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Structural Properties of Network Revenue Management models: an economic perspective

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Abstract

Many revenue management problems have a network aspect. In this paper, we argue that a network can be thought of as a system of substitutable and complementary products, and the value of a revenue management model should be supermodular or submodular in the availability of two resources as the resources are economic substitutes or complements. We demonstrate that this is true in the case of a two-resource dynamic stochastic revenue management model, and show how this applies for multi-resource deterministic static revenue management models.

1 Introduction

Revenue management (RM) is a subject which has attracted increasing attention from academics and business people over the last fifteen years or so. The central RM problem arises when a seller faces a stream of requests for a fixed resource from a segmented customer base, and where these requests can be accepted or rejected. The resource may be a network resource, for example, a network of flights.

A number of papers in the recent theoretic literature have focussed on exploring dynamic network models. While these models are intractable for problems of realistic size, it is hoped that by better understanding the behaviour of these models, the research community will be able to design new and better RM tools, using the coming technologies of dynamic stochastic optimization. Such technologies often depend critically on structural properties of the model in question. It is precisely such properties which we seek to elucidate in the present paper.

Our key insight is that the relationship between the flight legs in an airline network will often be one of economic complementarity or substitutability. One would therefore expect that the value of a dynamic stochastic model will be either supermodular or submodular in the capacities of these two flights. We show that this is indeed so for the two leg-case. Although we do not have a characterisation for the general n-leg dynamic stochastic case, we show that the value of a deterministic static RM model on a single hub network is supermodular in the capacity of unlike flights, and submodular in the capacity of like flights. We also show that a plausible conjecture for a three leg network is false in the deterministic case, and thus in the dynamic case also.

2 Literature Review

The literature on RM has burgeoned over the last fifteen years. Accordingly, we take as our starting point the special issue of Transportation Science of May, 1999. The reader interested in earlier literature is referred to the literature review of McGill and van Ryzin [29] and to the omnibus model of Lautenbacher and Stidham [27] (but also see Li and Oum [28]), in that special issue.

Recent papers include Secomandi et al. [31] and Boyd and Bilegan [3], who give fascinating overviews of some of the current issues in practice. Also worth mentioning in this connection is de Boer et al. [10], who provide an insightful discussion on the characteristics of some mathematical programming-based approaches to RM.

Other recent notable papers extend the classical single-leg dynamic model by incorporating, for example, passenger diversion between fare classes [42], or nonhomogeneous [41] or semi-Markov [4] stochastic processes. Related streams of research have been Chatwin's work on overbooking [5,6,7], the work of Feng and collaborators [12,13,14,15,16] which deals predominantly with RM in retail, and work of Kleywegt and Papastavrou [24,25], who study a generalisation of the RM and other problems, dubbed "the dynamic and stochastic knapsack problem".

A number of papers consider dynamic network RM [19,35,38]. The main result from this literature is that dynamic network models converge to deterministic models under a fluid scaling regime. Cooper [9] gives a useful presentation of the underlying convergence mechanism. Other authors [2,8] seek to develop algorithms for approximating the dynamic programming value function. Bertsimas and Popescu [2] comment on the desirability, and apparent unavailability, of monotonicity properties for this purpose.

The closest precursor of the present work is by You [40]. You identifies that, in the case of a two-leg flight network, the value function exhibits a monotonicity property which we shall call antidirectional supermodularity. Our contribution may be seen as extending and drawing the mathematical and conceptual context of You's result.

Outside of RM, we have found some other literatures insightful.

- One of these is the literature on transition monotonicity in queuing systems [23,39,34,26]. While the application context is very different, the structural properties under study are quite similar.
- Another literature explores the structural properties of (deterministic) constrained optimization systems [e.g. 17,22,43]. We shall draw on this literature in section 5 to show properties of deterministic RM models.
- Both these literatures draw on ideas from lattice programming [36,37]. While we have not found existing lattice programming results directly useful in the current context, the methods and philosophy of lattice programming has had considerable indirect influence on the current work.

The contributions we aim to make in this paper, then, are as follows:

- intuitive insight into plausible structural properties of RM systems

- the identification of a structural property, antidirectional supermodularity, which has not been systematically studied
- a (we hope) relatively easy and intuitive proof of the structural properties of dynamic network RM models with two substitutable and complementary flight legs respectively (the latter being the essence of You's result)
- a demonstration of the structural properties of a deterministic RM model with n flight legs on a single hub network structure
- a counterexample to a plausible hypothesis about a deterministic RM model with three flight legs (which also is a counterexample to the corresponding hypothesis about a dynamic RM problem with three flight legs)

3 Complementarity, substitutability and associated structural properties

The idea of two goods being economic complements or substitutes is such a natural one that is hard to say exactly what the defining characteristic of complementarity in fact is. There has historically been considerable discussion in economics about how best to understand complementarity, and it has been demonstrated that the intuitive idea is not free from contradictions [30].

Here we will merely content ourselves by noting two key aspects of the intuitive idea of complementarity. One is what Samuelson [30] calls the "either-or" nature of substitutability, and the "and-both" nature of complementarity: market demand may exist for tea or coffee (and so tea and coffee are substitutes) but tea and sugar (in which case tea and sugar are complements). The other aspect (at least in the case of revenue earning systems) is the notion of submodularity (supermodularity) of value: the value of tea and coffee (sugar) is less (more) than the value of tea by itself plus the value of coffee (sugar) by itself. The current work can be seen as an exploration of the idea that the first notion of complementarity implies the second.

A particular theme of the current work is that revenue complementarity and substitutability seem to be closely bound up with certain strong concavity properties. The close relationship between complementarity and some sort of a strong concavity principle was pointed out by Nicholas Georgescu-Roegen in the context of mathematical economics some time ago [20].

All of this provides us with a starting point for our discussion of structural properties. The function $f(\cdot)$ in the following discussion is understood to be a real-valued function defined on an n-dimensional integral domain $\mathbf{Z}^{\mathrm{n}}$, with e_i and e_j denoting unit vectors in dimensions i and j. We shall use the notation $\delta_j f(x)$ to refer to $f(x) - f(x - e_j)$; $\delta_{i+j} f(x)$ to refer to $f(x) - f(x - e_i - e_j)$; and $\delta_{i-j} f(x)$ to refer to $f(x) - f(x - e_i + e_j)$.

Definition 1

Consider the expression

$$\begin{aligned} \delta_i \delta_j f(x) &= \delta_j f(x) - \delta_j f(x - e_i) = \delta_i f(x) - \delta_i f(x - e_j) \\ &= f(x) - f(x - e_i) - f(x - e_j) + f(x - e_i - e_j) \end{aligned}$$

If, for all $x \in \mathbf{Z}^n$

$\delta_i \delta_j f(x) \leq (\geq) 0$ for distinct i,j, then $f(\cdot)$ is said to be *submodular (supermodular)* in i and j

$\delta_i \delta_i f(x) \leq (\geq) 0$ for some i, then $f(\cdot)$ is said to be *componentwise concave (convex)* in i

$\delta_i \delta_j f(x) \leq (\geq) 0$ for distinct i,j and $\delta_i \delta_i f(x) \leq (\geq) 0$ and $\delta_j \delta_j f(x) \leq (\geq) 0$ then $f(\cdot)$ is said to be *directionally concave (convex)* in i and j

All but the last are in common usage. The last is due to Shaked and Shantikumar [32].

