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Conference paper

Original citation:
Originally presented at: Proceedings of the 56th session of the International Statistical Institute, 22-29 August 2007, Lisbon, Portugal.

This version available at: http://eprints.lse.ac.uk/25217/

Available in LSE Research Online: September 2009

Locally stationary wavelet coherence with application to neuroscience

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Introduction

Time series analysis is used extensively in neuroscience in order to study the interdependence between two simultaneously recorded signals [Pereda et al. (2005)]. Neurophysiological time series are inherently non-stationary and so the covariance structure between the series may vary with time [Lachaux et al. (2002)]. The detection of these changes is very important as they reflect changes in the functional connectivity of the system and therefore allow us to make inferences on how segregated areas of the brain are interacting [Salinas & Sejnowski (2001)]. Our aim is to develop a method of localised coherence in order to analyse simultaneous recordings of neural activity taken from two areas of a rat's brain: the hippocampus and the prefrontal cortex, as in the experimental set-up of Jones & Wilson (2005).

While the cross-correlation function provides a natural estimate of the relationship between two series in the time domain, the cross-spectral density function, defined as the Fourier transform of the cross-covariance function, can be used similarly in the spectral domain [Brillinger (1975)]. The coherence function is derived from the normalisation of the cross-spectrum by the individual spectra and, roughly speaking, measures the correlation between the signals as a function of frequency. The main problem with this approach is that it assumes stationarity of the series and, therefore, is of limited relevance to our problem. An extension to Fourier analysis, to allow for non-stationarity, is windowed Fourier analysis which splits the signal into (possibly overlapping) sections [Daubechies (1992)] . Although this overcomes the assumption of global stationarity, it still requires stationarity within each segment.

Since wavelets are localised in both time and scale, they provide a natural approach to the modelling of series with time varying spectral characteristics (see Vidakovic (1999) for an introduction to wavelets). Unlike time resolved Fourier coherence which employs a constant window width for all frequencies, the wavelet transform uses shorter windows for higher frequencies, which leads to more "natural" localisation (see Daubechies (1992) for more on this topic). Wavelet coherence is an increasingly popular method in neuroscience, see for example Lachaux et al. (2002). In this paper we propose a new measure of wavelet coherence termed 'locally stationary wavelet coherence'. This is derived from the Locally Stationary Wavelet time series model of Nason et al. (2000). Following the work of Dahlhaus (1996), the model adopts the rescaled time principle, replacing the Fourier basis representation by a system of non-decimated wavelets. Due to the particular bias correction implied by the model, our new statistic differs significantly from wavelet coherence measures proposed previously.

Wavelet coherence using the LSW model

Definition 1. *The bivariate LSW process* $(X^{(1)}_{t,T}, X^{(2)}_{t,T})_{t=0,\dots,T-1}$, *for* $T = 2^j \geq 1$ *is a triangular stochastic array with mean-square representation*

$$X^{(1)}_{t,T} = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} W^{(1)}_j(k/T)\psi_{j,t-k}\xi^{(1)}_{j,k} \tag{1}$$

$$X^{(2)}_{t,T} = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} W^{(2)}_j(k/T)\psi_{j,t-k}\xi^{(2)}_{j,k} \tag{2}$$

where $\{\psi_{k,t}\}$ *are discrete, real valued, compactly supported, non-decimated wavelet vectors with scale and location parameters* $j \in \{-1,-2,...\}$ *and* $k \in \mathbb{Z}$, *respectively. For each* $j < -1$, *the functions* $W^{(i)}_j(k/T)$ *and* $\rho_j(k/T)$ *are assumed to be Lipschitz continuous, and are defined on rescaled time* $z = k/T \in [0,1]$ *which enables asymptotic estimation. Also,* $\xi^{(i)}_{j,k}$ *are zero mean orthonormal identically distributed random variables with the following properties*

- $cov(\xi^{(i)}_{j,k}, \xi^{(i)}_{j',k'}) = \delta_{j,j'}\delta_{k,k'}$
- $cov(\xi^{(1)}_{j,k}, \xi^{(2)}_{j',k'}) = \delta_{j,j'}\delta_{k,k'}\rho_j(k/T)$

where $\delta_{i,j}$ *is the Kronecker delta function, giving* $\delta_{i,j} = 1$ *for* $i = j$ *and* 0 *otherwise.*

The parameters $W^{(i)}_j(k/T)$ can be thought of as time and scale dependent transfer functions while the non-decimated wavelet vectors, ψ_j, can be thought of as building blocks analogous to Fourier exponentials in a spectral domain representation. Here the notation $j = -1$ denotes the finest scale wavelet, $j = -2$ the next finest scale and so forth.

This formulation parallels the univariate case of Nason et al. (2000), but in extending this to the bivariate setting we must allow for a potential correlation structure between the two series, given by $\rho_j(k/T)$. It is this quantity that we wish to estimate, with the functional sequence $\{\rho_j(k/T)\}_{j=-\infty}^{-1}$ providing a multiscale decomposition of the cross-correlation structure between $X^{(1)}_{t,T}$ and $X^{(2)}_{t,T}$.

The locally stationary wavelet coherence, $\rho_j(z)$, can be represented as $\rho_j(z) = \frac{C_j(z)}{\sqrt{S^{(1)}_j(z)S^{(2)}_j(z)}}$ where $C_j(z)$ is the locally stationary wavelet cross-spectrum $C_j(z) = W^{(1)}_j(z)W^{(2)}_j(z)\rho_j(z)$, and $S^{(1)}_j(z), S^{(2)}_j(z)$ are the evolutionary wavelet spectra defined as $S^{(i)}_j(z) = (W^{(i)}_j(z))^2$ as in Nason et al. (2000). The locally stationary wavelet coherence, $\rho_j(z)$, ranges from -1, indicating complete negative correlation, to +1 indicating complete correlation. A value of close to zero indicates a lack of correlation between the two series at the given scale and location.

