

Manolis Galenianos and [Philipp Kircher](#)
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On the Game-theoretic Foundations of Competitive Search Equilibrium*

By Manolis Galenianos and Philipp Kircher[†]

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Abstract

We provide a unified directed search framework with general production and matching specifications that encompasses most of the existing literature. We prove the existence of subgame perfect Nash equilibria in pure firm strategies in a finite version of the model. We use this result to derive a more complete characterization of the equilibrium set for the finite economy and to extend convergence results as the economy becomes large to general production and matching specifications. The latter extends the micro-foundations for the standard market-utility assumption used in competitive search models with a continuum of agents to new environments.

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1 Introduction

Models of directed search combine frictions, which are seen as an important feature of labor markets, with a significant role for pricing, which is mostly absent in models of random search. The main mechanism is that workers observe the offer of each firm before deciding where to look for employment and, as a result, they can direct their search towards jobs that they find more attractive. A common assumption in these models, known as the *market utility property*, is that a single firm's offer does not affect the workers' overall expected utility. This property facilitates equilibrium characterization because it allows firms to treat workers' expected utility parametrically; hence the moniker "competitive search" that is often given to this literature.

A natural question is what are the foundations of the market utility property? The underlying idea is that a single agent's actions do not affect aggregate outcomes in a market with a large number of participants and therefore any strategic interactions can be ignored. Ideally, of course, this is a property to be proved rather than assumed. The standard approach for doing so is to derive the equilibria of a finite economy, where strategic interactions are present and strategies and off-equilibrium payoffs are well-defined, in order to examine their limit as the number of agents becomes large. So far this analysis has been performed in very simple environments with risk-neutral agents, no informational or incentive problems beyond matching frictions, fixed productivity on the job and urn-ball matching (see Burdett, Shi and Wright (2001) for the case of homogeneous firms; see Peters (2000) for the case of heterogeneous firms).

However, the applied literature has moved on to questions that require more complicated environments in order to be dealt with in a satisfactory way. Examples of such environments include introducing risk-averse workers (Acemoglu and Shimer (1999)), match-specific private information (Guerrieri (2008)), endogenous choice of the intensive margin (hours) of work (Faig and Jerez (2004), Rocheteau and Wright (2005), Berentsen, Menzio and Wright (2010)) and moral hazard (Moen and Rozen (2007)). All of these papers use some version of the market utility property even though it has not been explicitly micro-founded in their environments. In addition, the empirical predictions of the urn-ball matching function perform poorly when

confronted with data (Petrongolo and Pissarides (2001)) and many authors have used more general matching functions which allow for a more flexible relation between the labor market tightness and the number of matches. In sum, the directed search literature has moved ahead of its foundations in terms of both the production and the matching technology.

This paper's contribution is two-fold. First, we propose a unified framework with flexible production and matching specifications and show that it encompasses most of the existing directed search literature including all of the aforementioned papers. Second, we show that such a framework retains sufficient tractability to analyze the finite economy where firms' strategic interactions are present and workers' expected utility is not taken parametrically. This analysis provides insights into the equilibrium of the finite market and, more importantly, it allows us to extend the micro-foundations of the market utility property to a very general environment.

We consider a finite economy with heterogeneous firms, homogeneous workers and general matching and production technologies.¹ As in the earlier literature, we assume that frictions arise from workers' lack of coordination. The hiring process is formalized as a game where every firm announces the payoffs that it offers and each worker decides how much effort to spend on searching for each of the jobs after observing all the announcements. Lack of coordination is captured by restricting attention to equilibria where workers follow symmetric strategies. In such equilibria some firms receive too many workers (i.e. more workers search for this firm than it has available vacancies) while others receive too few.

In our first Theorem we prove that there exist equilibria in pure firm strategies if the production function satisfies a simple condition, essentially concavity, and the matching function has some weak regularity properties. We combine existence in pure firm strategies with convergence theorems for the subgame of workers' applications (Peters (1997)) to show that the finite economy equilibria converge to the equilibria of the continuum economy with a market utility property as the number of agents grows (Theorem 4). Pure strategies allow us to side-step mixed strategy convergence which is much more involved and has only been performed in simple

¹See the conclusions for a discussion of models with heterogeneous workers.

environments with risk-neutral workers and fixed productivity on the job (Peters (2000)).

Additionally, we provide characterization and efficiency results for the finite economy that are currently lacking.² Existence in pure strategies allows us to evaluate a firm's strategy against its competitors' pure strategies which significantly reduces the complexity of characterizing equilibria. We prove that, under an additional condition on the production function, the compensation that a firm offers to its workers is increasing in its productivity (Theorem 2). Natural as this result appears, the strategic interaction prevalent in finite economies means that it is not immediate; indeed we provide an example where it fails when our additional condition is not satisfied. We also show that the pure strategy equilibrium is unique when firms are homogeneous (Theorem 3) proving that the equilibria characterized in Burdett, Shi and Wright (2001) are indeed unique. An additional application of our existence result can be found in Galenianos, Kircher and Virag (2010) where it is shown that constrained efficiency does not obtain in finite economies, unlike in continuum ones, at least for certain production specifications. This result is of interest because it illustrates that the efficiency results prevalent in the literature (Moen (1997), Shi (2001), Shimer (2005)) are due to the combination of directed search with a large market and that directed search by itself does not deliver efficiency. We expect additional comparative statics and characterization results to be within reach, and conjecture that adaptations of our approach can be used to extend related finite settings such as Camera and Selcuk (2009), Geromichalos (2008), Julien, Kennes and King (2005) and Lester (2010).

On a more technical level, we should add that the strategic interaction among the agents in a finite environment makes the equilibrium analysis non-trivial. Specifically, the action of a single firm affects the payoffs of all market participants, which means that we need to keep track of the full distribution of announcements when deriving the equilibrium conditions. Furthermore, it is not a priori obvious that equilibria in pure firm strategies exist. For instance, Acemoğlu and Ozdaglar (2007) show that equilibria in pure strategies need not exist in a related environment

²An exception is Burdett, Shi and Wright (2001) who characterize finite equilibria for the case where firms and workers are homogeneous. Montgomery (1991) examines a finite market but assumes that firms behave competitively, essentially using the market utility property.

where pricing and congestion interact non-trivially.³

Finally, finite directed search models resemble classical oligopoly problems. The demand curve for a firm consists of the expected number of workers that want its job. It is smooth in its “price” (i.e., the wage) due to the matching frictions. Even when there are more workers than firms, the firms do not extract all rents because an individual firm has an incentive to raise the wage in order to increase its probability of hiring. We contribute to the original motivation for directed search models (Peters 1984, 1991) by characterizing the smooth demand system (Lemma 1). This enables a deeper understanding of the interaction of competitive price setting and matching frictions in finite economies and provides the basis for the other results in this paper.

2 The General Model and Examples

We start with a description of the economic environment, strategies and equilibrium concept and then state our main existence theorem which is proved in Section 3. The model is presented in a sufficiently abstract way to encompass a number of environments. Section 2.2 elaborates on various applied examples in detail, illustrating how many of the production and matching specifications that have been used in the literature can be mapped into our setting.

2.1 The General Model

The economy is populated with a finite number of firms and workers, denoted by $M = \{1, \dots, m\}$ and $N = \{1, \dots, n\}$ respectively, where $m \geq 2$ and $n \geq 2$. For production to take place, a firm needs to hire a worker. All workers are ex ante identical and each of the (potentially heterogeneous) firms can hire at most one worker. The game starts with the hiring process. Then production takes place and payoffs are realized. The split of the surplus between worker

³In their model, prices and congestion interact additively while in directed search the congestion (probability of trade) interacts with the price multiplicatively. Existence obtains in our setting for a large class of functional forms for the trading probability.

and firm is determined during the hiring process according to the posting game described below. The payoff of being unmatched is normalized to zero for both firms and workers. Firms maximize their expected profits and workers maximize their expected utility.

The surplus generated when firm j fills its vacancy and provides utility v to its worker is denoted by $S_j(v)$.⁴ The firm's ex-post profits (i.e. conditional on a hire) are denoted by $\pi_j(v)$ so that $S_j(v) = \pi_j(v) + v$. Our first assumption presents the restrictions that we impose on the firms' profit functions.⁵ Illustrations of some economic environments that fall within Assumption 1 are presented in the next subsection.

Assumption 1 *We consider environments where for all $j \in M$:*

- i. $\pi_j(v)$ is weakly concave,*
- ii. $\pi_j(v)$ is twice continuously differentiable,*
- iii. there are unique \bar{v}_j and \underline{v}_j such that $\pi_j(\bar{v}_j) = 0$ and $\pi_j(\underline{v}_j) = \max_{v \geq 0} \pi_j(v)$.⁶*

The Pareto frontier between a worker and a firm is linear (strictly concave) when $\pi_j(v)$ is linear (strictly concave). In the case of strict concavity, utility is imperfectly transferable between workers and firms. Note that it is possible for the profit function to be increasing in the worker's payoff v at part of its domain, say when the worker has to exert costly effort (see example P6 in Section 2.2). It is easy to see that, under Assumption 1, no firm has an incentive to make an offer below \underline{v}_j or above \bar{v}_j and therefore the space of utilities that firms might offer to workers is $\mathcal{V} \equiv \times_{j=1}^m [\underline{v}_j, \bar{v}_j]$.

The hiring process has three stages. First, each firm simultaneously makes a public announcement: It commits to the utility that it will provide to the worker that it hires. Second, workers observe the announcements of all firms and each worker simultaneously applies to one

⁴In some environments, the worker's payoff within a match is stochastic. In that case, v represents the worker's expected utility conditional on getting the job. See Section 2.2 for illustrations.

⁵These conditions can be rewritten in terms of $S_j(\cdot)$. It turns out to be more convenient to work with $\pi_j(\cdot)$.

⁶Workers' individual rationality means that $v_j \geq 0$ is a necessary condition for a hire to occur.

firm. Last, each firm goes through a recruitment process in which it hires at most one of its applicants, and remains idle if it does not receive any application. Recruitment is anonymous, i.e., each applicant has the same chance to get hired.