It will be seen in the ensuing that there is an interesting parallelism between revenue earning systems (such as RM systems) and cost incurring systems (such as queueing systems) in this regard. Whereas revenue complementarity (substitutability) is associated with supermodularity (submodularity) and strong concavity, cost complementarity (substitutability) is associated with submodularity (supermodularity) and strong convexity in very much the same ways.

It is worth noting that we use the terms "submodularity" and "supermodularity" to mean what Topkis [36,37] means by decreasing (or antitone) and increasing (or isotone) differences, whereas Topkis defines sub- and supermodularity in terms of lattice meet and join. Our use of this alternative definition is justified by a key result of Topkis [36], which states that in the context of very general structures, these two properties are equivalent.

We will need a further definition in the same spirit as the above, this time our own.

Definition 2

If $\delta_i\delta_j f(x) \leq (\geq) 0$ and $\delta_i\delta_i f(x) \geq (\leq) 0$ and $\delta_j\delta_j f(x) \geq (\leq) 0$ for distinct i,j then we shall say $f(\cdot)$ is *antidirectionally convex (concave).*

Definition 3

Consider the expressions

$$\begin{aligned}\delta_{i-j}\delta_i f(x) &= \delta_i f(x) - \delta_i f(x - e_i + e_j) = \delta_{i-j} f(x) - \delta_{i-j} f(x - e_i) \\ &= f(x) - f(x - e_i + e_j) - f(x - e_i) + f(x - 2e_i + e_j)\end{aligned}$$

$$\begin{aligned}\delta_{i+j}\delta_i f(x) &= \delta_i f(x) - \delta_i f(x - e_i - e_j) = \delta_{i+j} f(x) - \delta_{i+j} f(x - e_i) \\ &= f(x) - f(x - e_i - e_j) - f(x - e_i) + f(x - 2e_i - e_j)\end{aligned}$$

If, for all $x \in \mathbf{Z}^{\mathrm{n}}$

$\delta_{i-j}\delta_i f(x) \leq(\geq) 0$ and $\delta_{j-i}\delta_j f(x) \leq(\geq) 0$ for distinct i,j, $f(\cdot)$ is said to be *subconcave (superconvex)* in i and j

$\delta_{i+j}\delta_i f(x) \leq(\geq) 0$ and $\delta_{i+j}\delta_j f(x) \leq(\geq) 0$ for distinct i,j, $f(\cdot)$ is said to be *superconcave (subconvex)* in i and j

This terminology is inspired by that of Koole [26] (who defines sub- and super-convexity, but not -concavity).

The reader will note that so far, all properties have been defined in terms of a pair of specified dimensions. If a function is submodular (supermodular, superconcave, etc) in all pairs of dimensions of its domain, we will simply say that it is submodular (supermodular, superconcave, etc). In the case of submodularity and supermodularity, Theorem 3.2 of Topkis [36] and Corollary 2.6.1 from Topkis [37] justifies that this usage is consistent with the more abstract lattice theoretic definition of sub- and super-modularity in the context of the domains and functions we will be concerned with in this paper.

A discussion in terms of an airline RM problem may be helpful for concreteness. Suppose that $f(\cdot)$ is a value function defined over a domain which represents different levels of resource (i.e. seat) availability on various different flights legs. In this application context, the δ terms represent the values of particular state transitions which correspond to acceptance of a request for a seat on the i flight leg (a "local request"), acceptance of a request for a seat on both i and j flight leg (a "through request") and acceptance of a request to switch a passenger from the j flight leg to the i flight leg (a "switch request"). Accordingly the sub-(super-)modularity and sub-(super-)concavity (convexity) properties can be taken to describe particular events (or alternatively, particular control decisions) being monotone in other RM events or control decisions. A similar interpretation in terms of queueing systems is possible.

The properties can be helpfully visualised graphically. In Figure 1, we show the differences on a fragment of the domain. The vertices of the figure are points in $\mathbf{Z}^2$ with the higher points being higher in the normal partial ordering on $\mathbf{Z}^2$. The righthand dimension is i, the lefthand j. The arrows represent the various differences: the head of the arrow is at the positive term in the difference and the tail at the negative term in the difference.

Figure 1 about here

Visualising the differences in this way makes it possible to derive properties on the diagrams instead of doing the algebra, which can be tedious. For example, in Figure 2, we show that if $f(\cdot)$ is submodular, and subconcave in i and j, then $f(\cdot)$ is also componentwise concave in i.

Figure 2 about here

Verifying algebraically, suppose $f(\cdot)$ is submodular and subconcave.

Then, for any $x^o \in \mathbf{Z}^n$,

1.$\delta_i \delta_j f(x^o) = f(x^o) - f(x^o - e_i) - f(x^o - e_j) + f(x^o - e_i - e_j) \leq 0$

(from submodularity)

2. $\delta_{i-j} \delta_i f(x^o - e_j) = f(x^o - e_j) - f(x^o - e_i) - f(x^o - e_i - e_j) + f(x^o - 2e_i) \leq 0$

(from subconcavity)

3. $\delta_i \delta_i f(x^o) = f(x^o) - 2f(x^o - e_i) + f(x^o - 2e_i) \leq 0$

(adding 1 and 2 gives componentwise concavity)

Now we shall introduce our final definition.

Definition 4 (Directional modularity properties)

a) If $f(\cdot)$ is sub-(super-)modular and subconcave (superconvex) in distinct dimensions i and j, it will be said to be *directionally sub-(super-)modular* in i and j.

b) If $f(\cdot)$ is sub-(super-)modular and subconvex (superconcave) in (distinct) dimensions i and j, it will be said to be *antidirectionally sub-(super-)modular* in i and j.

This family of four properties will do much of the work in the ensuing. In fact, although (anti-)directional concavity (convexity) properties are relatively better known and more immediately recognisable [32,40], the directional modularity properties of Definition 4 are in fact stronger and more fundamental (this is formally stated in Proposition 1). Moreover, the inductive argument which we use in this paper relies on the fact that certain sorts of dynamic programming operator propagate directional modularity, and it can be demonstrated (although it is tedious to do so) that (anti-)directional concavity (convexity) by themselves are not propagated by these same operators.

Four easy propositions will show some of the behaviours of these properties. Each part of these Propositions has a parallel version which is formed in a purely mechanical fashion by reversing inequalities and exchanging the words "submodular" and "supermodular"; "convex" and "concave"; "superconvex" and "subconcave"; and "subconvex" and "superconcave".

Proposition 1

a) If $f(\cdot)$ is directionally submodular in i and j, it is also directionally concave in i and j.

b) If $f(\cdot)$ is antidirectionally supermodular in i and j, it is also antidirectionally concave in i and j.

Proof

We have already shown above that if $f(\cdot)$ is submodular and subconcave in i and j, then it is also componentwise concave in i. This is a).

b) can be shown similarly. □

Proposition 2 (Reflection)

For $f(x) = f(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots\ x_n)$,

define $f^{\tilde{}j}(x) = f(x_1, x_2, \ldots, x_i, \ldots, -x_j, \ldots\ x_n)$.

