Revisited Multi-moment Approximate Option Pricing Models: A General Comparison (Part 1)*

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Abstract

After the seminal paper of Jarrow and Rudd (1982), several authors have proposed to use different statistical series expansion to price options when the risk-neutral density is asymmetric and leptokurtic. Amongst them, one can distinguish the Gram-Charlier Type A series expansion (Corrado and Su, 1996-b and 1997-b), the log-normal Gram-Charlier series expansion (Jarrow and Rudd, 1982) and the Edgeworth series expansion (Rubinstein, 1998). The purpose of this paper is to compare these different multi-moment approximate option pricing models. We first recall the link between the risk-neutral density and moments in a general statistical series expansion framework under the martingale hypothesis. We then derive analytical *formulae* for several four-moment approximate option pricing models, namely, the Jarrow and Rudd (1982), Corrado and Su (1996-b and 1997-b) and Rubinstein (1998) models. We investigate in particular the conditions that ensure the respect of the martingale restriction (see Longstaff, 1995) and consequently revisit the approximate option pricing models under study. We also get for these models the analytical expressions of implied probability densities, implied volatility smile functions and several hedging parameters of interest.

Keywords: Option Pricing Models, Skewness, Kurtosis.

JEL Classification: G.10, G.12, G.13.

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After the seminal paper of Jarrow and Rudd (1982), several authors have proposed to use different statistical series expansion to price options when the risk-neutral density is asymmetric and leptokurtic. Amongst them, one can distinguish the Gram-Charlier Type A series expansion (Corrado and Su, 1996-b and 1997-b), the log-normal Gram-Charlier series expansion (Jarrow and Rudd, 1982) and the Edgeworth series expansion (Rubinstein, 1998). The purpose of this paper is to compare these different multi-moment approximate option pricing models. We first recall the link between the risk-neutral density and moments in a general statistical series expansion framework under the martingale hypothesis. We then derive analytical formulae for several four-moment approximate option pricing models, namely, the Jarrow and Rudd (1982), Corrado and Su (1996-b and 1997-b) and Rubinstein (1998) models. We investigate in particular the conditions that ensure the respect of the martingale restriction (see Longstaff, 1995) and consequently revisit the approximate option pricing models under study. We also get for these models the analytical expressions of implied probability densities, implied volatility smile functions and several hedging parameters of interest.

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1 Introduction

The Black and Scholes (1973) *formula* is certainly one of the most used in finance, but presents some inconsistencies. In particular, several empirical studies[1] show that the model missprices deep out-of-the-money and deep in-the-money options. In other words, when the Black-Scholes *formula* is inverted, the implied volatilities estimates differ across exercise prices and maturities, and form patterns called “smile”, “smirk” or “sneers” depending on their shapes.[2] This result is generally attributed to the unrealistic hypothesis of a geometric Brownian motion for the underlying asset process or, equivalently, of a normally distributed continuous rate of return with constant volatility under an equivalent martingale measure. Indeed, if rare events are more frequent than it is assumed in the normal case, then the price of deep out-of-the-money options will be higher than the Black and Scholes (1973) model predicts. If, moreover, the log-return distribution is negatively skewed, prices of deep out-the-money put options will be higher than those of deep in-the-money call options and the implied volatility function will be downward biased.

In order to avoid these biases, different approaches have been proposed. A first one is to consider alternative stochastic processes than the geometric Brownian motion with or without additional stochastic factors. For instance, a jump-diffusion process is chosen by Merton (1976) and more recently by Bates (1991 and 1996-a), whilst Hull and White (1987), Stein and Stein (1991) and Heston (1993) consider stochastic volatility models. Bates (1996-

[1] See, for instance, MacBeth and Merville (1979), Rubinstein (1985 and 1994) and Bates (1996-c).

[2] Bates (2000) shows that “smiles” often appear before the crash of 1987 on the American market, whilst “sneer” patterns are more likely to be found since.

b and 2000) and Pan (2002) extend the jump-diffusion model to incorporate stochastic volatility to explain the structure of option prices, while Bakshi *et al.* (1997 and 2000) develop option pricing models that admit simultaneously stochastic volatility, stochastic interest rate and random jump.

A second approach is to use binomial - or trinomial - lattices in order to approximate the whole structure of market prices (see Rubinstein, 1994, Derman and Kani, 1994, Dupire 1994, Dernan *et al.*, 1996 and Jackwerth, 1997). In achieving an exact cross-sectional fit of option prices, trees can be constrained to reproduce moments of a prespecified implied density (see Rubinstein, 1998, Li, 2000 and Ang *et al.*, 2001).

Despite the fact that both approaches can yield skewed and leptokurtic risk-neutral density, they are not perfectly satisfactory. The most severe critics of these related models are the lack of parsimony (leading to possible overfitting), the choice of a deterministic volatility function (see Dumas *et al.*, 1998) or the existence of inadequate volatility term structure (see Das and Sundaram, 1999). Moreover estimation problems on illiquid markets are reported.

An alternative approach consists in specifying a functional form of the terminal risk-neutral density of the underlying asset price.[3] Amongst the distributional specifications investigated for the pricing kernel, one can firstly distinguish non-parametric - model-free - statistical methods that impose a very slight structure on the form of the distribution, such as kernel estimators (see Ait-Sahalia and Lo, 1998 and 2000), smoothed curve fitting methods of the pricing or the implied volatility function (see for instance Shimko, 1993, Rubinstein, 1994, Jackwerth and Rubinstein, 1996, Brown and Toft, 1997, Malz, 1997,

[3] Indeed for a given expiration date, there exists an infinite number of stochastic processes which are consistent with one particular risk-neutral distribution (see, for instance, Melick and Thomas, 1997 and Dupire, 1998).

Campa *et al.*, 1998 and Hartvig *et al.*, 1999) and maximum entropy estimation methods (see Buchen and Kelly, 1996, Stutzer, 1996, Guo, 2001 and Jondeau and Rockinger, 2002). Secondly, fully parametric models consider more flexible and general distributions than the normal, such as three parameter distributions (see Sherrick e*t al.*, 1996), four parameter distributions (see Posner and Milevsky, 1998, Lim *et al.*, 2000 and 2002, Theodossiou, 2000 and Corrado, 2001) and five parameter distributions - for instance mixtures of lognormal distributions (see for instance Ritchey, 1990, Bahra, 1996 and 1997, Malz, 1996 and 1997, Melick and Thomas, 1997 and Pirkner *et al.*, 1999). Thirdly, semi-parametric models consist in approximating the state price density using empirical counterparts of the implied moments. Initially developed by Jarrow and Rudd (1982), this last approach aims to approximate the risk-neutral density by a statistical series expansion such as a Gram-Charlier Type A series expansion (see Corrado and Su, 1996-b and 1997-b, Backus *et al.*, 1997, Bouchaud *et al.*, 1998, Brown and Robinson, 1999 and Knigth and Satchell, 2001), a lognormal Gram-Charlier series expansion (see Jarrow and Rudd, 1982, Turnbull and Wakeman, 1991, Corrado and Su, 1996-a and 1997-a, Jondeau and Rockinger, 2000 and Flamouris and Giamouridis, 2002) or an Edgeworth series expansion (Rubinstein, 1998 and Li, 2000).[4] The series are truncated to a finite order that usually gives a tractable closed-form expressions for option prices. In this last approach, the risk-neutral skewness and kurtosis of the underlying asset enter in option pricing in a very natural way since the coefficients of statistical series expansions are functions of moments of the given and approximating distributions.