The strategy of worker i specifies the probability with which he applies to each firm after observing some announcement $\mathbf{v} = (v_1, v_2, \dots, v_m) \in \mathcal{V}$. Let $p_j^i(\mathbf{v})$ denote the probability that worker i applies to firm j after observing \mathbf{v} . We focus our attention on equilibria where workers follow symmetric strategies: $p_j^i(\mathbf{v}) = p_j^l(\mathbf{v}) = p_j(\mathbf{v})$ for all $i, l \in N$. Such equilibria are intended to capture the frictions of labor markets. We denote the strategy of workers by the vector $\mathbf{p}(\mathbf{v}) = (p_1(\mathbf{v}), \dots, p_m(\mathbf{v}))$. When there is no possibility for confusion, we suppress the argument \mathbf{v} to keep notation simple.

We now specify the recruitment process, i.e., the mapping from the application strategies to the probabilities of filling a vacancy (for firms) and finding a job (for workers). The probability that a firm fills its vacancy when each worker applies there with probability p is denoted by $H(p)$. The probability that a worker is hired by a firm where every other worker applies with probability p is denoted by $G(p)$. We allow for general functional forms for $H(p)$ and $G(p)$ that encompass a variety of specifications including the commonly-used urn-ball matching (e.g. in Peters (2000) or Burdett, Shi and Wright (2001)). Several examples are illustrated in the next subsection. The next assumption summarizes the structure that we impose on the matching function.

Assumption 2 $H(p)$ and $G(p)$ satisfy the following conditions for $p \in [0, 1]$:

- i.* $H(p)$ is twice continuously differentiable, strictly increasing, concave and $H(p) \in [0, 1]$.
- ii.* $G(p)$ is twice continuously differentiable, strictly decreasing, convex and $G(p) \in [0, 1]$.
- iii.* $H(p) = npG(p)$.
- iv.* $\frac{1}{G(p)}$ is convex.

Furthermore, define $h(p) \equiv H'(p)$ and $g(p) \equiv G'(p)$. Parts *i* and *ii* ensure that $H(p)$ and $G(p)$ are probabilities and they behave nicely. Part *iii* guarantees the consistency of the matching function in expectation terms: the probability that a firm fills its vacancy is equal to the probability that a worker is hired by that firm times the average number of applicants to that firm. This condition links the probability that a firm hires with the probability the a worker gets the job and it also implies that a firm that attracts no applicants cannot hire ($H(0) = 0$). Part *iv* adds some structure to the relation between H and G . Specifically, it implies that a firm's hiring probability is concave in its applicants' probability of getting the job.⁷ This assumption is frequently used in the search literature (e.g. Shi (2009)) and it is satisfied in many common specifications for the meeting process, some of which we review below. In this paper, it is used to prove that workers' payoffs are quasi-concave (Lemma 3).

There are two reasons behind our choice of a general matching function: First, it strengthens our results by showing that they do not depend on the specifics of urn-ball matching. Second, and more important, this paper's aim is to provide micro-foundations for the applied work that assumes more general matching functions such as Moen (1997), Acemoglu and Shimer (1999), Rocheteau and Wright (2005), Guerrieri (2008), Menzio (2008) (see Section 5). It is worth emphasizing that the strategic interactions among agents are retained in our environment which is therefore strictly more general than the earlier literature.

A worker's expected utility from applying to firm j is given by $G(p_j)v_j$. Utility maximization leads to the following definition of the equilibrium in a subgame.

Definition 1 (Symmetric Subgame Equilibrium) *A symmetric equilibrium in the subgame that follows announcements \mathbf{v} is a vector $\mathbf{p}(\mathbf{v}) = (p_1(\mathbf{v}), \dots, p_n(\mathbf{v}))$ such that $\sum_j p_j(\mathbf{v}) = 1$ and for all $j \in M$*

$$p_j(\mathbf{v}) > 0 \Rightarrow G(p_j(\mathbf{v})) v_j = \max_{k \in M} G(p_k(\mathbf{v})) v_k. \quad (1)$$

⁷Let $p = G^{-1}(\hat{G})$ be the probability with which workers apply to a firm so that they get the job with probability $\hat{G} \in [0, 1]$. The firm's hiring probability is given by $\bar{H}(\hat{G}) = nG^{-1}(\hat{G})\hat{G}$, according to part *iii*. Using the inverse function theorem yields $\bar{H}'(\hat{G}) = nG^{-1}(\hat{G}) + n\hat{G}/G'(p)$ and $\bar{H}''(\hat{G}) = n[2G'(p) - \hat{G}G''(p)]/(G'^2)$. Finally, note that $\bar{H}''(\hat{G}) < 0 \Leftrightarrow 2G'(p) - \hat{G}G''(p) < 0$ which is equivalent to convexity of $1/G(p)$.

In words, for a worker to apply to firm j ($p_j > 0$), he needs to receive a level of expected utility that is at least as high as what he can get at any other firm.

Each announcement \mathbf{v} leads to a unique vector of application strategies if at least one firm offers strictly positive utility. That is, when workers follow symmetric strategies, the subgame equilibrium $\mathbf{p}(\mathbf{v})$ is *unique* given any \mathbf{v} with $v_j > 0$ for some $j \in M$ (Peters (1984), Proposition 1).⁸ When $\mathbf{v} = \mathbf{0}$ the workers' strategy is arbitrary. From now on we assume that $p_j(\mathbf{0}) = 1/m$ for all $j \in M$ but our results hold for any specification of $\mathbf{p}(\mathbf{0})$. We define *market utility* to be the expected utility that workers obtain in the subgame and denote it by $U(\mathbf{v})$.

We say that firm j is *active* when $p_j > 0$ and it is *inactive* when $p_j = 0$. In the former case the probability that the firm hires a worker is strictly positive; in the latter case it is zero. Let $A(\mathbf{v}) \equiv \{j \in M | p_j(\mathbf{v}) > 0\}$ denote the set of active firms for a given \mathbf{v} and note that it is non-empty. The set of inactive firms is denoted by $A^C(\mathbf{v})$. Following announcement \mathbf{v} we can without loss of generality reshuffle the firms' indexes so that $A(\mathbf{v}) = \{1, \dots, l\}$ and $A^C(\mathbf{v}) = \{l+1, \dots, m\}$ if $l < m$, or $A^C(\mathbf{v}) = \emptyset$ if $l = m$.

We now turn to the firms' problem in the first stage of the hiring process. Firm j takes as given the announcements of the other firms, \mathbf{v}_{-j} , and the response of workers in the subgame $\mathbf{p}(\mathbf{v})$. The expected profits of firm j are denoted by

$$\Pi_j(\mathbf{v}) \equiv H(p_j(\mathbf{v})) \pi_j(v_j), \quad (2)$$

where $p_j(\mathbf{v})$ solves (1). Profits are uniquely determined given \mathbf{v} since each announcement leads to a unique set of application probabilities in the subgame.

We now define the equilibrium of this game. A directed search equilibrium is a pure strategy Nash equilibrium in the game among firms with payoffs $\Pi_j(\mathbf{v})$. Formally:

⁸Peters (1984) proves this result for urn-ball matching but his proof can be extended in a straightforward way to our setting.

Definition 2 (Directed Search Equilibrium) *A directed search equilibrium is a vector of announcements $\mathbf{v} \in \mathcal{V}$ such that $\Pi_j(\mathbf{v}) \geq \Pi_j(v'_j, \mathbf{v}_{-j})$ for all $v'_j \in [\underline{v}_j, \bar{v}_j]$ and all $j \in M$ where the workers' strategies are given by the symmetric subgame equilibrium.*

We are ready to state our main result for the finite economy:

Theorem 1 *A directed search equilibrium exists when Assumptions 1 and 2 hold.*

The next sections provide examples, show how to prove this result and how to characterize such equilibria. Readers interested in the foundations for large economies can find those in Section 5.

2.2 Examples

This section illustrates that a number of production and matching environments that have been analyzed in the directed search literature are encompassed into our framework. We first look at the production side and Assumption 1 and then return to the matching side and Assumption 2.

Production: The following environments have appeared in the directed search literature and they differ with respect to workers' preferences, the production technology and the informational structure within a match.

P 1. **Canonical model.** The canonical example of the directed search literature is the linear production environment: workers are risk-neutral and firm j produces x_j if it fills its vacancy. In this environment each firm posts a wage w , the value to the worker who obtains this wage is $v = w$, the profits of firms j are given by $\pi_j(v) = x_j - v$ and the surplus created when firm j becomes matched is $S_j(v) = x_j$. In this example the Pareto frontier is linear. This environment is examined in Burdett, Shi and Wright (2001), Moen (1997),

Montgomery (1991) and Peters (2000).⁹

P 2. Risk Aversion. Workers are risk averse, production is deterministic, each firm posts a wage w and cannot insure workers against unemployment. Denote the utility of a worker who receives wage w by $v = \vartheta(w)$. The profits of firm j are given by $x_j - w$ and the surplus created when firm j fills its vacancy is $x_j - w + \vartheta(w)$. We can rewrite $\pi_j(v) = x_j - \vartheta^{-1}(v)$ and note that $\vartheta^{-1}(\cdot)$ is convex due to risk aversion. Together with the requirement that $x_j > \vartheta^{-1}(0)$, this environment satisfies Assumption 1. This model is analyzed in Acemoglu and Shimer (1999).¹⁰

P 3. Private match-specific information. Workers are ex-ante identical and privately draw their match-specific disutility of work after matching with a firm. Firms post wages. When the wage is w and the disutility is ϕ , the worker's net utility is $w - \phi$ and the worker's participation constraint implies that he will refuse to work if $\phi > w$. The worker's ex ante utility is $v = \int_{\phi \leq w_j} [w_j - \phi] d\Phi(\phi)$ where Φ is the disutility distribution. Under the standard monotone hazard rate condition for Φ one can invert this relationship such that $w_j(v)$ defines the wage that yields utility v to the worker. Profits are given by $\pi_j(v) = \int_{\phi \leq w_j(v)} [x_j - w_j(v)] d\Phi(\phi)$ and the surplus is $\pi_j(v) + v$. It is not hard to show that $\pi_j(v)$ is concave in v under the monotone hazard rate condition.¹¹ This environment is analyzed in Guerrieri (2008).