$f(\cdot)$ is antidirectionally supermodular in i and j iff $f^{\tilde{}j}(\cdot)$ is directionally submodular in i and j.

Proof

First, supermodularity. Suppose $f(\cdot)$ is supermodular in i and j.

Then, for any $x^o \in \mathbf{Z}^n$, suppose $x^o = (x_1^o, x_2^o, \ldots, x_i^o, \ldots, x_j^o, \ldots\ x_n^o)$and define $x^{o\tilde{}j}$ as $(x_1^o, x_2^o, \ldots, x_i^o, \ldots, -x_j^o, \ldots\ x_n^o)$.

$\delta_i\delta_j f(x^{o\tilde{}j} + e_j) = f(x^{o\tilde{}j} + e_j) - f(x^{o\tilde{}j} - e_i + e_j) - f(x^{o\tilde{}j}) + f(x^{o\tilde{}j} - e_i)$

$= f^{\tilde{}j}(x^o - e_j) - f^{\tilde{}j}(x^o - e_i - e_j) - f^{\tilde{}j}(x^o) + f^{\tilde{}j}(x^o - e_i)$

$= -\delta_i\delta_j f^{\tilde{}j}(x^o)$

But from supermodularity, $\delta_i\delta_j f(\cdot) \geq 0$, so $\delta_i\delta_j f^{\tilde{}j}(x^o) \leq 0$. So $f^{\tilde{}j}(\cdot)$ is submodular in i and j. Similarly, $f^{\tilde{}j}(\cdot)$ supermodular in i and j implies $f(\cdot)$ submodular in i and j.

Next, subconcavity. Suppose $f(\cdot)$ is subconcave in i and j

Then, for any $x^o \in \mathbf{Z}^n$,

$\delta_{i-j}\delta_i f(x^{o\tilde{}j}) = f(x^{o\tilde{}j}) - f(x^{o\tilde{}j} - e_i + e_j) - f(x^{o\tilde{}j} - e_i) + f(x^{o\tilde{}j} - 2e_i + e_j)$

$= f^{\tilde{}j}(x^o) - f^{\tilde{}j}(x^o - e_i - e_j) - f^{\tilde{}j}(x^o - e_i) + f^{\tilde{}j}(x^o - 2e_i - e_j)$

$= \delta_{i+j}\delta_i f^{\tilde{}j}(x^o)$

But from $f(\cdot)$ subconcave, $\delta_{i-j}\delta_i f(\cdot) \leq 0$, so $\delta_{i+j}\delta_i f^{\tilde{}j}(x) \leq 0$, so $f^{\tilde{}j}(\cdot)$ is superconcave. In a similar fashion, it can be shown that $f^{\tilde{}j}(\cdot)$ superconcave implies $f(\cdot)$ subconcave. □

Proposition 3

If $f(\cdot)$ is supermodular in i, j and k then the δ_k are partially ordered on the points a) $x, x-e_j, x-e_i, x-e_i-e_j$and b) $x, x-e_i, x-e_k, x-e_k-e_i$ as follows:

a) $\delta_k f(x) \geq \delta_k f(x-e_j), \delta_k f(x-e_i) \geq \delta_k f(x-e_i-e_j)$

b) $\delta_k f(x) \geq \delta_k f(x-e_i)$ and $\delta_k f(x-e_k) \geq \delta_k f(x-e_i-e_k)$

Proof

a) From the definition of supermodularity, note that $\delta_k f(x) \geq \delta_k f(x-e_j) \geq \delta_k f(x-e_i-e_j)$ and

$\delta_k f(x) \geq \delta_k f(x-e_i) \geq \delta_k f(x-e_i-e_j)$

b) also follows easily from the definition of supermodularity. $\square$

Proposition 4

If $f(\cdot)$ is separable (i.e. $f(x) = \sum_{k=1}^{n} f_k(x_k)$) and componentwise concave, then it is directionally submodular and antidirectionally supermodular.

Proof

It is well-known [37] that a separable function is both submodular and supermodular. To see superconcavity, note that:

$f(x) = f_i(x_i) + f_j(x_j) + \sum_{k=1}^{n/i,j} f_k(x_k)$

$f(x-e_i) = f_i(x_i-1) + f_j(x_j) + \sum_{k=1}^{n/i,j} f_k(x_k)$

$f(x-e_i-e_j) = f_i(x_i-1) + f_j(x_j-1) + \sum_{k=1}^{n/i,j} f_k(x_k)$

$f(x-2e_i-e_j) = f_i(x_i-2) + f_j(x_j-1) + \sum_{k=1}^{n/i,j} f_k(x_k)$

where $\sum_{k=1}^{n/i,j}$ is the summation over the indices 1,..., i-1,i+1,... j-1,j+1,...,n.

So, for any $x^o \in \mathbf{Z}^n$,

$\delta_{i+j}\delta_i f(x^o) = f_i(x_i^o) - f_i(x_i^o - 1) - f_i(x_i^o - 1) + f_i(x_i^o - 2) = \delta_i\delta_i f(x^o) \leq 0$

by componentwise concavity. $\square$

4 A dynamic two-leg revenue management model

In this section, we shall study a Markov Decision Process model of a two-leg RM problem. We suppose that we are concerned with a firm which offers two products for sale. The firm keeps track of its inventory using a two-dimensional state vector x. Requests (which may be accepted or rejected) to purchase, return or exchange units of resource arrive over a finite horizon starting at time T and ending at time 0. There is a penalty for overbooking which for convenience we shall we suppose to be linear. Time is discretised in our model sufficiently finely that the probability of more than one request occurring in a single time period is negligible, and, while the probability of a particular request occurring may change from one time period to another, this probability may *not* depend on the state. Acceptance of a request brings a "reward": in the case of a request for capacity, this reward is likely to be positive, and should be interpreted as a "fare"; in the case of a request for an exchange, it is likely to be positive and should be construed as a "charge" to the customer; and in the case of a request for cancellation, it is likely to be negative and should be understood as a "refund".

We define the following terms:

1. $J^*_{\mathcal{A},t}(\cdot)$ is the expected optimal value function of the DP.

2. $T_{c,j}$ is the dynamic programming operator for a particular transition request and reward.

3. X is the state space and $\{1, 2, \ldots, F\}$ is the index set of the dimensions

of the state space.

4. $\{0, 1, 2, \ldots, T\}$ is the set of discrete time intervals and is counted backwards from T at earliest booking time to 0 at flight departure

5. L is a lower and U an upper bound on resource, and C and D are vectors of very large penalties (expressed positively).

6. $\mathcal{A} = \{a_j\}$ is a set of transition vectors, indexed by $\{1, 2, ..., J\}$, perhaps including a null transition. $\{1, \ldots, K\}$ indexes a set of fare classes and each pair of a transition a_j and fareclass c has a real-valued reward, $r_{c,j}$ associated with it, except the null transition, which always brings reward 0. (Note that we assume that the number of fareclasses is the same for each transition: this is without loss because the probability of the arrival of a request for some fareclass/ transition combination could be 0).

