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A Model of Money
with Multilateral Matching[*]

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Abstract

We develop a model of decentralized monetary exchange that can be used to examine the distributional effects of inflation across heterogeneous agents who have private information. The private information can be about the productivity, preferences, or money holdings of the agents. Matching is multilateral and each seller is visited by a stochastic number of buyers. The good is allocated according a second-price auction in money. In equilibrium, homogeneous buyers hold different amounts of money leading to price dispersion. We find the closed-form solution for the distribution of money holdings. Entry of sellers is suboptimal except at the Friedman rule. When agents differ in productivity, inflation acts as a regressive tax, at least for moderate rates of money growth.

1　Introduction

We develop a model of decentralized monetary exchange that can be used to examine the distributional effects of inflation across heterogeneous agents who have private information. The private information can be about the productivity, preferences, or money holdings of

the agents. A version of the model with two productivity types predicts that inflation is a regressive form of taxation.

To study the distributional effects of inflation, it is useful to have an environment with private information about agents' types. Under complete information the results may be biased because an agent's type generally affects his terms of trade and hence the incentives for holding money. Most models in the monetary-search literature assume complete information about all the payoff relevant variables, e.g. productivity, preferences, or money holdings, to allow the terms of trade to be determined through bargaining (see Trejos and Wright (1995) and Shi (1995)). Since bilateral matching is typically assumed (Kiyotaki and Wright (1993)), bargaining is a natural allocation mechanism.

To deal with private information we model decentralized trade in a different way. In each period every agent who wants to consume (a buyer) visits at random one of the agents who can produce (a seller).[1] This results in stochastic demand at a given seller's location. We fix the supply of each seller at a single indivisible good in order to use auctions to determine the allocation. The number of sellers in the market is determined by a free entry condition. Anonymity makes fiat money essential for trade in these meetings. We embed this framework in the environment of Lagos and Wright (2005), henceforth LW.

We assume that the good is allocated according to a second-price auction.[2] This mechanism lends itself to the natural interpretation of intra-buyer competition for the good: the buyers who visit the same seller bid up the price, as in an ascending bid auction whose outcome is identical to a second-price auction. The result is that the buyer with most money purchases the good and the price he pays is equal to the second highest money holdings in the match. Stochastic demand conditions at a given location lead to uncertainty for buyers about the market-clearing price. As a result, buyers face a non-trivial choice of liquidity since the good can occasionally be purchased with little money.[3]

Our first set of results concern the case where agents are homogeneous. We show that in the unique equilibrium, despite homogeneity, buyers bring different amounts of money to the search market and we derive the closed-form solution for the distribution of money holdings.

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[1] A similar matching process is proposed in Goldberg (2007). Our framework does not suffer from the criticism that models in which buyers do not know where to purchase their desired good are not conducive to the study of monetary exchange (Howitt (2005)). The same criticism is dealt with in Corbae, Temzelides and Wright (2003) where matching is explicitly directed.

[2] Julien, Kennes and King (in press) consider second-price auctions in a setting with indivisible money and Green and Zhou (2002) use double auctions in bilateral matches. Camera and Selcuk (2005) develop a non-monetary model where the trading price depends on the stochastic number of buyers that a seller faces.

[3] More generally, in order to clear the market, prices need to adjust to realized demand conditions at each location. We propose the lowest price such that demand equals supply (for a similar interpretation of auctions see Demange, Gale and Sotomayor (1986)). In the conclusions we discuss the possibility of using different auctions for allocating the good. We conjecture that any standard auction yields the same qualitative results, but we use the second-price auction because it is easier to analyze.

Bringing more money increases the probability of consuming but it is costly because the value of any unspent balances depreciates over time due to inflation and discounting. Dispersion in money leads to dispersion in prices and we explicitly characterize the realized distribution of prices in the market.[4] The main efficiency result is that entry is suboptimal except when the money supply contracts at the rate of time preference – the Friedman rule.

When agents are permanently heterogeneous in their productivity levels we show that more productive buyers hold higher balances since they face lower utility costs of earning money. Higher inflation leads to lower seller entry (increased scarcity) which reduces welfare for all buyers. When inflation is below a threshold, low productivity buyers are hurt more from an increase in the rate of money growth despite the fact that they hold less money. Low productivity buyers are the first to be priced out of the market when scarcity increases: they are always outbid by their high productivity competitors who they face more often when scarcity is more acute. If the inflation rate is very high, the low productivity buyers hardly trade at all and therefore any additional increase in money growth has little effect on them.

Erosa and Ventura (2002) and Albanesi (2007) examine, and present evidence for, the regressive effect of inflation in models without explicit micro-foundations for fiat money. Private information is a feature of Faig and Jerez (2006) and Ennis (in press) where buyers are subject to idiosyncratic taste shocks and sellers screen customers by offering price-quantity menus. Both papers consider buyers who are ex ante homogeneous and are primarily interested in the distortions caused by inflation rather than its distributional effects on permanently heterogeneous agents. Satterthwaite and Shneyerov (2006) introduce multilateral matching and auctions to deal with private information in a non-monetary environment of decentralized trade. Our uniqueness and efficiency results are closely related to the competitive search market structure of Rocheteau and Wright (2005). Our setting is related to the budget constrained auctions examined in Che and Gale (1998) because buyers will, in general, choose to hold less money than their actual valuation for the good.

The rest of the paper is organized as follows. Section 2 describes the model and proves some preliminary results. The following section solves for the equilibrium when all agents are homogeneous and examines the efficiency properties of inflation. Section 4 introduces heterogeneity in productivity and examines the distributional effects of inflation on the different types of agents. Section 5 touches on robustness issues and concludes.

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[4]Dispersion of money holdings is typically a feature of models where an agent's money holdings *do* depend on his history of trades, e.g. Molico (2006), Green and Zhou (2002), or Camera and Corbae (1999). Price dispersion is also an equilibrium outcome of the monetary model of Head and Kumar (2005) where it is due to the informational asymmetries among buyers.

2 The Model

Time is discrete and runs forever. Each period is divided in two subperiods, following LW: a Walrasian market characterized by competitive trading and a search market characterized by trading frictions that are modeled explicitly. There is a continuum of infinitely-lived agents who differ across two dimensions. They have potentially different productivity in the Walrasian market and they belong to one of two groups in the search market, called *buyers* and *sellers*. In the Walrasian market all agents produce and consume but in the search market a buyer can only consume and a seller can only produce. Meetings in the search market occur between subsets of the population in a way described in detail below and they are characterized by two main frictions. First, all meetings are assumed to be anonymous which precludes credit. Hence all trades have to be *quid pro quo*. Second, there is no double coincidence of wants, as is clear from the assumptions on agents' types: some agents can only produce while others can only consume. Therefore, the agents cannot use barter to trade. These frictions mean that a medium of exchange is essential for trade (see Kocherlakota (1998) and Wallace (2001)).

There is a single storable object, fiat money, which can be used as a medium of exchange in the search market. The stock of money at time t is given by M_t^S and it is perfectly divisible. The money stock changes at gross rate γ, so that $M_{t+1}^S = \gamma\, M_t^S$, and new money is introduced, or withdrawn if $\gamma < 1$, via lump sump transfers to sellers in the Walrasian market.[5] We focus on policies with $\gamma \geq \beta\,\delta$, where $\beta\,\delta$ is the discount factor as discussed below, since it is easy to check that there is no equilibrium otherwise. To examine what happens when the rate of money growth is exactly equal to the discount factor (the Friedman rule) we take the limit of equilibria as $\gamma \to \beta\,\delta$.