⁹This environment has been extended to consider multiple applications by Albrecht, Gautier and Vroman (2006), Galenianos and Kircher (2009) and Kircher (2009). In models of finite economies, multiple applications lead to severe technical complications as shown in Albrecht, Gautier, Tan and Vroman (2005). See Julien, Kennes and King (2000) and Camera and Selcuk (2009) for models where wages are (potentially) renegotiated after matching.

¹⁰Notice that when workers are risk-averse, the optimal contract includes payments to workers who are not hired (Jacquet and Tan (2010)). Most of the literature, including this paper, ignores the possibility of such payments. One informal justification for this restriction on the contract space is the (unmodeled) existence of unqualified workers who are never hired but who would apply for jobs only to collect payments.

¹¹Profit $\pi_j(v)$ is concave if $w(v)$ is convex, which is equivalent with v being concave in w . Since $v'(w) = -\Phi(w)$ we have $v''(w) = -\Phi'(w) \leq 0$, because the density $\Phi'(w)$ is positive.

P 4. **Endogenous intensive margin.** Output is linear and disutility of work is convex in the hours of work. Firms post an hourly wage w and each hired worker decides how many hours to work. The worker's net utility is given by $v = wt - k(t)$ where t is the time spent working and $k(t)$ is a strictly convex function representing the disutility of work. This expression can be inverted to $w(v) = [v + k(t)]/t$ and implicitly define $t(v)$ when combined with $w(v) = k'(t)$ which is a necessary condition for optimal time allocation. When firm j employs a worker at hourly wage w it generates profits $\pi_j(v) = x_j t - wt = x_j t(v) - v - k(t(v))$ and surplus $S_j(v) = x_j t(v) - k(t(v))$. A sufficient condition for the profit function to be concave is $k'''(t) \geq 0$.¹² This environment is very similar to the product market model of Rocheteau and Wright (2005) and Berentsen, Menzio and Wright (2010) with buyers instead of workers and sellers instead of firms. Our framework does not address the cost of holding money which is a feature in these papers.

P 5. **Endogenous intensive margin with private information.** Consider the setting in Example P 2 with two differences. First, worker's disutility is $\phi k(t)$, where ϕ is a disutility shock that workers draw before deciding on the hours of work from some distribution Φ which satisfies the monotone hazard rate. Second, firms post a (possibly non-linear) wage schedule $w(t)$ that determines payments as a function of hours. Given the realization of ϕ the worker chooses $t(\phi)$ that maximizes $w(t) - \phi k(t)$ and his expected utility before observing ϕ is $v = \int [w(t(\phi)) - \phi k(t(\phi))] d\Phi(\phi)$. Given a level v that firms want to leave to the worker, they choose the contract $w(t)$ that fulfils the prior equality and maximizes their profits $\int [x_j t(\phi) - w(t(\phi))] d\Phi(\phi)$. The profit $\pi(v)$ is concave if $k'''(t) \geq 0$. Faig and Jerez (2006) examine this environment in a product market setting where a worker is a buyer and ϕ corresponds to his marginal valuation for the seller's (in our setting, firm's)

¹²Since $v - k(t(v))t(v) + k(t(v)) = 0$ defines $t(v)$, we have $t'(v) = [k''(t(v))t(v)]^{-1} \geq 0$ and $t''(v) = -[k''(t(v))t(v)]^{-3}[k'''(t(v))t(v) + k''(t(v))] \leq 0$. Then $\pi_j''(v) = [x - k'(t(v))]t''(v) - k''(t(v))t'(v)t(v)$, which is negative when $x - k'(t(v)) \geq 0$. This is the case everywhere on $[\underline{v}_j, \bar{v}_j]$. To see this, note that $k'(t(v))$ is equal to the wage that implements this utility, but only for $x_j - w \geq 0$ the firm makes weakly positive profits, which defined the range of possible offers $[\underline{v}_j, \bar{v}_j]$.

good.

P 6. **Moral hazard.** The firm does not observe the worker's effort t (moral hazard), output y within the match is stochastic and the firm posts an output-contingent wage schedule $w(y)$. Output is given by $y = xt + \phi$, where ϕ is drawn from some distribution Φ with increasing hazard rate. Only the worker observes ϕ and he then chooses $t(\phi)$ to maximize his net utility $w(y) - k(t)$, where $k(t)$ is a convex cost of effort. His expected utility from a schedule $w(y)$ is $v = \int [w(xt(\phi) + \phi) - k(t(\phi))]d\Phi(\phi)$. For a given v there is a contract that yields the highest profit $\pi(v)$ to the firm. Also, $k'''(t) > 0$ is a sufficient condition for $\pi(\cdot)$ to be concave. Moen and Rozen (2007) analyze this framework.

Matching: We provide several structural examples of matching functions that can be used in our framework. These examples differ in the elasticity of the hiring probability with respect to the number of firms in the economy and the elasticity of substitution between the expected number of applicants and the number of firms. Consider the case when all workers apply with probability p_j to firm j .

M 1. **Urn-ball.** Workers send their application to firm j with probability p_j . Assume that if a firm receives at least one application, it hires one of the applicants. This results in a Binomial distribution where firm j has n tries (n is the number of workers) and each try is successful with probability p_j (i.e. each worker applies to firm j with probability p_j). The probability that a firm has at least one applicant is $H(p_j) = 1 - (1 - p_j)^n$. This specification has been used in much of the literature, e.g. in Peters (1991, 2000), Montgomery (1991), Burdett, Shi and Wright (2001), Shi (2001) and Shimer (2005).

M 2. **Qualification shocks.** Extend the previous example with a match-specific shock that renders an applicant unqualified with probability τ (this could also represent the probability that the application is lost in the mail, etc). In this case a firm has a qualified applicant

with probability $H(p_j) = 1 - (1 - (1 - \tau)p_j)^n$, since the probability of a qualified application is $(1 - \tau)p_j$ rather than p_j as in the previous example. This example is described in Petrongolo and Pissarides (2001) to deal with some of the perceived short-comings of the standard urn-ball specification.

M 3. Limited interview capacity. Consider example M 2 and assume that a worker needs to be interviewed for a job in order to find out whether he is qualified, but a firm has only a limited number of interview slots. If the firm can interview no more than $\bar{n} < n$ applicants, then the probability of hiring is $H(p_j) = \sum_{i=1}^{\bar{n}-1} B(i, n, p_j)(1 - \tau^i) + \sum_{i=\bar{n}}^n B(i, n, p_j)(1 - \tau^{\bar{n}})$. Consider the first sum: $B(i, n, p_j) = \binom{i}{n} p_j^i (1 - p_j)^{1-i}$ is the Binomial probability that i applicants apply, and $1 - \tau^i$ is the probability that at least one of them is qualified. The second sum is similar, but due to limited interview capacity only \bar{n} of the i applicants can be evaluated. Such a process is examined in Wolthoff (2009).

M 4. Spatial search and CES matching. Other matching functions are feasible even though they have not been explicitly micro-founded. One example that satisfies Assumption 2 is $H(p_j) = np_j / (np_j + l)$ for $l > 0$ which approaches the well-known telephone-line matching function as the economy becomes large (see Section 5) and fits the specification in Rocheteau and Wright (2005) and Guerrieri (2008). One way to micro-found this matching function might be to think of p_j as the fraction of workers' search time in a particular geographic area in the proximity of firm j , and the owner of the firm hires if he meets one of the workers rather than one of the other l people that are also in the neighborhood. It is a special case of $H(p) = [(np)^{-\sigma} + 1]^{-1/\sigma}$ when $\sigma = 1$. This broader specification fulfills our assumptions for all $\sigma \in (0, 1)$ and resembles the popular CES matching function. We expect many other specifications to fit our framework as well.

3 Existence of Equilibrium

The following three subsections are devoted to the proof of Theorem 1. First, we examine the subgame that follows an arbitrary announcement by the firms and show that the workers' probability of applying to some firm j is quasi-concave in that firm's announcement. Then, we show that a firm's expected profits are quasi-concave in its announcement, v_j . Finally, we prove existence by using a fixed point argument which is extended to deal with the discontinuity in profits that often arises in models with a finite number of agents.

3.1 Analysis of the Subgame

In this section we characterize the workers' response to an arbitrary announcement by the firms \mathbf{v} , and we determine how that response changes when some v_j changes.

Characterization of Subgame: We characterize $\mathbf{p}(\mathbf{v})$ in two steps. First, we determine the set of active firms. Then we determine the exact probabilities with which workers visit the active firms.

Recalling that $U(\mathbf{v}) = \max_j G(p_j(\mathbf{v}))v_j$, we rewrite equation (1) as

$$\begin{aligned} G(p_j(\mathbf{v}))v_j &= U(\mathbf{v}), \quad \forall j \in A(\mathbf{v}), \\ G(p_j(\mathbf{v}))v_j &\leq U(\mathbf{v}), \quad \forall j \in A^C(\mathbf{v}). \end{aligned}$$

To determine whether firm j is active or inactive, compare v_j with $U(\mathbf{v})$. If $v_j > U(\mathbf{v})$, then $p_j > 0$. Equivalently, $v_j < U(\mathbf{v})$ implies that $p_j = 0$. Last, if the announcement of some firm j is exactly on the boundary ($v_j = U(\mathbf{v})$) then that firm is inactive ($p_j = 0$); if it were active then $G(p_j) < 1$ which leads to $G(p_j)v_j < U(\mathbf{v})$ contradicting subgame equilibrium.¹³

To summarize these results, note that the workers' market utility only depends on active firms: if $p_j(v_j, \mathbf{v}_{-j}) = 0$ then $U(v_j, \mathbf{v}_{-j}) = U(0, \mathbf{v}_{-j})$. The following condition determines whether a

¹³In other words, the correspondence $A(\mathbf{v})$ is lower hemi-continuous in \mathbf{v} .

firm is (in)active:

$$j \in A^C(\mathbf{v}) \Leftrightarrow v_j \leq \hat{v}_j(\mathbf{v}_{-j}) \equiv U(0, \mathbf{v}_{-j}). \quad (3)$$

We now focus on the active firms. In equilibrium, the exact probability with which a worker applies to each of the firms in $A(\mathbf{v})$ is determined by the requirement that he is indifferent across them:

$$G(p_k) v_k - G(p_l) v_l = 0, \quad \forall k \in A(\mathbf{v})/\{l\}, \quad (4)$$

$$\sum_{k \in A(\mathbf{v})} p_k - 1 = 0. \quad (5)$$

Equations (4) and (5) define a system \mathbf{F} of l equations with l exogenous and l endogenous variables. The announcements $\hat{\mathbf{v}} \equiv (v_1, \dots, v_l)$ of the active firms are the exogenous variables and the probabilities $\hat{\mathbf{p}} \equiv (p_1, \dots, p_l)$ are the endogenous variables.