7. $P_t = \{p_{c,j,t}\}$ are the probabilities of a request for a particular transition a_j with reward $r_{c,j}$ in a particular time period t. The probabilities in P_t sum to 1.

Assuming the firm wishes to maximise expected revenue, the optimal policy can be retrieved from a Dynamic Program characterised by the following value function, as discussed in [35]:

$$J^*_{\mathcal{A},t}(x) = \sum_{j=1}^{J} \sum_{c=1}^{K} p_{c,j,t} T_{c,j} J^*_{\mathcal{A},t-1}(x) \qquad \text{DynRM}(\mathcal{A})$$

$$T_{c,j} f(x) = \max_{u \in \{0,1\}} (f(x - ua_j) + ur_{c,j}) \qquad \text{for all } x \in X, \text{ for } t\text{=1 to T}$$

and the following initial conditions

$J^*_{\mathcal{A},0}(x) = \sum_{i=1}^{F}(C_i(x_i - L_i)^- - D_i(x_i - U_i)^+) \qquad \text{for all } x \in X$

We make a number of comments on our modelling choices in the forgoing:

1. This model is very much in the same spirit as other dynamic network models in the literature [2,9,35]. A notable limitation is that although we allow the arrival probabilities to be time non-homogeneous, the model is a discrete time model.

2. The notation DynRM($\mathcal{A}$) may appear somewhat unusual. The rationale for this is that we will be interested in a number of families of dynamic programs. Each family represents a collection of revenue management problems on a particular network structure. The families are indexed by sets of transition vectors - the $\mathcal{A}$ - and it is these sets which express network structure.

3. The penalties L and U can be thought of as a device to stop the controller moving into parts of the state space which are "off-limit" either because there is no more capacity to sell or because there are no more booked passengers to cancel. A similar device is used in Lautenbacher and Stidham [27]. The most natural alternative way of modelling – to declare that the state space has boundaries – complicates proofs without adding insight as one has to then take into account within the proof the possibility of a particular control action hitting a boundary. The reader will note that $J^*_{\mathcal{A},0}(x)$ is separable and componentwise concave as a function of x in each dimension of the state space.

4. Our model is a traditional fixed-price RM model, rather than a "dynamic pricing" model [18,19,40]. Fixed price RM models (like the one presently under consideration) assume that there are a number of streams of demand for each bundle of resource at different price levels, which are exogenous and fixed, and that the control lever available to the controller is to accept or reject bookings. Dynamic pricing models on the other hand assume that there is a single stream of demand for each bundle of resource, but that the controller can control the price of each resource and thus (through a demand curve) can control demand. The results which we prove here apply only to the former class of models. However, we can see no reason why the qualitative insight, which is that a network resource can be usefully thought of as a system of economic substitutes and complements, and that this translates to predictable behaviour of the value of the resource considered as a function of capacity, does not apply with equal force to dynamic pricing models.

We cannot derive properties for DynRM($\mathcal{A}$) without first specifying the set $\mathcal{A}$ of possible transitions. In this section, we restrict our attention to $F = 2$ and will explore two cases:

1. The complementary case, DynRM(C). Here,

$$\mathcal{A} = C = \{(0,0),(1,0),(0,1),(-1,0),(0,-1),(1,1),(-1,-1)\}$$

These events will be labelled a_0, a_1, a_2, a_{-1}, a_{-2}, a_{1+2}, and $a_{-(1+2)}$ respectively. These transitions represent either the null event (a_0), the arrival of a customer requesting capacity on a particular flight leg (a_1, a_2), the arrival of a customer requesting to cancel such a booking, (a_{-1}, a_{-2}), a request for a through booking, i.e. one seat on both flight legs 1 and 2 (a_{1+2}), and a cancellation of such a booking ($a_{-(1+2)}$).

2. The substitutable case, DynRM(S). Here,

$$\mathcal{A} = S = \{(0,0),(1,0),(0,1),(-1,0),(0,-1),(-1,1),(1,-1)\}$$

These events will be labelled a_0, a_1, a_2, a_{-1}, a_{-2}, a_{-1+2}, and a_{1-2}, respectively. These transitions represent either the null event (a_0), the arrival of a customer requesting capacity on a particular flight leg (a_1, a_2), the arrival of a customer wishing to cancel a request, (a_{-1}, a_{-2}) or a request to switch a customer from one flight leg to another (a_{-1+2}, a_{1-2}).

From an economic point of view, the legs in the complementary (substitutable) case are complementary (substitutable) products and so one would expect the value of the dynamic program to be supermodular (submodular) in the capacity of the two state spaces. In fact, as we shall show, $J^*_{\mathcal{A},t}(x)$ is antidirectionally supermodular (directionally submodular) in the complementary (substitutable) case.

First, however, we shall need two lemmata.

Lemma 1 (Preservation of supermodularity)

If $f(\cdot)$ is supermodular in the dimensions of X, then $g^-(\cdot)$ and $g^+(\cdot)$ are also supermodular in all dimensions of X where

$$g^-(x) = T_k^- f(x) = max(f(x - e_k) + s, f(x) + t)$$

$$g^+(x) = T_k^+ f(x) = max(f(x + e_k) + s, f(x) + t)$$

for some real-valued s and t.

Proof

See Appendix. $\square$

We note that this result is *not* implied by the well-known result of Topkis [36]. We also note that submodularity does not appear to be preserved under T_k^+/ T_k^- in the same way, although it *is* preserved if the max operator in the expression is replaced by the min operator.

Lemma 2 (Unfolding)

Consider a function $f{:}\mathbf{Z}^2 \to \mathbf{R}$. Construct a function $\widetilde{f}{:}\mathbf{Z}^3 \to \mathbf{R}$ by $\widetilde{f}(x) = \widetilde{f}(x_1, x_2, x_3) = f(x_1 - x_3, x_2 - x_3)$. Then $f(\cdot)$ is antidirectionally supermodular iff $\widetilde{f}(\cdot)$ is supermodular.

Proof

Trivially, $f(\cdot)$ supermodular in 1 and 2 iff $\widetilde{f}(\cdot)$ supermodular in 1 and 2

It remains to show that $f(\cdot)$ superconcave in 1 and 2 iff $\widetilde{f}(\cdot)$ supermodular in 1 and 3; and in 2 and 3.

Suppose $f(\cdot)$ is superconcave. Then, for any $x^o \in \mathbf{Z}^3$

$\delta_{1+2}\delta_1 f(x^o + e_1 + e_2) = f(x^o + e_1 + e_2) - f(x^o + e_2) - f(x^o) + f(x^o - e_1)$

$= \widetilde{f}(x^o - e_3) - \widetilde{f}(x^o - e_1 - e_3) - \widetilde{f}(x^o) + \widetilde{f}(x^o - e_1)$

$= -\delta_1\delta_3\widetilde{f}(x^o)$

But we know that $\delta_{1+2}\delta_1 f(x^o + e_1 + e_2) \leq 0$, so $\delta_1\delta_3\widetilde{f}(x^o) \geq 0$. The argument for the pair of dimensions 2 and 3 is symmetric, and the argument for iff is obtained by reversing the above proof. □

Theorem 1

The value of DynRM(C) is antidirectionally supermodular in its dimensions.