We denote the productivity distribution by $Y(\cdot)$ and the measure of buyers and sellers by B and S, respectively. An agent's type is given by $yk \in suppY(\cdot) \times \{b,s\}$ where y is his Walrasian market productivity and b and s denote a buyer and a seller, respectively. Let $W_t^{yk}(m)$ be the value of an agent of type yk who enters the Walrasian market at time t holding m units of money. His instantaneous utility depends on consumption, x, and hours of work, h. We assume that preferences are quasi-linear and take the form $U(x) - h$, where $U'(x) > 0$ and $U''(x) < 0$ for all x and $\lim_{x \to 0} U'(x) = \infty$, $\lim_{x \to \infty} U'(x) = 0$. An hour of work produces y units of the consumption good x. Let β be the discount rate between the Walrasian and search markets and denote the value of carrying m' dollars to the search market of period t by $V_t^{yk}(m')$. The agent's value function in the Walrasian market at time

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[5]In an earlier version of the paper we considered the case where both buyers and sellers receive transfers (see Galenianos and Kircher (2006)). This turns out not to matter for the equilibrium characterization but the analysis is somewhat simpler when buyers do not receive transfers.

t is

$$W_t^{yk}(m) \quad = \quad \max_{x,h,m'}[U(x) - h + \beta \ V_t^{yk}(m')] \tag{1}$$

$$\text{s.t.} \quad x \quad \leq \quad y \ h + \phi_t(\hat{T}_t^k + m - m'),$$

where ϕ_t is the value of money in consumption terms and $\hat{T}_t^k$ is the nominal monetary transfers to (or from) an agent, where $\hat{T}_t^s = (\gamma - 1) \ M_{t-1}/S$ and $\hat{T}_t^b = 0$.[6]

It will prove useful to first solve some of the non-monetary decisions of the agents and then concentrate on the more interesting choices relating to money holdings. Substituting the constraint with equality into equation (1) gives

$$W_t^{yk}(m) = \frac{\phi_t \ (\hat{T}_t^k + m)}{y} + \max_{x,m'}[U(x) - \frac{x + \phi_t \ m'}{y} + \beta \ V_t^{yk}(m)].$$

The quasi-linearity of preferences eliminates wealth effects simplifying the problem of the agent significantly: current balances, m, have no effect on the choices of consumption or future money balances.[7] Our assumptions on $U(\cdot)$ ensure that $U'(x_y^*) = 1/y$ is a necessary and sufficient condition for the optimal choice of x. We can rewrite the problem as

$$W_t^{yk}(m) \quad = \quad \frac{\phi_t \ (\hat{T}_t^k + m)}{y} + U_y^* + \max_{m'} \ [-\frac{\phi_t \ m'}{y} + \beta \ V_t^{yk}(m')], \tag{2}$$

where $U_y^* = U(x_y^*) - x_y^*$ and $U'(x_y^*) = 1/y$.

The choice of future money balances is more involved and we first describe the search market in order to see the relevant incentives.

The search market operates as follows. First, each seller decides whether to incur utility cost $K > 0$ in order to enter the search market. We interpret K as a production cost that has to be undertaken prior to matching with buyers and endows the seller with a single indivisible unit of the good. The good is perishable, can be transferred at zero cost and the utility to the seller of consuming his own good is zero while the utility that a buyer receives is given by $u > K$. Indivisibility aside, the other assumptions about the good are tailored so that the seller's reservation price is zero. Having a strictly positive reservation price does not significantly change our results but it complicates the analysis.

The sellers who enter the search market set up their shop at some physical location.

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[6]We only examine equilibria with $h > 0$ and hence we ignore the non-negativity constraints on h. One can impose conditions on the primitives to guarantee that this holds, as shown in LW.

[7]For the same reason, our assumption on how transfers are distributed is innocuous: it does not affect any of the monetary choices that we are interested in, although it does affect the agents' level of utility.

There is a continuum of locations each accommodating at most one seller. We denote the buyer-seller ratio at time t by λ_t. We assume that the measure of potential sellers, S, is large enough so that λ_t is determined by an indifference condition for entry. We normalize the measure of buyers to 1 so that $1/\lambda_t$ gives the measure of sellers who enter the search market. This normalization is without loss of generality, as the next paragraph makes clear.

We begin by analyzing the buyer's problem for a fixed λ_t and we then allow for the free entry of sellers. When entering the search market, a buyer visits at random one of the locations that are populated with sellers. This assumption can be formulated as an equilibrium of a broader game where buyers strategically choose which seller to visit: if every other buyer picks a seller at random, it is a best response to randomize as well.[8] Our matching process implies that the number of buyers that visit a particular seller is a random variable, and hence demand is stochastic while supply is fixed at one unit. Therefore, the good may get rationed and some of the buyers may end up not consuming. Before proceeding to describe the allocation process, note that we have urn-ball matching and so the number of buyers follows a Poisson distribution with parameter λ_t.[9] This matching function exhibits constant returns to scale and therefore the buyer-seller ratio is the only relevant statistic for the matching probabilities.

The way the good is allocated is the main innovation of this paper: we assume that a second-price auction takes place. The seller accepts any non-negative bids since his reservation price is zero. All buyers have the same valuation for the good, but they may hold different amounts of money. In a second-price auction a buyer's dominant strategy is to bid his money holdings, except when the money's value exceed his valuation of the good which we will show can never occur since the sole purpose of holding fiat money is to buy the search market good. We focus on equilibria where buyers follow dominant strategies. The outcome of the auction is that the buyer with most money (or one of the buyers with the highest money holdings in the case of a tie) buys the good and the price that he pays is equal to the second highest money holdings in the match. If a single buyer appears at some location then he gets the good at a price of zero. This mechanism balances demand and supply at the lowest price that clears the market, i.e. the lowest price such that exactly one unit of the good is demanded. The underlying idea is that a buyer makes price bids that can be matched by the other buyers who have visited the same location, as in an ascending bid (or second-price) auction. A difference with respect to bilateral matching is the seller receives a

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[8]We want to abstract from the repeated game effects that may arise if a buyer meets with the same seller at each period's search market. One way to do this is to assume sellers populate a random location every time they enter; alternatively, we can assume that it is not the same sellers who enter the market every period.

[9]Suppose k buyers are allocated randomly across l sellers. The number of buyers that visit a given seller follows a binomial distribution with probability $1/l$ and sample size k. As $k, l \to \infty$ keeping $k/l = \lambda$ the distribution converges to a Poisson distribution with parameter λ.

positive share of the surplus due to the presence of many buyers at the same location even though the winning buyer makes what amounts to a take-it-or-leave-it offer.

The incentives to hold money are now clear: holding more money increases the probability of consuming since it allows a buyer to outbid more of his potential competitors; on the other hand it is costly because the value of any unspent money balances depreciates over time due to discounting and inflation. It is also worth noting that the reason why an agent may not spend his fiat money is very different than in most of the monetary search literature. In this paper, the amount of money that a buyer ends up spending depends on how many other buyers visit the same location and on how much money they hold. Hence, despite the fact that a buyer is matched with some seller with probability one, he does not in general spend all of his money. In most of the literature the cost of liquidity arises from the buyer's failure to meet someone whose good he wants to buy.

We now characterize the buyers' value function. Whether a buyer transacts, and at what price, depends on how many other buyers have visited the same location and on how much money they hold. Hence we need to introduce some additional notation to describe the money holdings of all buyers: aggregating the choices of m' across buyers gives the distribution of money holdings at the end of Walrasian trading (or, equivalently, at the beginning of the search market) which we denote by $\hat{F}_t(\cdot)$.

