What Happens When You Regulate Risk?

Evidence from a Simple Equilibrium Model

By

Jón Daníelsson
And
Jean-Pierre Zigrand

DISCUSSION PAPER 393

October 2001

FINANCIAL MARKETS GROUP
AN ESRC RESEARCH CENTRE

LONDON SCHOOL OF ECONOMICS

ISSN 0956-8549-393

What Happens When You Regulate Risk? Evidence from a Simple Equilibrium Model*

Jón Daníelsson
Jean–Pierre Zigrand

Department of Accounting and Finance, and
Financial Markets Group
London School of Economics

Abstract

The implications of Value–at–Risk regulations are analyzed in a CARA–normal general equilibrium model. Financial institutions are heterogeneous in risk preferences, wealth and the degree of supervision. Regulatory risk constraints lower the probability of one form of a systemic crisis, at the expense of more volatile asset prices, less liquidity, and the amplification of downward price movements. This can be viewed as a consequence of the endogenously changing risk appetite of financial institutions induced by the regulatory constraints. Finally, the Value–at–Risk constraints may prevent market clearing altogether. The role of unregulated institutions (hedge-funds) is considered. The findings are illustrated with an application to the 1987 and 1998 crises.

Journal of Economic Literature classification numbers: G12, G18, G20, D50.

*An earlier version was presented at CERGE-EI, LSE, the Financial Stability Seminar at the Bank of England and at the European Finance Association Meetings. We thank Hyun Shin, Michel Habib and the participants for their helpful comments. Jean-Pierre Zigrand is the corresponding author, *j.p.zigrand@lse.ac.uk*. Our papers can be downloaded from *www.RiskResearch.org*.

1 Introduction

Financial institutions have traditionally been subjected to extensive regulations governing their behavior, where the form of regulations continually evolves. Ideally, the choice of regulatory instruments should represent the culmination of a consulting process involving theoretical, empirical, and political considerations. The political objectives of financial regulation have recently been stated as "limiting the costs to the economy from financial distress" and "protecting depositors." (Crockett(2000))[1]. In contrast, the empirical and especially the theoretical aspects of risk regulations are rather less studied. Our interest is the formal analysis of present risk regulations and its effects on the economy.

For market risk, the chosen regulatory instrument is internal risk modelling in the form of Value–at–Risk (VaR). The reason for the choice of VaR as a regulatory tool is somewhat obscure. Perhaps, at the time (early 1990s), it was felt that it represented the state–of–the–art in risk management techniques. Scant evidence exists however that the present market risk regulatory structure by means of Value–at–Risk Basle Committee on Banking Supervision(1996)) is based on any substantive theoretical and empirical analysis, since at the time of its adoption, no published formal economic arguments advocating VaR over other techniques appear to have been in existence[2]. Subsequent research by Artzner et al.(1999)Artzner, Delbaen, Eber, and Heath) indicates that VaR is not a desirable statistical measure because it fails to be subadditive. From Ahn et al.(1999)Ahn, Boudukh, Richardson, and Whitelaw) we note that the VaR measure may not be reliable because it is easy for a financial institution to legitimately manage reported VaR through options. Basak and Shapiro(2001)) analyze VaR based risk management and find that other risk management techniques are preferable.

We propose a general equilibrium model where risk regulations, despite being successful in reducing the probability of an ex-post systemic collapse, impose real costs on the economy during non–crisis periods, and may even be the cause of an ex-ante market breakdown by preventing financial institutions from absorbing the asset supply. Furthermore, in non–crisis periods, volatility is higher, while liquidity (and often prices as well) is lower in the presence of regulations.

In recent financial history, market risk and portfolio constraints have played

[1]Mr. Crockett is the General Manager of the Bank for International Settlements

[2]The choice of the multiplication factor three is most likely due to statistical arguments advanced by Stahl, ultimately published in Stahl(1997)).

a fundamental role during at least two crisis episodes: The 1987 crash and the 1998 crisis. In 1987, portfolio insurance was widely used to contain downside risk. A key feature of portfolio insurance is that complicated trading strategies must be executed when the markets are falling. During the 1987 crash, a large number of financial institutions attempted to execute similar trades simultaneously due to portfolio insurance, perhaps inevitably leading to large price swings and lack of liquidity. This may be a partial reason why the price drop was of such short duration. After the crisis, portfolio insurance ceases to be a factor, enabling prices to recover. Risk regulations played a role similar to the one attributed to portfolio insurance in the 1998 crisis. As volatility increased, banks only had two options, either to increase capital or dispose of risky assets. In the short run, this effectively implied that banks had to dispose of risky assets. This in turn may have contributed to the increase in volatility and the decrease in liquidity. We note however, that the relative importance of risk regulations during this episode is as of yet unclear. We analyze the 1998 crisis in Section 6.3 below.

Our model contains three types of agents in addition to the supervisory authorities: regulated financial institutions (RFI) such as banks and certain fund managers, unregulated financial institutions (UFI) such as hedge funds, and finally noise traders. We use a standard two period model where in the first period heterogeneous financial institutions have endowments of a number of risky and riskless assets which pay off normally distributed payoffs in the last period. The regulated institutions are subject to a risk constraint whereby the regulators impose a limit on VaR or extreme volatility. They are heterogeneous in risk aversion, and have CARA utility. The unregulated institutions also have CARA preferences, but they are free to hold any risk exposure they see fit. Noise traders on the other hand submit market orders. This in turn induces trading, price formation, and a price volatility level. Over the period, unless unbalanced risk–taking induces a systemic crisis, the assets pay off dividends, received in period one, and consumption occurs.

Imposing regulatory risk constraints on financial institutions has direct implications for market variables. These results are stated in Propositions 4 to 6. Using the unregulated economy as a benchmark, we demonstrate that binding risk regulations adversely affect prices, liquidity, and volatility when aggregate supply is sufficiently large. Furthermore, the tighter the constraint, the greater the adverse impact becomes. For easy visualisation, consider the single risky-asset economy, as illustrated on Figure 3 where we depict the equilibrium price of the risky asset as a function of the noise trades, ϵ. It turns out that the regulations cause the pricing function to become concave for typical trades, where the slope decreases with the severity of regulation.

Hence for a given change in demand, prices move more with regulation than without, implying higher (local) volatility and lower liquidity post regulation. In a crisis, we expect financial institutions to have to sell assets, e.g. to meet margin requirements. However, since in that case we move to the left in the figure, liquidity diminishes and (local) volatility shoots up.

Our results, outlined in Proposition 3, also have interesting implications for risk–appetite. The risk constraints prevent the less risk averse regulated institutions from holding the type and amount of risky assets they desire, inducing, via equilibrium price adjustments, the more risk averse financial institutions to purchase these risky assets. In effect, the institutions' *effective risk aversion* coefficient changes due to the constraints.

Despite the costs imposed upon the economy, regulations are usually thought to prevent *systemic collapses* or *systemic failures*. We note that no standard definition of systemic crisis exists (see De Bandt and Hartmann(2000)) for a survey), however a common notion of systemic crisis is when the entire banking system collapses. This is captured in our setting by assuming that in a systemic collapse the real output of all assets held drops to zero. Within our model, the probability of systemic crisis increases with the imbalance in risk-taking. This formulation may be viewed as a stylized way of capturing the domino effects resulting from the failure of several extremely levered financial institutions, where the externality of contributing towards the likelihood of a systemic crash is not internalized by the market participants.

Effectively, a systemic collapse happens in our model because dividends are intermediate inputs into the production process for consumption goods. The production process depends on inputs being appropriately distributed among agents. In particular, if specific assets are too concentrated by particular agents, production fails, and no consumption commodity is produced. This occurs when the less risk averse agents invest in the more risky, and hence more profitable inputs. Since no agent can individually affect the economy and hence risk distribution, the potential for systemic collapse does not factor in their decision making. It is the elimination of this externality that the supervisory authorities set out to achieve by means of regulations. We follow the regulators in choosing VaR as a regulatory instrument.

There exists however a possibly stronger form of an extreme event where equilibria are no longer defined due to the VaR constraints. Such an event, termed here *market breakdown*, is not possible in the absence of regulation since an equilibrium always exists in the unconstrained economy. The effect of VaR regulations is to reduce the capacity of investors to take on risk, introducing the possibility that markets cannot clear. Note that this is caused by the particular choice of a regulatory regime. As institutions try to sell their assets, the regulatory constraints effectively make it harder and harder for the market to absorb the supply up to the point where markets break down. There are economies where a reduction in the probability of one form of systemic crisis is thus only achieved by the introduction of the possibility of complete market breakdowns. The only way to prevent a market breakdown from happening is to allow (sufficiently many) financial institutions to remain unregulated. In other words, attempts to bring hedge funds under the regulatory umbrella may perversely lead to market breakdowns. Indeed, Brown et al.(1998)Brown, Goetzmann, and Park) have observed that the unregulated hedge funds actually helped in alleviating the Asian crisis.

The structure of the paper is as follows. In Section 2 we present the ingredients of the model. Section 3 discusses a financial institution's decision problem, and Section 4 establishes the necessary and sufficient conditions for the existence of equilibria. In Section 5 we study the impact on risk–taking, depth and volatility, and we analyze financial crises in Section 6 with special emphasis on the 1988 crisis. Section 7 concludes. Most proofs are contained in the Appendix, and all figures are at the end of the paper.

2 The Model

Our economy is a standard two period constant absolute risk–aversion model without asymmetric information and with stochastic asset supply. There are three families of agents: regulated financial institutions (RFI) that are subjected to regulatory risk constraints (e.g. banks), unregulated institutions (URI) (e.g. hedge funds), and noise traders. In the first period the UFIs and RFIs invest their endowments in both risky and riskless assets, and the noise traders submit an aggregate market order for assets. Consumption occurs at the second date.

The economy has $N+1$ assets, asset 0 is referred to as the *riskless asset* and promises to pays off d_0 . There are also N *risky assets* with promised normally distributed stochastic payoffs $\boldsymbol{d} \sim N\left(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}\right)$ at time 1. We denote this law of $\boldsymbol{d}$ by $\mathbb{P}^d$. Expectations taken with respect to this measure are denoted by E^d. We assume $\hat{\boldsymbol{\Sigma}}$ is positive definite (and in particular invertible, meaning that there are no redundant assets) and that the dividends are distributed independently of the noise trader demand $\boldsymbol{\epsilon}$. The vector of risky returns $\boldsymbol{R}$ is defined as the vector whose ith element is $R_i \equiv d_i/q_i$, the dividend of the ith asset divided by its price, q_i. We denote the price vector by $\boldsymbol{q}$. Returns are then normal with mean and variance $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ depend on $\boldsymbol{q}$.

The types of UFIs and RFIs, denoted by h, are distributed on the interval $[\ell, 1]$, $\ell > 0$[3] and have neither state–contingent endowments nor state–contingent preferences. Each type h is characterized by a constant coefficient of absolute risk aversion (CARA) α^h as well as an initial endowment of the riskless asset θ_0^h and of the risky assets $\boldsymbol{\theta}^h$. A fraction η of agents of each type h are regulated, the remaining fraction is unregulated.

We do not model the noise traders' utility explicitly, and only assume that they are hit by liquidity shocks at time 0 which cause them to submit an aggregate market order for $\boldsymbol{\epsilon}$ assets. This demand is assumed to be distributed on $\mathbf{E} \subset \mathbb{R}^N$ according to the law $\mathbb{P}^\epsilon$, for simplicity assumed to be independent of the law governing asset payoffs. Because the market order has to be absorbed by the UFIs and RFIs, prices depend upon $\boldsymbol{\epsilon}$.[4]

[3] The reason for this will become clear when we assume that their index h corresponds to their coefficient of absolute risk aversion.

[4] Notice that there are no income effects due to the CARA assumption and no informational effects due to the assumption that liquidity shocks bear no payoff relevant information.

3 Decision Problem of the Financial Institutions

Each FI is characterized by its type h, which determines risk-aversion and endowments, and by its regulation status t, which is either $t = \{r\}$ if the FI is regulated, or $t = \{u\}$ if it is unregulated. A FI (h, t) invests its initial wealth W_0^h in a portfolio comprising both riskless and risky assets, $(y_0^{h,t}, \boldsymbol{y}^{h,t})$. The time–zero wealth of an agent of type h (regulated or unregulated) comprises initial endowments in the riskless asset, θ_0^h, as well in risky assets, $\boldsymbol{\theta}^h$, so that $W_0^h \equiv q_0 \theta_0^h + \boldsymbol{q}' \boldsymbol{\theta}^h$. As a general rule, we denote aggregates over all types h by a superscript a, e.g. the aggregate riskless holdings by FIs are denoted as $y_0^a \equiv \eta \int_\ell^1 y_0^{h,r} dh + (1 - \eta) \int_\ell^1 y_0^{h,u} dh$, and the aggregate amount of outstanding risky assets is $\boldsymbol{\theta}^a \equiv \int_\ell^1 \boldsymbol{\theta}^h \mathrm{d}h$. Since the time-zero budget constraint $q_0 \theta_0^h + \boldsymbol{q}' \boldsymbol{\theta}^h \geq q_0 y_0^{h,t} + \boldsymbol{q}' \boldsymbol{y}^{h,t}$ is homogeneous of degree zero in prices, we can normalize, without loss of generality, the price of the riskless asset to $q_0 \equiv 1$, i.e. the riskless asset is used as the time–zero numéraire. We can write $R_f \equiv d_0 = d_0 / q_0$ for the return on the riskless asset. At time 1, the consumption commodity plays the role of the numéraire. Agents' preferences are:

Assumption 1 *The type characteristics $(1/\alpha^h, \theta_0^h, \boldsymbol{\theta}^h)$ are distributed via an integrable mapping $h \in [\ell, 1] \mapsto \left(1/\alpha^h, \theta_0^h, \boldsymbol{\theta}^h\right)$. A fraction of agents $\eta \in [0, 1]$ of each type h is regulated $(t = \{r\})$, while the complement $1 - \eta$ is unregulated* $(t = \{u\})$. ir c t p egu fn ol n which detd f m men l

In theory, a large number of possible regulatory environments exists for this purpose. In practice, we are not aware of any published research into the pros and cons of alternative market risk regulatory methodologies, and as a result, we adopt the standard market risk methodology, i.e., Value–at–Risk. The constraint takes the form (we drop the superscript r whenever no confusion arises):

$$\mathbb{P}^d\left[(E^d[W^h]-W^h)\geq VaR\right]\leq\bar{p},$$

i.e. the probability of a *loss* larger than the uniform regulatory number VaR is sufficiently small. Since the portfolio payoffs are normal, a sufficient statistic for portfolio risk is the volatility of W^h. The VaR constraint can therefore be stated as an exogenous upper bound $\bar{v}$ on portfolio variance,[5]

$$\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h\leq\bar{v}. \tag{1}$$

Each RFI solves

Problem 1 (Risk Constrained Problem)

$$\max_{\{\boldsymbol{y}^h,y_0^h\}} E^d[u^h(x^h)] \qquad \text{s.t.} \qquad \begin{aligned} &y_0^h+\boldsymbol{q}'\boldsymbol{y}^h\leq\theta_0^h+\boldsymbol{q}'\boldsymbol{\theta}^h\\ &x^h=W^h\equiv y_0^h d_0+\boldsymbol{d}'\boldsymbol{y}^h \quad \mathbb{P}^d\text{a.s.}\\ &\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h\leq\bar{v}\end{aligned}$$

In the Appendix we derive the optimal portfolio as summarized in the Lemma below.