Proof

The proof is by induction.

Base case. $J^*_{C,0}(x)$ is separable and componentwise concave, and by Proposition 4, this means that it is antidirectionally supermodular.

Inductive hypothesis. Suppose that $J^*_{C,t-1}(x)$ is antidirectionally supermodular.

Inductive step.

That antidirectional supermodularity propagates in the case of a null arrival is trivial. For the non-null arrivals, we can form an extension in the manner of Lemma 2 to get an extension $\widetilde{J}^*_{C,t-1}(x)$, with dimensions indexed as 1,2 and -(1+2). Note that by Lemma 2, $\widetilde{J}^*_{C,t-1}(x)$ is supermodular.

Now we can rewrite $T_{c,j}\widetilde{J}^*_{C,t-1}(x)$ for every c,j in one of the following forms:

$T^-_{c,j}\widetilde{J}^*_{C,t-1}(x) = max_{u\in\{0,1\}}(\widetilde{J}^*_{C,t-1}(x - ue_j) + ur_{c,j})$ for some unit vector e_j, or

$T^+_{c,j}\widetilde{J}^*_{C,t-1}(x) = max_{u\in\{0,1\}}(\widetilde{J}^*_{C,t-1}(x + ue_j) + ur_{c,j})$ for some unit vector e_j

In the terms of Lemma 1, in the case of a_1, a_2 and $a_{-(1+2)}$, the operator is a T^-, and in the case of a_{-1}, a_{-2} and a_{1+2}, the operator is a T^+.

Now we can apply Lemma 1, to see that all the $T_{c,j}\widetilde{J}^*_{C,t-1}(x)$ are supermodular.

It is trivial that supermodularity is preserved under convex combination, and so $\widetilde{J}^*_{C,t}(x)$ is supermodular and, by Lemma 2 again, $J^*_{C,t}(x)$ is antidirectionally supermodular. □

There is an interesting relationship between DynRM(C) and DynRM(S), which makes it possible to construct a proof of the structural properties of the latter without invoking Lemma 1 again. Specifically, it is possible to construct a partially reflected system, which is equivalent to DynRM(S), but in which we count bookings positively on one dimension and negatively on the other dimension. The Bellman equations of this system are the same as the Bellman equations of an instance of DynRM(C) and therefore, the value of this model is antidirectionally supermodular. By Proposition 2, the value of the original DynRM(S) must then be directionally submodular. This idea of seeing systems as partial reflections of one another was developed in quite a different setting by Gale and Politof [17] and by Topkis [37] in his proof of the modularity properties of the Transportation Problem.

Theorem 2

The value of DynRM(S) is directionally submodular in its dimensions.

Proof

Consider an arbitrary instance of DynRM(S).

We construct the equations for the value function, $\hat{J}^*_{\hat{S},t}(\cdot)$, of this system, by taking the initial conditions:

$$\hat{J}^*_{\hat{S},0}(x) = (C_1(x_1 - L_1)^- - D_1(x_1 - U_1)^+) + (D_2(U_2 - x_2)^- - C_2(L_2 - x_2)^+)$$

and taking the Bellman equations with S replaced everywhere by $\hat{S}$

where $\hat{S} = \{(0,0), (1,0), (0,-1), (-1,0), (0,1), (-1,-1), (1,1)\}$ and the members of $\hat{S}$ are labelled a_0, a_1, a_2, a_{-1}, a_{-2}, a_{1-2}, a_{2-1}.

It can be seen that $\hat{J}^*_{\hat{S},t}(\cdot)$ is equivalent to an instance of DynRM(C) and therefore by Theorem 1, it is antidirectionally supermodular.

It can also be easily shown by induction that $\hat{J}^*_{\hat{S},t}(x) = J^{*\tilde{}2}_{S,t}(x)$ in the sense of Proposition 2. In which case, the antidirectional supermodularity of $\hat{J}^*_{\hat{S},t}(x)$ entails that the value of DynRM(S) is directionally submodular. □

Some interpretation of these results may be helpful.

What we have shown is that the value of DynRM(C), where customers can request both flights, is antidirectionally supermodular in its dimensions. Let us revisit the meaning of antidirectional supermodularity. A function $f(x)$ is said to be antidirectionally supermodular if

$$\delta_i \delta_j f(x) \geq 0$$

$$\delta_{i+j} \delta_i f(x) \leq 0$$

So in this case, when one moves a unit in one dimension of state space, the value of the differences in the other dimension of state space increases; and furthermore, the value of the joint differences, $\delta_{i+j} f(x)$, decreases. That is to say, each seat increases in value the more seats are available on the other flight, and so the higher the single-leg fare needed to secure a booking, but contrariwise (and this is the "antidirectional" idea), the more seats on either flight, the lower the fare required to secure a through booking.

At the same time, we have shown that in DynRM(S), where customers can swap between flights, the value function is *directionally submodular* in its dimensions. This means that each seat decreases in value the more seats are available on the other flight. Moreover, we have shown that the switching curve is monotone: the fewer seats on one flight, the higher the minimum premium needed to secure a switch to that flight.

It could be objected that in an application context, one would expect the arrivals of such requests to depend on how many customers had already booked, yet these models do not to allow for state-independent cancellation and switch requests. There are three answers to such an objection.

1. The first answer is that a version of DynRM($C_{reduced}$) where $C_{reduced}$=$\{a_0, a_1, a_2, a_{1+2}\}$ would be quite realistic and would continue to exhibit the structural properties discussed above. An equivalent DynRM($S_{reduced}$) where $S_{reduced}$ does not contain switch requests would of course be rather vacuous, however.

2. The second answer is that Dynamic Programming-based approaches to RM are unlikely to be applied on the whole of an airline's network, as the curse of dimensionality would render such an approach prohibitive. Rather, they will be deployed on a tightly constrained subnetwork relatively close to flight departure, where the additional resolution DP ap-

proaches provide over deterministic approaches will give the greatest benefit. In such cases, the number of passengers booked already booked will be large relative to the number of passengers to come, and so modelling cancellations as state-independent may be a reasonable modelling assumption.

3. The last answer is pragmatic: it has been shown the modelling state-dependent cancellations in the single leg case destroys the concavity property of the DP [33]. Since the single leg model is a special case of our model, and since our directional submodularity/ antidirectional supermodularity properties imply componentwise concavity, it is evident that state-dependent cancellations would destroy the properties presently under study as well. Non-monotonicity may be the price of realism.

Let us relate these results back to the literature. Our Theorem 1 is essentially the same as Theorems 3.2 and 3.3 of You [40], although we feel our proof mechanism is more general and insightful. There is no direct analogue of Theorem 2 in the RM literature: however, in the queueing literature, Hajek [23] shows that the value function of a queueing system with two interacting queues has the property which we have called directional supermodularity. This result is qualitatively similar to the present result, and bears out our earlier claim that just as revenue substitutability is associated with directional submodularity, cost substitutability is associated with directional supermodularity. Koole [26], again in a queueing context, presents some results which are similar to Theorems 1 and 2, although expressed in cost, rather than revenue, terms, and developed within an alternative theoretical framework and from a different perspective.