Let $V_t^{yb}(m, n)$ denote the expected payoff of a productivity-y buyer who holds m dollars and meets n other buyers at the location that he visits. The probability that he meets exactly n competitors is $P_n^t = \lambda_t^n \, e^{-\lambda_t} / n!$ due to Poisson matching. The value of entering the search market with m dollars is thus given by

$$V_t^{yb}(m) = \sum_{n=0}^{\infty} P_n^t \, V_t^{yb}(m, n). \tag{3}$$

To calculate $V_t^{yb}(m, n)$, let $m_{(n)}$ be the highest money holdings among the n competitors that the current buyer faces. If $m < m_{(n)}$ the buyer with m dollars does not transact and he keeps all his money for the next Walrasian market. If $m > m_{(n)}$ the buyer with m dollars buys the good and pays $m_{(n)}$ to the seller. If $m = m_{(n)}$ there is a tie and the good is allocated at random to one of the buyers with $m_{(n)}$ dollars who then transfers his full money holdings to the seller. Therefore, $m_{(n)}$ is the only statistic we need in order to calculate the expected payoff of a buyer who faces n competitors.

The money holdings of each competitor is a random draw from $\hat{F}_t(\cdot)$ since buyers are allocated at random across sellers. The highest money holdings among the n other buyers is the highest order statistic among n *iid* draws from $\hat{F}_t(\cdot)$ which is distributed according

to $\hat{F}_t(\cdot)^n$.[10] Observe that the probability that two randomly chosen buyers hold exactly the same amount of money is zero unless $\hat{F}_t(\cdot)$ has a mass point at that level. To denote this possibility we define $\mu_t(m) \equiv \hat{F}_t(m) - \hat{F}_t(m^-)$, where $\hat{F}_t(m^-) = \lim_{\tilde{m} \nearrow m} \hat{F}_t(\tilde{m})$. This implies that $\mu_t(m) > 0$ if and only if there is a mass point at m. Conditional on all n competitors holding weakly less than m dollars, the number of buyers (out of n) who have exactly m dollars follows a binomial distribution with sample size n and probability $\mu_t(m)/\hat{F}_t(m)$.[11] Let $q_t^{nl}(m)$ denote the probability that l out of the other n buyers hold exactly m dollars conditional on none of them having more than m dollars. If there is no mass point at m, then $q_t^{n0}(m) = 1$ and $q_t^{nl}(m) = 0$ for $l \geq 1$.

The value of meeting n competitors at time t when holding m dollars is given by

$$
\begin{aligned}
V_t^{yb}(m, n) = {} & [1 - \hat{F}_t(m)^n] \, \delta \, W_{t+1}^{yb}(m) + \\
& \int_0^{m^-} [u + \delta \, W_{t+1}^{yb}(m - \tilde{m})] \, d\hat{F}_t(\tilde{m})^n + \\
& \hat{F}_t(m)^n \sum_{l=1}^n q_t^{nl}(m) \, [\frac{u + \delta \, W_{t+1}^{yb}(0)}{l+1} + \frac{l \, \delta \, W_{t+1}^{yb}(m)}{l+1}],
\end{aligned} \tag{4}
$$

where δ is the discount factor between the search and Walrasian markets. The term in the first square brackets gives the probability that at least one competitor holds *strictly more* money than m dollars, which means that the current buyer does not purchase the good and he keeps his money for next period's Walrasian market. The second term denotes the expected payoff when all other buyers hold *strictly less* money and hence the buyer with m dollars gets the good and pays the second highest amount. The integral gives the instantaneous utility from consuming the good, u, plus the continuation value after accounting for the payment. Finally, if none of the competitors bring more than m dollars but $l \geq 1$ of them hold exactly m dollars, then with probability $1/(l + 1)$ the current buyer gets the good, consumes, and continues to the next Walrasian market without any money; with probability $l/(l + 1)$ he does not purchase the good and he keeps all his money for the next period. It should be clear from this discussion that the probability of a purchase is discontinuous at m if $\hat{F}_t(\cdot)$ has a mass point at that level. Moreover, as mentioned above, the last term of equation (4) drops out if there is no mass point at m. We have completed the description of the buyer's problem.

We now turn to the sellers. Sellers can derive no benefit from holding money and therefore

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[10]This is a standard result from statistics. For instance, see Hogg and Craig (1994).

[11]Conditional on all buyers holding weakly less than m dollars, the money holdings of an agent is a random draw from $\hat{F}_t(\cdot)$ truncated at, and including, m. Hence, the probability that the result of any draw is exactly equal to m is given by $\mu_t(m)/\hat{F}_t(m)$. The binomial distribution follows since there are n draws.

they carry no money to the search market. We assume that $y = 1$ for all sellers since their productivity does not affect their decisions in any interesting way. A seller can choose whether to enter the search market or not. If he enters, he gives up K units of utility and may earn some revenues to spend in the following Walrasian market. Let $\hat{\Pi}_t$ be the seller's expected revenues if he enters the search market at time t. $\hat{\Pi}_t$ is a sufficient statistic for the value of entry because $W_t^{1s}(\cdot)$ is linear in money. If the seller does not enter, he continues to the following Walrasian market without money. As a result, the seller's value of the search market is given by

$$V_t^{1s} = \max\{-K + \delta\ W_{t+1}^{1s}(\hat{\Pi}_t),\ \delta\ W_{t+1}^{1s}(0)\}.$$

In equilibrium sellers are indifferent between the two options which means that, using equation (2), the following condition has to hold for all t:

$$\delta\ \phi_{t+1}\ \hat{\Pi}_t = K. \tag{5}$$

To determine $\hat{\Pi}_t$, note that the price a seller receives is equal to the second highest money holdings among the buyers that show up in his location. When n buyers visit a particular seller, the second highest order statistic is distributed according to $\hat{F}_t^{(n-1,\ n)}(m) = n\ \hat{F}_t(m)^{n-1}\ [1-\hat{F}_t(m)]+\hat{F}_t(m)^n$ (Hogg and Craig (1994)). The probability that n buyers show up is given by P_n^t and we define the distribution of prices by $\hat{G}_t(m) \equiv \sum_{n=1}^{\infty}\ P_n^t\ \hat{F}_t^{(n-1,\ n)}(m)$. $\hat{G}_t(m)$ denotes the probability that a seller receives no more than m dollars in the search market at time t, after summing over all the possible number of buyers. The expected revenues of a seller at t are given by $\hat{\Pi}_t = \int_0^{\infty}\ \tilde{m}\ d\hat{G}_t(\tilde{m})$.

Last, we need to define market clearing in the money market. Since sellers have zero balances, the demand for money at time t is given by the amount that buyers hold, i.e. $M_t^D = \int_0^{\infty}\ \tilde{m}\ d\hat{F}_t(\tilde{m})$. At time t, the money market clears if

$$M_t^D = M_t^S. \tag{6}$$

Turning to the equilibrium definition, note that there is always an equilibrium where fiat money is not valued, as is common in monetary models. Throughout this paper we focus on monetary equilibria with the property $\phi_t > 0\ \forall\ t$, and statements about non-existence of an equilibrium refer to monetary equilibria. Furthermore, we only examine stationary equilibria in the sense that real variables remain constant over time. In particular, we restrict attention to equilibria where $\lambda_t = \lambda_{t+1}$ and the real demand for money does not change, i.e.

$\phi_t \ M_t^D = \phi_{t+1} \ M_{t+1}^D$. Since the money supply grows at a constant rate γ and $M_t^D = M_t^S$ the latter condition implies that $\phi_t = \gamma \ \phi_{t+1}$. We define a stationary monetary equilibrium (henceforth an equilibrium) as follows.

