Monty Hall drives a wedge between Judy Benjamin and the Sleeping Beauty: a reply to Bovens

Monty Hall drives a wedge between Judy Benjamin and the Sleeping Beauty: a reply to Bovens

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(forthcoming in *Analysis*)

Bovens (2010) points out that there is a structural analogy between the Judy Benjamin problem (JB) and the Sleeping Beauty problem (SB). On grounds of this structural analogy, he argues that both should receive the same solution, viz. the posterior probability of the eastern region of the matrix in Table 1 should equal 1/3. Hence, $P^*(Red) = 1/3$ in the JB and $P^*(Heads) = 1/3$ in the SB.

	B *Ta*	*R* *He*
Q *Mo*		
S *Tu*		

Table 1

Bovens' argument rests on a standard error in implementing Bayesian updating, which is spelled out in Shafer (1985). When we are informed of some proposition, we do not only learn the proposition in question, but also that we have learned the proposition as one of the many propositions that we might have learned. The information is generated by a *protocol*, which determines the various propositions that we might learn. We should then update not on the proposition in question, but rather

on the fact that we learned this proposition as one of the many propositions that we might have learned.

A well-known application of this insight is the Monty Hall problem (MH) as Speed (1985: 276) points out in a discussion of Shafer (1985). As an illustration, let us apply Shafer's insight to the MH. In the MH, the contestant in a game show learns that there is a goat behind two of three doors *X*, *Y* and *Z* and a car behind one door. She is asked to pick one of the three doors. The contestant picks door *X*. Monty will then open one of the remaining doors, which he knows to have a goat behind it. Suppose Monty opens door *Y*. The contestant is then asked whether she wants to stick to the door she originally chose, i.e. door *X*, or whether she wants to switch to the other unopened door, i.e. door *Z*. Should the contestant switch doors, assuming that she wants to win the car rather than a goat?

If we naively update only on the content of the information, viz. that there is a goat behind door *Y*, then we reach the following conclusion:

$$(1) \qquad P(CarX \mid GoatY) = \frac{P(GoatY \mid CarX)P(CarX)}{P(GoatY)} = \frac{1 \times 1/3}{2/3} = 1/2$$

Hence $P(CarZ \mid GoatY) = 1/2$ as well and so, on this reasoning, it does not make any difference whether she does or does not switch doors. But, as is well-known, this reasoning is incorrect. We do not only learn that there is a goat behind door *Y*, we also learn that we learn this information as one of a range of possible items of information that Monty might have provided. If the protocol specifies that Monty *will* open one of the remaining doors with a goat behind it, then there are two items of

information that the contestant might receive, viz. "A goat is behind door *Y*" and the "A goat is behind door *Z*". Let us also specify that, as far as the contestant knows, Monty will randomise between doors *Y* and *Z* if both have goats behind them. We can now construct a table with conditional probabilities. Let "*INF*" be the variable that specifies the information provided by Monty and let "@" be the variable that specifies the actual location of the car. We construct the conditional probability table in Table 2.

$P(INF \mid @)$		@ =		
		CarX	*CarY*	*CarZ*
INF =	*GoatY*	1/2	0	1
	GoatZ	1/2	1	0

Table 2

In addition, the contestant has no reason to think one door more likely than another prior to Monty's information, i.e. $P(@ = CarX) = P(@ = CarY) = P(@ = CarZ) = 1/3$. We can now calculate:

(2) $P(@ = CarX \mid INF = GoatY) =$

$$\frac{P(INF = GoatY \mid @ = CarX)P(CarX)}{P(INF = GoatY)} = \frac{1/2 \times 1/3}{1/3(1/2 + 0 + 1)} = 1/3$$

So *P(@=CarZ | INF =GoatY)* = 2/3 and hence the contestant should switch doors.

Are there protocols on which the contestant has no reason to switch? Well, suppose that Monty just opens one of the remaining doors at random—it may or may not have the car behind it. On this protocol, the contestant may expect four possible items of information. We construct the conditional probability table in Table 3:

P(INF | @)		*@ =*		
		CarX	*CarY*	*CarZ*
INF =	*CarY*	0	1/2	0
	GoatY	1/2	0	1/2
	CarZ	0	0	1/2
	GoatZ	1/2	1/2	0

Table 3

We calculate:

(3) $P(@ = CarX \mid INF = GoatY) =$

$$\frac{P(INF = GoatY \mid @ = CarX)P(CarX)}{P(INF = GoatY)} = \frac{1/2 \times 1/3}{1/3(1/2 + 0 + 1/2)} = 1/2$$

Hence, on this protocol, it does not pay to switch doors. So, the moral of the MH is that the protocol is all important. Let us now investigate whether we can gain some mileage from this insight for the SB and the JB.