Equations (3), (4) and (5) fully describe the equilibrium of the subgame. As noted in Section 2.1, $\mathbf{p}(\mathbf{v})$ is uniquely defined when $v_j > 0$ for some $j \in M$ and we assume that $p_j(\mathbf{0}) = 1/m$.

Workers' reaction to a change in a firm's announcement: We now examine how the equilibrium of the subgame changes when the announcement of firm j is perturbed from v_j to some v'_j . Let \mathbf{v} denote the initial announcement and suppose that $v_k > 0$ for some $k \in M$. The case of $\mathbf{v} = \mathbf{0}$ is treated separately below. We will use the implicit function theorem on equations (4) and (5) but we first need to determine whether the set of active firms changes, i.e. whether $A(v'_j, \mathbf{v}_{-j})$ is the same as $A(v_j, \mathbf{v}_{-j})$.

Consider firm j with $j \in A(v_j, \mathbf{v}_{-j})$ and note that $U(v'_j, \mathbf{v}_{-j}) > U(v_j, \mathbf{v}_{-j}) \Leftrightarrow v'_j > v_j$. When

v'_j is “close enough” to v_j we have that $j \in A(v'_j, \mathbf{v}_{-j})$. Furthermore, given any $k \neq j$:

$$\begin{aligned} v_k > U(v_j, \mathbf{v}_{-j}) &\Rightarrow k \in A(v_j, \mathbf{v}_{-j}) \text{ and } k \in A(v'_j, \mathbf{v}_{-j}) \\ v_k < U(v_j, \mathbf{v}_{-j}) &\Rightarrow k \in A^C(v_j, \mathbf{v}_{-j}) \text{ and } k \in A^C(v'_j, \mathbf{v}_{-j}) \\ v_k = U(v_j, \mathbf{v}_{-j}) \text{ and } v'_j > v_j &\Rightarrow k \in A^C(v_j, \mathbf{v}_{-j}) \text{ and } k \in A^C(v'_j, \mathbf{v}_{-j}) \\ v_k = U(v_j, \mathbf{v}_{-j}) \text{ and } v'_j < v_j &\Rightarrow k \in A^C(v_j, \mathbf{v}_{-j}) \text{ and } k \in A(v'_j, \mathbf{v}_{-j}) \end{aligned}$$

When $j \in A^C(v_j, \mathbf{v}_{-j})$ we have two cases to consider. First, if $v_j < U(\mathbf{v})$ then firm j attracts no applicants after a small enough perturbation, the market utility remains unchanged ($U(v'_j, \mathbf{v}_{-j}) = U(v_j, \mathbf{v}_{-j})$) and therefore $A(v'_j, \mathbf{v}_{-j}) = A(v_j, \mathbf{v}_{-j})$. Second, when $v_j = U(\mathbf{v})$ then an increase in v_j means that firm j starts attracting applicants *and* the market utility increases:¹⁴ $v'_j > v_j \Rightarrow j \in A(v'_j, \mathbf{v}_{-j})$ and $U(v'_j, \mathbf{v}_{-j}) > U(v_j, \mathbf{v}_{-j})$. When $v_j = U(\mathbf{v})$ and $v'_j < v_j$, the market utility is not affected and the set of active firms remains unchanged.

Essentially, $A(\mathbf{v})$ is constant in v_j unless some firm is exactly on the boundary for being active. For a given \mathbf{v}_{-j} , this argument implies that there are at most m critical points for $v_j \in [\underline{v}_j, \bar{v}_j]$ where some firm (possibly including j) is exactly on the boundary. Let $\Psi_j(\mathbf{v}_{-j})$ denote the set of announcements by firm j where some firm is on the boundary, given \mathbf{v}_{-j} ; similarly, let $\Omega_j(\mathbf{v}_{-j})$ denote the set of announcements where $v_k \neq U(\mathbf{v})$ for all $k \in M$ (we occasionally omit the argument \mathbf{v}_{-j} for notational simplicity). The lemma summarizes our results.

Lemma 1 *The set $\Psi_j(\mathbf{v}_{-j})$ contains a finite number of points.*

Proof. See above. ■

We now characterize how \mathbf{p} changes in response to a change in v_j . We will show that $p_j(v_j, \mathbf{v}_{-j})$ is quasi-concave in v_j . We first focus on announcements in Ω_j and then generalize our results to the full domain $\Omega_j \cup \Psi_j$.

¹⁴Recall that $v_j = U(\mathbf{v})$ implies $p_j = 0$ and hence firm j is inactive.

Consider an announcement (v_j, \mathbf{v}_{-j}) where $v_j \in \Omega_j(\mathbf{v}_{-j})$ and some perturbation $\mathbf{v}' = (v'_j, \mathbf{v}_{-j})$. When v'_j is close enough to v_j the set of active firms does not change: $A(\mathbf{v}) = A(\mathbf{v}')$. If $v_j < U(\mathbf{v})$ then firm j is inactive both under \mathbf{v} and under \mathbf{v}' and therefore \mathbf{p} is not affected by a small change in v_j , i.e. $\partial p_k / \partial v_j = 0 \forall k$. If $v_j > U(\mathbf{v})$, we shall apply the implicit function theorem around $\mathbf{F}(\hat{\mathbf{p}}, \hat{\mathbf{v}}) = \mathbf{0}$. The Jacobian of \mathbf{F} with respect to (p_1, \dots, p_l) is given by

$$D_{\mathbf{p}}F = \begin{pmatrix} \xi_1(\mathbf{v}) & 0 & 0 & \dots & 0 & -\xi_l(\mathbf{v}) \\ 0 & \xi_2(\mathbf{v}) & 0 & \dots & 0 & -\xi_l(\mathbf{v}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \xi_{l-1}(\mathbf{v}) & -\xi_l(\mathbf{v}) \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix},$$

where $\xi_k(\mathbf{v}) \equiv g(p_k(\mathbf{v}))$ v_k denotes the change in the expected utility offered by firm k due to an increase in p_k . The rank of this matrix is l : the expected utility of applying to firm k decreases in p_k and therefore $\xi_k \neq 0$ for all $k \in A(\mathbf{v})$. As a result we can apply the implicit function theorem to show that $\partial p_j(\mathbf{v}) / \partial v_j$ exists locally around \mathbf{v} and that the matrix of partial derivatives is defined by $D_{\mathbf{v}}\mathbf{p} = -(D_{\mathbf{p}}\mathbf{F})^{-1}D_{\mathbf{v}}\mathbf{F}$. The following lemma describes our result:

Lemma 2 (Workers' response to a perturbation in the announcements) *When $v_j \in \Omega_j(\mathbf{v}_{-j})$ and $j \in A(\mathbf{v})$ a change in v_j leads to*

$$\frac{\partial p_j(\mathbf{v})}{\partial v_j} = T_j(\mathbf{v})^{-1} G(p_j(\mathbf{v})), \quad (6)$$

where $T_j(\mathbf{v}) = -\xi_j(\mathbf{v}) - [\sum_{k \in A(\mathbf{v}) \setminus \{j\}} \xi_k(\mathbf{v})^{-1}]^{-1}$.

Proof. See the appendix. ■

Finally, when $\mathbf{v}_{-j} = \mathbf{0}_{-j}$ we have $p_j(v_j, \mathbf{0}_{-j}) = 1/m$ for $v_j = 0$ and $p_j(v_j, \mathbf{0}_{-j}) = 1$ for $v_j > 0$. Similarly, for all $k \neq j$ we have $p_k(\mathbf{0}) = 1/m$ when $v_j = 0$ and $p_k(\mathbf{v}) = 0$ when $v_j > 0$. In other

words, $p_j(v_j, \mathbf{0}_{-j})$ is discontinuous at $v_j = 0$ and $0 \in \Psi_j(\mathbf{0}_{-j})$.

This characterization result is key to our analysis. It describes the change in the workers' probability of applying for a particular job when the firm changes its announcement. It has a clear economic interpretation. First, the response is stronger if the probability of getting the job $G(p_j)$ is higher. Clearly, a given increase in v translates into a higher gain for an individual worker when the job is easier to get in the first place. The response is negatively related to the marginal benefit $|\xi_j(\mathbf{v})|$. A large $|\xi_j(\mathbf{v})|$ means that an increase in the application probability at firm j diminishes the workers' utility from applying to firm j by a large amount. In that case a small increase of the application probabilities by workers is sufficient to equalize the expected utilities across all firms. Similarly, the strength of the response is negatively related to the marginal benefit $|\xi_k(\mathbf{v})|$ at some other firm k . When firm j improves its announcement, workers apply more to j and less to other firms. If the expected utility of applying to other firms improves quickly, then workers shift only little additional application probability to firm j before the expected utilities across firms is again equalized. Therefore, the response by workers is related in a tractable way to the change of expected utility of the current firm and its competitors. Note that the components of workers' response that relate to firm $k \neq j$ arise because of the strategic interactions across firms.

We use Lemma 2 to prove p_j is quasi-concave on the *full* domain of announcements. In particular when $\mathbf{v}_{-j} \neq \mathbf{0}_{-j}$ the application probability $p_j(v_j, \mathbf{v}_{-j})$ is equal to zero for $v_j \leq \hat{v}_j(\mathbf{v}_{-j})$ and it is strictly concave for $v_j \geq \hat{v}_j(\mathbf{v}_{-j})$. When $\mathbf{v}_{-j} = \mathbf{0}_{-j}$ the application probability is discontinuous at $v_j = 0$ with $p_j(0, \mathbf{0}_{-j}) = 1/m$ and $p_j(v_j, \mathbf{0}_{-j}) = 1$ for $v_j > 0$.