Definition 2.1 *An equilibrium is a list $\{W_t^{yk}, V_t^{yk}, \hat{F}_t, \phi_t, \lambda_t\}$ where W_t^{yk} and V_t^{yk} are the value functions, $\hat{F}_t$ is the distribution of money holdings at the beginning of the search market at t, ϕ_t is the price of money at t, and λ_t is the buyer-seller ratio at t such that the following conditions are satisfied for all t.*

1. *Optimality: given ϕ_t, any $m' \in supp\hat{F}_t$ solves (2).*

2. *Market Clearing: equation (6) holds.*

3. *Free entry: equation (5) holds.*

4. *Monetary Equilibrium and Stationarity: $\phi_t = \gamma \ \phi_{t+1} > 0$ and $\lambda_t = \lambda_{t+1}$.*

We now summarize the important decisions that agents make in our environment: buyers choose how much money to hold and sellers choose whether to enter the search market. More specifically, a buyer chooses his money holdings taking as given λ_t, ϕ_t and the decisions of other buyers. Aggregating across buyers yields $\hat{F}_t(\cdot)$ as a function of λ_t and ϕ_t. In equilibrium, the price of money is such that money demand equals M_t^S, which pins down ϕ_t as a function of the buyer-seller ratio and the supply of money. Free entry of sellers gives λ_t as a function of the cost of entry.

In the next section we solve the model for homogeneous buyers and we re-introduce heterogeneity in section 4.

3 The Case of Homogeneous Buyers

We now consider the special case where $y = 1$ for all buyers. We omit the productivity superscript for the rest of this section.

3.1 The Buyers' Problem

At the beginning of every period the problem of a buyer is to choose the optimal money holdings, taking as given the choices of all other agents and the price of money. It is immediate that the utility of consumption puts an upper bound on the range of the optimal fiat money decision. Let m_t^* be such that a buyer is indifferent between spending m_t^* to consume or keeping the full amount for the next Walrasian market. This amount exists since $u < \infty$ and

it is defined by the following equation: $u + \delta \, W_{t+1}^b(m - m_t^*) = \delta \, W_{t+1}^b(m) \Rightarrow m_t^* = u/(\delta \, \phi_{t+1})$, where m is a large enough number. In equilibrium a buyer never spends more than m_t^*, hence he never brings more than that amount to the search market. Letting $\underline{m}_t$ and $\overline{m}_t$ denote the infimum and supremum, respectively, of the support of $\hat{F}_t(\cdot)$ the discussion above implies that $0 \leq \underline{m}_t \leq \overline{m}_t \leq m_t^*$.

We can reformulate the buyer's problem on a period-by-period basis as

$$\max_{m \in [0, m_t^*]} \; -\phi_t \, m + \beta \, V_t^b(m), \tag{7}$$

taking $\hat{F}_t(\cdot)$, ϕ_t, and λ_t as given (stationarity implies that knowing the price of money for some t pins down the whole path of prices). The first proposition derives a set of properties that the distribution $\hat{F}_t(\cdot)$ satisfies in equilibrium.

Proposition 3.1 *In equilibrium $\hat{F}_t(\cdot)$ is non-atomic on $[0, m_t^*)$, the support of $\hat{F}_t(\cdot)$ is connected, and the infimum of the support is 0.*

Proof. Suppose that $\hat{F}_t(\cdot)$ has a mass point at some $\check{m} \in [0, m_t^*)$. Buying the good for $\check{m}$ dollars gives positive net utility since $\check{m} < m_t^*$. Equation (4) implies that the probability of buying is discontinuous at $\check{m}$, hence $V_t^b(\check{m}) < V_t^b(\check{m}^+)$. Bringing $\check{m} + \epsilon$ yields strictly higher payoff than $\check{m}$ since the cost of bringing infinitesimally more money is negligible and therefore in equilibrium a buyer never brings $\check{m}$ yielding a contradiction.

Suppose that there is no buyer whose money holdings belong to some interval (m_1, m_2), with $\underline{m}_t \leq m_1 < m_2 \leq \overline{m}_t$. The m_1-buyer trades in exactly the same events as the buyer with m_2 dollars since they both outbid exactly the same set of competitors (except when there is a mass point at m_2 which cannot occur unless $m_2 = m_t^*$, as shown above; if $m_2 = m_t^*$ the additional transactions that the m_2-buyer can perform do not yield any utility gains since he is indifferent between keeping his money or consuming). It is easy to verify that $V_t^b(m_2) = V_t^b(m_1) + \delta \, \phi_{t+1} \, (m_2 - m_1)$. Examining the initial decision of how much money to hold, we have that $-\phi_t \, m_1 + \beta \, V_t^b(m_1) - [-\phi_t \, m_2 + \beta \, V_t^b(m_2)] = (m_2 - m_1) \, [\phi_t - \beta \, \delta \, \phi_{t+1}]$ which is strictly positive since $\phi_t = \gamma \, \phi_{t+1}$ and $\gamma > \beta \, \delta$. This means that carrying m_1 dollars gives strictly higher value than holding m_2 which cannot happen in equilibrium.

Last, a buyer bringing $\underline{m}_t$ dollars can only transact when he does not meet any competitors, in which case the price he pays is equal to 0. Therefore $V_t(\underline{m}_t) = V_t(0) + \delta \, \phi_{t+1} \, \underline{m}_t$, which implies that $\underline{m}_t > 0$ cannot occur in equilibrium for the same reason as above. ∎

The intuition why $\hat{F}_t(\cdot)$ is non-atomic in its interior is straightforward (the next proposition proves that there is no mass point at the upper boundary either): If there is a mass

point in the distribution of money holdings, then it is very likely to meet some buyer holding exactly that amount of money. In that case, a buyer who brings infinitesimally more money faces a discretely higher probability of winning the auction for negligible additional cost. Therefore, this buyer enjoys a higher expected payoff which cannot happen in equilibrium.[12]

One important implication of this result is that the optimal decision of buyers is correspondence-valued.[13] In equilibrium there is a range of values of m that yield the same expected payoff and therefore buyers are willing to randomize over them. Equation (7) implies that $V_t(m)$ is linear in the domain of solutions and hence it is *not* strictly concave.

From now on we only consider $\hat{F}_t(\cdot)$ that are continuous on $[0, m_t^*)$ with $\hat{F}_t(0) = 0$ and $supp\hat{F}_t = [0, \overline{m}_t]$. We can rewrite equations (3) and (4) as follows:

$$V_t^b(m) = \delta \ W_{t+1}^b(m) + \sum_{n=0}^{\infty} P_n^t \ [u \ \hat{F}_t(m)^n - \delta \ \phi_{t+1} \int_0^m \tilde{m} \ d\hat{F}_t(\tilde{m})^n]. \tag{8}$$

This expression is very intuitive: the first term is the value that the buyer can guarantee himself without a purchase; inside the square brackets, the first term is the probability of buying the good times the instantaneous utility of consumption while the second term is the payment from a purchase. Note that we do not account for the event where the buyer has $m = m_t^*$ dollars and he meets another buyer holding exactly the same amount, which could occur since we have not yet ruled out the possibility of a mass point at m_t^*. However, in that event the price is m_t^* which means that the buyer is indifferent between buying the good or continuing with all his money. Therefore, the value of holding m_t^* is still given by equation (8).

We now turn to the explicit characterization of the solution to the buyer's problem. In equilibrium, $-\phi_t \ m + \beta \ V_t^b(m)$ is constant on $[0, \overline{m}_t]$. We construct $\hat{F}_t(\cdot)$ such that this condition holds.

Proposition 3.2 *In equilibrium the distribution of money holdings is uniquely defined by*

$$\hat{F}_t(m) = \frac{1}{\lambda_t} \ \log[1 - e^{\lambda_t} \ (\gamma/(\beta \ \delta) - 1) \ \log(1 - \frac{\delta \ \phi_t \ m}{\gamma \ u})]. \tag{9}$$

Furthermore, $\overline{m}_t < m_t^*$.

---

[12]The logic of this proof is similar to the no mass point proof of Burdett and Judd (1983) and Burdett and Mortensen (1998) though the context is very different.

[13]In contrast, in LW there is always a symmetric equilibrium where all agents bring the same amount of money. Equilibria with asymmetric money holdings are possible but they can be ruled out under relatively mild assumptions.