We structure the SB so that we can invoke the mechanism of protocols. Let *the structure of the game* be the proposition that awakenings can occur in all four world-time quadrants (*Ta-Mo, …*), except for *He-Tu*. In the original SB, Beauty learns this information on Su and retains it throughout. In Bovens' SB` (2010), Beauty learns on Su that all world-time quadrants are possible and is then told the complete structure of the game upon awakening, i.e. she is told that *He-Tu* is actually ruled out. Beauty also knows that amnesia of this additional information is induced when she is put back to sleep. This variation, he argues, cannot make for a difference to the solution of the SB.

To bring in protocols, let us parse the structure of the game as follows. Suppose that on Su, Beauty is informed that one and only one world-time quadrant is impossible, but she is not told which one, and that she will be told upon awakening which world-time quadrant is impossible. Subsequently, when she awakens, she is informed that it is *He-Tu* that is impossible. She retains the information that she received on Su, but amnesia of the information provided upon awakening is induced. Let us call this the SB``. This variation cannot make a difference to the solution either. The information that Beauty has at her disposal is simply parsed up differently in the SB, in the SB` and in the SB``. After being informed upon awakenings in both the SB` and the SB``, she knows exactly the same in the SB.

Just like in the MH, we can now construct a conditional probability table representing Beauty's credences upon awakening in Table 4. @ is the variable that takes as its values the particular world-time quadrant that Beauty is in upon awakening:

P(INF | @)		*@ =*			
		Ta-Mo	*Ta-Tu*	*He-Mo*	*He-Tu*
INF =	¬*Ta-Mo*	0	1/3	1/3	1/3
	¬*Ta-Tu*	1/3	0	1/3	1/3
	¬*He-Mu*	1/3	1/3	0	1/3
	¬*He-Tu*	1/3	1/3	1/3	0

Table 4

We calculate:

(4) $P^*(He) = P(@=He\text{-}Mo \mid INF = \neg He\text{-}Tu) =$

$$\frac{P(INF = \neg He - Tu \mid @ = He - Mo)P(@ = He - Mo)}{P(INF = \neg He - Tu)} =$$

$$\frac{1/3 \times 1/4}{1/4(1/3 + 0 + 1/3 + 1/3)} = 1/3$$

So when Beauty is informed of the full structure of the game in the SB``, she will update her credence for *He* to 1/3. How the information is parsed does not make any

difference to Beauty's credence. Hence, 1/3 should also be Beauty's posterior credence for *Heads* in the SB.

Let us now turn to the JB. What protocol might yield the information $R \rightarrow \neg S$? Consider the following protocol. The informer assesses the region R and only the region R in order to exclude one area—i.e. to indicate an area where Judy cannot possibly be. E.g. she may have access to technology that permits her to provide exactly one quadrant that yields a true negative for the presence of Judy. If this is the case, then the protocol may yield two items of information, viz. $INF = R \rightarrow \neg S$ or $INF = R \rightarrow \neg Q$. Let @ be the variable which takes the actual location of SB as its values, so @ = *BQ, BS, RQ,* or *RS*. We spell out the conditional probabilities in Table 5

$P(INF \mid @)$		@=			
		BQ	*BS*	*RQ*	*RS*
$INF =$	$R \rightarrow \neg S$	1/2	1/2	1	0
	$R \rightarrow \neg Q$	1/2	1/2	0	1

Table 5

We calculate:

(5) $P^*(R) = P(@ = RQ \mid INF = R \rightarrow \neg S) =$

$$\frac{P(INF = R \rightarrow \neg S \mid @ = RQ)P(@ = RQ)}{P(INF = R \rightarrow \neg S)} = \frac{1 \times 1/4}{1/4(1/2 + 1/2 + 0 + 1)} = 1/2$$

So if we take into account the protocol, we see that, although the SB and the JB have *structural* similarities, careful attention to the *protocol* teaches us that $P^*(He) = 1/3$ whereas $P^*(R) = 1/2$.

One might ask, Is it possible to spell out alternative protocols so that $P^*(He) = 1/2$ and $P^*(R) = 1/3$? We can do so and, indeed, it will be instructive to evaluate such protocols.

Let us start with the SB. Consider the following protocol. Suppose that Beauty is put to sleep on Sunday and suppose she is told that a fair coin has been flipped, that there will be awakenings on both Monday and Tuesday independently of the coin flip, and that amnesia will be induced after an awakening. Furthermore, when she awakens someone will inform her of one time quadrant *in a Heads world* in which she is *not* located. Further, suppose that she is being informed that she is not in *He-Tu*. Then the protocol precisely mirrors the JB protocol in Table 5.

P(INF | @)		@=			
		Ta-Mo	*Ta-Tu*	*He-Mo*	*He-tu*
INF =	¬*He-Mo*	1/2	1/2	1	0
	¬*He-Tu*	1/2	1/2	0	1

Table 6

We can calculate that $P^*(He) = 1/2$. But this is far away from the original SB. This protocol captures a game that has a radically different structure than the original SB.