Lemma 3 *The application probability $p_j(v_j, \mathbf{v}_{-j})$ is quasi-concave in v_j for given \mathbf{v}_{-j} .*

Proof. See the appendix. ■

3.2 Analysis of firms' strategies

We now analyze how profits change when a firm's announcement is perturbed. The goal is to prove the quasi-concavity of expected profits.

Consider firm j and fix the other firms' announcement \mathbf{v}_{-j} . We first focus on $v_j \in \Omega_j(\mathbf{v}_{-j})$ and we describe how to extend our results to $v_j \in \Psi_j(\mathbf{v}_{-j})$ below (the case of $\mathbf{v}_{-j} = \mathbf{0}$ is treated separately). If $v_j < \hat{v}_j(\mathbf{v}_{-j})$ then firm j is inactive, its expected profits are zero and $\partial \Pi_j(\mathbf{v}) / \partial v_j = 0$. If $v_j > \hat{v}_j(\mathbf{v}_{-j})$ then firm j is active and the first derivative of its expected profits with respect to its own announcement is

$$\frac{\partial \Pi_j(v_j, \mathbf{v}_{-j})}{\partial v_j} = H(p_j(v_j, \mathbf{v}_{-j})) \frac{d\pi_j(v_j)}{dv_j} + h(p_j(v_j, \mathbf{v}_{-j})) \pi_j(v_j) \frac{\partial p_j(v_j, \mathbf{v}_{-j})}{\partial v_j}. \quad (7)$$

The second derivative is

$$\begin{aligned} \frac{\partial^2 \Pi_j(v_j, \mathbf{v}_{-j})}{\partial v_j^2} &= H(p_j(v_j, \mathbf{v}_{-j})) \frac{d^2 \pi_j(v_j)}{dv_j^2} + 2 h(p_j(v_j, \mathbf{v}_{-j})) \frac{d\pi_j(v_j)}{dv_j} \frac{\partial p_j(v_j, \mathbf{v}_{-j})}{\partial v_j} \\ &+ h'(p_j(v_j, \mathbf{v}_{-j})) \left(\frac{\partial p_j(v_j, \mathbf{v}_{-j})}{\partial v_j} \right)^2 \pi_j(v_j) + h(p_j(v_j, \mathbf{v}_{-j})) \pi_j(v_j) \frac{\partial^2 p_j(v_j, \mathbf{v}_{-j})}{\partial v_j^2} \end{aligned} \quad (8)$$

It is not hard to see that equation (8) is negative. The first term is weakly negative since π_j is weakly concave. The second term is weakly negative since π_j is weakly decreasing on $[\underline{v}_j, \bar{v}_j]$, $h(p_j) > 0$ and $\partial p_j / \partial v_j > 0$. The third term is non-positive since $h'(p_j) \leq 0$, and the fourth term is strictly negative because of $\partial^2 p_j / \partial v_j^2 < 0$. Therefore, expected profits Π_j are strictly concave on $(\hat{v}_j(\mathbf{v}_{-j}), \bar{v}_j) \cap \Omega_j(\mathbf{v}_{-j})$. This result can be extended to the elements in $\Psi_j(\mathbf{v}_{-j})$ using the same arguments as in the proof of Lemma 3. When $\mathbf{v}_{-j} = \mathbf{0}$, the expected profits of firm j are discontinuous at $v_j = 0$ due to the discontinuity of p_j at $\mathbf{v} = \mathbf{0}$. More specifically, $\Pi_j(v_j, \mathbf{0}_{-j}) = \pi_j(0)/m$ when $v_j = 0$ and $\Pi_j(v_j, \mathbf{0}_{-j}) = \pi_j(v_j)$ when $v_j > 0$.

We have established that a firm's expected profits are quasi-concave in its announcement. In particular we have shown that when $\mathbf{v}_{-j} \neq \mathbf{0}_{-j}$ the expected profits of firm j are continuous, equal to zero for $v_j \in [\underline{v}_j, \hat{v}_j(\mathbf{v}_{-j})]$ and strictly concave for $v_j \in [\hat{v}_j(\mathbf{v}_{-j}), \bar{v}_j]$; therefore, $\Pi_j(v_j, \mathbf{v}_{-j})$

is quasi-concave on $[0, \bar{v}_j]$. When $\mathbf{v}_{-j} = \mathbf{0}_{-j}$, the expected profits are discontinuous at $v_j = 0$ with $\Pi_j(0, \mathbf{0}_{-j}) = \pi_j(0)/m$ and $\Pi_j(v_j, \mathbf{0}_{-j}) = \pi_j(v_j)$ for $v_j \in (0, \bar{v}_j]$.

Lemma 4 *Expected profits $\Pi_j(v_j, \mathbf{v}_{-j})$ are quasi-concave in v_j for given \mathbf{v}_{-j} .*

Proof. See above. ■

It is worth remarking that this lemma is not sufficient to rule out mixed strategy equilibria. The quasi-concavity of firm j 's expected profits is shown when *the other firms follow pure strategies*. Under mixed strategies, the profits of firm j from posting v_j is given by the weighted sum of the expected profits that result from each realization of the other firms' announcement where the weights are equal to each realization's probability. Since the sum of quasi-concave functions is not necessarily quasi-concave, we cannot rule out that firm j 's best response to mixed strategies is also a mixed strategy.

3.3 Finding a Fixed Point

The final step to prove the existence of a directed search equilibrium is to find a fixed point in firms' strategies. The strategy space, \mathcal{V} , is compact and the expected profit function is quasi-concave. However, as show above, profits are discontinuous at $\mathbf{v} = \mathbf{0}$.

When \mathcal{V} does not include $\mathbf{0}$, i.e. if $\underline{v}_j > 0$ for some j , then existence follows by standard fixed point arguments: the expected profit function is continuous and therefore the best response correspondence of the firms is upper hemi-continuous by Berge's Theorem. Quasi-concavity of profits leads to a convex-valued best-response correspondence and Kakutani's fixed point theorem ensures the existence of an equilibrium.

However, when $\mathbf{0} \in \mathcal{V}$ we have to deal with the resulting discontinuity. To prove existence we use the concept of Better-Reply Security of Reny (1999). In our environment Better-Reply Security means the following. Consider any $\mathbf{v} \in \mathcal{V}$ that is not an equilibrium announcement and any sequence $\mathbf{v}_h \in \mathcal{V}$ such that $\mathbf{v}_h \rightarrow \mathbf{v}$ as $h \rightarrow \infty$ with limit payoff vector $(\bar{\Pi}_1, \bar{\Pi}_2, \dots, \bar{\Pi}_m) = \lim_{h \rightarrow \infty} (\Pi_1(\mathbf{v}_h), \Pi_2(\mathbf{v}_h), \dots, \Pi_m(\mathbf{v}_h))$. The game among firms is Better-Reply Secure if there exists

a player j and an action \tilde{v}_j such that $\Pi_j(\tilde{v}_j, \tilde{\mathbf{v}}_{-j}) > \bar{\Pi}_j$ for all $\tilde{\mathbf{v}}_{-j}$ in the neighborhood of \mathbf{v}_{-j} . That is, if the original announcement is not an equilibrium then there exists a firm that can always do strictly better even if the other firms slightly deviate from the profile. When profits are continuous around \mathbf{v} , this is trivially the case.

We only have to check the condition for the case when all firms offer zero, i.e. at $\mathbf{v} = \mathbf{0}$. For any sequence of \mathbf{v}_h converging to zero there is some firm j that in the limit has an application probability below the average, i.e. $p_j \leq 1/m$ and its payoffs are $\bar{\Pi}_j \leq H(1/m)\pi_j(0)$. If firm j offers $\tilde{v}_j = \varepsilon$, then all workers apply to firm j as long as $v_k < \varepsilon/n$ for all $k \neq j$. So for every ε there is a neighborhood around the strategy of the other firms such that firm j hires with probability one. By the continuity of the ex post profit function, firm j can ensure itself a payoff close to $\pi_j(0)$ for ε small enough. This is strictly higher than $\bar{\Pi}_j$ because the firm can now hire for sure and hence the game is Better-Reply Secure. As a result, an equilibrium exists by the fixed point Theorem 3.1 in Reny (1999).

This concludes the proof of Theorem 1.

4 Characterization of the Equilibrium Set

In this section we characterize the equilibrium set. We show that more productive firms will in equilibrium offer higher utility to workers under an additional assumption on the production technology. Additionally, we provide an example where our assumption does not hold and the more productive firm offers lower utility. We then show that the directed search equilibrium is unique when firms are homogeneous.

We first need to rank firms by their productivity. We will use the following definition and only consider environments where the firms can be ranked accordingly.

Definition 3 *We say that firm j is more productive than firm k if*

$$\pi_j(0) \geq \pi_k(0) \text{ and} \tag{9}$$

$$d\pi_j(v)/dv \geq d\pi_k(v)/dv \forall v. \tag{10}$$

If one of the inequalities is strict, we say that firm j is strictly more productive than firm k . If both (9) and (10) hold with equality, then we say that firms j and k are equally productive.

Equation (9) states that when workers receive zero utility the profits of firm j are weakly higher than the profits of firm k . Equation (10) states that the profits of firm j increase faster (or drop more slowly) than k 's when workers' utility increases. It immediately follows that for a given level of worker utility, firm j makes higher profits than k . For example, in the linear profit functions $\pi_j(v) = x_j - v$ of Montgomery (1991) and Burdett, Shi and Wright (2001), Definition 3 translates into our usual notion of being more productive ($x_j \geq x_k$) because the slopes of the profit functions are identical. Note, however, that Definition 3 is a strictly stronger requirement than $\pi_j(v) \geq \pi_k(v)$ for all v .

Proving that more productive firms offer higher utility to prospective employees is straightforward in the context of a continuum economy. One need only establish the following simple single-crossing condition between the probability of hiring, H , and the utility that is offered to workers, v : to "gain" a unit increase in H , a more productive firm is always willing to raise v by a larger amount than a less productive firm. In a continuum economy, this argument is sufficient to show that more productive firms offer higher utility to workers.