Proof. Equation (7) implies that $V_t^{b'}(m) = \phi_t/\beta$ for $m \in [0, \overline{m}_t]$. For $V_t^b(\cdot)$ to be differentiable, any equilibrium $\hat{F}_t(\cdot)$ has to be differentiable on $(0, \overline{m}_t)$. We start by assuming differentiability and we then verify that our solution satisfies this property.

Taking the derivative of (8) with respect to m we get (using Leibniz's rule and noting that m does not enter the integrand)

$$
\begin{aligned}
V_t^{b'}(m) &= \delta \; \phi_{t+1} + \sum_{n=0}^{\infty} P_n^t \; [u \; n \; \hat{F}_t(m)^{n-1} \; \hat{F}_t'(m) - \delta \; \phi_{t+1} \; m \; n \; \hat{F}_t(m)^{n-1} \; \hat{F}_t'(m)] \\
&= \delta \; \phi_{t+1} + (u - \delta \; \phi_{t+1} \; m) \; \hat{F}_t'(m) \; \lambda_t \; e^{-\lambda_t \; (1-\hat{F}_t(m))},
\end{aligned}
\tag{10}
$$

which follows from the fact that $n \sim Po(\lambda_t)$.

Equating (10) with ϕ_t/β and rearranging yields the following differential equation:

$$
\lambda_t \; \hat{F}_t'(m) \; e^{\lambda_t \; \hat{F}_t(m)} = e^{\lambda_t} \; \frac{\delta \; \phi_{t+1} \; i_t}{u - \delta \; \phi_{t+1} \; m},
$$

where $i_t \equiv \phi_t/(\phi_{t+1} \; \beta \; \delta) - 1$ is the nominal interest rate at t. Integrating both sides over m, using the initial condition $\hat{F}_t(0) = 0$, and recalling that $\phi_t = \phi_{t+1} \; \gamma$ yields (9).

The maximum money balances, $\overline{m}_t$, can be calculated by using $\hat{F}_t(\overline{m}_t) = 1$:

$$
\overline{m}_t = \frac{u}{\delta \; \phi_{t+1}} (1 - e^{-\frac{1-e^{-\lambda_t}}{i_t}}).
\tag{11}
$$

Since $m_t^* = u/(\delta \; \phi_{t+1})$ it is clear that all buyers bring less money than m_t^* and hence there is no mass point in the distribution of money holdings. ∎

The next step is to close the buyers' side of the model by finding the equilibrium price of money, ϕ_t, which equates the demand of money with exogenous supply M_t^S.

Proposition 3.3 *There is a unique equilibrium price ϕ_t^* such that $M_t^D = M_t^S$.*

Proof. We first show that money demand at time t, M_t^D, is decreasing in ϕ_t. We can use equation (9) to define money demand at t as a function of ϕ_t, $M_t^D(\phi_t)$. It is easy to check that $\partial \hat{F}_t(m)/\partial \phi_t > 0$. As a result the money distribution that results from a low ϕ_t first order stochastically dominates the one that results from high ϕ_t which proves that $\partial M_t^D/\partial \phi_t < 0$.

To complete the proof we need to show that $M_t^D(\infty) < M_t^S < M_t^D(0)$ for some arbitrary M_t^S. Note that $\lim_{\phi_t \to \infty} \overline{m}_t = 0 \Rightarrow \lim_{\phi_t \to \infty} M_t^D(\phi_t) = 0$. Also, $\lim_{\phi_t \to 0} \overline{m}_t = \infty$ and $\lim_{\phi_t \to 0} \hat{F}_t(m) = 0$, $\forall m < \overline{m}_t$ imply that $\lim_{\phi_t \to 0} M_t^D(\phi_t) = \infty$. ∎

Corollary 3.1 *The price of money is determinate. The buyer-seller ratio, λ_t, uniquely determines the distribution of money holdings of buyers.*

To simplify notation, we redefine all variables in real terms. We express a dollar in terms of its consumption value in the search market. Equation (8) implies that m dollars are worth $z_t = \delta \ \phi_{t+1} \ m$ units of utility at time t, i.e. we convert the m dollars to utility terms at the price of the following Walrasian market (when they can next be used) and then discount that utility to the present search market terms. Together with the stationarity condition $\phi_t = \gamma \ \phi_{t+1}$, this means that the real value of any unspent balances depreciates at rate γ: in period $t + 1$ the m dollars are worth $z_{t+1} = \delta \ \phi_{t+2} \ m = z_t/\gamma$. We denote real transfers by $T_t^k = \delta \ \phi_{t+1} \ \hat{T}_t^k$ and expected real revenues by $\Pi_t = \delta \ \phi_{t+1} \ \hat{\Pi}_t$. Last, we dispense with the time subscript since we are in a stationary environment.

Making the relevant substitutions into our value functions we obtain

$$W^b(z) \ = \ z \ \gamma/\delta + U^* + \max_{z'} \ \{-z' \ \gamma/\delta + \beta \ V^b(z')\} \tag{12}$$

$$V^b(z) \ = \ \delta \ W^b(z/\gamma) + \sum_{n=0}^{\infty} P_n \ [u \ F(z)^n - \int_0^z \tilde{z} \ dF(\tilde{z})^n] \tag{13}$$

$$W^s(z) \ = \ (z + T^s) \ \gamma/\delta + U^* + \beta \ V^s \tag{14}$$

$$V^s \ = \ \max\{\delta \ W^s(\Pi/\gamma) - K, \ \delta \ W^s(0)\}. \tag{15}$$

We can also rewrite the distribution of real balances as

$$F(z) = \frac{1}{\lambda} \ \ln[1 - e^\lambda \ i \ \ln(1 - \frac{z}{u})], \tag{16}$$

where $i = \gamma/(\delta \ \beta) - 1$ and define the distribution of real revenues $G(\cdot)$ accordingly. The highest real money holdings are given by $\overline{z} = u \ (1 - e^{-(1-e^{-\lambda})/i})$.
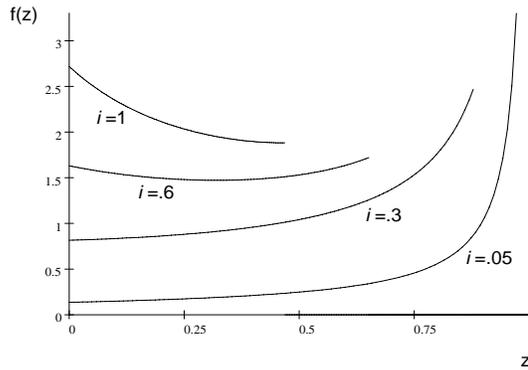
Figure 1: Density of real money balances for different levels of the interest rate i and $u = 1$, $\lambda = 1$.

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Figure 1 shows the density of real money holdings for different levels of the interest rate and $\lambda = 1$. At very high interest rates the density is decreasing. In the intermediate range it is U-shaped. For low interest rate it is increasing.

Note that the distribution of real balances collapses to a mass point at u as the rate of money growth approaches the Friedman rule ($\gamma \to \beta \, \delta$ or, equivalently, $i \to 0$): $F(z) \to 0$ for any $z < \bar{z}$ and $\bar{z} \to u$ as $i \to 0$. This means that at the Friedman rule the real balances of *every* buyer is equal to his valuation for the good. Furthermore, it is easy to show that the mean of the money distribution increases and its variance decreases as the inflation rate falls.

3.2 The Sellers' Problem

Section 3.1 established that the buyers' distribution of real money holdings (and hence expected real revenues) is uniquely determined by the buyer-seller ratio. Therefore, from now on we write $\Pi(\lambda)$. In equilibrium, free entry requires that $\Pi(\lambda) = K$. We show that an equilibrium exists if and only if the inflation rate is below a certain threshold. Furthermore, we prove that if an equilibrium exists then there is a unique λ^* satisfying the free entry condition. The uniqueness of equilibrium is interesting because typically there are multiple stationary equilibria in models with bilateral matching and bargaining. We elaborate on the reasons that lead to this difference at the end of the section.