The purpose of this article is to focus on this last field of literature. We aim to present, in an unified framework, the theoretical foundations of the option pricing models based on

[4] While these expansions are the most popular in the literature, others have also been considered such as Laguerre series expansions (Brenner and Eom, 1997 and Dufresne, 2000) and Kummer functions (Abadir and Rockinger, 1997).

statistical series expansion methods, namely, the Jarrow and Rudd (1982), the Corrado and Su (1996-b and 1997-b) and the Rubinstein (1998) models.

Our study provides several contributions. Firstly, we investigate the conditions that ensure the respect of the martingale restriction (see Longstaff, 1995). This gives us crucial insights on approximations involved in the multi-moment approximate option pricing models. Indeed, while it is showed that the martingale restriction is fulfilled in the Jarrow and Rudd (1982) model, the Corrado and Su (1996-b and 1997-b) and the Rubinstein (1998) models do not conform to it and need then to be revisited. We also establish the link between these models and alternative option pricing models such as the Black and Scholes (1973) and the Hermite polynomial models (see Madan and Milne, 1994 and Abken *et al.*, 1996). Next, we provide analytical *formulae* for implied density function and we generalize the approach of Backus *et al.* (1997) regarding the volatility smile functions. We finally provide hedging parameters of interest following Corrado and Su (1997-a), Hull and White (1997), and Knigth and Satchell (2001).

The paper is organized as follows. In section 2, we review the statistical foundations and the pricing *formulae* of the Jarrow and Rudd (1982), Corrado and Su (1996-b and 1997-b) and Rubinstein (1998) models. In section 3, we present the implied probability density and the implied volatility smile functions. We also compute the Greeks - namely, the Delta, Gamma, Vega, Khi and Psi[5]. Section 4 summarizes and concludes. Main proofs (Appendixes 1 to 10) and Figures (Appendix 11) are collected at the end of the article.

[5] The two last one - proposed by Hull and White (1997) - measure respectively changes in the option price with respect to changes in skewness and kurtosis.

2 Pricing of Options when Risk-neutral Densities are Skewed and Leptokurtic

When pricing an option, several elements of interest are involved. We start by defining variables under consideration, the no-arbitrage conditions and the general expression of the option price. We then recall main statistical series expansion that lead to revisited - because of the martingale restriction - multi-moment approximate option pricing models.

2.1 Option Pricing and Martingale Restriction

The first element of interest in option pricing is the conditional distribution of the terminal price of the underlying asset. Let $_\tau$ be the τ- period log-return on the underlying asset defined such as:

$$\begin{aligned} {}_\tau &= \ln\left(\frac{{}_T}{{}_t}\right) \\ &= \sum_{i=1}^{N} \ln\left[\frac{{}_{t+i\Delta}}{{}_{t+(i-1)\Delta}}\right] \\ &= \sum_{i=1}^{N} {}_i \end{aligned} \tag{1}$$

where $_T$ and $_t$ are respectively the terminal and the actual price of the underlying asset, $= \tau \quad \Delta$ is the number of unit time intervals of length Δ during a period $\tau = (\quad - \quad)$ and $_i$ is the instantaneous log-return on the underlying asset. Rearranging terms, we obtain:

$$\ln \, {}_T = \ln \, {}_t + \sum_{i=1}^{N} {}_i \tag{2}$$

then:

$${}_T = {}_t \exp\left(\sum_{i=1}^{N} {}_i\right) \tag{3}$$

and the conditional distribution of the terminal price of the underlying asset depends on that of $_i$. If we assume that $_i$ are random variables with finite variance, it follows by application of the central limit theorem and the definition of a lognormal random variable[6] that when tends to infinity, the underlying asset terminal log-price is conditionally normally distributed and the underlying asset terminal price is conditionally log-normally distributed.

The second element of interest when valuing options is the determination of the fair price in a risk-neutral framework. An European call option is a contract which confers on its holder the right, with no obligation, to purchase an underlying asset, which current price is noted $_t$, for a prescribed amount, known as the exercise or strike price, denoted , at the expiration date, . Under the assumptions of (dynamically) complete market and no arbitrage opportunity, and if we suppose that the risk-free rate of interest, denoted , is constant, the theoretical price of a call option is the present value of the expected payoff at expiry, given by the following pricing kernel (see Harrison and Kreps, 1979):

$$
\begin{aligned}
= & \; [\;_t \quad \tau \quad\quad _T\ \theta] = {}^{-r\tau}\ _Q[\quad (\;_T - \quad 0)] \\
= & \; {}^{-r\tau} \int_{S_T=K}^{+\infty} (\;_T - \quad)\ (\;_T)\quad _T \qquad (4)
\end{aligned}
$$

where $_Q[\,]$ is the expectation under the risk-neutral probability measure, θ is a vector of parameters - the first moments - characterizing the risk-neutral density of underlying asset terminal price $(\;_T)$.

The third element of interest is linked with the martingale restriction implied by the no-arbitrage condition. Under this condition, the expected price under the correct probability measure should be equal to the current asset price compounded at the risk-free rate.

[6] A random variable x is said to be log-normal if $\ln(x)$ is normally distributed. For a study of the log-normal distribution, see, for instance, Aitchinson and Brown (1966).

Accordingly, the probability measure to be considered must satisfy the so-called martingale restriction (see Longstaff, 1995):

$$_{Q}[\ _{T}] = \ ^{r\tau}\ _{t} \tag{5}$$

and then the density of underlying asset terminal price $(\ _T)$ must respect:

$$\ln\{\ _{Q}[\ _{T}]\} = \ln\left[\int_0^{+\infty}\ _T\ (\ _T)\ \ _T\right] = \ \tau + \ln(\ _t) \tag{6}$$

depending on the shape of the chosen density as a *proxy* for the "true" underlying risk-neutral density.

Finally, a closed-form for the option *formula* can be obtained if we assumed a lognormal distribution for the terminal price of the asset, as in Black and Scholes (1976), or if we use a statistical series expansion for the conditional density of the price of the asset, as in Jarrow and Rudd (1982) or for the conditional density of the related continuously compounded return, as in Corrado and Su (1996-b and 1997-b) and Rubinstein (1998).

2.2 Risk-neutral Density and Moments

The problem is then to get an analytical expression for the risk-neutral density function. One way of doing that is, following Jarrow and Rudd (1982), to use a statistical series expansion[7] of the state price density in order to get an approximation used in (4) when replacing $(\ )$ by the rigth-hand side of the following equation:

$$(\ ) = \ (\ \ \theta) + \varepsilon(\ ) \tag{7}$$

where $(\ )$ is a fitted density, the random variable under interest - terminal price or log-return - θ is a vector of moments characterizing the "true" risk-neutral density, $(\ )$ a statistical series expansion and $\varepsilon(\ )$ a residual.