Methodologically, our proof of Lemma 1 is somewhat similar to Weber and Stidham's [39] proof of multimodularity properties of a network of queues. There are a number of differences occasioned by the RM context, in particular in the handling of boundaries.

It may seem natural to seek a generalisation of these results to a multi-leg context. However, it is not immediately clear how the above proof mechanism should be generalised, nor just what form a generalisation should take, as we will explore in the next section.

5 Multi-leg deterministic RM models

In this section, we shall study a constrained deterministic linear optimization model, which we shall call DetRM($\mathcal{A}$). DetRM($\mathcal{A}$) is a Multi-Commodity Network Flow where each commodity has a single routing ("path") and can be thought of as optimally assigning demand for transportation over a constrained network: such models are sometimes used in RM practice, either to generate allocations which can then be used as booking limits, or more commonly to obtain estimates of the marginal values of different resources, which can then be used as "bid prices".

The terms of our model are defined as follows:

1. As in DynRM($\mathcal{A}$), f indexes flights, j indexes transition vectors, and c indexes fare classes.

2. $A = [a_1| \dots |a_j| \dots \ |a_J]$, for all column vectors $a_j \in \mathcal{A}$ i.e. A is the $F \times J$ arc-path matrix of the underlying network.

3. $D = \begin{bmatrix} D_1 \\ D_2 \\ \dots \\ D_c \\ \dots \\ D_K \end{bmatrix}$ where $D_c = \begin{pmatrix} D_{c,1} \\ D_{c,2} \\ \dots \\ D_{c,j} \\ \dots \\ D_{c,J} \end{pmatrix}$. $D \in \mathbf{R}^{\mathrm{C} \times \mathrm{J}}$ can be interpreted as a column vector of the demand for up to C fare classes on J paths.

4. $d = \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_c \\ \dots \\ d_K \end{bmatrix}$ where $d_c = \begin{pmatrix} d_{c,1} \\ d_{c,2} \\ \dots \\ d_{c,j} \\ \dots \\ d_{c,J} \end{pmatrix}$. $d \in \mathbf{R}^{\mathrm{C} \times \mathrm{J}}$ (the decision vector) is a column vector of allocations of demand for C fare classes on J paths.

5. $r = [r_1|r_2| \dots |r_c| \dots |r_K]$ and $r_c = (r_{c,1}, \dots, r_{c,j}, \dots, r_{c,J})$. $r \in \mathbf{R}^{\mathrm{C} \times \mathrm{J}}$ is a row vector of fares.

6. $x \in \mathbf{R}^{\mathrm{F}}$ is a vector of capacities.

We present here the so-called arc-path formulation [1, p666] of DetRM($\mathcal{A}$)

$$z(x) = \max_{d} rd \qquad \text{DetRM}(\mathcal{A})$$

s.t.

1) $\quad 0 \leq A(\sum_{c=1}^{K} d_c) \leq x$

2) $0 \leq d \leq D$

DetRM($\mathcal{A}$) models the situation where demand is known with certainty and the problem is merely one of assigning demand to flights. The inequalities 1) are capacity constraint, which ensure that the total demand assigned cannot exceed capacity, and the inequalities 2) are demand constraints, which ensure that demand cannot be allocated unless that demand does indeed exist. Note that we have written the value of DetRM($\mathcal{A}$) as a function of x, since we be varying x parametrically. (x is *not* the decision variable here.)

It may be helpful to expand on what the constraints 1) are doing. Let us focus on a single constraints in that set of constraints. The RHS will be the scalar x_i. x_i represents the capacity on the ith arc/ flight leg in the network. Consider the vector of the total flow from all commodities on each individual path through the network ($\sum_{c=1}^{K} d_c$). Then, from that vector, pick out the entries representing flow on paths which pass through flight leg i and add them up (the multiplication by the arc path matrix A, which has arcs as its rows, paths as its columns, and 1s in cells where a given path contains a given arc, 0s otherwise). This number – the total flow on the ith arc – has to be less than the capacity of the ith arc.

Given an instance of DynRM($\mathcal{A}$), it is possible to find a corresponding instance of DetRM($\mathcal{A}$) by taking $D_{c,j} = \sum_{t=0}^{T} p_{c,j,t}$, as the expectation of demand to come. It is well-known that for $x \geq 0$, and where all the a_j are non-negative, the particular instance of DynRM($\mathcal{A}$) converges to the corresponding instance of DetRM($\mathcal{A}$) under an appropriate fluid scaling regime [9].

We shall study the structural properties of DetRM($\mathcal{A}$) for two distinct problems.

Definition 5

If $\mathcal{A}$ is such that $\{1, .., F\}$ can be partitioned into $\mathcal{M}$ and $\mathcal{N}$ such that for all non-null $a_j \in \mathcal{A}$, $a_j = e_m + e_n$ or e_m or e_n for some $m \in \mathcal{M}$, $n \in \mathcal{N}$, we will say that this DetRM($\mathcal{A}$) is a *partitionable problem* and write it as DetRM(PP).

In somewhat less formal terms, this means that we can number the flight legs such that each customer request which is not for a single flight leg will be for a pair of flights with one odd and one even number. For an airline with out-and-back cycles emanating from a single hub, the obvious way to do this is to number the incoming flights odd and the outgoing flights even (many airlines do indeed number their flights in this way). For an airline with two hubs, A and B, and with out-and-back cycles emanating from each hub, and single leg flights from hub to hub, the solution is to number A-incoming (outgoing) flights and B-outgoing (incoming) flights odd (even) and the condition is fulfilled as long as no passenger undertakes a three leg journey passing through both hubs. A network with three (completely connected) hubs cannot be accommodated in this framework, but such airlines are relatively rare outside the United States.

In order to explore the properties of DetRM(PP), we shall need a result from the theory of network flows.

Theorem 3 (Gale and Politof)

The value of a maximum weight circulation problem on a directed graph is supermodular (submodular) in the capacity space of two arcs α and β, if every cycle on the network containing both α and β orients α and β in the same (opposite) direction; and neither supermodular or submodular if there exists at least two cycles, one orienting α and β in the same direction, and the other orienting α and β in the opposite direction.

Proof

See Gale and Politof [17] □

In other words, where all cycles direct α and β in the same (opposite) direction, the incremental value of an increase in the capacity of both α and β is not less than (not greater than) the sum of the value of an increase in the capacity on α alone plus the value of an increase in the capacity of β alone.

This is useful because we may be able to represent DetRM(PP) as a maximum weight circulation problem on a directed graph, so that the above result applies. We present a plausible representation in Figure 3. The heavy arcs are the flight arcs: a flow on these arcs represents passengers carried on a particular flight. The light arcs are demand arcs: a flow on these arcs represents passengers from a particular market segment served. The flight arcs are directed into ($\mathcal{M}$) and out of ($\mathcal{N}$) a central node, and the demand arcs lead from the head of one flight arc to the tail of the same or another flight arc.