We begin by characterizing the distribution of real prices.

Proposition 3.4 *The distribution of prices is given by*

$$G(z) = (1 + \lambda - \lambda \, F(z)) \, e^{-\lambda \, (1 - F(z))}, \tag{17}$$

where $F(z)$ is defined in (16).

Proof. Recall that $G(z) = \sum_{n=0}^{\infty} P_n \, F^{(n-1,n)}(z)$ and $F^{(n-1)}(z) = n \, F(z)^{n-1} \, [1 - F(z)] + F(z)^n$. Equation (17) follows. $\blacksquare$

Proposition 3.5 *If an equilibrium exists, then it is unique.*

Proof. For uniqueness, it is sufficient to show that $\partial \Pi(\lambda)/\partial \lambda > 0$. To prove that the expected profits (prices) are strictly increasing in the number of buyers per seller we show that $\partial G(z)/\partial \lambda < 0$ which means that the distribution of prices for high λ first order stochastically

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dominates the one for a low λ. Using equation (16), one can show that

$$
\begin{aligned}
\frac{\partial F(z)}{\partial \lambda} & = -\frac{F(z)}{\lambda} + \frac{-e^{-\lambda} \ i \ \ln(1 - z/u)}{\lambda \ [1 - e^{-\lambda} \ i \ \ln(1 - z/u)]} \\
& = \frac{1}{\lambda} \ [1 - F(z) - e^{-\lambda \ F(z)}].
\end{aligned} \tag{18}
$$

The last step is to note that

$$
\begin{aligned}
\frac{\partial G(z)}{\partial \lambda} & = -\lambda \ (1 - F(z)) \ [1 - F(z) - \lambda \ \frac{\partial F(z)}{\partial \lambda}] \ e^{-\lambda \ (1 - F(z))} \\
& = -\lambda \ (1 - F(z)) \ e^{-\lambda} < 0,
\end{aligned} \tag{19}
$$

where the second equality results from inserting equation (18). This completes the proof. ■

Proposition 3.6 *Given K, an equilibrium exists if and only if $\gamma < \bar{\gamma}(K)$, where $\bar{\gamma}(K)$ is defined by $K = u \ (1 - e^{-1/\bar{i}})$ and $\bar{i} = \bar{\gamma}(K)/(\beta \ \delta) - 1$.*

Proof. It easy to check that $\lim_{\lambda \to 0} \Pi(\lambda) = 0$. Therefore, if $\lim_{\lambda \to \infty} \Pi(\lambda) > K$ the (unique) equilibrium exists. As $\lambda \to \infty$ a seller is visited by some buyer with probability 1. Furthermore, the seller's revenues converge to the highest money holdings, $\bar{z}$. Therefore,

$$
\lim_{\lambda \to \infty} \Pi(\lambda) = \lim_{\lambda \to \infty} \bar{z} = u \ (1 - e^{-1/i}).
$$

Noting that the maximum profits are decreasing in the inflation rate $(\partial \Pi(\infty)/\partial \gamma < 0)$ and that $\Pi(\infty) = K$ if and only if $\gamma = \bar{\gamma}(K)$ completes the proof. ■

Decreasing the inflation rate acts as a mean-preserving spread for the distribution of real prices due to adjustment at the extensive margin. The mean of the distribution (expected revenues) is constant regardless of inflation because of the response of seller entry. The price is zero when a seller has zero or one buyer and this event occurs more frequently when inflation decreases due to higher entry. When two or more buyers show up, the distribution of prices converges to a mass point at u as the rate of money growth approaches the Friedman rule. Thus, the variance of real prices increases as inflation approaches the Friedman rule.

The uniqueness of equilibrium is worth commenting on, especially since Rocheteau and Wright (2005) find that in a model with bilateral matching there are generically multiple equilibria. In both models, increased competition for the search market goods among buyers (higher λ) may reduce the buyers' incentives to bring money since their balances are less likely to be used: in the bilateral matching framework every buyer brings less money in response to an increase of the buyer-seller ratio; in our model, the same thing happens for
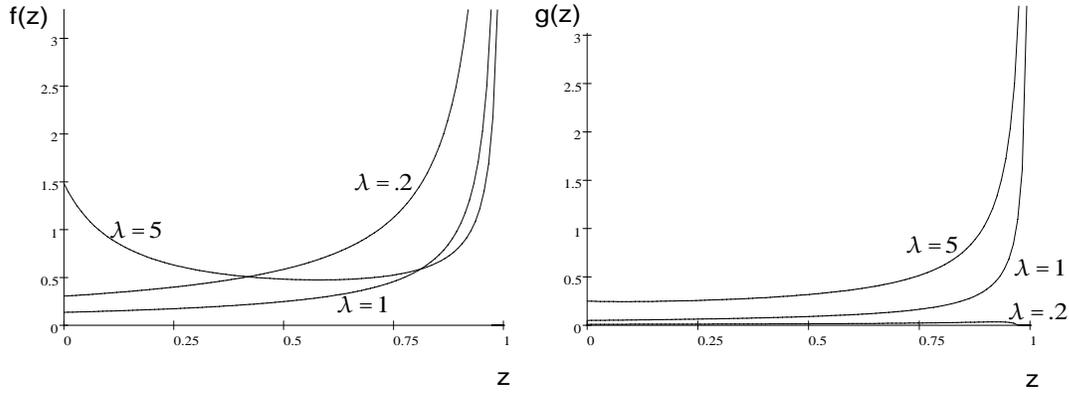
Figure 2: Density of real balances and prices for different levels λ and $u = 1$, $i = 0.05$.

at least some of the buyers, as can be seen in the left graph of figure 2 where buyers at the bottom end of the money distribution choose to hold *less* money as λ increases from 1 to 5. Nevertheless, in our model an increase in λ leads to higher expected revenues for sellers since more buyers visit each seller on average.[14] In a bilateral matching framework, however, a seller is only matched with a single buyer at a time and an increase in λ leads to greater probability of trade but for lower revenues. Whether expected profits (probability of trade times revenues) increase or not depends on parameter values, leading to generic multiplicity of equilibria. It is therefore the multilateral nature of matching that leads to the uniqueness result. Indeed, when Rocheteau and Wright (2005) consider a search market structure based on competitive search they find uniqueness of equilibrium.

3.3   Efficiency and Inflation

In this section we examine the effects of inflation on efficiency. Since every meeting between a seller and some buyers results in the efficient outcome (a trade), the question of interest is whether the efficient number of sellers enter into the market. We show that efficiency is attained only when the inflation rate is at the Friedman rule, i.e. the stock of money decreases at the rate of time preference. Again, this feature is shared with the competitive search market structure of Rocheteau and Wright (2005).

Proposition 3.7 *The level of entry is efficient at the Friedman rule. When $\gamma > \beta \, \delta$ entry is suboptimal.*

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[14]More specifically, a high value of λ leads to a price distribution that first order stochastically dominates the one that results from a low λ, as can be seen in figure 2. The figure omits the mass point at a zero price which results when zero or one buyers show up.

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Proof. We start by solving for the level of entry that maximizes surplus in the search market. A planner chooses λ to maximize the following objective function:

$$\mathcal{W} = \frac{(1 - e^{-\lambda})\, u}{\lambda} - \frac{K}{\lambda}. \tag{20}$$

The first term gives the surplus that is generated from a trade, u, times the number of trades (number of sellers, $1/\lambda$, times the probability a seller trades, $1 - e^{-\lambda}$). The second term gives the entry cost of $1/\lambda$ sellers. Setting the first order conditions with respect to λ to zero yields

$$(1 - e^{-\lambda^P} - \lambda^P\, e^{-\lambda^P})\, u \;=\; K. \tag{21}$$

It is easy to check that the second derivative is negative, hence equation (21) is both necessary and sufficient for optimality.