[7] Statistical series expansion are conceptually similar to a Taylor series expansion: a given density is approximated by an expansion around a prespecified distribution.

In this case, estimation of parameters included in the vector of moments θ are sufficient to recover a parametric approximation of the risk-neutral density[8]. More formally, any robust class of density $(\ )$ can be written as (see Johnson *et al.*, 1994, p.28 and Appendix 1):

$$(\ ) = (\ ) + \sum_{i=1}^{+\infty} \frac{1}{!} \left[\sum_{j=1}^{N-1} (-1)^j \frac{\ _j \quad ^j}{!} \right]^i (\ ) + \varepsilon(\ ) \tag{8}$$

where $(\ )$ is an arbitrary density, $\kappa_j(\ ) \quad = [1 \quad -1]$ its cumulants, $_j = [\kappa_j(\ ) - \kappa_j(\ )]$ $\kappa_1(\ ) = \ _1(\ )\ \ \kappa_2(\ ) = \ _2(\ )\ \ \kappa_3(\ ) = \ _3(\ )\ \ \kappa_4(\ ) = \ _4(\ ) - 3\ _2(\ )^2$ with $_j \quad = [1 \quad 4]$ the centered moments of order , is the differentiation operator such as $^j(\ ) = \ ^j(\ ) \quad ^j$ and $\varepsilon(\ )$ is a residual.[9]

In the last *formula*, terms in $(\ )$ represent a traditional general statistical series expansion. Some restrictions could be added on existence of moments[10] and on the fact that the distribution could be uniquely defined using its moments[11]. Specific ordering of terms and special choices about the form of the approximating distribution lead to several expressions of equation (8).

In particular, the way terms are ordered in the general form (8) lead to different statistical series expansion as presented hereafter. Indeed, developing and collecting terms determined

[8] Some of the others common approximation techniques of density by their moments include Cornish-Fisher series expansion and Johnson family of curves.

[9] The cumulants of f (x) are defined as coefficients of $(\ j)^{-1}$ $d^j g(x)/dx^j$ in equation (8), whether or not f (x) ≥ 0. So, in general, expression (8) will not constitute a proper probability density function (see Kendall and Stuart, 1977, pp.168-171 and Johnson *et al.*, 1994, pp.25-30). Nevertheless, this problem can be solved by imposing restrictions on the domain of variation of the moments (see for instance, Barton and Dennis, 1952, Balitskaia and Zolotuhina, 1988, and Jondeau and Rockinger, 2001). Another problem that can arise is that, even if for all x, f (x) ≥ 0, the density may display multimodality (see Barton and Dennis, 1952). Despite these limitations, it is often possible to obtain from statistical series expansion useful approximate expression of a distribution with known moments.

[10] In a financial framework, expansions usually consider only the first four moments.

[11] That is not the case for the log-normal distribution for instance.

by successive derivatives of $(\ )$ in (8), up say to the fourth order, leads to:

$$
\begin{aligned}
(\ ) &= \upsilon_{GC}(\quad \theta) + \varsigma(\ ) \\
&= (\ ) - {}_1 \frac{(\ )}{} + \left[\frac{{}_2 + (\ {}_1)^2}{2!} \right] \frac{{}^2\ (\ )}{{}_2} \\
&\quad - \left[\frac{{}_3 + 3\ {}_1\ {}_2 + 3(\ {}_1)^3}{3!} \right] \frac{{}^3\ (\ )}{{}_3} \\
&\qquad + \left[\frac{{}_4 + 4\ {}_3\ {}_1 + 3(\ {}_2)^2 + 6(\ {}_1)^2\ {}_2 + (\ {}_1)^4}{4!} \right] \frac{{}^4\ (\ )}{{}_4} \\
&\qquad\quad + \varsigma(\ )
\end{aligned}
\tag{9}
$$

where ${}_j$, with $= [1 \quad 4]$, are defined as previously and $\varsigma(\ )$ is an error term.

The state price density is then a linear combination of $(\ )$ and its derivatives. The collection of terms in $(\ )$ is called a Gram-Charlier series expansion (see for instance Johnson *et al.*, 1994, p.28)[12]. Second, third, fourth and fifth terms in equation (9) allow to adjust $(\ )$ according to the gap between, respectively, the mean, the variance, the skewness and the kurtosis of the approximated distribution and that of the approximating density function (each term being weighted by the first, second, third and fourth derivatives of the approximating density function). The last part of equation (9) - the residual $\varsigma(\ )$ - captures terms neglected in the expansion.

If we moreover assume that is a standardized random variable and $(\ )$ a Gaussian distribution, then equation (9) becomes:

$$
\begin{aligned}
(\ ) &= \upsilon_{GC}(\quad \varphi \quad \theta) + \zeta(\ ) \\
&= \varphi(\ ) + \frac{\kappa_3(\ )}{3!}\ {}_3(\ )\, \varphi(\ ) + \frac{\kappa_4(\ )}{4!}\ {}_4(\ )\, \varphi(\ ) + \zeta(\ )
\end{aligned}
\tag{10}
$$

where $\varphi(\ ) = (2\pi\tau)^{-1/2} \exp(-\ {}^2\ 2)$ is the standard normal density function, $\kappa_j(\varphi) = \kappa_j(\ )$ for $= [1\ 2]$ and $\kappa_j(\varphi) = 0$ for $= [3\ 4]$, ${}_i(\ )$ denotes the - Hermite polynomial defined

[12] Some authors refer to it also as a Bruns-Charlier Expansion (see Hall, 1997).

by Rodrigues' $formula$[13] $_i(\ ) = (-1)^n \varphi(\ )^{-1}\ {}^i\varphi(\ )\qquad {}^i$ and $\zeta(\ )$ is a residual.[14] The equation (10) corresponds to Gram-Charlier Type A series or Hermite polynomial series expansion.[15]

For practical purposes, expression (8) is usually truncated up to the fourth order, and the remainder $\varepsilon(\ )$ is dropped. Since the successive terms in a Gram-Charlier expansion are not necessarily in decreasing order of importance, $\varsigma(\ )$ in (9) may not converge uniformly to zero as more terms are added. However, if is a normalized sum of n independent and identically distributed random variables $_i$, with $=[1\qquad]$ that is:

$$= {}^{-1/2}\sigma^{-1}\sum_{i=1}^{n}(\ _i -\ ) \tag{11}$$

it is possible to sort differently terms in equation (8) such as to ensure that it constitutes a proper asymptotic series expansion[16]. The ordering is based on the fact that, for a sum

[13] See Abramowitz and Stegun (1972).