Lemma 1 (Optimal Portfolio) *The optimal portfolio of risky assets for RFI (h,t) has the mean-variance form*

$$\boldsymbol{y}^{h,t}=\frac{1}{\alpha^h+\phi^{h,t}}\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}}-R_f\boldsymbol{q}) \tag{2}$$

where $\phi^{h,u}\equiv 0$ and $\phi^{h,r}\equiv\frac{2\lambda^{h,r}}{E^d[u^{h\prime}(W^h)]}\geq 0$, with $\lambda^{h,r}$ being the Lagrange multiplier of the VaR constraint. The effective degree of risk-aversion, $\alpha^h+\phi^{h,r}$ is independent of the initial wealth W_0^h and only depends on α^h, $\boldsymbol{q}$ and $\bar{v}$.

[5]Indeed, denoting the cumulative standard normal distribution function by $N(\cdot)$, $\mathbb{P}^d\left[(E^d[W^h]-W^h)\geq VaR\right]\leq\bar{p}$ iff $N\left(\frac{-VaR}{\mathrm{Std}^d(W^h)}\right)\leq\bar{p}$ iff $\mathrm{Std}^d(W^h)\leq\frac{VaR}{-N^{-1}(\bar{p})}$ iff $\mathrm{Var}^d(W^h)\leq\left(\frac{VaR}{-N^{-1}(\bar{p})}\right)^2\equiv\bar{v}$.

Whereas the coefficient of absolute risk-aversion is constant for unrestricted FIs, it is effectively endogenous for the FIs subjected to the VaR regulations and larger than their utility-based coefficient, $\alpha^h + \phi^{h,r} \geq \alpha^h$. This is one way of capturing the often-heard expression among practitioners that "risk-aversion went up." This is reminiscent of the effet of portfolio insurance on optimal asset holdings found in Grossman and Zhou(1996)). Also see Basak and Shapiro(1995)) and Gennotte and Leland(1990)).

Suppose that equilibria exist. We shall provide necessary and sufficient conditions for existence in the next section. Market clearing prices solve $\eta \int_\ell^1 \boldsymbol{y}^{h,r} \mathrm{d}h + (1-\eta) \int_\ell^1 \boldsymbol{y}^{h,u} \mathrm{d}h + \boldsymbol{\epsilon} = \boldsymbol{\theta}^a$, equivalently they satisfy the implicit relation (implicit since Ψ itself depends on $\boldsymbol{q}$):

$$\boldsymbol{q} = \frac{1}{R_f} \left[\hat{\boldsymbol{\mu}} - \Psi \hat{\boldsymbol{\Sigma}} (\boldsymbol{\theta}^a - \boldsymbol{\epsilon}) \right] \tag{3}$$

where

$$\Psi^{-1} \equiv \eta \int_\ell^1 \frac{1}{\alpha^h + \phi^{h,r}} \mathrm{d}h + (1-\eta) \int_\ell^1 \frac{1}{\alpha^h} \mathrm{d}h \tag{4}$$

The factor Ψ can be understood as a *market–price of risk* scalar, since (3) can be rewritten as

$$\hat{\boldsymbol{\mu}} - R_f \boldsymbol{q} = \hat{\boldsymbol{\Sigma}} \left[\Psi (\boldsymbol{\theta}^a - \boldsymbol{\epsilon}) \right]$$

which shows that $\Psi(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})$ is the *market–price of risk* vector, the product of Ψ, a scalar common to all N sources of risk, and the residual market portfolio (RM) $\boldsymbol{\theta}^a - \boldsymbol{\epsilon}$. Alternatively, Ψ can be viewed as the reward-variability ratio of the (residual) market, $\Psi = \frac{\hat{\mu}_{\mathrm{RM}} - R_f q_{\mathrm{RM}}}{\hat{\sigma}^2_{\mathrm{RM}}}$. Compared to an economy without any VaR constraints ($\eta = 0$) where the market-price of risk scalar is $\gamma \equiv \left(\int_\ell^1 \frac{1}{\alpha^h} \mathrm{d}h \right)^{-1}$, we have $\Psi \geq \gamma$. In other words, the market price of risk is higher in a constrained economy than in an unconstrained one. For completeness, we could also derive the CAPM relation (with the residual market portfolio $\boldsymbol{\theta}^a - \boldsymbol{\epsilon}$ as opposed to the market portfolio),

$$\boldsymbol{\mu} - R_f \mathbf{1} = \boldsymbol{\beta}_{RM} (\mu_{RM} - R_f) \tag{5}$$

We can use (3) to express asset demands at equilibrium as

$$\boldsymbol{y}^{h,t} = \varpi^{h,t}(\boldsymbol{q}) (\boldsymbol{\theta}^a - \boldsymbol{\epsilon}) \tag{6}$$

where $\varpi^{h,t}(\boldsymbol{q}) \in \mathbb{R}_+$ is the effective risk tolerance of (h,t) as a fraction of the aggregate risk tolerance,

$$\varpi^{h,t}(\boldsymbol{q}) \equiv \frac{\Psi}{\alpha^h + \phi^{h,t}} = \frac{\left(\frac{1}{\alpha^h+\phi^{h,t}}\right)}{\eta \int_\ell^1 \left(\frac{1}{\alpha^k+\phi^{k,r}}\right) dk + (1-\eta) \int_\ell^1 \left(\frac{1}{\alpha^k}\right) dk}.$$

We summarize these results in the following proposition.

Proposition 1 *Assume an equilibrium exists. Each FI* (h,t) *holds a fraction of the residual market portfolio, equalling their share of the aggregate effective risk tolerance,* $\varpi^{h,t}(\boldsymbol{q})$.

If subjected to binding r

To allow for closed-form solutions, we maintain in the rest of this paper the assumption that the FIs' risk aversion is uniformly distributed on $[\ell, 1]$, $\ell \geq 0$: $\alpha^h = h$. We establish the following result in the Appendix:

Proposition 2 (On the Existence and Uniqueness of Equilibria) *If $\eta < 1$, there exists a unique competitive equilibrium for any $(\boldsymbol{\epsilon}, \bar{v}, \ell) \in \boldsymbol{E} \times [0, \infty) \times (0, 1]$.*

If $\eta = 1$, there exists an equilibrium for $\ell \in [0, 1]$ and for $(\bar{v}, \boldsymbol{\epsilon})$ satisfying $\boldsymbol{\epsilon} \in \mathcal{E}(\bar{v}, \ell) \equiv \{\boldsymbol{\epsilon} \in \mathbf{E} : [(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})]^{1/2} \leq (1 - \ell)\sqrt{\bar{v}}\}$. For $(\bar{v}, \boldsymbol{\epsilon})$ s.t. $\boldsymbol{\epsilon} \in$ int $\mathcal{E}(\bar{v}, \ell)$, the equilibrium (the pair comprising the price and the allocation) is unique, while for $(\bar{v}, \boldsymbol{\epsilon})$ s.t. $\boldsymbol{\epsilon} \in \partial\mathcal{E}(\bar{v}, \ell)$ asset prices and consumption allocations are indeterminate (within a certain range of prices) but the allocation of risky assets is not.

In the case where $\eta = 1$, the necessary condition implies that there can only be an equilibrium if the residual aggregate payoff variability accrues to investors. Given our assumptions, this implies that a typical RFI, is actually permitted to assume per–capita risk, i.e. $\sqrt{(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})}/(1-\ell) \leq \sqrt{\bar{v}}$. Introducing some more notation, define the number of agents over which the risk needs to be evenly spread by $\kappa(\boldsymbol{\epsilon}; \bar{v}) \equiv \sqrt{\frac{(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})}{\bar{v}}}$, in which case equilibria exist iff $\kappa \leq 1 - \ell$. Alternatively, if we define the critical level of regulation $v_*(\boldsymbol{\epsilon}) \equiv \frac{(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})}{(1-\ell)^2}$, the condition $\boldsymbol{\epsilon} \in \mathcal{E}(\bar{v}, \ell)$ is equivalent to $\bar{v} \geq v_*(\boldsymbol{\epsilon})$, whereas $\boldsymbol{\epsilon} \in \partial\mathcal{E}(\bar{v}, \ell)$ is equivalent to $\bar{v} = v_*(\boldsymbol{\epsilon})$.

4.1 On Hedge Funds and Market Clearing

In the present regulatory environment, some FIs, e.g. hedge funds, are exempted from the risk regulations. Following the collapse of LTCM, some policymakers have discussed whether to bring hedge funds under the regulatory umbrella. We can analyze the impact of regulating hedge funds within the model and see how this affects market outcomes. Our starting point is a situation where a fraction $1 - \eta > 1$ of FIs is unregulated. We know that equilibria exist in this case.

Any large market order of risky assets $\boldsymbol{\epsilon} \in \boldsymbol{E}$ has to be absorbed by the FIs. Intuitively, as η is raised further towards 1, more and more FIs are subjected

$\eta)x^{h,u})dh = x^a$. Walras' Law at times 0 and 1 says that $(y_0^a - \theta_0^a + \epsilon_0) + \boldsymbol{q}'(\boldsymbol{y}^a - \boldsymbol{\theta}^a + \boldsymbol{\epsilon}) = 0$ [W0] and $x^a + x^\epsilon = d_0(y_0^a + \theta_0^\epsilon + \epsilon_0) + \boldsymbol{d}'(\boldsymbol{y}^a + \boldsymbol{\theta}^\epsilon + \boldsymbol{\epsilon})$ [W1]. So assume that $\boldsymbol{y}^a - \boldsymbol{\theta}^a + \boldsymbol{\epsilon} = 0$. Then by [W0] the market for the riskless asset clears as well, and by [W1] we immediately have $x^a + x^\epsilon = d_0(\theta_0^a + \theta_0^\epsilon) + \boldsymbol{d}'(\boldsymbol{\theta}^a + \boldsymbol{\theta}^\epsilon)$ under "normal market conditions."

to a maximum level of risk they can take on, and prices adjust to guide the supply towards the institutions whose constraint is not binding yet. But as $\eta \to 1$, there are no such institutions left. So for noise trades sufficiently different from $\boldsymbol{\theta}^a$, no price will be able to induce market-clearing.

Figure 1 illustrates this phenomenon in an economy with two assets and different levels of tightness $\bar{v}$. Each level of tightness determines an ellipsoid set of noise supplies that can be supported by a competitive equilibrium.

Given a noise trade $\boldsymbol{\epsilon}$, define $v^*(\boldsymbol{\epsilon})$ as the weakest level of regulation for which there is an agent whose risk-taking constraint is binding. For $\bar{v}$ larger than the critical level $\sup_{\boldsymbol{\epsilon} \in \boldsymbol{E}} v^*(\boldsymbol{\epsilon})$ (whose expression is derived in the appendix), regulations are so loose that no institution hits its risk-constraint, no matter which market orders the noise traders submit. This implies that the ellipsoid extends to the whole of $\mathbb{R}^2$.

For the regulatory level $\bar{v}_1 < \sup_{\boldsymbol{\epsilon} \in \boldsymbol{E}} v^*(\boldsymbol{\epsilon})$ on the other hand, equilibria can be supported for noise trades in the ellipsoid $\mathcal{E}(\bar{v}_1)$ only. For $\boldsymbol{\epsilon}$ outside of this ellipsoid, FIs cannot absorb the supply as described earlier, and markets break down. And for a tighter regulatory level $\bar{v}_2 < \bar{v}_1$, the set of supportable supplies shrinks even further, $\mathcal{E}(\bar{v}_2) \subset \mathcal{E}(\bar{v}_1)$.

It is interesting to note that it is not true that equilibria exist for small noise trades $\boldsymbol{\epsilon}$, but only for noise trades around the aggregate outstanding supply $\boldsymbol{\theta}^a$. This seems to be an unlikely outcome in the real world as one would suspect the support of $\boldsymbol{\epsilon}$, $\boldsymbol{E}$, to be a small neighbourhood of $\mathbf{0}$, at least during "normal" market conditions. In particular for tight regulations such as $\bar{v}_2$, there is no equilibrium if noise traders don't trade or trade only a little: $\mathbf{0} \notin \mathcal{E}(\bar{v}_2)$. This suggests the following corollary:

Corollary 1 *Assume $\eta = 1$ and assume that not all assets are in zero net supply. Furthermore, also suppose that the support of noise trades $\boldsymbol{E}$ is a small enough neighbourhood of $\mathbf{0}$. If the supervisory authorities impose stringent risk limits (in the sense that $\bar{v}$ is small enough to lead to $\mathcal{E}(\bar{v}) \not\ni \mathbf{0}$, i.e. $\bar{v} < \frac{\boldsymbol{\theta}^{a\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{\theta}^a}{(1-\ell)^2}$), some agents need to be exempted from those constraints for markets to clear.*

The immediate implication is that UFIs (e.g. hedge funds), are needed to ensure market clearing ($\eta < 1$ needed), suggesting that demands for the regulation of hedge funds may be misguided, at least for assets that are not in zero net supply. For derivatives, however, $\boldsymbol{E} \subseteq \mathcal{E} \ni \mathbf{0}$, and no exemptions are required as long as regulations are not too strict.