We now compare λ^P with the equilibrium buyer-seller ratio. First note that an inefficiency arises whenever the inflation rate is high enough to prevent any entry to the search market because (21) means the market should operate (i.e. $\lambda^P < \infty$) whenever $u > K$. When entry is positive, the number of sellers is still suboptimal, except at the Friedman rule. The equilibrium buyer-seller ratio is determined by the free entry condition $\Pi(\lambda) = K$ and therefore efficiency obtains only if expected profits are given by $\Pi(\lambda) = (1 - e^{-\lambda} - \lambda\, e^{-\lambda})\, u$. At the Friedman rule the real balances of all buyers are equal to their valuation of the good, u, and the seller appropriates the full surplus of a match if two or more buyers show up and he receives zero in the complementary case. The probability of the former event is given by $1 - P_0 - P_1 = 1 - e^{-\lambda} - \lambda\, e^{-\lambda}$ and hence the expected revenues of the seller are equal to the left hand side of (21) leading to efficient entry of sellers. When the inflation rate is higher than the Friedman rule, the sellers appropriate a strictly lower share of the surplus and therefore entry is suboptimal. ■

4   The Distributional Effects of Inflation

In this section we re-introduce buyer heterogeneity to examine how the inflation burden is allocated across agents of different productivity. We consider the case of two types of buyers: proportion α_H of buyers produce y_H units of output per hour of work in the Walrasian market and the complementary proportion, α_L, produce y_L where $y_H > y_L$.[15] We do not go over the sellers' part of the model because it is identical to the previous section.

We measure the cost of inflation in *leisure-equivalent* units.[16] Since leisure enters linearly,

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[15]It is not hard to generalize our characterization to a continuous $Y(\cdot)$. See Galenianos and Kircher (2006).

[16]A couple of caveats are in order. Our distributional results are clearly affected by who is the beneficiary

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to determine which type loses more from an increase in inflation we compare the change in the value function of each type as γ varies. Our main result is that low productivity buyers are hurt more from an increase in the rate of money growth if the inflation rate is below a certain threshold. Any further increase of the inflation rate above that threshold hurts the high productivity buyers more. We show that a buyer's per-period utility only depends on the buyer-seller ratio because buyers adjust their real money holdings in response to inflation. Higher inflation leads to lower seller entry which reduces the utility of all buyers. Buyers of higher productivity are better insulated against inflation-induced scarcity since they face lower costs in utility terms of holding fiat money. Holding more money allows them to better deal with increased competition for the search market goods which leads to lower utility losses from inflation. If there are very few sellers in the market (e.g. because the inflation rate is already very high) the low productivity buyers consume very infrequently and hence any further increase in the rate of money growth hurts primarily the high productivity buyers.

From now on a buyer is distinguished by a superscript $j \in \{L, H\}$ denoting his productivity. The value functions of a buyer are given by (corresponding to (12) and (13)):

$$W^j(z) = z\,\gamma/(\delta\,y_j) + U_j^* + \max_{z'} \left[-z'\,\gamma/(\delta\,y_j) + \beta\,V^j(z') \right], \tag{22}$$

$$V^j(z) = \delta\,W^j(z/\gamma) + \sum_{n=0}^{\infty} P_n \left[u\,F(z)^n - \int_0^z \tilde{z}/y_j\,dF(\tilde{z})^n \right]. \tag{23}$$

where $\delta\,W^j(z/\gamma) = z/y_j + \delta\,W^j(0)$, $U_j^* = U(x_j^*) - x_j^*$, and x_j^* is the solution to $U'(x_j^*) = 1/y_j$. Notice that $W^j(z)$ is still linear in z but the slope now depends on y_j.

We take advantage of the stationarity of our environment to express the value of an agent in terms of his period-by-period decentralized market value. This allows us to do explicit comparative static exercises with respect to the inflation rate. Letting $P(z)$ be the probability that a buyer holding z dollars purchases the search market good and $Q(z)$ be the expected price that he pays, we have $V^j(z) = u\,P(z) + \delta\,W^j[(z - Q(z))/\gamma]$ for $j \in \{L, H\}$. Entering this expression into equation (22) gives

$$W^j(z) = z\,\gamma/(\delta\,y_j) + U_j^* - z'\,\gamma/\delta + \beta\,\{u\,P(z') + \delta\,W^j[(z' - Q(z'))/\gamma]\}, \tag{24}$$

where z' is an optimal solution to the money holding problem. Recalling that $\delta\,W^j(z/\gamma) =$

$z/y_j + \delta\, W^j(0)$ and $i = \gamma/(\beta\,\delta) - 1$, it is straightforward to re-arrange (24) as

$$(1 - \beta\,\delta)\, W^j(0) \;=\; U^*_j + \beta\, X_j, \tag{25}$$

$$\text{where } X_j \;\equiv\; u\, P(z) - \frac{1}{y_j}\,(z\, i + Q(z)). \tag{26}$$

Equation (25) describes the per-period utility of a type-j buyer. The first term reflects the utility arising from Walrasian market consumption. Equation (26) gives the expected utility of search market consumption net of cost (price as well as liquidity cost due to the depreciation of the value of money).

In the next proposition we characterize the equilibrium distribution of money holdings with heterogeneous buyers. Let $F_j(\cdot)$ denote the distribution of real balances and $Z_j \equiv suppF_j(\cdot) \subset [\underline{z}_j, \overline{z}_j]$ denote the support of that distribution for each type j. The aggregate distribution of money balances is given by $F(z) = \alpha_H\, F_H(z) + \alpha_L\, F_L(z)$ where $suppF \subset [0, \overline{z}]$. Similarly, define the ratio of type-j buyers to sellers by $\lambda_j \equiv \alpha_j\, \lambda$.

Proposition 4.1 *In equilibrium $\overline{z}_L = \underline{z}_H$, $suppF_L(\cdot) = [0, \overline{z}_L]$ and $suppF_H(\cdot) = [\overline{z}_L, \overline{z}_H]$. The equilibrium distributions of real money holdings are given by*

$$F_L(z) \;=\; \frac{1}{\lambda_L}\,\log[1 - e^{\lambda_L}\, i\, \ln(1 - \frac{z}{y_L\, u})], \quad \forall\, z \in Z_L = [0, \overline{z}_L], \tag{27}$$

$$F_H(z) \;=\; \frac{1}{\lambda_H}\,\log[1 - e^{\lambda_H}\, i\, \ln(\frac{y_H\, u - z}{y_H\, u - \overline{z}_L})] \quad \forall\, z \in Z_H = [\overline{z}_L, \overline{z}_H], \tag{28}$$

where $\overline{z}_L = y_L\, u\, (1 - e^{-e^{-\lambda_H}(1 - e^{-\lambda_L})/i})$ and $\overline{z}_H = y_H\, u\, (1 - e^{-(1 - e^{-\lambda_H})/i}) + \overline{z}_L\, e^{-(1 - e^{-\lambda_H})/i}$.

Proof. Define $\hat{W}^j(z) \equiv y_j\, W^j(z)$ and $\hat{V}^j(z) \equiv y_j\, V^j(z)$ to obtain the system of equations

$$\hat{W}^j(z) \;=\; z\,\gamma/\delta + y_j\, U^*_j + \max_{z'}\,[-z'\,\gamma/\delta + \beta\, \hat{V}^j(z')]$$

$$\hat{V}^j(z) \;=\; z + \delta\, \hat{W}^j(0) + \sum_{n=0}^{\infty} P_n\, [y_j\, u\, F(z)^n - \int_0^z \tilde{z}\, dF(\tilde{z})^n].$$

This is identical to an economy where the valuations in the search market are given by $\hat{u}_j = (y_j\, u)$, except for Walrasian market consumption $(y_j\, U^*_j)$ which, however, does not affect any monetary decisions.