[14] The Hermite polynomials through the fourth order are (see Kendall and Stuart, 1977, p.163):

$$\begin{cases} \mathsf{H}_0(\mathsf{x}) = 1 \\ \mathsf{H}_1(\mathsf{x}) = \mathsf{x} \\ \mathsf{H}_2(\mathsf{x}) = (\mathsf{x}^2 - 1) \\ \mathsf{H}_3(\mathsf{x}) = (\mathsf{x}^3 - 3\mathsf{x}) \\ \mathsf{H}_4(\mathsf{x}) = (\mathsf{x}^4 - 6\mathsf{x}^2 + 3) \\ \mathsf{H}_5(\mathsf{x}) = (\mathsf{x}^5 - 10\mathsf{x}^3 + 15\mathsf{x}) \\ \mathsf{H}_6(\mathsf{x}) = (\mathsf{x}^6 - 15\mathsf{x}^4 + 45\mathsf{x}^2 - 15) \end{cases}$$

[15] While $formula$ (10) is one of the most commonly used in statistical theory, it must be emphasized that Gram-Charlier expansion based on a standard beta, standard gamma, poisson, log-normal (see below) and t-student distributions have also been developped.

[16] An asymptotic expansion is defined to be a expansion which has the property that when truncated at some finite number r, the remainder is of smaller order than the last term that has been included (see for instance, Hall, 1992, p.45 and Spanos, 1986, pp.205-206).

of n standardized random variables[17], the j-th cumulant is proportional to $^{1-j/2}$ with ≥ 2 (see Appendix 2). After developing and collecting terms of equal order in $^{-1/2}$ in (8), say up to $^{-1}$order, $(\ )$ can then be expressed as:

$$\begin{aligned} (\ ) &= \upsilon_E(\quad \theta) + \xi(\ ) \qquad (12) \\ &= (\ ) - \ ^{-1/2} \frac{_{i,3}}{3!} \frac{^{3}(\ )}{^{3}} \\ &\quad + \ ^{-1}\left[\frac{_{i,4}}{4!} \frac{^{4}(\ )}{^{4}} + 10 \frac{(\ _{i,3})^2}{6!} \frac{^{6}(\ )}{^{6}} \right] \\ &\quad + \xi(\ ) \end{aligned}$$

where $_{i,j} = [\kappa_{i,j}(\ ) - \kappa_{i,j}(\ )]$ with $\kappa_{i,j}$, now the - cumulant of the standardized random variable $\sigma^{-1}(\ _i - \ )$, $(\ \times\ ) = (1 \quad) \times (1 \quad 4)$ with $\kappa_{i,1} = 0$ and $\kappa_{i,2} = 1$ and $\xi(\ )$ is a residual with $\xi(\ ) = \ (\ ^{-1})$ where $(\ )$ corresponds to the landau notation.

In the last formulation, the group of terms in $(\ )$ is known as an Edgeworth series expansion (see, for instance, Johnson *et al.*, 1994, p.28)[18]. Second and third terms in equation (12) allow to adjust $(\ )$ according to the gap between the skewness and the kurtosis of the risk-neutral distribution function and that of the approximating density (successive terms being now weighted by $^{-1/2}$ and $^{-1}$). The last part of equation (12) - the residual $\xi(\ )$ - takes into account terms in the development based on higher order cumulants.

If we assume again that $(\ )$ is a standard normal density, equation (12) becomes:

$$\begin{aligned} (\ ) &= \upsilon_E(\quad \varphi \quad \theta) + \eta(\ ) \\ &= \varphi(\ ) + \frac{\kappa_3(\ )}{3!} \ _3(\ )\,\varphi(\ ) \qquad (13) \\ &\quad + \left[\frac{\kappa_4(\ )}{4!} \ _4(\ ) + 10 \frac{[\kappa_3(\ )]^2}{6!} \ _6(\ ) \right] \varphi(\ ) \\ &\quad + \eta(\ ) \end{aligned}$$

[17] When standardized random variables are not an IID sequence see Kochard (1999).

[18] Some authors refer to it also as a Edgeworth-Sargan (Mauleon and Perote, 2000).

where $\varphi(\ )$ $_i(\ )$ and $\eta(\ )$ denotes respectively the standard normal density, the - Hermite polynomial and a residual. This form is called a normal Edgeworth series expansion (see, for instance, Spanos, 1986).

Note that none of the expression (12) or (13) possess a general theoretical superiority over the equation (9) or (10), since they depend on a particular assumption about the orders of magnitude of successive cumulants (see Johnson *et al.*, 1994, p.28).

2.3 Fourth-moment Option Pricing Models

The statistical series expansion methodologies recalled, we present the derivation of the multi-moment approximate option pricing models, depending on the choice of the approximating distribution of the risk-neutral density. While the martingale restriction is nothing else than a rescaling of the risk-neutral density, it is shown nevertheless that the restriction is model dependent. This thus leads to revisit some of multi-moment approximate option pricing models under review.

2.3.1 The Black and Scholes (1973) Model

Black and Scholes (1973) model assumes that the dynamics of the underlying asset follows a geometric Brownian motion:[19]

$$_t = \left(\alpha + \frac{\sigma^2}{2}\right) \ _t \ + \sigma \ _t \ _t \tag{14}$$

where α is the expected value of the log-return, σ represents the related volatility and $_t$ is a standard Brownian motion under the physical measure.

When markets are complete, Harrison and Pliska (1981) show that there exists a risk-

[19] For the stochastic differential equation notation, see Baxter and Rennie, (1996), p.85.

neutral transformation that leads to the following expression:

$$_t = \quad _t \quad + \sigma \quad _t \quad {}_t^Q \tag{15}$$

where ${}_t^Q$ is a Brownian motion under the risk-neutral probability measure.

It follows from Itô's lemma that the risk-neutral density of the terminal price of the underlying asset is lognormal, that is:

$$(\quad _T) = \frac{1}{_T \sigma \sqrt{2\pi\tau}} \exp \left\{ - \frac{\left[\ln \left(\frac{S_T}{S_t} \right) - \left(\quad - \frac{1}{2}\sigma^2 \right) \tau \right]^2}{2\sigma^2\tau} \right\} \tag{16}$$

or, by definition, that of the asset log-return is normal, that is:

$$(\ln \quad _T) = \frac{1}{\sigma\sqrt{2\pi\tau}} \exp \left\{ - \frac{\left[\ln \quad _T - \left(\ln \quad _t + \left[\quad - \frac{1}{2}\sigma^2 \right] \tau \right) \right]^2}{2\sigma^2\tau} \right\} \tag{17}$$

so the price of an European call option under the Black and Scholes (1973) assumptions can be written as:

$$\begin{aligned} _{BS} &= \quad [\quad _t \quad \tau \quad \upsilon \quad \quad \theta] \\ &= \quad ^{-r\tau} \int_{S_T = K}^{+\infty} (\quad _T - \quad) \, \upsilon \, [\quad (\quad _t \; \theta) \quad (\quad _t \; \theta)] \quad _T \end{aligned} \tag{18}$$

where $\upsilon \, (\quad \theta)$ is defined - in the particular case of Black and Scholes (1973) - such as:

$$\begin{cases} \upsilon \, (\,) = \\ \quad (\,) = \quad (\,) \\ \quad = \quad _T \\ \quad (\,) = \quad (\,) \\ \theta = \sigma \end{cases}$$

with $(\,)$ the lognormal distribution function.