As an illustration, assume in Figure 1 with a regulatory level $\bar{v}_2$ that noise traders dump assets (an irrational panic, say), leading to $\boldsymbol{\epsilon} \in \mathbb{R}^2_-$. No matter how low prices fall, there is no price level low enough for RFIs to be able to absorb this supply since they are all prevented from holding the risk. Hedge funds on the other hand are the natural buyers for undervalued assets, and if given the opportunity to buy into the selling, they restore equilibrium. This scenario is reminiscent of aspects of the Asian crisis, especially the crucial four month period of June through September 1997, such as for instance exposited in the empirical study by Brown et al.(1998)Brown, Goetzmann, and Park):

> The ringgit dropped by 10% over this period... The hedge funds appeared to be unwinding their negative positions in the ringgit or its correlates beginning in June. In fact, the figure suggests they were buying into the ringgit crash from June through August. An interpretation of this activity is that the hedge fund managers were supplying liquidity to a rapidly falling market. It is tempting to suggest that they cushioned the rapid fall of the ringgit, rather than hastened it.

One may advance the hypothesis that without the hedge-funds, markets would not have cleared, and the Asian economies would have had to resort much more to capital controls.

4.2 Special Cases when all Institutions are Regulated

Two interesting results arise in the special case when all FI's are regulated, i.e. $\eta = 1$. The first relates to the indeterminacy that arises when $\eta = 1$ and $\bar{v} = v_*(\boldsymbol{\epsilon})$, while the second result exhibits an economy where equilibria exist in a constrained economy but not in a unconstrained economy.

Remark 1 (On Indeterminacy) Consider the following basic intuition behind the indeterminacy result that holds if $\eta = 1$. When $\boldsymbol{\epsilon} \in \partial\mathcal{E}$, the RFIs hold the maximal risk compatible with the regulations. Compensating them more for their risk will not induce any of them to hold more of the risky asset. So raising $\Psi \geq (1-\ell)^{-1}$ will scale prices, but will not affect the allocation of the risky assets. Since they hit their constraint anyway, any price that would induce them to hold a yet riskier portfolio must yield the same effective risky demand. This is reminiscent of the indeterminacy of no–arbitrage prices of non–redundant assets.

Formally, this can be seen from the individual portfolio: for $\boldsymbol{\epsilon} \in \partial\mathcal{E}$, it turns out that $\frac{\alpha^h+\phi^h}{1-\ell} = \Psi$, so that $\boldsymbol{y}^h = \frac{\Psi}{\alpha^h+\phi^h}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon}) = \frac{1}{1-\ell}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})$, independent of Ψ. The effective risk aversion due to regulation rises proportionately with Ψ, not inducing any change in the allocation of assets. Still, since $W^h = y_0^h R_f + \boldsymbol{d}'\boldsymbol{y}^h$ and $y_0^h = \theta_0^h + \boldsymbol{q}'[\boldsymbol{\theta}^h - \frac{1}{1-\ell}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})]$, we can see that both the holdings of the riskless asset as well as the consumption choices do depend on Ψ via $\boldsymbol{q}$. Since $\boldsymbol{q} = R_f^{-1}[\hat{\boldsymbol{\mu}} - \Psi\hat{\Sigma}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})]$, raising Ψ affects the distribution of y_0^h and of consumption as long as $\boldsymbol{\theta}^a \neq \boldsymbol{\epsilon}$ (by the full rank assumption on $\hat{\Sigma}$) and not all h have the same endowments of risky assets, $\boldsymbol{\theta}^h - \frac{1}{1-\ell}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon}) \neq 0$ for some h.

Similar indeterminacy has also been noted by Grossman and Zhou(1996)). More fundamentally, the general theory behind the real indeterminacy arising from constrained net trades has been studied (albeit in a model with finitely many states and with linear net trade restrictions) by Polemarchakis and Siconolfi(1998)).

Remark 2 (On Existence and Risk-Neutrality) In contrast to the previous discussion, here we consider a situation where equilibria exist in the presence of strict regulations, $\eta = 1$, but not without. The reason is that the unregulated fringe is too risk–neutral in aggregate but too risk-averse as individuals:

Corollary 2 *Assume $\ell = 0$. In a risk-constrained economy where $\bar{v} < \infty$, and for $\boldsymbol{\epsilon} \in \mathcal{E}$, an equilibrium exists if $\eta = 1$, while no equilibrium exists in the less constrained economy $\eta < 1$ (except in the, presumably unlikely, event that $\boldsymbol{\epsilon} = \boldsymbol{\theta}^a$).*

The reason for this is clear. In the economy with $\eta < 1$, Ψ, the inverse of the aggregate effective risk tolerance defined in (4), is zero, and the only equilibrium price candidate is the risk neutral price vector $\boldsymbol{q} = R_f^{-1}\hat{\boldsymbol{\mu}}$. But if that's the case, then almost no FI, regulated or unregulated, holds any assets, since their risk aversions are almost all strictly positive. Thus demand cannot equal supply, unless by chance the whole supply is met by noise traders, $\boldsymbol{\epsilon} = \boldsymbol{\theta}^a$. On the other hand we saw in Proposition 2 that equilibria exist for all $\boldsymbol{\epsilon} \in \mathcal{E}$ in the constrained economy where $\eta = 1$.

The intuition as to why equilibria exist if $\eta = 1$ is as follows. In the risk-constrained economy the inverse of the aggregate risk tolerance is effectively $\Psi(\ell)^{-1} = \int_\ell^1 \frac{1}{\alpha^h + \phi^{h,r}} dh$. Assume first that $\boldsymbol{\epsilon} \neq \boldsymbol{\theta}^a$. The inverse of aggregate risk-tolerance is bounded away from zero, $\Psi > 0$ at $\ell = 0$, so that the economy's risk-tolerance is not infinite, and an equilibrium exists. Next assume that $\boldsymbol{\epsilon} = \boldsymbol{\theta}^a$. Pricing is then risk-neutral, but this is exactly what an equilibrium requires, since FIs must not hold any assets.

5 Impact on Risk–Taking, Depth and Volatility

The imposition of the VaR constraints affects the equilibria directly, with interesting results on risk-taking, liquidity, and volatility. We present our main results in a series of Propositions, with all proofs relegated to the Appendix.

5.1 Risk–Taking

Consider the implications of the VaR constraint on those agents who are directly affected by the constraint and those only indirectly affected by the constraint. We reserve the term *risk-aversion* to their CARA coefficients α^h. We call $\alpha^h + \phi^{h,r}$ their coefficient of *effective risk-aversion*, and we call its inverse *risk appetite*. It is shown in the Appendix that $\alpha^h + \phi^{h,r} = \max\left\{\alpha^h, \kappa\Psi\right\}$, where κ is the number of agents needed to spread the risk without violating anyones's VaR constraint, and $\kappa\Psi$ is the index of the marginal FI whose constraint is barely binding.

Proposition 3 (Effects on Risk–Taking)

(i) Less risk averse inframarginal RFIs, $h \in [0, \Psi\kappa)$, have less risk appetite in the presence of VaR constraints, $\alpha^h + \phi^{h,r} = \kappa\Psi > \alpha^h$, while the risk appetite of the more risk averse, $h \in [\Psi\kappa, 1]$, remains unchanged at α^h.

(ii) The lower the admissible risk-taking level $\bar{v}$, the more RFIs hit their risk-taking constraints, i.e. the index of the marginal RFI, $\Psi\kappa$, rises.

(iii) The more risk-averse RFIs hold riskier portfolios in the presence of VaR constraints than they would otherwise. Risk-taking is therefore more uniform in a regulated economy. For a given $\boldsymbol{\epsilon}$, in the limit $\bar{v} = v_*(\boldsymbol{\epsilon})$ and all constrained FIs hold the same risky portfolio.

Item (i) is intuitive: considering the RFI's maximization program, we see that the effective risk-aversion of RFI h is $\alpha^h + \phi^h$. Now since less risk-averse RFIs hold riskier portfolios, they hit the risk-constraint earlier than more risk-averse RFIs. And once the VaR constraint for RFI h binds, its effective risk-aversion equals $\kappa\Psi$, which is the same for all RFI's whose VaR constraint is binding. We see that the risk appetite is endogenous and fluctuates with $\boldsymbol{\epsilon}$. the more extreme a realization of $\boldsymbol{\epsilon}$ is, the more RFI's hit their constraint, lowering their risk appetite. And the degree to which they become effectively more risk-averse depends on the level of regulation, $\bar{v}$.

Items (ii) and (iii) show that as regulations are tightened, more agents hit the allowed risk-limit. Since the aggregate risk is unaffected by regulations (recall that we are ignoring the possibility of a systemic collapse at this stage), the tighter the regulatory risk constraint, the more risk is shifted from the less risk averse agents (the agents in the interval $[\Psi\kappa, \Psi\kappa + \frac{\partial \Psi\kappa}{\partial \bar{v}} d\bar{v}]$) via an appropriate price change to the unconstrained and the constrained but more risk-averse institutions. In other words, a binding regulatory risk constraint implies that financial institutions effectively become more uniform in behavior since their attitudes to risk becomes more homogeneous.

In the limit for a very tight policy, $\bar{v} = v_*(\boldsymbol{\epsilon})$, all hold the same risky portfolio. While in an unconstrained economy only the less risk-averse agents would hold a portfolio as risky or riskier than $\bar{v}$, the economy with risk-taking constraints forces each regulated investor to hold the same (maximal) amount of risk $\bar{v}$, no matter what his degree of risk-aversion $\alpha^h \in [\ell, 1]$. This is because each RFI's effective risk-aversion becomes identical, so that the ratios of their effective risk-aversion to the aggregate risk-aversion, ϖ^h, are identical.

This points out a potentially perverse implication of otherwise well–intentioned prudential regulations: By limiting the allowable level of extreme risk that can be taken, at equilibrium, the agents may be induced to hold portfolios that are riskier than the ones they would otherwise have held. If one does interpret RFIs close to ℓ as banks (since they effectively take deposits), then a stricter VaR regulation limits the natural role of these institutions (e.g. taking deposits and assuming risk), and optimal risk-sharing is compromised.

5.2 Depth and Volatility

The risk constraint affects the depth of the markets directly. In our context, *depth* (defined below) is an appropriate measure of liquidity, or alternatively the inverse depth, or *shallowness*. From (3, 4) and Definition 1 we know that the equilibrium pricing mapping is

$$Q(\boldsymbol{\epsilon}, \bar{v}) = \frac{1}{R_f} \left[\hat{\boldsymbol{\mu}} - \Psi(\kappa(\boldsymbol{\epsilon}, \bar{v})) \hat{\boldsymbol{\Sigma}} (\boldsymbol{\theta}^a - \boldsymbol{\epsilon}) \right]$$

with $\kappa(\boldsymbol{\epsilon}, \bar{v}) \equiv \sqrt{\frac{(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})' \hat{\boldsymbol{\Sigma}} (\boldsymbol{\theta}^a - \boldsymbol{\epsilon})}{\bar{v}}}$. The inverse depth, or shallowness, of the entire market is defined as the maximal extent to which an additional (unit-size) market order for a portfolio impacts its price, formally

$$\mathcal{S}(\boldsymbol{\epsilon}, \bar{v}) \equiv \max_{\boldsymbol{\theta} \text{ s.t. } ||\boldsymbol{\theta}||=1} |\boldsymbol{\theta}' dQ| = \max_{\boldsymbol{\theta} \text{ s.t. } ||\boldsymbol{\theta}||=1} |\boldsymbol{\theta}' (\partial_{\boldsymbol{\epsilon}} Q) \boldsymbol{\theta}|$$

We now state three of the main results of this paper with formal proofs in the Appendix.

Proposition 4 (Depth) *Depth ("liquidity") is lower the tighter the constraint (i.e. the smaller $\bar{v}$), $\frac{\partial \mathcal{S}(\boldsymbol{\epsilon},\bar{v})}{\partial \bar{v}} < 0$ for all $\boldsymbol{\epsilon} \in \mathcal{E}$. In particular, depth is lower in the regulated economy than in the unconstrained economy for any $\boldsymbol{\epsilon} \in \mathcal{E}$.*

Refer to Figure 3 for an illustration. No RFI's risk taking constraint is binding for $\epsilon \in [\underline{\theta}^a(\bar{v}), \bar{\theta}^a(\bar{v})]$, and $Q(\cdot, \bar{v})$ is defined on $\mathcal{E}(\bar{v}) \equiv [\underline{\epsilon}(\bar{v}), \bar{\epsilon}(\bar{v})]$, with $\underline{\epsilon}(\bar{v}) \equiv \theta^a - \frac{\sqrt{\bar{v}}}{\sigma}(1-\ell)$ and $\bar{\epsilon}(\bar{v}) \equiv \theta^a + \frac{\sqrt{\bar{v}}}{\sigma}(1-\ell)$. We have not made any assumptions regarding the distribution of $\boldsymbol{\epsilon}$. However, in most cases we expect $\boldsymbol{\epsilon} < \boldsymbol{\theta}^a$ because otherwise the noise traders' aggregate demand exceeds the value of all assets in the economy, i.e. in aggregate, the noise traders corner the market, in which case they really cease to be noise traders. In the two Propositions that follow, we assume that $\boldsymbol{\epsilon} < \boldsymbol{\theta}^a$ and that $N = 1$. This implies that the pricing function is concave over the relevant domain, and in most interesting cases (large negative noise trades, or restrictive regulations) the pricing function is strictly concave.

Proposition 5 (Bulls v.s. Bears) *Assume $N = 1$ and that regulations are sufficiently strict so that some agents are hitting the regulatory constraint at $\boldsymbol{\epsilon} = 0$, $\bar{v} < \left(\frac{\sigma\theta^a}{\ell \ln \ell^{-1}}\right)^2$. Also, assume that $\mathbb{P}^\epsilon([\theta^a, \infty)) = 0$.*

Then, inflows by the UFIs raise prices less than outflows lower them.