An argument similar to proposition 3.1 shows that $suppF = [0, \overline{z}]$ and $F(\cdot)$ is non-atomic on $[0, z^*_L) \cup (z^*_L, z^*_H)$. Buyer optimization implies that $\hat{V}^{j\prime}(z) = \gamma/(\beta\,\delta)$ when $z \in Z_j$ and (as in (10)) the derivative of $\hat{V}^j(\cdot)$ on the interior of Z_j is given by:

$$\hat{V}^{j\prime}(z) \;=\; 1 + (\hat{u}_j - z)\, \lambda\, F'(z)\, e^{-\lambda\,(1 - F(z))}, \quad \text{for } z \in \text{int}(Z_j).$$

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It is immediate that $\hat{V}^{H\prime}(z) > \hat{V}^{L\prime}(z)$ which implies that every low type buyer holds less money that any high type buyer and hence $Z_L = [0, \bar{z}_L]$ and $Z_H = [\bar{z}_L, \bar{z}]$.

The search market value functions for the two types of agents are given by

$$\hat{V}^L(z) = \delta \hat{W}^L(z) + e^{-\lambda_H} \sum_{n=0}^{\infty} P_n^L \left[u_L \ F_L(z)^n - \int_0^z \tilde{z} \ dF_L(\tilde{z})^n \right]$$

$$\hat{V}^H(z) = \delta \hat{W}^H(z) + e^{-\lambda_H} \sum_{n=0}^{\infty} P_n^L \left[u_H - \int_0^{\bar{z}_L} \tilde{z} dF_L(\tilde{z})^n \right] + \sum_{n=1}^{\infty} P_n^H \left[u_H F_H(z)^n - \int_0^z \tilde{z} dF_H(\tilde{z})^n \right]$$

where P_n^j is the probability that n buyers of type j visit the same seller. To get (27) and (28) we replicate the steps of proposition 3.2 noting that the initial conditions and upper bounds for the two distributions are given by $F_L(0) = 0$, $F_L(\bar{z}_L) = 1$, $F_H(\bar{z}_L) = 0$ and $F_H(\bar{z}_H) = 1$. ∎

The following proposition states and proves our distributional results. Note that so long as the entry cost K is not too high then the buyer-seller ratio at the Friedman rule is above the threshold derived in the proposition.

Proposition 4.2 *Per-period utility declines with inflation for all buyers. It decreases more for high type buyers than for low type buyers if and only if the buyer-seller ratio is below a threshold.*

Proof. The fact that low productivity buyers are indifferent between holding 0 or $\bar{z}_L$ dollars implies $X_L = u \ P(0) = u \ P(\bar{z}_L) - [\bar{z}_L \ i + Q(\bar{z}_L)]/y_L$. Rearranging yields

$$\bar{z}_L \ i + Q(\bar{z}_L) = y_L \ u \ [P(0) - P(\bar{z}_L)] \tag{29}$$

Since both L- and H-buyers are willing to hold $\bar{z}_L$ dollars we can insert (29) into (26) and evaluate at $z = \bar{z}_L$ for both types:

$$X_L = u \ P(\bar{z}_L) - u \ [P(\bar{z}_L) - P(0)] \tag{30}$$

$$X_H = u \ P(\bar{z}_L) - u \ [P(\bar{z}_L) - P(0)] \frac{y_L}{y_H} \tag{31}$$

Equations (30) and (31) are useful because they incorporate the equilibrium decisions of money holdings in a succinct way.

Note that $P(0) = e^{-\lambda}$ and $P(\bar{z}_L) = e^{-\alpha_H \lambda}$ since a buyer with no money can only purchase if no one else visits the same seller, while a buyer with $\bar{z}_L$ dollars can outbid all L-buyers but is outbid by all H-buyers. As a result, *the buyers' period utility is a function of the buyer-seller ratio alone* which means that a higher inflation rate entails no utility loss for

buyers if the number of sellers is fixed. Since sellers have a participation decision, higher inflation implies that there are fewer sellers in the market and hence a higher λ. In this case, all buyers are worse off because the probability of consuming decreases regardless of type or money holdings.

To compare which type is hurt more by inflation we take the derivative of per-period utility with respect to λ:

$$\frac{dX_L}{d\lambda} = -u \ e^{-\lambda} \quad \text{and} \quad \frac{dX_H}{d\lambda} = -u \ [e^{-\lambda} \ \frac{y_L}{y_H} + e^{-\alpha_H \lambda} \ (1 - \frac{y_L}{y_H}) \ \alpha_H].$$

The last step is to compare

$$\frac{dX_L}{d\lambda} < \frac{dX_H}{d\lambda} \Leftrightarrow 1 > \alpha_H \ e^{(1-\alpha_H) \ \lambda} \equiv T(\lambda). \tag{32}$$

Clearly, $T(0) = \alpha_H < 1$, $T(\infty) = \infty > 1$, $T'(\lambda) > 0$ which means that there is a unique $\lambda(\alpha_H)$ such that the two derivatives are equal. When $\lambda < \lambda(\alpha_H)$, the low-productivity buyers are hurt more from inflation than high-productivity buyers and vice versa. Also, $\lim_{\alpha_H \to 0} \lambda(\alpha_H) = \infty$ and condition (32) always holds when $\lambda < 1$ since $\alpha_H \ e^{(1-\alpha_H)} \leq 1$ for all $\alpha_h \in [0, 1]$. ∎

5   Conclusions

We develop a monetary model that can be used to examine the effects of inflation across heterogeneous agents by accommodating the presence of private information in a tractable way. Applying our framework to an environment with two productivity types we show that inflation is a regressive form of taxation, at least for moderate levels of money growth.

We model decentralized exchange in a different way from most of the literature. Agents match multilaterally, allowing auctions to be used as a way of allocating goods leading to the accommodation of private information. We use second-price auctions for convenience. We conjecture that our qualitative results obtain when sellers use any standard auction. The possibility of a mass point in the money distribution can be disproved in much the same way as in this paper. The fact that there are no liquidity costs at the Friedman rule means that revenue equivalence across auctions holds leading to efficient entry regardless of which auction is used. In fact, if sellers can post any direct mechanism in order to compete for buyers, at the Friedman rule they would (in equilibrium) all post a second price auction without reserve (see McAfee (1993) or Peters (1997)). Regardless of the auction type, higher inflation results in positive liquidity costs which leads agents to lower their money holdings

giving rise to suboptimal entry. Our results on the distributional effects of inflation do not substantially depend on the precise pricing mechanism, which leads us to believe that they will not be affected by changing the auction format.

An important assumption of our model is the indivisibility of goods (multiple indivisible goods per seller can be accommodated by multi-unit auctions as in Demange, Gale and Sotomayor (1986), for which we expect qualitatively similar results). The underlying idea is that for many products, especially large ticket items, the non-convexities that are present in consumption lead to indivisibilities. One way to introduce an intensive margin is to make the quality of the good an endogenous choice for the seller at the match and allow buyers to offer price-quality bids, competing in terms of the surplus that they leave to the seller. To the extent that the highest bidder is awarded the good (of now endogenous quality), we believe that our principal results will not be substantially affected. An alternative is suggested by Dutu, Julien and King (2007) who let sellers choose and advertise their quality level in a directed search framework and then run a second price auction with the buyers who visit them. A different route is to introduce lotteries. This would be a useful exercise since the trade-offs are quite different than in the current model. However, lotteries are not well-studied in the context of auctions and hence we leave this topic to future research.

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