Performing the following change of variable on $_T$ in integral (18):

$$= \frac{\log \left(\frac{S_T}{S_t} \right) - \quad \tau}{\sigma\sqrt{\tau}} \tag{19}$$

where τ and $\sigma\sqrt{\tau}$ respectively represent the expected value and the volatility of the log-return under the risk-neutral measure, leads to the Black and Scholes *formula* (1973), that is:

$$
\begin{aligned}
{}_{BS} &= {}^{-rt}\int_{z=\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} (\quad {}_{T} - \quad)\,\varphi(\quad) \\
&= {}_{t}\Phi(\quad) - \quad {}^{-r\tau}\Phi\left(\quad - \sigma\sqrt{\tau}\right)
\end{aligned}
\tag{20}
$$

with:

$$
= \frac{\log\left(\ {}_{t} \qquad {}^{-r\tau}\right) + \sigma^{2}\tau \ \ 2}{\sigma\sqrt{\tau}}
$$

where $\varphi(\ )$ and $\Phi(\ )$ are respectively the standard normal density function and the standard normal distribution.

The main advantage of this model is that all parameters, except the volatility, are directly observable. However, empirical evidence against the hypothesis that returns are homoskedastic and normally distributed, and the existence of some anomalies on option markets reported in several studies (see for instance Rubinstein, 1994) lead to the development of option pricing models based upon alternative risk-neutral density function.

Whilst Black-Scholes (1973) model supposes that the continuous underlying asset return is normally distributed, Jarrow-Rudd (1982) have proposed a method based on statistical series expansions for pricing options when densities are skewed and leptokurtic. The Black-Scholes (1973) model is then a special case of the Jarrow-Rudd (1982) model. The unknown state price density of the underlying asset return is approximated by using the information of skewness and kurtosis departures from Gaussianity. In this approach, only the first moments of the risk-neutral distribution are needed and can be approximated using their empirical counterparts estimated on the data.

2.3.2 The Jarrow and Rudd (1982) Model

Following Jarrow and Rudd (1982), we assume that the approximate distribution of the asset price $(\ _T)$ is the lognormal distribution $(\ _T)$, with the two first centered moments equal to the "true" ones[20], that is:

$$\kappa_1(\ \ ) = \kappa_1(\ ) \text{ and } \kappa_2(\ \ ) = \kappa_2(\ ) \tag{21}$$

using a Gram-Charlier series expansion, the risk-neutral density function can be written as:

$$(\ _T) = \ (\ _T) - \frac{_3}{3!}\frac{^3(\ _T)}{^3_T} + \frac{_4}{4!}\frac{^4(\ _T)}{^4_T} + \varepsilon(\ _T) \tag{22}$$

where $_3 = \kappa_3(\ \ ) - \kappa_3(\ )$, $_4 = \kappa_4(\ \ ) - \kappa_4(\ )$ and $\varepsilon(\ _T)$ is a residual.

Substituting this expression into the risk-neutral valuation operator (4), yields the following theorem.

Theorem 1 (Jarrow and Rudd, 1982). *Under the hypotheses of existence of the first five non-central moments of the underlying asset terminal price density, the choice of the lognormal as the approximate density of the underlying asset terminal price density and perfection and completeness of financial markets, the fair price of an European call option* $_{JR}$ *written on a stock* $_t$ *with strike price* *is:*

$$\begin{aligned} _{JR} &= \ [\ _t \quad \tau \quad \upsilon_{GC} \quad _T \ \sigma \ \kappa_3 \ \kappa_4] \\ &= \ ^{-r\tau}\int_{S_T=K}^{+\infty} (\ _T - \ \ ) \left[(\ _T) - \frac{_3}{3!}\frac{^3(\ _T)}{^3_T} \right. \\ &\qquad \left. + \frac{_4}{4!}\frac{^4(\ _T)}{^4_T} \right] \ _T + \varsigma(\ _T) \end{aligned} \tag{23}$$

where $\varsigma(\ _T)$ *is a residual.*

Proof: *see previous discussion.*

[20] These restrictions are justified by an heuristic argument of goodness-of-fit of the approximating density to the approximated one.

Developing equation (23), the Jarrow and Rudd European call option price can be expressed as:

$$
\begin{aligned}
{}_{JR} \; = \; {}_{BS} - {}^{-r\tau}\frac{3}{3!}\int_{S_T=K}^{+\infty}({}_T - \;)\frac{{}^3(\;{}_T)}{{}^3_T}\;{}_T \\
+ \; {}^{-r\tau}\frac{4}{4!}\int_{S_T=K}^{+\infty}({}_T - \;)\frac{{}^4(\;{}_T)}{{}^4_T}\;{}_T + \varsigma(\;{}_T)
\end{aligned}
\qquad (24)
$$

where ${}_{BS}$ is the price of an European call and corresponds to the standard moneyness measure under the Black and Scholes (1973) hypotheses.

The second term of the equation (24) corrects the pricing error due to the asymmetry of the original distribution function, whilst the third allows to take into account the phenomenon of heavy tails and the fourth term is a residual depending on the strike price. This statistical series expansion could obviously be based on higher moments, but one can think that moments higher than the fourth one, if they exist, would bring no supplementary valuable information. If the risk-neutral density of the underlying asset price is lognormal, then ${}_j = 0$ for $= [3\ 4]$ and equation (24) collapses to the Black and Scholes (1973) *formula*.

Recalling that $\kappa_1(\;) = \kappa_1(\,) = {}_t\,{}^{\mu\tau+\frac{\sigma^2\tau}{2}}$, $\kappa_2(\;) = \kappa_2(\,) = [\kappa_1(\,)]^2\left({}^{\sigma^2\tau} - 1\right)$, $\kappa_3(\,) = {}_3(\,)\;\; \kappa_4(\,) = {}_4(\,) - 3\;{}_2(\,)^2$ and using the martingale restriction (see Appendix 3), that is:

$$\tau = \;\tau - \frac{\sigma^2\tau}{2}$$

we obtain the following explicit *formula* for the price of an European call option.