However, this logic does not apply in a finite economy because a single firm's action affects market outcomes and, in particular, the probability of hiring when making a given offer. Consider two firms (say 1 and 2) that currently offer different levels of utility (v_1 and v_2) and are both contemplating a deviation to some \hat{v} . The hiring probability that firm 1 faces if it offers \hat{v} is different from the one that firm 2 faces because the overall distribution of offers will be different: if firm 1 deviates to \hat{v} then the distribution includes \hat{v} and v_2 but not v_1 ; if firm 2 deviates,

the distribution includes \hat{v} and v_1 but not v_2 . Therefore the hiring probability when offering \hat{v} depends on which firm is making that offer. As a result, single-crossing in terms of preferences is not enough because the “technology” by which a firm can convert the utility that it offers into the probability of hiring differs for the different firms. Maybe the easiest way to see that our main result in Theorem 2 is non-trivial due to the strategic interactions is the observation that one can construct environments with equilibria that are not characterized by first order conditions where higher productivity firms indeed pay lower wages (see Example 1 below).

We prove our result for equilibria that are characterized by first order conditions, because our proof relies on a direct comparison of these conditions. However, it is not necessary for the equilibrium to be characterized by the first order conditions and we provide an additional condition which guarantees that this first order approach is valid. The reason why the first order conditions need not hold in equilibrium is that a firm’s expected profits may contain kinks. To see this, consider a firm (say, firm 1) that offers v_1 and is active and suppose that some other firm (say, firm 2) offers v_2 and is on the boundary for being active. Think of how the expected profits of firm 1 are affected by a change in v_1 : If firm 1 reduces its announcement the market utility will fall and firm 2 will become active, adding a competitor for workers’ services; this makes the supply of workers more elastic with respect to the announcement. Formally, in (6) the strictly negative term $\xi_2(v_1, v_2) = g(0)v_2$ is additionally introduced when firm 1 reduces its announcement.¹⁵ If firm 1 increases its offer the market utility will increase, firm 2 will remain inactive and the supply of workers will be less elastic with respect to v_1 . This means that the additional term does not appear in (6). This creates a kink in the expected profits of firm 1, and therefore its the optimal choice may not be characterized by a first order condition.

The following assumption is sufficient to rule out the scenario described above by guaranteeing that all firms are active. More precisely, it states that every firm is active in equilibrium, even

¹⁵The term $g(0)$ is strictly negative: Since $G(p)$ is strictly decreasing and continuously differentiable, we have $g(0) = \lim_{p \searrow 0} G'(p) \leq 0$. Moreover, the convexity of $G(p)$ rules out that $g(0) = 0$ as otherwise $g(p) \geq 0$ for $p > 0$, violating the assumption that $G(p)$ is strictly decreasing. Finally, in the example $v_1 > 0$ (as otherwise firm 2 could not be inactive, but would be active at any weakly positive announcement), and so for firm 2 to be on the brink of becoming active it has to be that $v_2 > 0$.

when all of its competitors offer the maximum individually rational utility.

Assumption 3 $p_j(\bar{\mathbf{v}}) > 0$ for all j where $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_m)$.

It is easy to show that Assumption 3 holds as long as the maximum utilities that firms are willing to offer are not too far apart, i.e. there exists parameter $\gamma < 1$ such that Assumption 3 holds whenever $\min_j \bar{v}_j > \gamma \max_j \bar{v}_j$. Note that we only rely on Assumption 3 for the characterization proof of Section 4 and this assumption is not necessary for our other results.

We now prove that if a low productivity firm's first order conditions hold and it offers higher utility than a high productivity firm then the high productivity firm's first order conditions are not satisfied. While our equilibrium definition focuses on pure strategies, note that it does not restrict identical firms to offer the same utility to workers. This is one implication of the following theorem which shows that in equilibrium a more productive firm necessarily offers higher utility to workers.

Theorem 2 *If Assumption 3 holds, then in any directed search equilibrium $v_j > v_k$ if firm j is strictly more productive than firm k and $v_j = v_k$ if firm j is equally productive to firm k .*

Proof. See Appendix. ■

We now provide an example where Assumption 3 does not hold and there is an equilibrium where a high productivity firm offers a lower wage than a low productivity firm. We construct it in the canonical setting of the directed literature with linear production as outlined in Example 1 in Section , which has been the focus e.g. in Moen (1997), Montgomery (1991) and Peters (2000). Since Assumption 3 holds when all firms are identical, and since it is easy to show that with two firms and linear production the equilibrium is always characterized by first order conditions, we resort to an example which in the end features more than two firms and firm heterogeneity.

Example 1 *To set up the example, consider first a simple environment with two risk-neutral workers and two identical firms who produce 1 when matched and zero otherwise. The profit of firm j is given by $\pi_j(v) = 1 - v$. It is straightforward to show that the unique directed search equilibrium has utility offers $v_1 = v_2 = 1/2$ and expected utility for workers of $U(1/2, 1/2) = 3/8$.*

Now choose $\epsilon > 0$ and $\kappa > \epsilon$ such that at wage profile $(\hat{v}_1, \hat{v}_2) = (1/2 + \epsilon, 1/2 + \kappa)$ both firms individually prefer to reduce their offers. Choose both ϵ and κ small enough such that the incentives to reduce the wage are small. These parameters exist due to the convexity of the firms best response function. If the firms offer \hat{v}_1 and \hat{v}_2 then workers obtain some expected utility \hat{U} .

Next, introduce a third firm with profit function $\pi_3 = \hat{U} - v$ that offers wage $\hat{v}_3 = \hat{U}$. In this extended environment none of the original firms has any longer an incentive to lower their utility offer since workers would start applying to the third firm (the function $p_j(v_j, v_{-j})$ can be shown to be non-differentiable at \hat{v} because firm 3 has a non-negligible impact). Therefore, in the extended environment \hat{v}_1, \hat{v}_2 and \hat{v}_3 constitutes an equilibrium, and the original two firms pay different wages despite the fact that their profit functions are identical.

By standard upper-hemicontinuity arguments we can slightly improve the productivity of firm 1 and obtain an equilibrium arbitrarily close to \hat{v}_1, \hat{v}_2 and \hat{v}_3 . Since $\hat{v}_1 < \hat{v}_2$ we end up with an equilibrium where the higher productivity firm posts a strictly lower utility. Note that the proof crucially relies on the non-differentiability of the profit function at the equilibrium offers.

Examples of this type can be constructed in any setting that fulfills our assumptions on production and matching. We can first look at the case where two firms have exactly identical and therefore announce the same value to the workers according to Theorem 2, then introduce a third firm with productivity slightly above the announcement of the original firms and let it offer its full productivity, then adjust the announcements of the other firms upward slightly to set the third firm exactly at the point of becoming active and therefore none of the original firms wants to reduce its offers due to the resulting discontinuity,¹⁶ and finally since preferences are

¹⁶In the absence of the third firm, it is easy to see for example from the proof of the following Theorem 3 that the original firms would like to reduce their announcement if we increase it upward from the equilibrium level, so only the kink induced by the third firm holds them back.

strict we can adjust the productivities of the original firms slightly to unequal levels.

Theorem 2 holds when firms are homogeneous and it can be used to prove that there is a unique equilibrium in such a case. In the only related result, Burdett, Shi and Wright (2001) prove that there is only one equilibrium where all firms offer the same wage in an environment with linear production and urn-ball matching. However, they do not examine asymmetric strategies by the (identical) firms, except for the special 2-firm 2-worker case. Our previous theorem establishes that there cannot be equilibria in asymmetric strategies when firms are homogeneous and, therefore, the equilibrium in Burdett, Shi and Wright (2001) is unique. Our proof still includes some additional steps to show that the result holds for general matching functions and general production technologies.

Theorem 3 *When all firms are equally productive, the directed search equilibrium is unique.*

Proof. See Appendix. ■

5 Competitive Search as a Limit

In this section we present the standard one-shot version of a directed search economy with a continuum of agents under the market utility property and show that it is the limit of the finite game as the number of agents becomes large. This setup encompasses the models described in Section 2.2. Our exposition is closely related to Peters (1997).

Consider an economy with measure one of firms and measure b of workers. The workers are homogeneous and firms are potentially heterogeneous with types distributed on $\Theta = [0, 1]$ according to probability measure P . When a firm of type $\theta \in \Theta$ fills its vacancy and pays v to its worker it makes profits $\pi_\theta(v)$, where π_θ satisfies Assumption 1 and $\bar{v} \equiv \sup_{\theta \in \Theta} \bar{v}_\theta < \infty$.

The timing of the model is the same as in the finite case: firms post announcements, workers decide where to apply for a job, matching occurs and payoffs are realized. The workers' strategies

result in an expected *queue length* λ which represents the ratio of the expected number of applications per firm at each announcement level v and corresponds to np_j in the finite case. The probability that a firm facing queue length λ hires a worker is given by $r_f(\lambda)$ and the probability that a worker who applies to such a firm finds a job is $r_w(\lambda)$, where $r_w(\lambda) = r_f(\lambda)/\lambda$. Additionally, r_f is strictly increasing and concave, r_w is strictly decreasing and convex and they are both twice continuously differentiable.

The queue length across different announcements is determined by the market utility property which is an indifference condition, similar to equation (1), stating that a worker receives at least the market utility U when applying to a firm. An important additional element is that this relation holds both on and *off* the equilibrium path, i.e. it determines a firm's hiring probability from offering some v that is not posted by anyone else:

$$\text{If } v > U \text{ then } \lambda \text{ is s.t. } r_w(\lambda)v = U, \text{ otherwise } \lambda = 0. \quad (11)$$

As in the finite case, an announcement that is too low ($v \leq U$) receives no applicants ($\lambda = 0$) and a firm is active only if $v > U$. Let $\lambda(v, U)$ be the queue length defined by (11). Each firm anticipates this relation between the queue length and its announcement, and solves the problem

$$\max_v r_f(\lambda(v, U))\pi_\theta(v) \quad (12)$$

Definition 4 (Competitive Search Equilibrium) *A competitive search equilibrium comprises the workers' market utility U^* and a cumulative distribution of announcements Y^* such that for all intervals $[v_l, v_h] \subset \mathbb{R}$:*

$$Y^*(v_h) - Y^*(v_l) \leq P\{\theta \in \Theta : \text{some } v \in (v_l, v_h] \text{ solves (12) for } \theta\}, \quad (13)$$

and

$$\int \lambda(v, U^*) dY^*(v) = b. \quad (14)$$

The left hand side of equation (13) gives the equilibrium measure of offers in $(v_l, v_h]$. The right hand side gives the proportion of firms that find it optimal to make an announcement in $(v_l, v_h]$. If every firm has a unique announcement, then (13) holds with equality.¹⁷ Equation (14) ensures that the worker-firm ratio integrated across all firms actually adds up to the measure of workers in the economy. It ensures that the utility that the workers obtain indeed reflects their scarcity.