The single-asset intuition can then be extended to an arbitrary finite number of assets:

Proposition 6 (Volatility and Risk Constraints) *Assume that $\mathbf{E} \subseteq \mathcal{E}(\bar{v})$, and that $\bar{v}' > \bar{v}$. Equilibrium prices are more volatile in the economy with tighter regulations, $\bar{v}$, than in economy $\bar{v}'$. In particular, there is more volatility in the constrained economy than in the unconstrained economy. This follows from the fact that uniform shallowness implies ex-ante volatility.*

The basic intuition behind these results is that the less risk–averse RFIs (i.e. h close to ℓ), who are the first ones to hit the risk–taking constraint, have greater impact on pricing in the unconstrained economy. Therefore, the risk averse have more weight on the impact of the risk averse agents on price formation post regulations. The UFIs who were previously absorbed by the more risk–neutral RFIs now have to be absorbed by the more risk–averse. However, the risk-averse are less willing to absorb these (additional) market orders. Hence the imposition of the risk constraint reduces market depth, and the market impact of a market order is larger. Since the arrival of market orders is random, this generates more volatile asset prices.

6 Systemic Crises and Risk Regulations

Above, we side-stepped the issue of systemic crisis, the raison d'être for regulatory VaR constraints. From a more normative point of view it is obvious that these regulations lead to a Pareto inferior allocation, and hence it would be desirable to explicitly model systemic failures. Unfortunately, no existing systemic crisis model can be straightforwardly integrated into our model in a manner that allows for closed-form solutions (see Allen and Gale(2000)) for an attempt). Furthermore, since our main objective is the analysis of the positive asset pricing implications of risk regulations, our framework is clearly too abstract to capture many aspects of actual systemic crisis.

It is however possible to analyze some aspects of systemic crisis in our model. In specifying what constitutes a systemic crash, we follow a common definition, i.e., that the financial and credit system collapses and all real promises become effectively worthless. Furthermore, we can capture the contribution factors to systemic risk by following views expressed the regulators. In particular, Crockett(2000)), has argued that regulations are necessary because unregulated markets lead to output losses due to financial instability. He says that upswings bear the seed for a crisis eventually leading to lost output since in upswings "too many resources" flow into risky investments in a particularly unbalanced fashion. Presumably this implies that an excessive (due to a free-riding externality) amount of risk is concentrated on a small but significant number of investors. In turn, this imbalance may create, probably via a series of defaults, a systemic crisis in which total output collapses.

6.1 Modelling Systemic Crisis

Within our context, a systemic collapse implies the bankruptcy of the entire financial sector. We capture this by assuming that in a systemic collapse the real output of all assets, including the risk–less, is zero. This happens when the distribution risk along the FIs becomes extremely skewed, with the least risk averse assuming the bulk of financial risk. The failure of extremely levered institutions may trigger a domino effect, effectively causing a systemic collapse. The fear of this happening is behind the common view that some FIs are "too big to fail" and as a result, the supervisory authorities need to control risk taking. Embedded in this view of the world is the notion that FI's optimize without regard to systemic risk, and hence impose externalities on the financial system by taking on excessive risk.

Three main ingredients are needed for systemic collapse to occur. First, a sufficient number of FIs need to acquire extremely risky positions. Second, the financial sector realizes adverse earnings such that a sufficient number of these institutions default. Finally, this in turn triggers a domino effect. Within this context, the probability of the systemic collapse increases as the distribution risk becomes more uneven.

We measure the imbalance in the distribution of risk by a Gini coefficient, $G \in [0, 1]$, defined as

$$G(\boldsymbol{\epsilon}, \bar{v}) \equiv \frac{1}{1/2(1+\ell^2)-\ell} \int_\ell^1 \left[(h-\ell) - (1-\ell) \frac{1}{\int_\ell^1 M^k dk} \int_\ell^h M^k dk \right] dh$$

The inequality or distribution variable M is chosen to be $M^k = \|\boldsymbol{q}'\boldsymbol{y}^{1+\ell-k}\|$, $k \in [\ell, 1]$, the amount of resources invested in risky assets.[7] Note that this amount does not depend on initial wealth due to the CARA assumption.

Production breaks down if inequality is too large, $G > g$, where g is an exogenous constant which is part of the definition of the production technology, and to make the problem interesting, we assume that $\mathbb{P}^\epsilon(G > g) \in (0, 1)$ at $\bar{v} = +\infty$. Output therefore satisfies the equation

$$\text{output} = \begin{cases} 0 & \text{if } G(\boldsymbol{y}, \boldsymbol{q}; \bar{v}) > g \\ d_0 y_0^a + \boldsymbol{d}'\boldsymbol{y}^a & \text{otherwise} \end{cases}$$

Even though the technology is common knowledge, each institution is negligible and rationally disregards the effect of their own investment behavior on aggregate output and on the probability of a breakdown, $\mathcal{P}(\boldsymbol{\epsilon}, \bar{v}) \equiv \mathbb{P}^\epsilon(G > g)$. Markets are thus incomplete since no asset pays off in the state of the systemic collapse.

6.2 Program of Regulated Financial Institutions

In the second pe **0** d⊆o

collapse happens with probability $\mathcal{P}(\boldsymbol{\epsilon}, \bar{v})$. The probability $\mathcal{P}(\boldsymbol{\epsilon}, \bar{v})$ depends on the distribution of risk among the agents as discussed above.

The RFI's ex-ante program (before $\boldsymbol{\epsilon}$ is realized) consists in choosing demand schedules to solve

Problem 2 (Risk Constrained Ex-ante Problem)

$$
\begin{aligned}
\max_{\{\boldsymbol{y}^h, y_0^h\}} \quad & \mathcal{P}(\boldsymbol{\epsilon}, \bar{v}) u^h(0) + (1 - \mathcal{P}(\boldsymbol{\epsilon}, \bar{v})) E[u^h(x^h) \| G(\boldsymbol{\epsilon}, \bar{v}) < g] \\
\text{s.t.} \quad & y_0^h + \boldsymbol{q}'\boldsymbol{y}^h \leq \theta_0^h + \boldsymbol{q}'\boldsymbol{\theta}^h \qquad \mathbb{P}^\epsilon \text{ a.s.} \\
& x^h = W^h \equiv y_0^h d_0 + \boldsymbol{d}'\boldsymbol{y}^h \qquad \mathbb{P}^\epsilon \text{ and } \mathbb{P}^d \text{ a.s.} \\
& \boldsymbol{y}^{h\prime} \hat{\boldsymbol{\Sigma}} \boldsymbol{y}^h \leq \bar{v} \qquad \mathbb{P}^\epsilon \text{ a.s.}
\end{aligned}
$$

Since individual institutions are negligible, this formulation gives rise to a free-riding problem. Each financial institution knows that a systemic crisis follows if the imbalance between risky and riskless investments becomes too large, but they also know that they themselves have no effect on aggregates, and therefore choose to neglect the effect of their actions on $\mathcal{P}(\boldsymbol{\epsilon}, \bar{v})$. Given the manner by which we capture systemic risk, the RFI's Problem 2 is therefore equivalent, at each given $\boldsymbol{q}$, to Problem 1 solved before, and the demand schedules and equilibrium prices derived there remain unchanged.

Finally, Proposition 7 implies that the VaR regulations are effective in reducing the probability of a systemic crash.

Proposition 7 *Assume $\eta = 1$. For a given $\boldsymbol{\epsilon}$, consider the regulatory levels $\bar{v} \in [v_*(\boldsymbol{\epsilon}), \infty)$.*

Imbalance G is reduced as $\bar{v}$ is lowered, and $\lim_{\bar{v} \to v_*+} G = 0$.

The regulator faces a typical cost-benefit tradeoff. Imposing more stringent rules (a drop in $\bar{v}$)induces suboptimal risk-sharing, more shallow asset markets and more volatile allocations during normal market conditions, but it reduces the probability of a systemic failure, $\text{Prob}(G(\boldsymbol{\epsilon}, \bar{v}) > g)$, which would have led to a total loss of output and consumption.

Since it seems natural to assume (and in line with the Basel Agreements) that $\bar{v}$ cannot be fine-tuned after observing $\boldsymbol{\epsilon}$, the socially optimal $\bar{v}$ does depend on things such as the distribution of $\boldsymbol{\epsilon}$ and the gains from risk-sharing.

6.3 Analysis of Financial Crisis

The Propositions above can now readily be applied to the analysis of financial crisis. Consider the special case where market volatilities are increasing and unregulated funds need to sell a large amount of assets, i.e., $\boldsymbol{\epsilon}$ is negative, large, and variable. We can differentiate three distinct *categories* of financial crises within the context of our model:

- **Non–catastrophic event**: a large sell–off, leading to sharp drops in asset prices, but not to a breakdown or collapse, $(\bar{v}, \boldsymbol{\epsilon})$ s.t. $\bar{v} \geq v_*(\boldsymbol{\epsilon})$ and $G(\boldsymbol{\epsilon}; \bar{v}) \leq g$.
- **Systemic collapse**: the financial imbalances cause the collapse, at time 1, of the real productive sector, and (for simplicity) all output drops to zero, $(\bar{v}, \boldsymbol{\epsilon})$ s.t. $\bar{v} \geq v_*(\boldsymbol{\epsilon})$ and $G(\boldsymbol{\epsilon}; \bar{v}) > g$.
- **Market breakdown**: financial markets at time zero cannot clear, $(\bar{v}, \boldsymbol{\epsilon})$ s.t. $\bar{v} < v_*(\boldsymbol{\epsilon})$. We have seen that a market breakdown can only occur if all FIs are regulated, $\eta = 1$.

Proposition 7 showed that market risk regulations can be effective in lowering the probability of a systemic crash. On the other hand, market risk-regulations may lead, in extreme cases, to the breakdown of all markets, including financial markets at time zero. And similarly, the VaR constraints impose costs in non–crisis periods, as seen in Propositions 6 and 4. In particular, it may exacerbate the effects on markets of a sudden selling by reducing the ability of FIs to absorb the selling orders:

Proposition 8 (Volatility in a non–catastrophic event) *More volatile asset payoffs $\mathbf{d}$ or more volatile UFI trades $\boldsymbol{\epsilon}$ create more shallow asset markets and more volatile asset prices in a regulated economy than in a non–regulated economy.*

At the onset of financial crisis, when volatilities are increasing, a RFI currently has two courses of action open in order to remain compliant with the regulations: it can either dispose of risky assets, or increase capital. Raising capital takes time, and may be impossible in crisis, so in practice, a sell–off of risky assets is the only avenue open in a crisis.

However, since many institutions are in the same situation, this leads to precisely the scenarios shown in Propositions 5, 6, and 8. Thus, the VaR constraints may reduce the probability of an ex-post systemic collapse by restraining risk-taking, but these constraints on risk-taking exacerbate the ex-ante symptoms of crises: poor liquidity, high volatility and a downward spiral of falling asset prices. They may in some cases even prevent market clearing altogether.

6.4 The 1998 Crisis

Consider specifically the Russia crisis of 1998. Prior to the crisis, volatility had been increasing, and many financial institutions were disposing of volatility with LTCM being a frequent buyer. It is not precisely known what role regulations played at the time, i.e., with institutions disposing of volatility for regulatory or other reasons, but the violation of regulatory constraints certainly was a factor in this. As Proposition 8 suggests, the initial pick up in volatility translated into a volatility of asset prices that was compounded by the regulatory risk-limits which resulted in more shallow markets. In a normal course of events, higher volatility inevitably leads to more margin calls. When Russia defaulted, many financial institutions, most prominently LTCM, were facing large losses, and unable to meet margin calls out of the remaining capital, leading to sell–offs of other assets. However, LTCM was not the only hedge-fund in that situation, and furthermore it seems that speculative capital was greatly reduced in general before and during the incident (in our model this may be viewed as an increase of η), exacerbating the symptoms by leaving few FIs able or willing to buy into the selling frenzy.

The fear that led to the rescue of LTCM (the rescue package effectively lowered $\boldsymbol{\epsilon}$) was that the sell-offs (and the resulting defaults induced by margin calls) would be so strong as to cause a melt-down of production and the real economy at time 1. In our model, the sell–off pushes risky stock into the RFIs' portfolios, which may induce $G > g$, a systemic collapse. The rationale for regulatory limits to risk-taking is that the VaR constraints create more uniform risk-taking among banks (Proposition 3), thereby reducing the risk imbalance G and the (unmodelled) risk of domino-style defaults.

And, due to the effective rise of η to a number close to 1, if the sell-off had been even larger, $\boldsymbol{\epsilon} \notin \mathcal{E}(\bar{v}, \ell)$, asset markets at time zero would not have been able to clear and the economy would have collapsed altogether.

7 Conclusion

The ultimate aim of financial risk regulations is the reduction of systemic risk. In general, this objective necessitates restricting financial institutions in their ability to assume risk. Even if this objective is laudable, in the real world specific policy instruments must be employed in the regulation of financial institutions. This goal may readily be achieved by proper application of Value–at–Risk regulations, or indeed any number of alternative regulatory regimes.

We consider the present VaR regulations in a two period general equilibrium Model. Our results indicate that a naïve implementation of risk constraints may have unintended adverse consequences.

A major flaw in the VaR based market risk regulations is uniformity of application. It may be justified in the name of fairness (exempting hedge funds may penalize regulated institutions), however this very uniformity of application leads to the adverse effects of VaR regulations. In general, we feel that heterogeneity and flexibility in regulatory policy should be a fundamental element in an effective regulatory environment. This however appears to run counter to current trends in regulatory policy which appears to be increasing in scope and uniformity, see the 2001 Basel Committee proposals, which recommends that credit, operational, and liquidity risk be regulated by means of modelling, just as market risk is now. The problem of uniformity may even be exacerbated in environments that conform to the structure of a global game, see for instance Morris and Shin(1999)).

In addition, the present regulations and the new proposals fail to consider the fact that the risk is endogenous. The lesson here is similar to the Lucas critique, Lucas(1976)). We demonstrate that regulating risk-taking changes the statistical properties of financial risk, rendering risk modelling all the more challenging. In particular, during crisis, VaR constraints change the risk appetite of financial institutions, effectively harmonizing their preferences. It is this effect which is most damaging, since during crisis it leads to higher volatility, larger drops in prices, and lower liquidity than would be realized in the absence of risk regulations.