Corollary 1 (Corrado and Su, 1996-a). *Under the hypotheses of existence of the first five non-central moments of the underlying asset terminal price density, the choice of the lognormal as the approximate density of the underlying asset terminal price density and perfection and completeness of financial markets, the fair price of an European call option* ${}_{JR}$ *written on a stock* ${}_t$ *with strike price can also be written as:*

$$ {}_{JR} \simeq {}_{BS} + \lambda_1\;{}_3 + \lambda_2\;{}_4 \qquad (25)$$

with:

$$\begin{cases} {}_3 = (\ _t \ ^{r\tau})^3 \left(\ ^{\sigma^2\tau} - 1\right)^{3/2} \frac{e^{-r\tau}}{3!} \ _1 (\ ) \frac{l(K)}{K\sigma\sqrt{\tau}} \\ {}_4 = (\ _t \ ^{r\tau})^4 \left(\ ^{\sigma^2\tau} - 1\right)^{2} \frac{e^{-r\tau}}{4!} [\ _2 (\ ) + 2 \ _1 (\ ) \sigma\sqrt{\tau} - \sigma^2\tau] \frac{l(K)}{K^2\sigma^2\tau} \\ {}_1 (\ ) = 2\sigma\sqrt{\tau} - \\ {}_2 (\ ) = \ ^2 - 3 \ \ \sigma\sqrt{\tau} + 3\sigma^2\tau - 1 \end{cases}$$

and:

$$\lambda_1 = [\gamma_1 (\ ) - \gamma_1 ( )]$$

$$\lambda_2 = [\gamma_1 (\ ) - \gamma_1 ( )]$$

where $_1 ( )$ and $_2 ( )$ *are polynomials respectively of first and second order,* $\gamma_1 ( )$ *and* $\gamma_2 ( )$ *are the Fisher parameters for skewness and kurtosis*[21]*:*

$$\gamma_1 ( ) = \frac{_3 ( )}{_2^{3/2} ( )} \quad and \quad \gamma_2 ( ) = \frac{_4 ( )}{_2^2 ( )} - 3$$

and the remainder term $\varsigma (\ _T)$ *have been neglected in (25).*

Proof: *see Appendix 3.*

The coefficients $[\gamma_1 (\ ) - \gamma_1 ( )]$ and $[\gamma_2 (\ ) - \gamma_2 ( )]$ measure, respectively, the excess skewness and the excess of excess kurtosis of the true risk-neutral density, and characterize the gap between the distribution function of the underlying asset price and the lognormal one. Parameters $_3$ and $_4$, because they also depend on the exercise price relative to options and the standard deviation of the underlying asset, represent the sensitivities of the price of a specific option to departures from log-normality. The difference between Black-Scholes and Jarrow-Rudd induced option prices is then a non-linear function of the excess moments, the level of the volatility of the market and the specific exercise price of the option

[21] In the case of the log-normal density, Fisher parameters are equal to:

$$\begin{cases} \gamma_1 (\mathsf{l}) = 3\left(\mathsf{e}^{\sigma^2\tau} - 1\right)^{\frac{1}{2}} + \left(\mathsf{e}^{\sigma^2\tau} - 1\right)^{\frac{3}{2}} \\ \gamma_2 (\mathsf{l}) = 16\left(\mathsf{e}^{\sigma^2\tau} - 1\right) + 15\left(\mathsf{e}^{\sigma^2\tau} - 1\right)^2 + 6\left(\mathsf{e}^{\sigma^2\tau} - 1\right)^3 + \left(\mathsf{e}^{\sigma^2\tau} - 1\right)^4 \end{cases}$$

considered. Figure 1 of Appendix 11 displays the sensitivities of the option price to the excess moments which represent the value of parameters $- \;_3$ and $\;_4$ as a function of the moneyness of the specific option under valuation (see also Corrado and Su, 1996-a).

- Please insert Figure 1 somewhere here -

Simulations done by Jarrow and Rudd (1982) show that their *formula* constitutes a good approximation of the option price when the underlying asset follows a Brownian process with jumps. Moreover, Jarrow and Rudd (1983) test their relation for pricing individual stock options with market data, and confirm that the use of third and fourth moments seem to improve in-sample the European call option pricing. The same conclusion has been drawn by Corrado and Su (1996-a, 1997-a) who test the Jarrow-Rudd *formula* on S&P 500 index options traded on the Chicago Board Option Exchange (CBOE). Using optimization techniques to obtain implicit parameter values in-sample, they conclude to a better fit of Jarrow and Rudd (1982) *formula* out-of-sample.

2.3.3 The Revisited Corrado and Su (1996-b and 1997-b) Model

While the Jarrow and Rudd (1982) model leads to a closed-form solution for option pricing when densities are skewed and leptokurtic, this approach remains nevertheless muddily complex since its expression involves the computation of the lognormal distribution derivatives. Following Madan and Milne (1994), an alternative approach is to work with Hermite polynomials series in which the conditional distribution of the underlying asset price log-return - rather than the price itself - is considered, and a standard normal density is used as the approximating distribution[22].

[22] Hermite polynomials have also been used in the context of American options to provide an efficient numerical integration scheme, denoted the Gauss-Hermite integration, for the compound option approxi-

Let the τ-period log-return of the underlying asset $\ _\tau$ has a conditional mean $\ \tau$ and a standard deviation $\sigma\sqrt{\tau}$, and define the standardized variable as:

$$= \frac{\log\left(\frac{S_T}{S_t}\right) - \ \tau}{\sigma\sqrt{\tau}} \tag{26}$$

Using a Gram-Charlier type A series expansion, the risk-neutral density function for is now:

$$(\ ) = \varphi(\ ) + \frac{\kappa_3(\ )}{3!} \ _3(\ )\,\varphi(\ ) + \frac{\kappa_4(\ )}{4!} \ _4(\ )\,\varphi(\ ) + \varepsilon(\ ) \tag{27}$$

where $\varphi(\ )$ is the standard normal density function and the standard normal cumulative density, $\kappa_j(\varphi) = \kappa_j(\ )$ for $= [1\ 2]$ and $\kappa_j(\varphi) = 0$ for $= [3\ 4]$ and $\ _i(\ )$ denotes the - Hermite polynomial.

Substituting (27) into the risk-neutral valuation operator (4), after the change of variable (19) have been performed in (4), Corrado and Su **(**1996-b and 1997-b) show that the value for an European call option can be obtained from the following theorem.

Theorem 2 (Corrado and Su, 1996-b and 1997-b). *Under the hypotheses of existence of the five first non-central moments of the underlying asset log-return density, the choice of the normal as the approximate density of the continuous compound return density and perfection and completeness of financial markets, the fair price of an European call option*

---

mate valuation when early exercise is continuously optimal (see Omberg, 1988) and in semi-nonparametric econometric estimation approaches (see for instance, Gallant and Nychka, 1987, Gallant and Tauchen, 1989, Gallant *et al.*,1990 and Lee and Tse, 1991).

$_{CS}$ written on a stock $_t$ with strike price is (with previous notation):

$$
\begin{aligned}
_{CS} &= \quad [\; _t \quad \tau \quad \upsilon_{GC} \quad \varphi \quad \sigma \; \kappa_3 \; \kappa_4] \\
&= \; ^{-rt} \int_{z=\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \left(\; _t \; ^{\mu\tau+\sigma\sqrt{t}z} - \; \right) \\
& \quad \left[1 + \frac{\kappa_3\,(\;)}{3!} \; _3\,(\;) + \frac{\kappa_4\,(\;)}{4!} \; _4\,(\;) \right] \varphi\,(\;) \\
& \quad + \varsigma \left(\frac{\ln(\quad _t) - \;\tau}{\sigma\sqrt{\tau}} \right)
\end{aligned}
\tag{28}
$$

where $\varsigma\,(\;)$ is a residual.