For some of the convergence results it is more useful to talk about a firm's rank in the distribution. We define a firm as being of rank $x \in [0, 1]$ if a fraction x of other firms has a weakly lower type. We can back out the actual type of the firm that has rank x as $\tau(x) = \sup\{\theta \in \theta | P([0, \theta]) \leq x\}$. Let Π_x^* denote the expected profit of a firm of rank x in the competitive equilibrium.

We will now explore the connection of this limit game to games of the finite economy that we analyzed in Section 3. Consider a finite economy with m firms and $n = bm$ identical workers. In what follows, we index the variables that refer to the finite economy by m . We label firms in the finite economy by their rank in the productivity distribution, so that firm j is of rank j/m . Furthermore, we assume that the rank remains unchanged as the economy grows in that it coincides with that of firm of type $\tau(j/m)$ in the limit economy. Therefore, by construction the distribution of types in the finite economy converges weakly to the type distribution in the limit economy. Theorem 1 proves that the finite economy has a pure strategy equilibrium. Let Y_m denote the distribution of announcements for that equilibrium, U_m the market utility of the workers and $\Pi_{m,x}$ the expected profit of firm $j = mx$.

In the finite game we have some trading probabilities given by $H(p)$ and $G(p)$ when workers apply with probability p to a firm, where H and G fulfill Assumption 2. The matching probabilities change when we increase the number of workers n , and to make this dependence obvious

¹⁷In principle, a firm could earn maximum profits from several distinct announcements, which is why (13) has a weak inequality. To see that (13) always holds with equality if each firm type has a unique optimum, observe the following. If the inequality were strict for some interval $[v_l, v_h]$ then for the union of $[v_l, v_h]$ and $[0, \bar{v}] \setminus [v_l, v_h]$ the left hand side of (13) is 1 but the right hand side would have to add to more than 1, violating the requirement that P is a probability measure.

we can write $H(n, p)$ and $G(n, p)$.¹⁸ Intuitively, np reflects the expected number of workers at this firm. We will consider matching functions r_w and r_f that can be approached as the limits of H and G as $n \rightarrow \infty$ keeping $np = \lambda$. It is easy to see that any pair r_w and r_f that fulfills Assumption 2 (when p is replaced by λ) can be approached by some sequence of functions $H(n, p)$ and $G(n, p)$ that fulfill Assumption 2. Since Assumption 2 is quite general, this includes most matching functions that have been used in the literature. In particular, the limit matching functions of the examples in section 4 are included, which in particular rationalizes the following different limit matching technologies that have both different levels and elasticities:

$$\text{Example M1 : } r_f(\lambda) = 1 - e^{-\lambda} = \lim_{n \rightarrow \infty | np = \lambda} 1 - (1 - p)^n;$$

$$\text{Example M2 : } r_f(\lambda) = 1 - e^{-(1-\tau)\lambda} = \lim_{n \rightarrow \infty | np = \lambda} 1 - (1 - p)^n;$$

$$\text{Example M3 (for } \bar{n} = 2): r_f(\lambda) = (1 - e^{-\lambda} - \lambda e^{-\lambda})(1 - \tau^2) + \lambda e^{-\lambda}(1 - \tau);$$

$$\text{First Example M4 : } r_f(\lambda) = \frac{\lambda}{l + \lambda} = \lim_{n \rightarrow \infty | np = \lambda} \frac{np}{l + np};$$

$$\text{Second Example M4 : } r_f(\lambda) = (1 + \lambda^{-\sigma})^{-1/\sigma} = \lim_{n \rightarrow \infty | np = \lambda} (1 + (np)^{-\sigma})^{-1/\sigma}.$$

We will show that an allocation that can be supported for the limit of finite games constitutes a competitive search equilibrium, and vice versa. The following result shows the payoffs of workers and firms converge for large m to those in the limit economy, which implicitly means that the equilibrium matching probabilities converge.

Theorem 4 *For any convergent subsequence of equilibria such that $Y_m \rightarrow Y^*$ there exists U^* such that $\{U^*, Y^*\}$ constitutes a competitive search equilibrium, and expected utilities converge ($U_m \rightarrow U^*$) as well as expected profits ($\Pi_{m,x} \rightarrow \Pi_x^*$). Conversely, for any competitive search equilibrium $\{U^*, Y^*\}$ there exists a subsequence of equilibria such that $Y_m \rightarrow Y^*$, $U_m \rightarrow U^*$, and $\Pi_{m,x} \rightarrow \Pi_x^*$.*

Proof. The analysis for the subgame against a convergent distribution $Y_m \rightarrow Y^*$ of (possibly

¹⁸It is more convenient to index these probability by n . Of course, this is identical to indexing them by m since $n = bm$.

non-equilibrium) offers follows directly from Peters (1997), Theorem 3 and Theorem 4.¹⁹ He characterizes the payoffs for the firms that offer any of the wages in Y_m . Peters (1997, p. 256) lays out that his equivalence theorems extend directly to convergence of finite *equilibria* if the finite equilibria exist in pure posting strategies (because in this case the equilibrium can be represented as a step function Y_m). Our Theorem 1 establishes such existence in pure posting strategies. ■

6 Conclusions

In this paper we consider finite directed search economies with heterogeneous firms, homogeneous workers and general production and matching structures. We characterize the response by workers to changes in the offers by firms and prove the existence of subgame perfect Nash equilibria in pure firm strategies. In addition to being interesting in its own right, this result is useful in a number of ways. Proving the convergence of finite equilibria to the continuum economies becomes relatively straightforward (Section 5), showing that the competitive search models that have been considered in the literature have solid micro-foundations. Furthermore, a more complete characterization of the equilibrium set is feasible (Section 4) and examining the efficiency properties of the finite economy becomes easier (Galenianos, Kircher and Virag (2010), for the special case of linear production).

A number of questions remain open for this class of models. The cardinality of the pure strategy equilibrium set has not been characterized (especially as concerns uniqueness) while the existence of non-degenerate mixed strategy equilibria has not been proved or disproved. A different research direction would be to introduce heterogeneity on the worker side. With two-sided heterogeneity one can address questions regarding the sorting patterns between workers and firms. This question has been examined in continuum models by Shi (2001), Shimer (2005) and Eeckhout and Kircher (2010) but, to our knowledge, only Peters (2009) has made progress

¹⁹The proofs in Peters (1997) work with the function $H(n, p) = 1 - (1 - p)^n$, but straightforward replacement by the general functional form $H(n, p)$ shows convergence for more general matching functions.

in analyzing a finite economy.²⁰

Manolis Galenianos, Pennsylvania State University, USA.

Philipp Kircher, London School of Economics and Political Science, UK, and University of Pennsylvania, USA.

²⁰Peters (2009) considers the game among heterogeneous workers for given wage offers by firms, while strategic decisions of the firms are not analyzed for finite numbers. He does integrate firms' decisions in a limit game.

7 Appendix

Lemma 2.

Proof. We show that the partial derivatives translate into equation (6). (See Korn and Korn (1968) for the relevant matrix algebra). $D_{\mathbf{p}}\mathbf{F}$ is a matrix with elements $\alpha_{ss} = \xi_s(\mathbf{v})$ and $\alpha_{sl} = -\xi_l(\mathbf{v})$ for $s \in \{1, \dots, l-1\}$, $\alpha_{ls} = 1$ for $s \in \{1, \dots, l\}$ and $\alpha_{sk} = 0$ otherwise. To calculate the determinant $|D_{\mathbf{p}}\mathbf{F}|$ we use Laplace's development to expand the last row and obtain $|D_{\mathbf{p}}\mathbf{F}| = \sum_{s=1}^l \Lambda_{ls}$, where Λ_{ls} is the cofactor to element α_{ls} . That is, $\Lambda_{ls} = (-1)^{l+s}|Q_{ls}|$, where Q_{ls} is the matrix resulting from $D_{\mathbf{p}}\mathbf{F}$ by elimination of the l th row and the s th column. Since Q_{ll} is a diagonal matrix we have $|Q_{ll}| = \prod_{k \in L(\mathbf{v}) \setminus \{l\}} \xi_k(\mathbf{v})$. For $s < l$ we expand the s th row of $|Q_{ls}|$ which yields $|Q_{ls}| = (-1)^{l-1+s}(-\xi_l(\mathbf{v}))|B_{ls}|$, where B_{ls} is a $(l-2)^2$ -dimensional diagonal matrix with diagonal elements $\xi_k(w)$ for all $k \in A(\mathbf{v}) \setminus \{s, l\}$. We therefore have $|Q_{ls}| = (-1)^{l+s} \prod_{k \in A(\mathbf{v}) \setminus \{s\}} \xi_k(\mathbf{v})$, which yields that $|D_{\mathbf{p}}\mathbf{F}| = \sum_{s=1}^l \prod_{k \in A(\mathbf{v}) \setminus \{s\}} \xi_k(\mathbf{v})$.