The situation is less univocal in the JB. Let us first consider two protocols on which $P^*(R) = 1/3$. Suppose that the informer examines all quadrants and will provide JB with one true negative, i.e. one quadrant in which Judy is not. We spelled out the conditional probability table for this protocol in Table 7. Or suppose that the informer examines *RS* and is able to provide her with either a true positive or a true negative. We spelled out the conditional probability table for this protocol in Table 8. In each case, Judy does receive the information that $\neg RS$.

$P(INF \mid @)$		$@ =$			
		BQ	*BS*	*RQ*	*RS*
INF =	$\neg BQ$	0	1/3	1/3	1/3
	$\neg BS$	1/3	0	1/3	1/3
	$\neg RQ$	1/3	1/3	0	1/3
	$\neg RS$	1/3	1/3	1/3	0

Table 7

P(INF | *@)*		*@=*			
		BQ	*BS*	*RQ*	*RS*
INF =	*RS*	0	0	0	1
	¬*RS*	1	1	1	0

Table 8

Table 7 is structurally analogous to Table 4 and so $P^*(R) = 1/3$. For Table 8, we calculate

(6) $P^*(R) = P(@ = RQ \mid INF = \neg RS) =$

$$\frac{P(INF = \neg RS \mid @ = RQ)P(@ = RQ)}{P(INF = \neg RS)} = \frac{1 \times 1/4}{1/4(1+1+1+0)} = 1/3$$

So there do exist protocols on which $P^*(R) = 1/3$. But how plausible are these protocols, given the original formulation of the JB? In the original JB puzzle, Judy receives information of the form 'If you are in *R*, then the probability that that you are in *Q* is *p*'. (vanFraassen, 1982: 366-7) A limiting case of this information is 'If you are in R, then the probability that you are in *Q* is 1,' or, in other words, 'If you are in *R*, then you are not in *S*.' Now, the choice of the conditional as a mode of expressing the information carries a conversational implicature that one has checked *R*, rather than the whole area (as in the protocol underlying Table 7) or rather than *R*S (as in the

protocol underlying Table 8). For this reason, the protocol underlying Table 5 is more in line with the choice of the conditional in the informer's message.

Is there no protocol that would warrant the use of the conditional and that would yield $P^*(R) = 1/3$? Imagine the following scenario. Suppose that the informer is intent on checking the area *R*. However, due to heavy cloud cover, he can get no information about *RQ*. He is able to report a false positive or a false negative about *RS*. If this is the protocol then we are back to Table 8 and indeed $P^*(R) = 1/3$. Now, arguably, it might not be unnatural for the informer to say 'If you are in *R*, then you are not in *S*' in such a case, since he was indeed intent on checking R and this comes through in the choice of the conditional in his message.

But, as we know from everyday life, much misunderstanding is due to misreading conversational implicatures. So what is Judy supposed to do? In the absence of a clear protocol, the problem is simply underdetermined. Judy may have a subjective probability distribution over alternative protocols – some yielding $P^*(R) = 1/3$ and some yielding $P^*(R) = 1/2$. If this is so, then, given her credences, she will need to calculate a weighted average and $P^*(R)$ will take on a determinate value in the range [1/3, 1/2]. Or she may face radical uncertainty with respect to the protocols. In this case, there is no more to be said than that the problem is underdetermined and that $P^*(R)$ has upper bound 1/2 and lower bound 1/3. And in the face of limited uncertainty with respect to protocols, Judy can determine a more narrow range of values within [1/3, 1/2].

What makes an appeal to protocols so inviting is that it provides us not only with a correct treatment of the SB and the JB, but also with an error theory of all the confusion in this area. Simply recall the early confusion around the MH. The MH is a case in which the relevance of the protocol is straightforward and still, the erroneous solution of $P^*(CarX) = 1/2$ in the actual MH was much argued for due to the complete disregard for protocols. What underlies all the confusion about the SB and the JB is the same disregard for protocols.

Bovens (2010) has the virtue of recognising a certain similarity between the SB and the JB. But it fails to recognise the dissimilarity between underlying protocols. Protocols are expressed in conditional probability tables that spell out the probability of coming to learn various propositions conditional on the actual state of the world. The principle of total evidence requires that we not update on the content of the proposition learned but rather on the fact that we learn the proposition in question. Now attention to protocols drives a wedge between the SB and the JB. We have shown that the solution to a close variant of the SB which involves a clear protocol is $P^*(He) = 1/3$ and since Beauty's has precisely the same information at her disposal in the original SB at the time that she is asked to state her credence for *Heads*, the same solution should hold. The solution to the JB, on the other hand, is dependent on Judy's probability distribution over protocols. One reasonable protocol yields $P(R) = 1/2$, but Judy could also defend alternative values or a range of values in the interval [1/3, 1/2] depending on her probability distribution over protocols.

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[1] **Analysis Advance Access published online on September 28, 2009**
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