A Proofs

Proof of Lemma 1 The program consists in solving

$$\max_{\{\boldsymbol{y}^h, y_0^h\}} E^d \left[u^h(d_0[\theta_0^h + \boldsymbol{q}'\boldsymbol{\theta}^h - \boldsymbol{q}'\boldsymbol{y}^h] + \boldsymbol{d}'\boldsymbol{y}^h)\right] - \lambda^h \left[\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h - \bar{v}\right]$$

The FOC (the program is strictly convex, so the FOC are both necessary and sufficient) are

$$E^d \left[u^{h\prime}(W^h)(\boldsymbol{d} - d_0\boldsymbol{q})\right] = 2\lambda^h \hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h$$

or equivalently

$$\mathrm{Cov}^d(u^{h\prime}(W^h), \boldsymbol{d}) + E^d \left[u^{h\prime}(W^h)\right] E[\boldsymbol{d}] - d_0 E^d \left[u^{h\prime}(W^h)\right] \boldsymbol{q} = 2\lambda^h \hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h$$

Next, by Stein's Lemma[8] and the fact that $\mathrm{Cov}^d(\boldsymbol{d}, W^h) = \mathrm{Cov}^d(\boldsymbol{d}, \boldsymbol{d}'\boldsymbol{y}^h) = \hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h$ we get that:

$$\boldsymbol{y}^h = \frac{1}{\alpha^h + \phi^h} \hat{\boldsymbol{\Sigma}}^{-1} \left[\hat{\mu} - d_0\boldsymbol{q}\right]$$

where we also used the fact that in this CARA–Normal setup $\frac{-\mathrm{E}^d[u^{h\prime\prime}]}{\mathrm{E}^d[u^{h\prime}]} = \alpha^h$, and where we defined $\phi^h \equiv \frac{2\lambda^h}{E^d[u^{h\prime}]}$.

Finally, we'll derive the expression for $\alpha^h + \phi^h$ and show that it does not depend on the wealth of the institution. In order to accomplish this, we first need to find an expression for ϕ^h. To simplify expressions, define

$$\rho \equiv (\hat{\boldsymbol{\mu}} - R_f\boldsymbol{q})' \hat{\boldsymbol{\Sigma}}^{-1} (\hat{\boldsymbol{\mu}} - R_f\boldsymbol{q}) \tag{7}$$

It can easily be established that[9]

$$\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h = \bar{v} \quad (\text{and } \lambda^h \geq 0) \Rightarrow \alpha^h + \phi^h = \sqrt{\frac{\rho}{\bar{v}}} \tag{8}$$

$$\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h < \bar{v} \quad (\text{so } \lambda^h = 0) \Rightarrow \alpha^h + \phi^h = \alpha^h \tag{9}$$

[8] Stein's Lemma asserts that if x and y are multivariate normal, if g is everywhere differentiable and if $E[g'(y)] < \infty$, then $\mathrm{Cov}(x, g(y)) = E[g'(y)]\mathrm{Cov}(x, y)$.

[9] Indeed, assume that $\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h = \bar{v}$. Since $y^h = \frac{1}{\alpha^h+\phi^h}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\mu}} - R_f\boldsymbol{q})$, this expression becomes $\left(\frac{1}{\alpha^h+\phi^h}\right)^2 \rho = \bar{v}$. Of course, if $\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h < \bar{v}$ then $\lambda^h = 0$ and thus $\phi^h = 0$.

This implies that $\alpha^h + \phi^h$ is independent of W_0^h for given prices,

$$\alpha^h + \phi^h = \max\left\{\alpha^h, \sqrt{\frac{\rho}{\bar{v}}}\right\} \tag{10}$$

Indeed, assume first that $\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h < \bar{v}$. Then by (9) we have that $\alpha^h+\phi^h = \alpha^h$, so we need to show that $\alpha^h \geq \sqrt{\frac{\rho}{\bar{v}}}$. Now since $\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h = {\alpha^h}^{-2}\rho$, we know that ${\alpha^h}^{-2}\rho < \bar{v}$, so that indeed $\alpha^h > \sqrt{\frac{\rho}{\bar{v}}}$. Next, assume that $\boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h = \bar{v}$. Then from (8) $\alpha^h + \phi^h = \sqrt{\frac{\rho}{\bar{v}}}$. So we need to establish that $\alpha^h \leq \sqrt{\frac{\rho}{\bar{v}}}$, which follows from $\phi^h \geq 0$. ■

Proof of Proposition 2 (On the Existence and Uniqueness of Equilibria)

We need to exhibit a solution to the fixed-point problem. Fix some $\boldsymbol{\epsilon} \in \mathbf{E}$ and assume first that $\ell > 0$. Recall from (4) that

$$\begin{aligned}\Psi^{-1} &= \eta \int_\ell^1 \frac{1}{\alpha^h + \phi^{h,r}} dh + (1-\eta)\int_\ell^1 \frac{1}{\alpha^h} dh \\ &= \eta \int_{I_1} \frac{1}{\alpha^h} dh + \eta \int_{I_2} \sqrt{\frac{\bar{v}}{\rho}} dh + (1-\eta)\int_\ell^1 \frac{1}{\alpha^h} dh \end{aligned} \tag{11}$$

where $I_1 \equiv \left\{h \in [\ell, 1] : \alpha^h > \sqrt{\frac{\rho}{\bar{v}}}\right\}$ and $I_2 \equiv \left\{h \in [\ell,1] : \alpha^h \leq \sqrt{\frac{\rho}{\bar{v}}}\right\}$.
In order to solve for the equilibrium, we can either express Ψ (from (11)) as a function of $\boldsymbol{q}$ and then solve (3) for $\boldsymbol{q}$, or we can use (3) to express $\boldsymbol{q}$ as a function of Ψ and then solve (11) for Ψ. We chose the latter approach for obvious reasons.

For convenience, we establish some preliminary calculations and notation. First, insert the pricing relation (3) into the definition of ρ from (7) to get the expression $\sqrt{\rho} = \Psi\sqrt{(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})}$. Second, define the parameter $\kappa(\boldsymbol{\epsilon}) \equiv \sqrt{\frac{(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})}{\bar{v}}} = \Psi^{-1}\sqrt{\frac{\rho}{\bar{v}}}$. $\kappa(\boldsymbol{\epsilon})$ represents the ratio of the standard deviation of the dividends of the residual market portfolio $\boldsymbol{\theta}^a - \boldsymbol{\epsilon}$ to the maximal allowable standard deviation of the payoffs of individual portfolios. By our assumption that $\alpha^h = h$, we can then define the ranges $I_1 = \{h \in [\ell, 1] : \alpha^h = h > \Psi\kappa(\boldsymbol{\epsilon})\}$ and $I_2 = \{h \in [\ell,1] : \alpha^h = h \leq \Psi\kappa(\boldsymbol{\epsilon})\}$ to get the functional equation

$$\Psi^{-1} = \eta \int_{I_1} h^{-1}\mathrm{d}h + \eta|I_2|(\Psi\kappa(\boldsymbol{\epsilon}))^{-1} + (1-\eta)\int_\ell^1 h^{-1}\mathrm{d}h \tag{12}$$

For simplicity, we drop the explicit dependence of κ upon $\boldsymbol{\epsilon}$ wherever no confusion arises. Let us first concentrate on the first element on the RHS where we have to distinguish 3 cases:

$$\int_{I_1} \frac{1}{h} dh = \begin{cases} \int_{\Psi\kappa}^{1} \frac{1}{h} dh = -\ln(\Psi\kappa) & ; \Psi\kappa \in [\ell, 1] \\ \int_{1}^{1} \frac{1}{h} dh = 0 & ; \Psi\kappa > 1 \\ \int_{\ell}^{1} \frac{1}{h} dh = -\ln(\ell) & ; \Psi\kappa < \ell \end{cases}$$

The second element can be rewritten as

$$I_2 = \begin{cases} [\ell, \Psi\kappa] & ; \Psi\kappa \in [\ell, 1] \\ [\ell, 1] & ; \Psi\kappa > 1 \\ \emptyset & ; \Psi\kappa < \ell \end{cases}$$

so that

$$|I_2| = \begin{cases} \Psi\kappa - \ell & ; \Psi\kappa \in [\ell, 1] \\ 1 - \ell & ; \Psi\kappa > 1 \\ 0 & ; \Psi\kappa < \ell \end{cases}$$

The equilibrium relations thus become

$$\Psi^{-1} = \begin{cases} -\eta \ln(\ell) + (1-\eta) \ln \ell^{-1} & ; \Psi\kappa < \ell \\ \eta \big[-\ln(\Psi\kappa) + (\Psi\kappa - \ell)(\Psi\kappa)^{-1} \big] + (1-\eta) \ln \ell^{-1} & ; \Psi\kappa \in [\ell, 1] \\ \eta(1-\ell)(\Psi\kappa)^{-1} + (1-\eta) \ln \ell^{-1} & ; \Psi\kappa > 1 \end{cases}$$

We rewrite the system in terms of a fixed-point problem via a mapping T.

$$T(\Psi) \equiv \begin{cases} T_1(\Psi) \equiv \frac{1}{\ln \ell^{-1}} & ; \Psi\kappa < \ell \\ T_2(\Psi) \equiv \frac{1+\eta\ell\kappa^{-1}}{\eta - \eta \ln(\Psi\kappa) + (1-\eta) \ln \ell^{-1}} & ; \Psi\kappa \in [\ell, 1] \\ T_3(\Psi) \equiv \frac{1-(1-\ell)\kappa^{-1}\eta}{(1-\eta) \ln \ell^{-1}} & ; \Psi\kappa > 1 \end{cases} \tag{13}$$

Before we proceed to proving that there is a unique Ψ^* satisfying $\Psi^* = T\Psi^*$, we need to establish some preliminary properties of T. The three subdomains of T depend on whether $\Psi < \ell\kappa^{-1}$, $\Psi \in [\ell\kappa^{-1}, \kappa^{-1}]$ or $\Psi > \kappa^{-1}$.

P1 On its domain, $T_1 > \ell\kappa^{-1}$ iff $\kappa > \ell \ln \ell^{-1}$.

P2 On its domain, $T_3 < \kappa^{-1}$ iff $\kappa < (1-\ell)\eta + (1-\eta)\ln \ell^{-1}$.

P3 $T_3 < T_1$ iff $\kappa < 1-\ell$.

P4 $\kappa < 1-\ell \Rightarrow \kappa < (1-\ell)\eta + (1-\eta)\ln \ell^{-1}$. To see this, notice that $1-\ell < (1-\ell)\eta + (1-\eta)\ln \ell^{-1}$ iff $1 + \ln \ell < \ell$, which holds for all $\ell \in \mathbb{R}_+$.

P5 $T_2(\ell\kappa^{-1}) = T_1$, iff $\kappa = \ell \ln \ell^{-1}$, i.e. T is continuous around $\ell\kappa^{-1}$.

P6 $T_2(\ell\kappa^{-1}) < T_1$ iff $\kappa > \ell \ln \ell^{-1}$

P7 $T_2(\kappa^{-1}) < \kappa^{-1}$ iff $\kappa < \eta(1-\ell) + (1-\eta)\ln \ell^{-1}$.

P8 $T_2(\ell\kappa^{-1}) > \ell\kappa^{-1}$ iff $\kappa > \ell \ln \ell^{-1}$.

P9 If $\kappa = \eta(1-\ell) + (1-\eta)\ln \ell^{-1} \Rightarrow T_2(\kappa^{-1}) = T_3$, i.e. T continuous around κ^{-1}.

P10 $T_2(\Psi)$ is continuous and monotonically increasing on its domain.

P11 The slope of T_2 is less than 1 at any fixed point. Indeed, at a fixed point, $\frac{\partial T_2}{\partial \Psi} = \frac{\eta}{\eta - \eta \ln(\Psi\kappa) + (1-\eta)\ln \ell^{-1}} < 1$ since $-\eta \ln(\Psi\kappa) + (1-\eta)\ln \ell^{-1} > 0$.

P12 $\eta(1-\ell) + (1-\eta)\ln \ell^{-1} > \ell \ln \ell^{-1}$ for all $\eta \in (0,1)$ and $\ell \in (0,1)$.

Thus, we can distinguish 5 cases that can arise, depending on κ (i.e. ultimately depending on $\boldsymbol{\epsilon}$ and $\bar{v}$). We treat the case $\eta < 1$ first.

$\kappa < \ell \ln \ell^{-1}$. Then by P12 we also know that $\kappa < \eta(1-\ell) + (1-\eta)\ln \ell^{-1}$. The fixed point occurs on the T_1 segment only. T_2 remains below the 45 degree line over its domain, since by P8 $T_2(\ell\kappa^{-1}) < \ell\kappa^{-1}$ and by P7 $T_2(\kappa^{-1}) < \kappa^{-1}$, while by P11 T_2 cannot cross the 45 degree line. Also, $T_2(\ell\kappa^{-1}) > T_1$. The unique equilibrium is illustrated on Figure 4.

$\kappa = \ell \ln \ell^{-1}$. By P5 the unique equilibrium is as on Figure 5.

$\kappa \in (\ell \ln \ell^{-1}, \eta(1-\ell) + (1-\eta)\ln \ell^{-1})$. We know by P8, $T_2(\ell\kappa^{-1}) > \ell\kappa^{-1})$. There can't be a fixed point on T_1, since by P6 $T_1 > T_2(\ell\kappa^{-1})$. There also cannot be a fixed point on T_3 by P2. That there needs to be a fixed point follows from P10, and uniqueness is guaranteed by P11. Refer to Figure 6.

$\kappa = \eta(1-\ell) + (1-\eta)\ln \ell^{-1}$. P9 implies Figure 7.

$\kappa > \eta(1-\ell) + (1-\eta)\ln \ell^{-1}$. By P12, $\kappa > \ell \ln \ell^{-1}$ as well, and the unique equilibrium occurs on T_3, as seen on Figure 8.

The case $\eta = 1$ can be treated similarly, please refer to Figures 9, 10, 11.