Figure 3 about here

Examination of Figure 3 shows that that every $\mathcal{M}$ flight arc is adjacent to, and oriented in the same direction as, every $\mathcal{N}$ flight arc. Therefore it is not possible that there is any cycle which orients such a pair of arcs in the opposite direction. Moreover, every $\mathcal{M}$ ($\mathcal{N}$) flight arc is adjacent to, and oriented in the opposite direction to, every other $\mathcal{M}$ ($\mathcal{N}$) flight arc. Therefore it is not possible that there is any cycle which orients such a pair of arcs in the same direction. Therefore, the value of the maximum weight cycle problem on this graph is supermodular in every pair of dimensions (m, n) and submodular in every pair of dimensions (m_i, m_j) and (n_i, n_j) for m, m_i, $m_j \in \mathcal{M}$ and n, n_i, $n_j \in \mathcal{N}$.

While we are studying the figure, it is interesting to note that while no pair of flight arcs can be oriented in both the same and opposite directions by different cycles, the same is not true of the demand arcs. This can be seen by examining the arcs α and β: the cycle consisting of arcs marked with numbers prime orients α and β in the same direction, and the cycle consisting of arcs marked with numbers double prime orients α and β in the opposite direction.

It remains to show that DetRM(PP) can indeed be represented as a maximum weight cycle problem on such a graph. We will do this in the proof of Theorem 4.

Theorem 4

The value of DetRM(PP) is supermodular in every pair of dimensions (m, n) and submodular in every pair of dimensions (m_i, m_j) and (n_i, n_j) for m, m_i, $m_j \in \mathcal{M}$ and n, n_i, $n_j \in \mathcal{N}$.

Proof

We assume without loss of generality for all (m, n) $\in \mathcal{M} \times \mathcal{N}$, e_m, e_n and $e_m + e_n \in \mathcal{A}$. (Even though the transition is in $\mathcal{A}$, we can always set the probability of a request for this transition to 0.)

Rewrite DetRM(PP) with additional variables l_f representing the total passenger load on leg f. The F-vector l can be organised as $\begin{pmatrix} l_{\mathcal{M}} \\ l_{\mathcal{N}} \end{pmatrix}$ where $l_{\mathcal{M}}=(l_{m_1},\ l_{m_2},\ldots,\ l\ _{m_M})^T$ and $l_{\mathcal{N}}=(l_{n_1},\ l_{n_2},\ldots,\ l_{n_N})^T$, M and N being the cardinalities of $\mathcal{M}$ and $\mathcal{N}$ respectively. The matrix A likewise be divided vertically into $A_{\mathcal{M}}$ (consisting of the rows relating to the flights in $\mathcal{M}$), and $A_{\mathcal{N}}$ (consisting of the rows relating to the flights in $\mathcal{N}$).

By substituting, DetRM(PP) can now be written as

$z(x) = \quad \max_{d,l} rd$

s.t.

1') $\quad 0 \leq l \leq x$

2) $\quad 0 \leq d \leq D$

3) $\quad A_{\mathcal{M}}(\sum_{c=1}^{K} d_c) - l_{\mathcal{M}} = 0$

4) $\quad -A_{\mathcal{N}}(\sum_{c=1}^{K} d_c) + l_{\mathcal{N}} = 0$

This revised program can be seen to be a maximum weight circulation problem on the graph of Figure 3. The objective function maximises the sum of weighted flows on the demand arcs, i.e. the revenues. The new constraints 3) and 4) can be interpreted as the node balance constraints for the nodes on the left-hand and right-hand of Figure 3, respectively. The revised constraints 1') can be reinterpreted as the capacity constraint on the flight arcs and the constraints 2) as the capacity constraints on the demand arcs. The node constraint for the central node (which is redundant) can be derived by adding up 3) and 4) together. □

This result does not seem to be presented here in its fullest possible generality. Glover and colleagues [21] have claimed without proof that a similar maximum weight circulation problem can be derived whenever the underlying network is acyclic. As all time-space networks are acyclic, this would provide a way to demonstrate super- and sub-modularity results for more general network structures.

This result is entirely consistent with the economic interpretation which we advanced at the outset of this paper. This says, for example, if we have an airline which operates a single hub network between cities in the east and cities in the west, flights out of (into) cities in the east (west) will be substitutable with each other, but complimentary with flights into (out of) cities in the west (east).

However, once the network structure becomes more complicated, this sort of pleasing result breaks down. To see this, we will consider another sort of instance of DetRM($\mathcal{A}$).

Definition 6

If we have an instance of DetRM($\mathcal{A}$) such that F=3 and $\mathcal{A}$={(1,1,0),(0,1,1) and (1,1,1)} we say that this DetRM($\mathcal{A}$) is a *chain problem*, and write it as DetRM(Ch).

Airline networks containing such sequences of legs and demand patterns are by no means uncommon. One might naturally expect to find that since these three legs are complements in the “and-both” sense, the value function would be supermodular in all three pairs of dimensions. However, this is not the case.

To see the counterexample, take DetRM(Ch) with the following parameters:

$r_{1+2} = \$80$

$r_{2+3} = \$70$

$D_{1+2}, D_{2+3}, D_{1+2+3} = \infty$

Now evaluate this program for four values of x under two cases.

Case 1. $\mathrm{r}_{1+2+3} = \$200$

Case 2. $r_{1+2+3} = \$120$

In Case 1, the optimal solution and the value of the problem are as shown in Table 1.

Table 1 about here

Checking for the sign of $\delta_1\delta_3 z(x)$, we find:

$$z(x) - z(x - e_1) - z(x - e_3) + z(x - e_1 - e_3) = \$50 > 0$$

However, in Case 2, the optimal solution and the value of the problem are as shown in Table 2.

Table 2 about there

Now, checking for the sign of $\delta_1\delta_3 z(x)$ gives quite a different picture:

$$z(x) - z(x - e_1) - z(x - e_3) + z(x - e_1 - e_3) = -\$30 < 0$$

So DetRM(Ch) is neither supermodular nor submodular in general. Note that as DetRM(Ch) is the deterministic special case of DynRM(Ch), this means that DynRM(Ch) is not supermodular or submodular either.

6 Conclusion

In this paper, we have discussed plausible structural properties of network revenue management problems, based an intuitive interpretation of network resource as a system of economic substitutes and complements. We have shown that this interpretation is borne out by structural properties of a two-leg system in the dynamic case, and shed some light on how these properties apply for more complex network structures in the multi-leg deterministic case.

The natural next question to ask is whether the results which are obtainable for the n-resource static deterministic problem can be generalised to the dynamic stochastic case. Our research into this question is still ongoing, but it does seem to be true that this is not the case in any simple sense. Of course, from a more practical standpoint, it would also be good to know more about how such properties which may be available can be exploited algorithmically. Lastly, we have tried in this paper to develop a closer linkage between RM and the theory of queueing networks: strengthening and deepening this linkage may, we feel, be a way to stronger and more insightful results about the structural properties of stochastic systems in general.

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7.2 Appendix. Proof of Lemma 1

There are two main cases to consider: $g^{-}(\cdot)$ (I) and $g^{+}(\cdot)$ (II).

We shall deal with I first. Consider an arbitrary pair of distinct dimensions i and j.

There are three possible subcases to consider: i, j and k are all distinct, or i=k, and j is distinct, or j=k and i is distinct. The latter two are obviously symmetrical, so we have effectively two subcases.