Proof: *see previous discussion.*

Developing this expression, the call option price can be written as:

$$
\begin{aligned}
_{CS} &= \; ^{-rt} \int_{z=\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \left(\; _t \; ^{\mu\tau+\sigma\sqrt{t}z} - \; \right) \varphi\,(\;) \\
& \quad + \; ^{-rt} \frac{\kappa_3\,(\;)}{3!} \int_{z=\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \left(\; _t \; ^{\mu\tau+\sigma\sqrt{t}z} - \; \right) \; _3\,(\;)\, \varphi\,(\;) \\
& \quad + \; ^{-rt} \frac{\kappa_4\,(\;)}{4!} \int_{z=\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \left(\; _t \; ^{\mu\tau+\sigma\sqrt{t}z} - \; \right) \; _4\,(\;)\, \varphi\,(\;) \\
& \quad + \zeta \left(\frac{\ln(\quad _t) - \;\tau}{\sigma\sqrt{\tau}} \right)
\end{aligned}
\tag{29}
$$

where $\zeta\,(\;)$ is a residual.

The second and the third terms of the equation take into account the pricing error due to the skewness and the kurtosis deviations from normality.

Recalling that $\kappa_1\,(\;) = \kappa_1\,(\varphi) = 0$, $\kappa_2\,(\;) = \kappa_2\,(\varphi) = 1$, $\kappa_3\,(\;) = \gamma_1\,(\;)$ $\kappa_4\,(\;) = \gamma_2\,(\;)$ - where $\gamma_1\,(\;)$ and $\gamma_2\,(\;)$ denote the Fisher parameters - and using the martingale restriction with the Gram-Charlier series expansion (see Backus *et al.*, 1997, Kochard, 1999 and Appendix 4):

$$
\tau = \; \tau - \frac{1}{2}\sigma^2\tau - \ln\left[1 + \frac{\gamma_1\,(\;)}{3!}\,\sigma^3\tau^{3/2} + \frac{\gamma_2\,(\;)}{4!}\,\sigma^4\tau^2\right] \tag{30}
$$

we obtain the following corollary for the price of an European call option.

Corollary 3 (Corrado and Su, 1996-b and 1997-b). *Under the hypotheses of existence of the first five non-central moments of the underlying asset log-return density, the choice of the normal as the approximate density of the continuous compound return density and perfection and completeness of financial markets, the fair price of an European call option* $_{CS}$ *written on a stock* $_t$ *with strike price can also be written as:*[23]

$$_{CS} \simeq \ ^{*}_{BS} + \gamma_1 (\ ) \ ^{\prime}_{3} + \gamma_2 (\ ) \ ^{\prime}_{4} \tag{31}$$

with (using previous notation):

$$\begin{cases} ^{\prime}_{3} = [3!\,(1+\omega)]^{-1} \ _t\,\sigma\sqrt{\tau} \ _1(\ ^{*})\,\varphi(\ ^{*}) \\ ^{\prime}_{4} = [4!\,(1+\omega)]^{-1} \ _t\,\sigma\sqrt{\tau} \ _2(\ ^{*})\,\varphi(\ ^{*}) \end{cases}$$

and:

$$\begin{cases} ^{*} = (\sigma\sqrt{\tau})^{-1} \left[\ln(\ _t \ \ ^{-r\tau}) + \frac{1}{2}\sigma^2\tau - \ln[(1+\omega)]\right] \\ \omega = \frac{\gamma_1(f)}{3!}\,\sigma^3\tau^{3/2} + \frac{\gamma_2(f)}{4!}\,\sigma^4\tau^2 \end{cases}$$

where $^{*}_{BS}$ *is the Black and Scholes price evaluated at a corrected standardized moneyness level denoted* *, ω *is a constant and the remainder term* $\zeta\left(- \ + \sigma\sqrt{\ }\right)$ *have been neglected in (31).*

Note that the previous expression[24] differs from those of Corrado and Su (1996-b and 1997-b) because we explicitly used the martingale restriction presented in Sub-section 2.1. Parameters $^{\prime}_{3}$ and $^{\prime}_{4}$ do not anymore represent the true marginal effects of the non-normal log-return skewness and kurtosis on the option price since terms depending on kurtosis

[23] This formula is also consistent with the Hermite polynomial option pricing model developped by Madan and Milne (1994).

[24] Kochard, (1999), developp an expression like (31) using an expansion of an infinite order. Moreover, if we neglect terms in $\sigma^3\tau^{3/2}$ and $\sigma^4\tau^2$, this *formula* is thus consistent with those presented by Backus *et al.* (1997), that is: $C_{CS} = C_{BS} + \gamma_1(f)\,Q_3^{b} + \gamma_2(f)\,Q_4^{b}$ with $Q_3^{b} = \frac{1}{3!} S_t\,\sigma\sqrt{\tau}\,(2\sigma\sqrt{\tau} - d)\,\varphi(d)$ and $Q_4^{b} = \frac{1}{4!} S_t\,\sigma\sqrt{\tau}\,\left(d^2 - 3d\,\sigma\sqrt{\tau} - 1\right)\varphi(d)$.

(skewness) appear in $'_3$ ($'_4$). Nevertheless, the Figure 2 of Appendix 11 indicates that the modified parameters are close - even if different - from the original ones. In that sense, $'_3$ remains mainly related to skewness whilst $'_4$ seems to be strongly linked with kurtosis when realistic values are considered.

- Please insert Figure 2 somewhere here -

Simulations done by Backus *et al.* (1997) show that the Corrado and Su *formula* constitutes a good approximation of the option price when the underlying asset follows a jump-diffusion process. Moreover, Corrado and Su (1996-b) and Brown and Robinson (1999) test the model by using, respectively, S&P 500 index options traded on the Chicago Board Option Exchange (CBOE) and SPI index future options traded on the Sydney Futures Exchange. They show that the use of higher moments seems to improve significantly the in-sample option pricing accuracy. Corrado and Su (1997-a) also conclude to a better fit of their *formula* on an out-of-sample basis, using actively traded individual equity options on the Chicago Board Option Exchange (CBOE), while Kochard (1999) document on the Chicago Mercantile Exchange S&P 500 index future options market an in-sample and out-sample pricing improvement for this model.

While the option pricing model based on Gram-Charlier series expansion leads to analytic expressions for the option price, as it has been pointed previously, the successive terms that appear in the series expansion of the risk-neutral density are not necessarily in decreasing order of importance, so that the expansion may not converge regularly. The use of an Edgeworth series expansion can attenuate this problem.

2.3.4 The Revisited Rubinstein (1998) Model

Following Rubinstein (1998), we consider a normal Edgeworth series expansion as a natural candidate for approximating the "true" risk-neutral density of the underlying asset log-return. In this case, recalling that the density expansion is:

$$\begin{aligned} (\ ) &= \varphi(\ ) + \frac{\kappa_3(\ )}{3!} \quad {}_3(\ )\, \varphi(\ ) \\ &\quad + \left\{ \frac{\kappa_4(\ )}{4!} \quad {}_4(\ ) + 10\, \frac{[\kappa_3(\ )]^2}{6!} \quad {}_6(\ ) \right\} \varphi(\ ) \\ &\quad + \varepsilon(\ ) \end{aligned} \tag{32}$$

where is defined as in (19), $\varphi(\ )$ $_i(\ )$ and $\varepsilon(\ )$ denotes respectively the standard normal density, the - Hermite polynomial and a residual.