Notice that by construction the equilibrium Ψ^* satisfies $\Psi^* \geq \gamma$. Since $\boldsymbol{\epsilon}$ affects Ψ only in as far as it affects κ, it is useful to point out that the mapping $\kappa \mapsto \Psi(\kappa; \eta, \ell)$ (we often drop the dependency on η and ℓ if no ambiguity arises and write the mapping as $\Psi(\kappa)$), can be characterized as follows. If $\eta < 1$ and $\ell > 0$, then

$$\Psi(\kappa) = \begin{cases} \frac{1}{\ln \ell^{-1}} & ; \kappa \in [0, \ell \ln \ell^{-1}] \\ -\frac{\kappa + \eta\ell}{\eta\kappa W_{-1}(-(\kappa\eta^{-1}+\ell)\exp(-\ln \ell - 1 + \eta^{-1}\ln \ell))} & ; \kappa \in (\ell \ln \ell^{-1}, \eta(1-\ell) + (1-\eta)\ln \ell^{-1}) \\ \frac{1-(1-\ell)\kappa^{-1}\eta}{(1-\eta)\ln \ell^{-1}} & ; \kappa \geq \eta(1-\ell) + (1-\eta)\ln \ell^{-1} \end{cases}$$

where $W_{-1}(\cdot)$ is the non-principal (lower) branch of the Lambert W-correspondence.[10] Notice that the mapping Ψ is continuous. The fact that no equilibrium exist if $\ell = 0$ is shown subsequently in the proof of Corollary 2.

If $\eta = 1$, then

$$\Psi(\kappa) = \begin{cases} \frac{1}{\ln \ell^{-1}} & ; \kappa \in [0, \ell \ln \ell^{-1}] \\ -\frac{\kappa + \ell}{\kappa W_{-1}(-(\kappa+\ell)e^{-1})} & ; \kappa \in (\ell \ln \ell^{-1}, 1-\ell) \\ \text{any number} \geq \frac{1}{1-\ell} & ; \kappa = 1-\ell \\ \text{undefined} & ; \kappa > 1-\ell \end{cases}$$

Over the entire domain the correspondence $\Psi(\kappa)$ is illustrated in figure (2).

We will find the partial derivatives of the equilibrium Ψ useful, and in particular their signs. As preliminaries, we establish some properties on the domain $\kappa \in (\ell \ln \ell^{-1}, \eta(1-\ell) + (1-\eta)\ln \ell^{-1})$ of the fixed-point system $F(\Psi, \kappa, \ell, \eta) \equiv \kappa\Psi[\eta - \eta\ln(\Psi\kappa) + (1-\eta)\ln \ell^{-1}] - [\kappa + \eta\ell]$.

D1 $\frac{\partial F}{\partial \Psi} = \Psi^{-1}[\kappa + \eta\ell] - \kappa\eta > 0$. Indeed, the sign is positive iff $\kappa + \eta\ell - \Psi\kappa\eta > 0$. Now at a fixed-point, we know that $\kappa\Psi = \frac{\kappa + \eta\ell}{\eta - \eta\ln(\Psi\kappa) + (1-\eta)\ln \ell^{-1}}$, so that $\kappa + \eta\ell = \kappa\Psi\left(\eta - \eta\ln(\Psi\kappa) + (1-\eta)\ln \ell^{-1}\right)$. Plugging this into the expression above, we get that the sign is positive iff $-\eta\ln(\Psi\kappa) + (1-\eta)\ln \ell^{-1} > 0$, which holds for both terms are positive.

[10] The Lambert W correspondence is defined as the multivariate inverse of the function $w \mapsto we^w$.

D2 $\frac{\partial F}{\partial \kappa} = \eta \ell \kappa^{-1} - \eta \Psi = \eta[\ell \kappa^{-1} - \Psi] < 0$.

D3 $\frac{\partial F}{\partial \eta} = \kappa \Psi[1 - \ln(\Psi \kappa) - \ln \ell^{-1}] - \ell < 0$, since we can simplify this inequality to $\frac{1}{\ln \ell^{-1}} - \Psi < 0$. The latter holds since $\frac{1}{\ln \ell^{-1}} = \gamma < \Psi$.

D4 $\frac{\partial F}{\partial \ell} = -\kappa \Psi (1 - \eta) \ell^{-1} < 0$.

We can deduce from the implicit function theorem that the slopes, for $\ell > 0$ and $\eta < 1$ are given by

$$\frac{\partial \Psi}{\partial \kappa} = \begin{cases} 0 & ; \kappa \in (0, \ell \ln \ell^{-1}] \\ -\frac{\Psi(\kappa \eta \Psi - \eta \ell)}{\kappa^2(\eta \Psi - (1 + \kappa^{-1} \ell \eta))} & ; \kappa \in (\ell \ln \ell^{-1}, \eta(1 - \ell) + (1 - \eta) \ln \ell^{-1}) \\ \frac{\eta(1 - \ell)}{\kappa^2 (1 - \eta) \ln \ell^{-1}} & ; \kappa \geq \eta(1 - \ell) + (1 - \eta) \ln \ell^{-1} \end{cases}$$

So $\frac{\partial \Psi}{\partial \kappa} \geq 0$, and > 0 if $\eta \in (0, 1)$ and $\kappa > \ell \ln \ell^{-1}$. It is easy to verify that the mapping Ψ is indeed differentiable on the entire $\mathbb{R}_+$.

For $\eta = 1$, we find

$$\frac{\partial \Psi}{\partial \kappa} = \begin{cases} 0 & ; \kappa \in (0, \ell \ln \ell^{-1}] \\ -\frac{\Psi(\kappa \Psi - \ell)}{\kappa^2(\Psi - (1 + \kappa^{-1} \ell))} > 0 & ; \kappa \in (\ell \ln \ell^{-1}, 1 - \ell) \\ \text{undefined} & ; \kappa \in [1 - \ell, +\infty) \end{cases}$$

Notice that $\lim_{d \to 0+} \frac{\Psi(\kappa + d) - \Psi(d)}{d} = \lim_{d \to 0-} \frac{\Psi(\kappa + d) - \Psi(d)}{d} = 0$ at $\kappa = \ell \ln \ell^{-1}$.

Repeating the exercise for the derivative with respect to η, we find for $\ell > 0$ and $\eta < 1$

$$\frac{\partial \Psi}{\partial \eta} = \begin{cases} 0 & ; \kappa \in (0, \ell \ln \ell^{-1}] \\ -\frac{\Psi \kappa (1 - \ln \ell^{-1} \Psi)}{\eta(\kappa + \eta \ell - \Psi \kappa \eta)} > 0 & ; \kappa \in (\ell \ln \ell^{-1}, \eta(1 - \ell) + (1 - \eta) \ln \ell^{-1}) \\ \frac{\ell \kappa^{-1}}{(1 - \eta)^2 \ln \ell^{-1}} > 0 & ; \kappa > \eta(1 - \ell) + (1 - \eta) \ln \ell^{-1} \end{cases}$$

Finally,

$$\frac{\partial \Psi}{\partial \bar{v}} = \frac{\partial \Psi}{\partial \kappa} \frac{\partial \kappa}{\partial \bar{v}} \leq 0, \quad < 0 \quad \text{if } \boldsymbol{\theta}^a \neq \boldsymbol{\epsilon}, \ \eta > 0 \text{ and } \kappa > \ell \ln \ell^{-1}$$

■

Proof of Proposition 3 Most results follow from the properties of the index of the marginal regulated investor, $\Psi \kappa$. Define $v_*(\boldsymbol{\epsilon})$ as the weakest level of regulation for which all RFIs hit their VaR constraints,

$$v_*(\boldsymbol{\epsilon}) \equiv \frac{(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})}{[\eta(1-\ell) + (1-\eta)\ln \ell^{-1}]^2}$$

If $\eta = 1$, $v_*(\boldsymbol{\epsilon})$ is also the strictest level of regulation, given $\boldsymbol{\epsilon}$, that can still be supported by an equilibrium. Similarly, define $v^*(\boldsymbol{\epsilon})$ as the weakest level of regulation for which there is an agent whose risk-taking constraint is binding,[11]

$$v^*(\boldsymbol{\epsilon}) \equiv \begin{cases} \frac{(\boldsymbol{\theta}^a-\boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a-\boldsymbol{\epsilon})}{(\ell \ln \ell^{-1})^2} & \text{if } \eta > 0 \\ 0 & \text{if } \eta = 0 \end{cases}$$

We summarize some properties of $\Psi\kappa$ first. They are proved for $\eta < 1$, but they hold for $\eta = 1$ as well as long as the parameters are such that $\bar{v} \geq v_*(\boldsymbol{\epsilon})$, the necessary and sufficient condition for an equilibrium to exist.

L1 $\Psi\kappa$ is differentiable in $\bar{v}$.

L2 $\frac{\partial \Psi\kappa}{\partial \bar{v}} \leq 0$, < 0 iff $\boldsymbol{\theta}^a \neq \boldsymbol{\epsilon}$, in particular for $\bar{v} \in (v_*(\boldsymbol{\epsilon}), v^*(\boldsymbol{\epsilon}))$.

Indeed, for the index of the marginal RFI $\Psi\kappa$, $\frac{\partial \Psi\kappa}{\partial \bar{v}} = \left[\Psi + \kappa \frac{\partial \Psi}{\partial \kappa}\right] \frac{\partial \kappa}{\partial \bar{v}} \leq 0$, < 0 iff $\boldsymbol{\theta}^a \neq \boldsymbol{\epsilon}$ (since $\frac{\partial \kappa}{\partial \bar{v}} = 0$ iff $\boldsymbol{\theta}^a = \boldsymbol{\epsilon}$). Notice that if $\boldsymbol{\theta}^a = \boldsymbol{\epsilon}$, then the interval $(v_*(\boldsymbol{\epsilon}), v^*(\boldsymbol{\epsilon}))$ is empty.

L3 $\Psi\kappa = 1$ at $\bar{v} = v_*(\boldsymbol{\epsilon})$.

Indeed, $\lim_{\kappa \to \eta(1-\ell)+(1-\eta)\ln \ell^{-1}} \kappa\Psi(\kappa) = \lim_{\kappa \to \eta(1-\ell)+(1-\eta)\ln \ell^{-1}} \frac{\kappa-(1-\ell)\eta}{(1-\eta)\ln \ell^{-1}} = 1$

L4 $\Psi\kappa = \ell$ at $\bar{v} = v^*(\boldsymbol{\epsilon})$.

Indeed, $\lim_{\kappa \to \ell \ln \ell^{-1}} \kappa\Psi(\kappa) = \lim_{\kappa \to \ell \ln \ell^{-1}} \frac{\kappa}{\ln \ell^{-1}} = \ell$.

The proof of (i) is obvious, and (ii) is simply (L2).

As to (iii), denote the payoff variance of investor h by $v^h \equiv \boldsymbol{y}^{h\prime}\hat{\boldsymbol{\Sigma}}\boldsymbol{y}^h = \frac{1}{(\alpha^h+\phi^h)^2}\Psi^2\bar{v}\kappa^2$ $= \frac{1}{(\alpha^h+\phi^h)^2}\Psi^2(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})$. We need to show that $\frac{\partial v^h}{\partial \bar{v}} < 0$ if $h > \Psi\kappa$.

[11] Almost no regulated investor's constraint is binding iff $v^h \leq \bar{v}$ all h, i.e. iff $\left(\frac{1}{\alpha^h+\phi^{h,r}}\right)^2 \Psi^2\kappa^2\bar{v} \leq \bar{v}$ $(\forall h)$, iff $\left(\frac{1}{\alpha^h+\phi^{h,r}}\right)^2 \Psi^2(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta} - \boldsymbol{\epsilon}) \leq \bar{v}$ iff $\left(\frac{\frac{1}{\alpha^h}}{\int_\ell^1 \frac{1}{\alpha^k}dk}\right)^2 (\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon}) \leq \bar{v}$ for all h. Now this holds for all h iff it holds for $h = \ell$, and plugging in $\frac{\frac{1}{\ell}}{\int_\ell^1 \frac{1}{k}dk} = \left[\ell \ln \ell^{-1}\right]^{-1}$ we get the stated result.

So assuming that $h > \Psi\kappa$ (so that in particular $\Psi\kappa < 1$), it is indeed easy to see that

$$\begin{aligned}\frac{\partial v^h}{\partial \bar{v}} &= \frac{2}{h^2}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\Psi\frac{\partial \Psi}{\partial \bar{v}} \\ &\leq 0; \quad < 0 \text{ if } \boldsymbol{\theta}^a \neq \boldsymbol{\epsilon}, \eta > 0 \text{ and } \kappa > \ell \ln \ell^{-1}\end{aligned}$$

Taken together with (ii), this shows that as regulations are tightened marginally, the risk is taken away from the agents in the interval $[\Psi\kappa, \Psi\kappa + \frac{\partial \Psi\kappa}{\partial \bar{v}} d\bar{v}]$ and transferred (via an appropriate price change) to the unregulated and to the regulated but more risk-averse.

For a given $\boldsymbol{\epsilon}$, let us lower $\bar{v}$ towards $v_*(\boldsymbol{\epsilon})$. Observations L1 to L4 taken together say that as $\bar{v}$ becomes smaller, the marginal RFI tends to 1, in which case all investors' risk-taking constraint is binding, and thus all hold the same risky portfolio. This is so because each RFI's effective risk-aversion is identical in that situation, so that the ϖ^h, their risk-aversions relative to the aggregate risk-aversion, are identical. But we saw that $\boldsymbol{q}'\boldsymbol{y}^h = \varpi^h \boldsymbol{q}'(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})$, and the right-hand side is in the limit independent of h. ∎

Proof of Proposition 4

As preliminaries, let us record the following useful results.

J1 $\frac{\partial \kappa}{\partial \bar{v}} = -\frac{1}{2}\frac{\kappa}{\bar{v}}$, from the definition of κ, and $\partial_{\boldsymbol{\epsilon}}\kappa = \kappa^{-1}\bar{v}^{-1}\hat{\boldsymbol{\Sigma}}(\boldsymbol{\epsilon} - \boldsymbol{\theta}^a)$.