I.A. i,j,k distinct

By proposition 3, we know that $\delta_k f(x) \geq \delta_k f(x-e_j), \delta_k f(x-e_i) \geq \delta_k f(x-e_i-e_j)$.

We consider the case $\delta_k f(x) \geq \delta_k f(x-e_j) \geq \delta_k f(x-e_i) \geq \delta_k f(x-e_i-e_j)$. The case $\delta_k f(x) \geq \delta_k f(x-e_i) \geq \delta_k f(x-e_j) \geq \delta_k f(x-e_i-e_j)$ is obviously symmetrical.

There are now 5 subsubcases to consider.

I.A.1. $\delta_k f(x) \geq \delta_k f(x-e_i) \geq \delta_k f(x-e_j) \geq \delta_k f(x-e_i-e_j) \geq s-t$

I.A.2. $\delta_k f(x) \geq \delta_k f(x-e_i) \geq \delta_k f(x-e_j) \geq s-t \geq \delta_k f(x-e_i-e_j)$

I.A.3. $\delta_k f(x) \geq \delta_k f(x-e_i) \geq s-t \geq \delta_k f(x-e_j) \geq \delta_k f(x-e_i-e_j)$

I.A.4. $\delta_k f(x) \geq s-t \geq \delta_k f(x-e_i) \geq \delta_k f(x-e_j) \geq \delta_k f(x-e_i-e_j)$

I.A.5. $s-t \geq \delta_k f(x) \geq \delta_k f(x-e_i) \geq \delta_k f(x-e_j) \geq \delta_k f(x-e_i-e_j)$

We are interested in the sign of $\delta_i \delta_j g^-(x) = g^-(x) - g^-(x-e_i) - g^-(x-e_j) + g^-(x-e_i-e_j)$ in each case. Observe that the result follows trivially in case I.A.1 and I.A.5, and that the expressions for $\delta_i \delta_j g^-(x)$ in cases I.A.2 and I.A.4 are greater than the expressions for $\delta_i \delta_j g^-(x)$ in cases I.A.1 and I.A.5.

Finally, note that in case I.A.3.,

$\delta_i \delta_j g^-(x) = g^-(x) - g^-(x-e_i) - g^-(x-e_j) + g^-(x-e_i-e_j)$

$= f(x) + t - f(x-e_i) - t - f(x-e_j-e_k) - s + f(x-e_i-e_j-e_k) + s$

$= f(x) - f(x-e_i) - f(x-e_j-e_k) + f(x-e_i-e_j-e_k)$

$= f(x) - f(x-e_i) - f(x-e_k) + f(x-e_i-e_k)$

$+ f(x-e_k) - f(x-e_i-e_k) - f(x-e_j-e_k) + f(x-e_i-e_j-e_k)$

Now since $\delta_i \delta_j f(x) = f(x) - f(x-e_i) - f(x-e_k) + f(x-e_i-e_k) \geq 0$

and

$\delta_i \delta_j f(x - e_k) = f(x - e_k) - f(x - e_i - e_k) - f(x - e_j - e_k) + f(x - e_i - e_j - e_k) \geq 0$

by the hypothesis of the supermodularity of $f(x)$, it follows that $\delta_i \delta_j g^-(x) \geq 0$.

I.B. $i \neq j = k$

By proposition 3, we have

$\delta_k f(x) \geq \delta_k f(x - e_i)$and $\delta_k f(x - e_k) \geq \delta_k f(x - e_i - e_k)$

So there are nine subcases altogether, as shown in Table A1.

Table A1 about here

Now we are interested in the sign of $\delta_i \delta_j g^-(x)$.

I.B1.1, I.B1.3, and I.B.3.1 and I.B.3.3 are trivial. I.B1.2, I.B2.1 and I.B3.2 are greater than or equal to I.B.1.3, I.B1.1, and I.B3.3 respectively.

I.B2.2 and I.B2.3 are a little more subtle and will be exhibited here.

I.B.2.2 $\delta_k f(x) \geq s - t \geq \delta_k f(x - e_i)$ and $\delta_k f(x - e_k) \geq s - t \geq \delta_k f(x - e_i - e_k)$

$\delta_i \delta_j g^-(x) = g^-(x) - g^-(x - e_i) - g^-(x - e_k) + g^-(x - e_i - e_k)$

$= f(x) + t - f(x - e_i - e_k) - s - f(x - e_k) - t + f(x - e_i - e_k - e_k) + s$

$= f(x) - f(x - e_i - e_k) - f(x - e_k) + f(x - e_i - 2ek)$

$= \delta_k f(x) - \delta_k f(x - e_i - e_k)$

But the hypotheses of I.B.2.2 imply $\delta_k f(x) - \delta_k f(x - e_i - e_k) \geq 0$

So $\delta_i \delta_j g^-(x) \geq 0$, as required.

I.B.2.3 $s - t \geq \delta_k f(x) \geq \delta_k f(x - e_i)$ and $\delta_k f(x - e_k) \geq s - t \geq \delta_k f(x - e_i - e_k)$

$\delta_i \delta_j g^-(x) = g^-(x) - g^-(x - e_i) - g^-(x - e_k) + g^-(x - e_i - e_k)$

$= f(x - e_k) - f(x - e_i - e_k) - f(x - e_k) + f(x - e_i - e_k - e_k) + (s - t)$

$= -f(x - e_i - e_k) + f(x - e_i - e_k - e_k) + (s - t)$

$= (s - t) - \delta_k f(x - e_i - e_k) \geq 0$ by the hypothesis of I.B.2.3.

Case II is symmetrical and can be dealt with in a similar manner.

The full workings for these proofs are available from the author on request.

Table 1

x	d_{1+2}	d_{2+3}	d_{1+2+3}	f_1	f_2	f_3	$z(x)$
(1,1,1)	0	0	1	1	1	1	\$200
(0,1,1)	0	1	0	0	1	1	\$70
(1,1,0)	1	0	0	1	1	0	\$80
(0,1,0)	0	0	0	0	0	0	0

Table 2

x	d_{1+2}	d_{2+3}	d_{1+2+3}	f_1	f_2	f_3	$z(x)$
(1,1,1)	1	0	0	1	1	0	\$120
(0,1,1)	0	1	0	0	0	1	\$70
(1,1,0)	1	0	0	1	1	0	\$80
(0,1,0)	0	0	0	0	0	0	0

Table A1

	$\delta_k f(x-e_k) \geq \delta_k f(x-e_i-e_k) \geq s-t$	$\delta_k f(x-e_k) \geq s-t \geq \delta_k f(x-e_i-e_k)$	$s-t \geq \delta_k f(x-e_k) \geq \delta_k f(x-e_i-e_k)$
$\delta_k f(x) \geq \delta_k f(x-e_i) \geq s-t$	$B.1.1$	$B.2.1$	$B.3.1$
$\delta_k f(x) \geq s-t \geq \delta_k f(x-e_i)$	$B.1.2$	$B.2.2$	$B.3.2$
$s-t \geq \delta_k f(x) \geq \delta_k f(x-e_i)$	$B.1.3$	$B.2.3$	$B.3.3$