Next, consider the matrix $D_{\mathbf{v}}\mathbf{p} = -(D_{\mathbf{p}}\mathbf{F})^{-1}D_{\mathbf{v}}\mathbf{F}$ of partial derivatives. As an implication of Cramer's Rule $(D_{\mathbf{p}}\mathbf{F})^{-1} = |D_{\mathbf{p}}\mathbf{F}|^{-1}C$, where C is the matrix with elements $\gamma_{js} = \Lambda_{sj}$. The Jacobian with respect to the exogenous variables $D_{\mathbf{v}}\mathbf{F}$ evaluated at $(\mathbf{p}(\mathbf{v}), \mathbf{v})$ is simply a diagonal matrix except for the last column, with elements $\beta_{ss} = G(p_s(\mathbf{v}))$ and $\beta_{sl} = -G(p_l(\mathbf{v}))$ for $s \in \{1, \dots, l-1\}$ and zeros elsewhere. We therefore have $\partial p_j(\mathbf{v})/\partial v_j = -\Lambda_{jj}|D_{\mathbf{p}}\mathbf{F}|^{-1}G(p_j(\mathbf{v}))$. This follows immediately for $j \in \{1, \dots, l-1\}$, and holds for $j = l$ by symmetry which is cumbersome but straightforward to verify analytically. Since the cofactor Λ_{jj} has a similar structure as the determinant $|D_{\mathbf{p}}\mathbf{F}|$ only with row and column j missing, we have $\Lambda_{jj} = \sum_{s \in A(\mathbf{v}) \setminus \{j\}} \prod_{k \in A(\mathbf{v}) \setminus \{j, s\}} \xi_k(\mathbf{v})$, and we obtain

$$\frac{\partial p_j(\mathbf{v})}{\partial v_j} = -\frac{\sum_{s \in A(\mathbf{v}) \setminus \{j\}} \prod_{k \in A(\mathbf{v}) \setminus \{j, s\}} \xi_k(\mathbf{v})}{\sum_{s \in A(\mathbf{v})} \prod_{k \in A(\mathbf{v}) \setminus \{s\}} \xi_k(\mathbf{v})} G(p_j(\mathbf{v})). \quad (15)$$

Equation (6) follows then from simple algebraic manipulations. ■

Lemma 3.

Proof. Fix \mathbf{v}_{-j} . We first consider $\hat{v}_j \in \Psi_j(\mathbf{v}_{-j})$, i.e. points where the workers reaction is not

differentiable. We have already established there is only a finite number of such points. At these points the concavity of $p_j(v_j, \mathbf{v}_{-j})$ follows trivially because a decrease in the announcement by firm j increases other firms' expected number of applicants, while an increase does not. That is, by continuity of $p_j(\cdot)$, $\xi_j(\cdot)$ and $G(\cdot)$ equation (15) implies that $\lim_{v_j \nearrow \hat{v}_j} \partial p_j(v_j, \mathbf{v}_{-j}) / \partial v_j < \lim_{v_j \searrow \hat{v}_j} \partial p_j(v_j, \mathbf{v}_{-j}) / \partial v_j$.

The remaining task is to show that $p_j(v_j, \mathbf{v}_{-j})$ is strictly concave for $\hat{v}_j \in \Omega_j(\mathbf{v}_{-j})$. Recall that $T_j(\mathbf{v}) = -\xi_j(\mathbf{v}) - X_j(\mathbf{v})$ where $X_j(\mathbf{v}) = 1 / \sum_{k \in A(\mathbf{v}) \setminus \{j\}} \xi_k(\mathbf{v})$. We differentiate (15) with respect to v_j to obtain the following:

$$\frac{\partial^2 p_j}{\partial v_j^2} = -\frac{1}{T_j^2} \left(g(p_j) \frac{\partial p_j}{\partial v_j} [X_j + v_j] - G(p_j) \left[g'(p_j) \frac{\partial p_j}{\partial v_j} v_j + g(p_j) + \frac{\partial X_j}{\partial v_j} \right] \right), \quad (16)$$

where \mathbf{v} is omitted for brevity. We now show that (16) is strictly negative. We split the term in the round bracket into three parts, B_1 , B_2 and B_3 , and show that each is non-negative.

The first part is given by $B_1 = g(p_j) [\partial p_j / \partial v_j] X_j$ and it is strictly positive because $g(p_j)$ and X_j are strictly negative. Part B_2 is given by

$$B_2 = g(p_j) \frac{\partial p_j}{\partial v_j} v_j - G(p_j) \left[g'(p_j) \frac{\partial p_j}{\partial v_j} v_j + g(p_j) \right].$$

Rearranging the above and using (15) yields

$$B_2 = G(p_j) v_j [2g(p_j)^2 - g'(p_j) G(p_j)] + X_j g(p_j) G(p_j).$$

The last term is positive so we only need to show that term in the square bracket is positive, which holds exactly when $1/G(p)$ is convex.

Finally, consider $B_3 = -G(p_j) [\partial X_j / \partial v_j]$. Note that

$$\frac{\partial X_j}{\partial v_j} = X_j^2 \left[\sum_{k \in A(\mathbf{v}) \setminus \{j\}} \frac{g'(p_k)}{g(p_k)^2 v_k} \frac{\partial p_k}{\partial v_j} \right],$$

Since $\partial p_k / \partial v_j \leq 0$ for $k \neq j$ and $g'(p_k) \geq 0$, due to the convexity of $G(p)$, we have shown that B_3 is non-negative. ■

Theorem 2.

Proof. Under Assumption 3, $A(\mathbf{v}) = M$ and the announcement of every firm is characterized by its first order condition:

$$\frac{\partial \Pi_j}{\partial v_j} = H(p_j) \frac{d\pi_j(v_j)}{dv_j} + h(p_j) \pi_j(v_j) \frac{\partial p_j}{\partial v_j} = 0 \quad \forall j \in M. \quad (17)$$

From now on we focus on firms 1 and 2 without loss of generality. Let firm 1 be strictly more productive than firm 2. The proof proceeds by contradiction. Assume $v_1 \leq v_2$ (the proof for equal productivities and $v_1 < v_2$ is analogous). Under this assumption we will show that $\partial \Pi_2 / \partial v_2 = 0$ and then $\partial \Pi_1 / \partial v_1 > 0$, which contradicts profit maximization for firm 1 and proves that $v_1 > v_2$ is a necessary condition for equilibrium.

We proceed by assuming $v_1 \leq v_2$. To compare the first order conditions of firms 1 and 2 we can work with the following two sets of inequalities:

$$\frac{d\pi_1(v_1)}{dv_1} \geq \frac{d\pi_2(v_1)}{dv_2} \geq \frac{d\pi_2(v_2)}{dv_2}, \quad (18)$$

$$\pi_1(v_1) \geq \pi_2(v_1) \geq \pi_2(v_2). \quad (19)$$

The first inequality of equations (18) and (19) is due to firm 1 being more productive and at least one of them has to hold strictly (according to Definition 3). The second inequality of equation (18) is due to the (weak) concavity of $\pi_j(\cdot)$. The second inequality of equation (19) is due to the fact that $\pi_j(v_j)$ is decreasing in v_j in the relevant range.

Rearranging equation (17) yields

$$\frac{d\pi_j(v_j)}{dv_j} + \frac{h(p_j)}{H(p_j)} \frac{\partial p_j}{\partial v_j} \pi_j(v_j) = 0. \quad (20)$$

If the term multiplying $\pi_j(v_j)$ is higher for firm 1 than for firm 2, then the first derivative of firm

1 is strictly positive when $v_1 \leq v_2$ which proves our result. Using equation (6) we can rewrite:

$$\frac{h(p_j)}{H(p_j)} \frac{\partial p_j}{\partial v_j} = -\frac{h(p_j)}{H(p_j)} G(p_j) \frac{\sum_{s \in M \setminus \{j\}} \prod_{k \in M \setminus \{j,s\}} \xi_k}{\sum_{s \in M} \prod_{k \in M \setminus \{s\}} \xi_k}. \quad (21)$$

Note that the last term has the same denominator for all j . Therefore we need only show that

$$\frac{h(p_1)G(p_1)}{H(p_1)} \sum_{s \neq 1} \prod_{k \notin \{1,s\}} |g(p_k)|v_k \geq \frac{h(p_2)G(p_2)}{H(p_2)} \sum_{s \neq 2} \prod_{k \notin \{2,s\}} |g(p_k)|v_k \quad (22)$$

recalling that $\xi_k \equiv g(p_k)v_k$ and $g(p_k) < 0$. The assumption that $v_1 \leq v_2$ implies $p_1 \leq p_2$ and hence $h(p_1) \geq h(p_2)$, $H(p_1) \leq H(p_2)$ and $G(p_1) \geq G(p_2)$. The term $\prod_{k \notin \{1,2\}} |g(p_k)|v_k$ is contained inside the summation in both sides of inequality (22). It is therefore sufficient to show:

$$\frac{h(p_1)G(p_1)}{H(p_1)} |g(p_2)|v_2 \geq \frac{h(p_2)G(p_2)}{H(p_2)} |g(p_1)|v_1. \quad (23)$$

Subgame equilibrium implies that $v_2/v_1 = G(p_1)/G(p_2)$. Together with $G(p_j) = H(p_j)/(np_j)$ and $|g(p_j)| = [G(p_j) + h(p_j)/n]/p$, inequality (23) reduces to

$$\frac{G(p_2) + h(p_2)/n}{G(p_2)h(p_2)/n} \geq \frac{G(p_1) + h(p_1)/n}{G(p_1)h(p_1)/n}.$$

If $R(p) \equiv G(p)^{-1} + nh(p)^{-1}$ is strictly increasing in p we have our result. Differentiation yields $R'(p) = -G(p)^{-2}g(p) - nh(p)^{-2}h'(p)$ which is strictly positive for any $p \in (0, 1)$ because $h'(p) \leq 0$ and $g(p) < 0$. ■

Theorem 3.

Proof. When all firms are equally productive Assumption 3 holds and in equilibrium all firms offer the same level of utility by Theorem 2. As a result, $p_j = 1/m$ for all $j \in M$ in all possible equilibria. Suppose there are two candidate equilibria A and B where firms offer v_A and $v_B > v_A$, respectively, and consider the firms' first order conditions. The terms $H(p)$ and $h(p)$ are the same in both candidate equilibria. The concavity of the profit function implies that

$d\pi(v_A)/dv_A \geq d\pi(v_B)/dv_B$. Profits are a decreasing function of offered utility in \mathcal{V} which implies that $\pi(v_A) > \pi(v_B)$. Finally, $\partial p_j/\partial v_A > \partial p_j/\partial v_B$ follows from equation (15): $G(p)$ and $g(p)$ are the same in both equilibria and $T_j(\mathbf{v}_A) < T_j(\mathbf{v}_B)$. ■

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