J2 $\partial^2_{\boldsymbol{\epsilon},\bar{v}}\Psi = -\frac{1}{2\bar{v}}\left[\kappa\frac{\partial^2\Psi}{\partial\kappa^2} + \frac{\partial\Psi}{\partial\kappa}\right]\partial_{\boldsymbol{\epsilon}}\kappa$. Indeed, since $\partial_{\bar{v}}\Psi = \frac{\partial\Psi}{\partial\kappa}\frac{\partial\kappa}{\partial\bar{v}}$, we know from J1 that $\partial^2_{\boldsymbol{\epsilon},\bar{v}}\Psi = \frac{d}{d\boldsymbol{\epsilon}}\left(\frac{\partial\Psi}{\partial\kappa}\frac{\partial\kappa}{\partial\bar{v}}\right) = \frac{\partial\kappa}{\partial\bar{v}}\frac{\partial^2\Psi}{\partial\kappa^2}\partial_{\boldsymbol{\epsilon}}\kappa + \frac{\partial\Psi}{\partial\kappa}\partial_{\boldsymbol{\epsilon}}\left(-\frac{1}{2}\frac{\kappa}{\bar{v}}\right)$.

J3 $\partial_{\boldsymbol{\epsilon}}Q$ is positive definite (downward-sloping equilibrium inverse demand). Indeed, $\partial_{\boldsymbol{\epsilon}}Q = R_f^{-1}\left[\Psi\hat{\boldsymbol{\Sigma}} - \hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\partial_{\boldsymbol{\epsilon}}\Psi)'\right] = R_f^{-1}\left[\Psi\hat{\boldsymbol{\Sigma}} + \hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}\kappa^{-1}\bar{v}^{-1}\frac{\partial\Psi}{\partial\kappa}\right]$, positive definite.

The idea of the proof is to show that $\partial^2_{\boldsymbol{\epsilon},\bar{v}}Q$ is negative definite (ND). Intuitively, we want to show that the market impact of a trade goes up as the regulation is tightened, i.e. that $\frac{\partial}{\partial\bar{v}}|(d\boldsymbol{q})'(d\boldsymbol{\epsilon})| = \frac{\partial}{\partial\bar{v}}[(d\boldsymbol{q})'(d\boldsymbol{\epsilon})] < 0$ since $(d\boldsymbol{q})'(d\boldsymbol{\epsilon}) > 0$ as $\partial_{\boldsymbol{\epsilon}}Q$ is PD by J3. Now this expression equals $\frac{\partial}{\partial\bar{v}}[(d\boldsymbol{\epsilon})'\partial_{\boldsymbol{\epsilon}}Q(d\boldsymbol{\epsilon})] = (d\boldsymbol{\epsilon})'\partial^2_{\boldsymbol{\epsilon},\bar{v}}Q(d\boldsymbol{\epsilon}) < 0$ for all $d\boldsymbol{\epsilon} \neq 0$, but that's the definition of negative definiteness.

Before we show that $\partial^2_{\boldsymbol{\epsilon},\bar{v}}Q(d\boldsymbol{\epsilon})$ is negative definite, we want to relate this idea with the definition of shallowness given in the text, $\mathcal{S}(\boldsymbol{\epsilon},\bar{v}) \equiv \max_{\boldsymbol{\theta}}|\boldsymbol{\theta}'(\partial_{\boldsymbol{\epsilon}}Q)\boldsymbol{\theta}|$

s.t. $\|\boldsymbol{\theta}\| = 1$, namely that $\frac{\partial \mathcal{S}}{\partial \bar{v}} < 0$ iff $\partial^2_{\boldsymbol{\epsilon},\bar{v}} Q$ negative definite. Indeed, pick any $\boldsymbol{\theta}$ s.t. $\|\boldsymbol{\theta}\| = 1$, then it is immediate that $\frac{\partial(\boldsymbol{\theta}'\partial_{\boldsymbol{\epsilon}} Q \boldsymbol{\theta})}{\partial \bar{v}} = \boldsymbol{\theta}' \left(-\partial_{\boldsymbol{\epsilon},\bar{v}} Q\right) \boldsymbol{\theta}$, which proves the claim. In some sense, a tighter $\bar{v}$ makes $\partial_{\boldsymbol{\epsilon}} Q$ "more positive definite."

The pricing function is $Q(\boldsymbol{\epsilon}, \bar{v}) = R_f^{-1}\left[\hat{\boldsymbol{\mu}} - \Psi \hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\right]$, from which we can deduce that $\partial_{\bar{v}} Q = -R_f^{-1} \hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\frac{d\Psi}{d\bar{v}}$, and furthermore that $\partial^2_{\boldsymbol{\epsilon},\bar{v}} Q = R_f^{-1}\hat{\boldsymbol{\Sigma}}\frac{d\Psi}{d\bar{v}} - R_f^{-1}\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})\partial^2_{\boldsymbol{\epsilon},\bar{v}}\Psi$. This expression can be simplified, using J2, to

$$\partial^2_{\boldsymbol{\epsilon},\bar{v}} Q = -\frac{1}{2} R_f^{-1} \frac{\partial \Psi}{\partial \kappa} \frac{\kappa}{\bar{v}} \hat{\boldsymbol{\Sigma}} - \frac{1}{2} R_f^{-1} \bar{v}^{-2} \left[\frac{\partial^2 \Psi}{\partial \kappa^2}\kappa + \frac{\partial \Psi}{\partial \kappa}\right] \left[\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}\right] \kappa^{-1}$$

The first term is negative definite, while the second one is negative semidefinite. Indeed, it can be shown that the expression $\left[\frac{\partial^2 \Psi}{\partial \kappa^2}\kappa + \frac{\partial \Psi}{\partial \kappa}\right]$ is strictly positive, while the term $\left[\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})(\boldsymbol{\theta}^a - \boldsymbol{\epsilon})'\hat{\boldsymbol{\Sigma}}\right]$ is clearly positive semidefinite. This concludes the proof that $\partial^2_{\boldsymbol{\epsilon},\bar{v}} Q$ is negative definite. ■

Proof of Proposition 5 We would like to establish Figure 3. Assume that $N = 1$, so that there is a single risky asset. We can deduce the following three characteristics of the pricing function, for $\bar{v}' > \bar{v}$:

(i) We have

$$\begin{aligned} Q(\epsilon, \bar{v}) < Q(\epsilon, \bar{v}') &\quad \text{on } \epsilon < \theta^a - \frac{\sqrt{\bar{v}}}{\sigma}\ell \ln \ell^{-1} \\ Q(\epsilon, \bar{v}) = Q(\epsilon, \bar{v}') &\quad \text{on } \epsilon \in \left[\theta^a - \frac{\sqrt{\bar{v}}}{\sigma}\ell \ln \ell^{-1}, \theta^a + \frac{\sqrt{\bar{v}}}{\sigma}\ell \ln \ell^{-1}\right] \\ Q(\epsilon, \bar{v}) > Q(\epsilon, \bar{v}') &\quad \text{on } \epsilon > \theta^a + \frac{\sqrt{\bar{v}}}{\sigma}\ell \ln \ell^{-1} \end{aligned}$$

(ii) $\frac{\partial Q}{\partial \epsilon} > 0$, and

(iii) $\frac{\partial Q}{\partial \epsilon}(\bar{v}) > \frac{\partial Q}{\partial \epsilon}(\bar{v}')$.

(iv) The graph is symmetrical with center $(\theta^a, Q(\theta^a, \infty))$.

We briefly prove (i) here. First, assume that $\epsilon < \theta^a - \frac{\sqrt{\bar{v}}}{\sigma}\ell \ln \ell^{-1}$. Then $\kappa = \frac{\sigma|\theta^a - \epsilon|}{\sqrt{\bar{v}}} = \frac{\sigma(\theta^a - \epsilon)}{\sqrt{\bar{v}}} > \ell \ln \ell^{-1}$. It follows that $\Psi > \gamma$ and $\frac{\partial \Psi}{\partial \kappa} > 0$, so that $\Psi(\epsilon, \bar{v}) > \Psi(\epsilon, \bar{v}')$ since $\bar{v}' > \bar{v}$ implies that $\kappa(\epsilon, \bar{v}') < \kappa(\epsilon, \bar{v})$. Since $Q(\epsilon, \bar{v}) = R_f^{-1}\left[\bar{\mu} - \Psi\sigma^2(\theta^a - \epsilon)\right]$, this results in $Q(\epsilon, \bar{v}) < Q(\epsilon, \bar{v}')$ for all $\epsilon < \theta^a - \frac{\sqrt{\bar{v}}}{\sigma}\ell \ln \ell^{-1}$.

Next, assume that $\epsilon \in \left[\theta^a - \frac{\sqrt{\bar{v}}}{\sigma}\ell \ln \ell^{-1}, \theta^a + \frac{\sqrt{\bar{v}}}{\sigma}\ell \ln \ell^{-1}\right]$, so that $\theta^a - \epsilon \in [-\frac{\sqrt{\bar{v}}}{\sigma}\ell \ln \ell^{-1}, +\frac{\sqrt{\bar{v}}}{\sigma}\ell \ln \ell^{-1}]$, which implies that $\frac{\sigma(\theta^a-\epsilon)}{\sqrt{\bar{v}}} \in [-\ell \ln \ell^{-1}, \ell \ln \ell^{-1}]$ and finally that $\kappa = \frac{\sigma|\theta^a-\epsilon|}{\sqrt{\bar{v}}} \in [0, \ell \ln \ell^{-1}]$. Then $\Psi = \gamma$, and $Q(\epsilon, \bar{v}) = Q(\epsilon, +\infty)$, and in particular that $Q(\epsilon, \bar{v}) = Q(\epsilon, \bar{v}')$. This completes the proof of item (i).

In order to prove (iv), we fix any $\bar{v}$ and we need to verify $Q(\theta^a+\eta) - Q(\theta^a) = Q(\theta^a) - Q(\theta^a - \eta)$, $\eta > 0$. This equation boils down to $\Psi(\kappa(\theta^a + \eta)) = \Psi(\kappa(\theta^a - \eta))$. But $\kappa(\theta^a + \eta) = \frac{\sigma|\eta|}{\sqrt{\bar{v}}} = \kappa(\theta^a - \eta)$ completes the proof of (iv).

Taken together, the equilibrium correspondence looks as on figure 3. ■

Proof of Proposition 8 We show the claim for more volatile asset payoffs, the claim for fund trades being evident. Using the results of Proposition 6 we find that in order to show $\frac{\partial \mathrm{Var} Q}{\partial \sigma} > 0$ it is sufficient to show $\frac{\partial Q^2}{\partial \sigma \partial \epsilon} > 0$.

It follows from J3 and the fact that $\frac{d\kappa}{d\sigma} = \frac{|\theta^a - \epsilon|}{\sqrt{\bar{v}}}$ that

$$\frac{\partial Q^2}{\partial \sigma \partial \epsilon} = 2\sigma R_f^{-1} \left[\Psi + \kappa \left(4\frac{\partial \Psi}{\partial \kappa} + \kappa \frac{\partial^2 \Psi}{\partial \kappa^2}\right)\right] > 0$$

since both terms in brackets are positive. ■

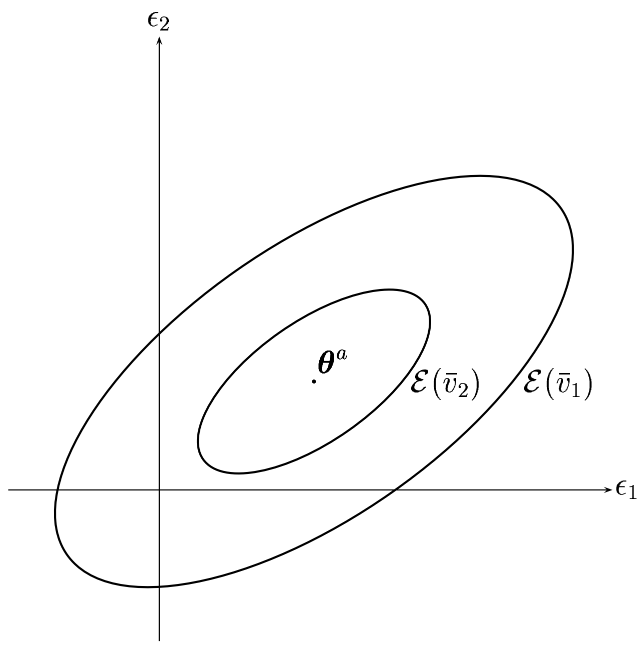

Figure 1: Equilibrium ellipsoids with increasingly restrictive risk constraints

In this scenario there are two assets, and in the absence of any regulations, equilibria exist for $\boldsymbol{\epsilon} \in \mathbb{R}^2$. When the risk constraint is $\bar{v}_1$, the set of $\boldsymbol{\epsilon}$ that can be supported by an equilibrium is the larger ellipsoid, and includes zero noise trader demand. However a more restrictive constraint $\bar{v}_2$ does not include zero net demand, and hence equilibria do not exist if noise trades are zero.

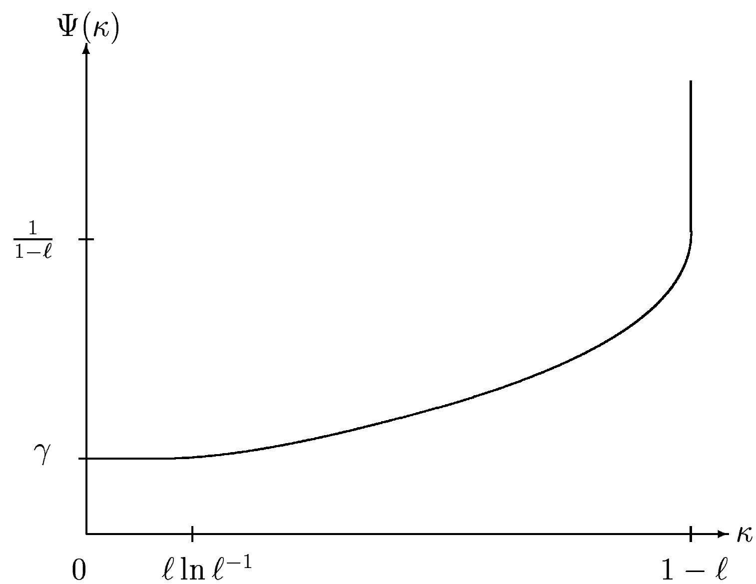

Figure 2: Illustration of the reward-to-risk function $\Psi(\kappa)$ when $\eta = 1$ and $\ell > 0$.