Estimation Theory

Definition 2. *For the LSW processes* $X^{(i)}_{t,T}$ *for* $i = 1, 2$, *constructed using the wavelet system* ψ, *the empirical non decimated wavelet coefficients are given by*

$$d^{(i)}_{j,t,T} = \sum_s X^{(i)}_{s,T}\psi_{j,s-t} \tag{3}$$

Although the use of other types of wavelets is possible, we use Haar wavelets for our estimator, following the theory of Nason et al. (2000). The wavelet coefficients are used to construct the cross-wavelet periodogram and wavelet periodogram, defined as follows.

Definition 3. *The wavelet periodograms for the LSW processes $X_{t,T}^{(i)}$, for $i = 1, 2$, are given by*

$$I_{j,t,T}^{(i)} = |d_{j,t,T}^{(i)}|^2 \tag{4}$$

The wavelet cross-periodograms is given by

$$I_{j,t,T}^{(1,2)} = d_{j,t,T}^{(1)} d_{j,t,T}^{(2)} \tag{5}$$

Proposition 1. *The expectation of the cross-periodogram, $I_{j,t,T}^{(1,2)}$, is given by*

$$\mathbb{E} I_{j,t,T}^{(1,2)} = \sum_{i=-\infty}^{-1} W_i^{(1)}(t/T) W_i^{(2)}(t/T) \rho_i(t/T) A_{ij} + O(T^{-1} 2^{-j}) \tag{6}$$

Also, the variance is given by

$$\begin{aligned} Var I_{j,t,T}^{(1,2)} &= \sum_{i=-\infty}^{-1} S_i^{(1)}(t/T) A_{i,j} \sum_{i=-\infty}^{-1} S_i^{(2)}(t/T) A_{i,j} \\ &+ \Big(\sum_{i=-\infty}^{-1} W_i^{(1)}(t/T) W_i^{(2)}(t/T) \rho_i(t/T) A_{i,j} \Big)^2 + O(2^{-2j} T^{-1}) \end{aligned}$$

where $A_{i,j}$ is the autocorrelation wavelet inner product matrix $A_{i,j} = \sum_\tau \Psi_i(\tau) \Psi_j(\tau)$.

We can see from Proposition 1 that the expectation of the wavelet cross-periodogram is composed of the sum of wavelet cross-spectra, $C_j(z)$. The cross-periodogram is therefore a natural estimator of the wavelet cross-spectrum, but we first need to correct for the bias incurred by the matrix $A_{i,j}$. Also, since the cross-periodogram has non-vanishing variance, it needs to be smoothed to obtain consistency. For this we use simple moving average smoothing. Other, more advanced smoothing techniques (see for example Nason et al. (2000)) are potentially viable and will be considered in future work. The estimator is therefore constructed by first smoothing the periodogram to give $\widetilde{I}_{j,t,T}^{(1,2)} = \frac{1}{2M+1} \sum_{m=-M}^{M} I_{j,t+m,T}^{(1,2)}$, and then correcting the smoothed periodogram using $\widehat{C}_l(t/T) = \sum_{j=-J*}^{-1} \widetilde{I}_{j,t,T}^{(1,2)} A_{l,j}^{-1}$ for some $J^* < \log_2(T)$ to be specified later, chosen to ensure the consistency of $\widehat{C}_l(z)$.

Proposition 2. *Let $J^* = \alpha \log_2(T)$ where $\alpha \in (0, 1)$. The estimator $\widehat{C}_l(t/T)$ converges in probability to $W_l^{(1)}(t/T) W_l^{(2)}(t/T) \rho_l(t/T)$ provided that $MT^{\alpha-1} \to 0$ as $T \to \infty$ and $M \to \infty$ for each fixed scale l.*

The wavelet periodograms, $I_{j,t,T}^{(i)}$ for $i = 1, 2$ are smoothed and corrected similarly to give $\widetilde{I}_{j,t,T}^{(i)} = \frac{1}{2M+1} \sum_{m=-M}^{M} I_{j,t+m,T}^{(i)}$, and $\widehat{S}_l(t/T) = \sum_{j=-J*}^{-1} \widetilde{I}_{j,t,T}^{(i)} A_{l,j}^{-1}$.

Proposition 3. *Let $J^* = \alpha \log_2(T)$ where $\alpha \in (0, 1)$. Then $\widehat{S}_l^{(i)}(t/T)$ converges in probability to $S_l^{(i)}(t/T)$ provided that $MT^{\alpha-1} \to 0$ as $T \to \infty$ and $M \to \infty$ for each fixed scale l.*

Given estimates of the cross-spectrum, $\widehat{C}_l(t/T)$, and individual spectra, $\widehat{S}_l^{(i)}(t/T)$ of each process and provided that $S_l^{(1)}(t/T) > 0$ and $S_2^{(1)}(t/T) > 0$ the estimator of the locally stationary wavelet coherence given by

$$\widehat{\rho}_l(t/T) = \frac{\widehat{C}_j(t/T)}{\sqrt{\widehat{S}_j^{(1)}(t/T) \widehat{S}_j^{(2)}(t/T)}} \tag{7}$$

converges in probability to $\rho_l(t/T)$ by Slutsky's theorem [Davidson (1994)].

Having demonstrated how to estimate $\rho_l(t/T)$, the result provides us with a multiscale decomposition of the local dependence structure between the two series. The decomposition allows us to distinguish between fine-scale and coarse-scale dependence.

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