2\operatorname{Cov}\{\widehat{F}(0, m), \bar{\zeta}\bar{\xi}\} + \operatorname{Var}(\bar{\zeta}\bar{\xi}).$$

The proof proceeds by showing that

$$\begin{aligned}\lim_{n\to\infty} n^{2(1-\alpha-\beta)}\operatorname{Cov}\{\widehat{F}(0, m), \bar{\zeta}\bar{\xi}\} &= f_{\eta\theta}^2(0)Q(\alpha,\beta,\beta,\alpha)\\ &\quad + f_{\eta\eta}(0)f_{\theta\theta}(0)Q(\alpha,\alpha,\beta,\beta),\\ \lim_{n\to\infty} n^{2(1-\alpha-\beta)}\operatorname{Var}(\bar{\zeta}\bar{\xi}) &= f_{\eta\theta}^2(0)R(\alpha,\beta,\beta,\alpha)\\ &\quad + f_{\eta\eta}(0)f_{\theta\theta}(0)R(\alpha,\alpha,\beta,\beta).\end{aligned}$$

These proofs follow very closely the previous pattern, where we established them first for $m = \tilde{n}$ and then showed that the effect of taking $m < \tilde{n}$ makes no difference, the details being so similar as not to be worth reporting. □

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