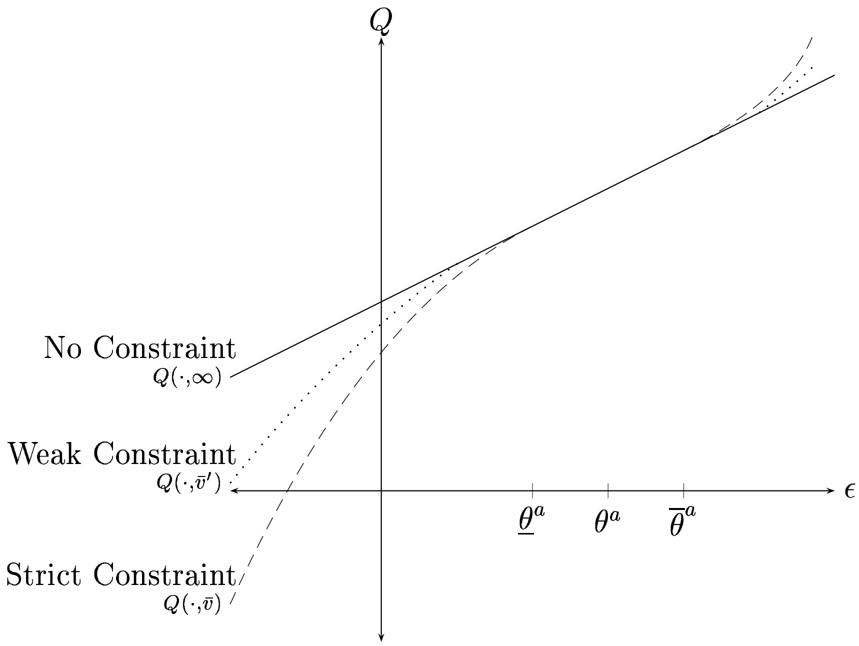

Figure 3: Pricing Function

The pricing function without constraints and with increasingly binding constraints, $\infty > \bar{v}' > \bar{v}$. The downside effects become more pronounced as the constraint becomes stricter.

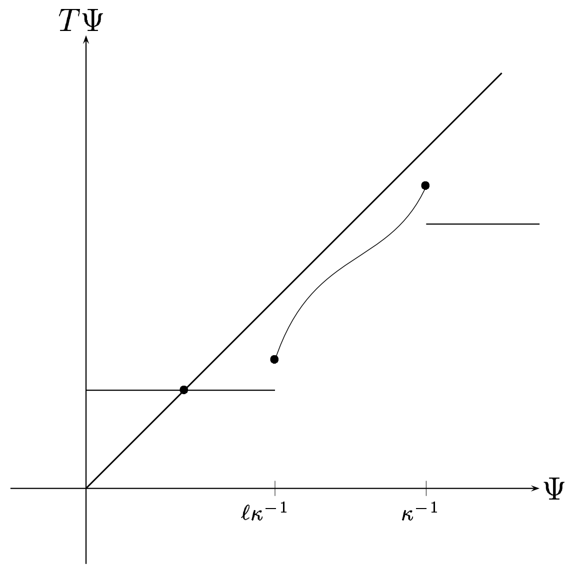

Figure 4: Fixed point for $\kappa < \ell \ln \ell^{-1}$.

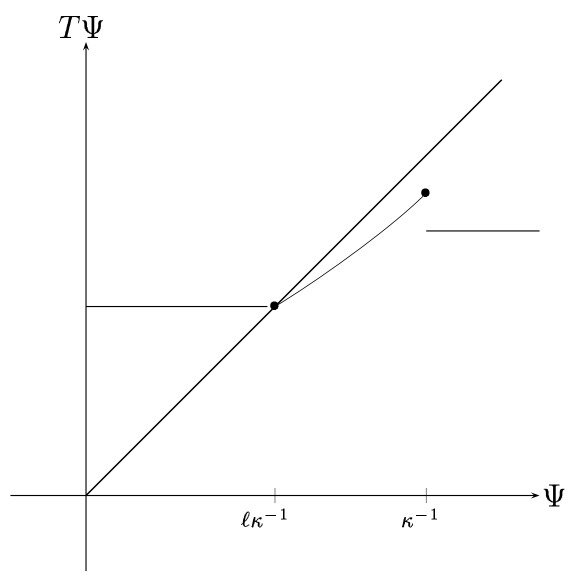

Figure 5: Fixed point for $\kappa = \ell \ln \ell^{-1}$.

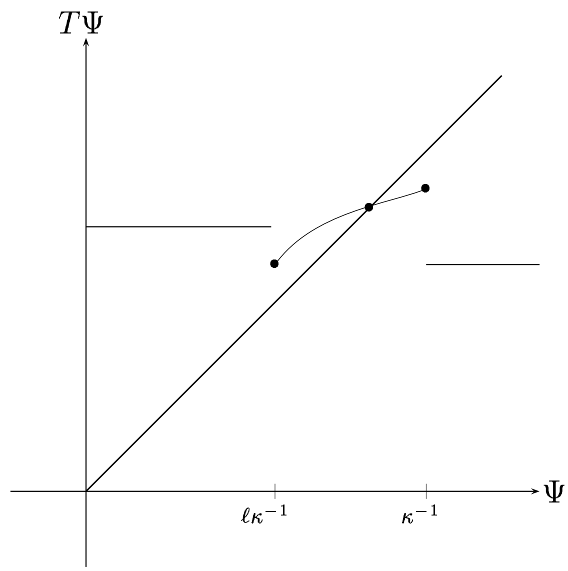

Figure 6: Fixed point for $\kappa \in (\ell \ln \ell^{-1}, \eta(1-\ell) + (1-\eta)\ln \ell^{-1})$.

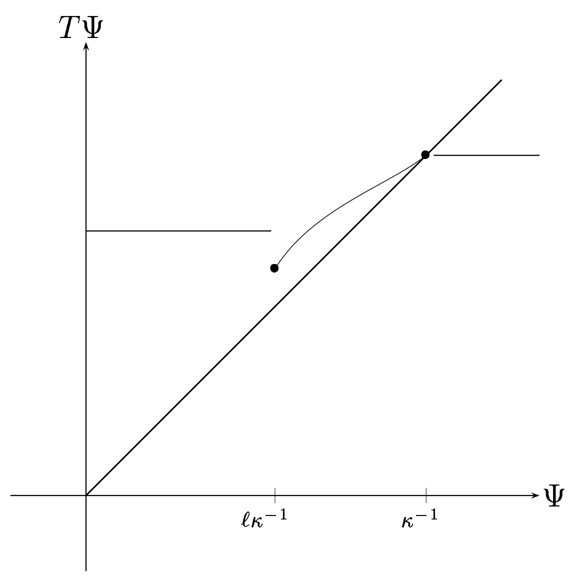

Figure 7: Fixed point for $\kappa = \eta(1-\ell) + (1-\eta)\ln \ell^{-1}$.

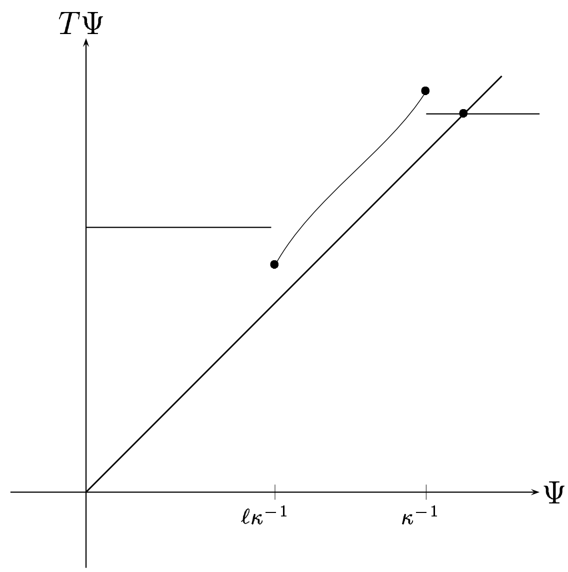

Figure 8: Fixed point for $\kappa > \eta(1-\ell) + (1-\eta)\ln \ell^{-1}$.

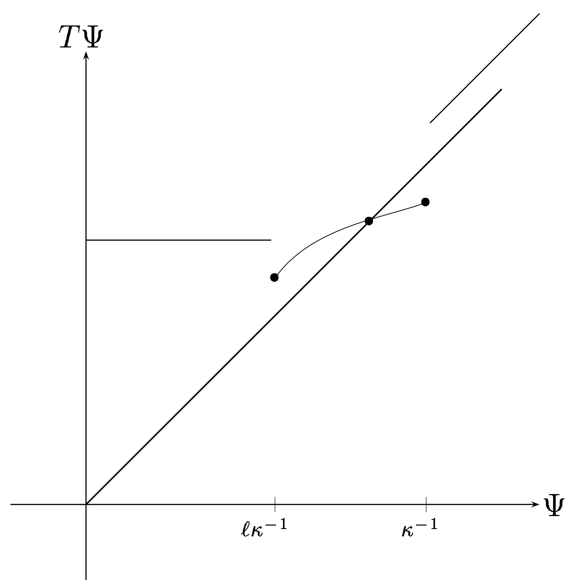

Figure 9: Economy with $\eta = 1$. Unique Fixed Point for $\kappa \in (\ell \ln \ell^{-1}, 1-\ell)$.

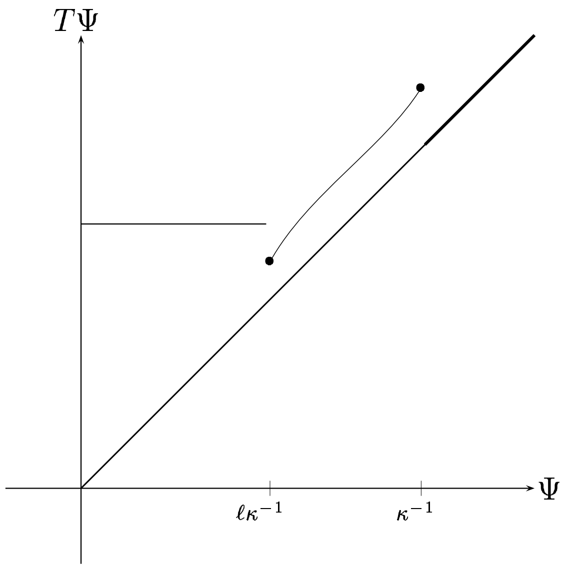

Figure 10: Economy with $\eta = 1$. Indeterminacy of Fixed Points if $\kappa = 1 - \ell$.

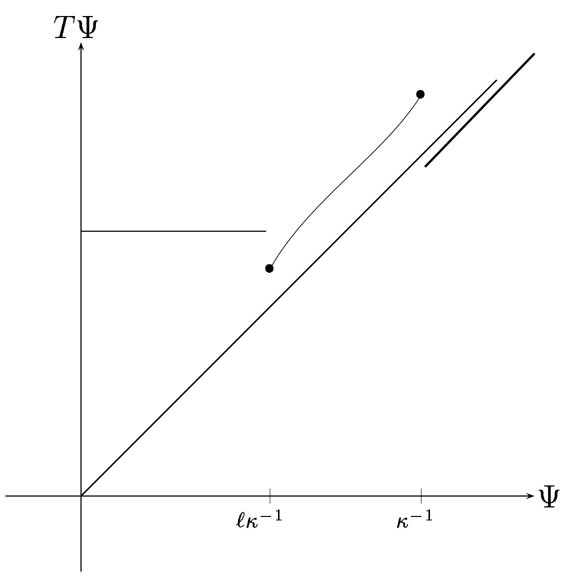

Figure 11: Economy with $\eta = 1$. No Fixed Point if $\kappa > 1 - \ell$.

References

D. Ahn, J. Boudukh, M. Richardson, and R. Whitelaw. Optimal risk management using options. *Journal of Finance*, 54(1):359–375, 1999.

F. Allen and D. Gale. Financial contagion. *Journal of Political Economy*, 108(1):1–33, 2000.

P. Artzner, F. Delbaen, J. Eber, and Heath. Coherent measure of risk. *Mathematical Finance*, 9(3):203–228, 1999.

S. Basak and A. Shapiro. A general equilibrium model of portfolio insurance. *Review of Financial Studies*, 8:1059–1090, 1995.

S. Basak and A. Shapiro. Value–at–risk based risk management: Optimal policies and asset prices. *Review of Financial Studies*, 14:371–405, 2001.

. Basle Committee on Banking Supervision. *Overview of the Amendment to the Capital Accord to Incorporate Market Risk*. 1996.

S. Brown, W. Goetzmann, and J. Park. Hedge funds and the asian currencu crisis of 1997. Technical Report 6427, NBER Working Paper, 1998.

A. Crockett. Marrying the micro- and macro prudential dimensions of financial stability. Mimeo, Bank of International Settlements, `http://www.bis.org/review/rr000921b.pdf`, 2000.

O. De Bandt and P. Hartmann. Systemic risk: a survey. Discussion paper series, no. 2634, CEPR, 2000.

G. Gennotte and H. Leland. Market liquidity, hedging, and crashes. *American Economic Review*, 00:999–1021, 1990.

S. Grossman and Z. Zhou. Equilibrium analysis of portfolio insurance. *Journal of Finance*, 51(4):1379–1403, 1996.

R. E. Lucas. Econometric policy evaluation: A critique. *Journal of Monetary Economics*, 1.2 Supplementary:19–46, 1976.

S. Morris and H. S. Shin. Risk management with interdependent choice, 1999. *Oxford Review of Economic Policy*, Autumn 1999, 52 - 62, reprinted in the Bank of England *Financial Stability Review*, 7, 141 - 150, `http://www.bankofengland.co.uk/fsr/fsr07art5.pdf`.

H. Polemarchakis and P. Siconolfi. Prices, asset markets and indeterminacy. *Journal of Economic Theory*, 82:46–64, 1998.

G. Stahl. Three cheers. *Risk*, 10:67–69, 1997.