Recalling that $\kappa_1(\ ) = \kappa_1(\varphi) = 0$, $\kappa_2(\ ) = \kappa_2(\varphi) = 1$, $\kappa_3(\ ) = \gamma_1(\ )$ $\kappa_4(\ ) = \gamma_2(\ )$ and using the martingale restriction with the Edgeworth series expansion (see Backus *et al.*, 1997, Kochard, 1999 and Appendix 4):

$$\tau = \ \tau - \frac{1}{2}\sigma^2\tau - \ln\left[1 + \frac{\gamma_1(\ )}{3!}\sigma^3\tau^{3/2} + \frac{\gamma_2(\ )}{4!}\sigma^4\tau^2 + 10\,\frac{\gamma_1(\ )^2}{6!}\sigma^6\tau^3\right] \tag{33}$$

the Edgeworth series expansion based option price can be expressed in the following theorem.

Theorem 3 (Rubinstein, 1998). *Under the hypotheses of existence of the five first non-central moments of the underlying asset log-return density, the choice of the normal as the approximate density of the continuous compound return density, and perfection and completeness of financial markets, the fair price of an European call option* $_R$ *written on a stock* $_t$ *with strike price can be written as:*

$$\begin{aligned} {}_R &\simeq \quad [\ {}_t \quad \tau \quad \upsilon_E \quad \varphi \quad \sigma\ \kappa_3\ \kappa_4] \\ &= \quad {}^{**}_{BS} + \gamma_1(\ ) \quad {}''_3 + \gamma_2(\ ) \quad {}''_4 + \gamma_1(\ )^2 \quad {}''_5 \end{aligned} \tag{34}$$

with (using previous notation):

$$
\begin{cases}
{}''_{3} = [3!\,(1 + \quad)]^{\,-1} \quad {}_{t}\,\sigma\sqrt{\tau} \quad {}_{1}(\quad {}^{**})\,\varphi(\quad {}^{**}) \\
{}''_{4} = \ [4!\,(1 + \quad)]^{-1} \quad {}_{t}\,\sigma\sqrt{\tau} \quad {}_{2}(\quad {}^{**})\,\varphi(\quad {}^{**}) \\
{}''_{5} = 10\,[6!\,(1 + \quad)]^{-1} \quad {}_{t}\,\sigma\sqrt{\tau} \quad {}_{4}(\quad {}^{**})\,\varphi(\quad *
\end{cases}
$$

- Please insert Figures 3 and 4 somewhere here -

2.4 Implied Probability Densities, Implied Volatility Smile Functions and Greeks

with:

$$\begin{cases} \frac{dl(S_T)}{dS_T} = -\left(1 + \frac{\ln(S_T/S_t)-\mu\tau}{\sigma^2\tau}\right)\frac{l(S_T)}{S_T} \\ \frac{d^2l(S_T)}{dS_T^2} = -\left(2 + \frac{\ln(S_T/S_t)-\mu\tau}{\sigma^2\tau}\right)\frac{1}{S_T}\frac{dl(S_T)}{dS_T} - \frac{l(S_T)}{S_T^2\sigma^2\tau} \\ \frac{d^3l(S_T)}{dS_T^3} = -\left(3 + \frac{\ln(S_T/S_t)-\mu\tau}{\sigma^2\tau}\right)\frac{1}{S_T}\frac{d^2l(S_T)}{dS_T^2} - \frac{2}{S_T^2\sigma^2\tau}\frac{dl(S_T)}{dS_T} + \frac{l(S_T)}{S_T^3\sigma^2\tau} \\ \frac{d^4l(S_T)}{dS_T^4} = -\left(4 + \frac{\ln(S_T/S_t)-\mu\tau}{\sigma^2\tau}\right)\frac{1}{S_T}\frac{d^3l(S_T)}{dS_T^3} - \frac{3}{S_T^2\sigma^2\tau}\frac{d^2l(S_T)}{dS_T^2} + \frac{3}{S_T^3\sigma^2\tau}\frac{dl(S_T)}{dS_T} - \frac{2\,l(S_T)}{S_T^4\sigma^2\tau} \end{cases}$$

where ( ) is the lognormal density function.

Proof: *see previous discussion.*

The implied risk-neutral density function can then be expressed as a linear function of the excess skewness and excess of excess kurtosis of the underlying asset price.

Theorem 5. *When the European call market price is given by the Corrado and Su (1996-b and 1997-b) formula, the implied risk-neutral density function of the continuous compounded asset return can be written such as:*

$$(\ ) \simeq \varphi(\ )\left[1 + \frac{\gamma_1(\ )}{3!}(\ ^3 - 3\ ) + \frac{\gamma_2(\ )}{4!}(\ ^4 - 6\ ^2 + 3)\right] \quad (36)$$

where $\varphi(\ )$ is the standard normal density function and is defined as in (19).

Proof: *see previous discussion.*

The implied state price distribution function is a linear function of the skewness and the excess kurtosis of the asset log-return.

Theorem 6. *When the European call market price is given by the Rubinstein (1998)* formula, *the implied risk-neutral density function of the continuous compounded asset return can be written such as:*

$$\begin{aligned}(\ ) \simeq\ & \varphi(\ )\left[1 + \frac{\gamma_1(\ )}{3!}(\ ^3 - 3\ ) + \frac{\gamma_2(\ )}{4!}(\ ^4 - 6\ ^2 + 3)\right. \\ & \left. +10\frac{[\gamma_1(\ )]^2}{6!}(\ ^6 - 15\ ^4 + 45\ ^2 - 15)\right]\end{aligned} \quad (37)$$

where $\varphi(\ )$ is the standard normal density function and is defined as in (19).

Proof: *see previous discussion.*

As with the Gram-Charlier Type A series expansion, the implied risk-neutral density function can be written as a function of the skewness and excess kurtosis of the underlying asset log-return. Nevertheless, the relation is no longer linear but quadratic.

In Figure 5 of Appendix 11, we display simultaneously the Black-Scholes and the Jarrow-Rudd, the Corrado-Su and the Rubinstein probability density functions with realistic skewness and kurtosis.

- Please insert Figure 5 somewhere here -

2.4.2 Implied Volatility Smile Functions

Following the approach of Backus *et al.,* (1997) and Bakshi *et al.* (2002), we can also provide the implied standard deviation function that corresponds to a volatility, denoted Ψ, that equates the market price of the option to the value given by the Black-Scholes (1973) *formula*, other values and parameters fixed. Using Jarrow-Rudd (1982), Corrado-Su (1996-b and 1997-b) and Rubinstein (1998) European call option pricing models, we get the following expressions for the implied volatility smile functions[26].

Theorem 7. *When the European call market price is given by the Jarrow and Rudd (1982)* formula*, the implied volatility smile function can be written such as*[27]*:*

$$\begin{aligned} {}_{JR} &= \Psi(\ _t \quad \tau \quad \upsilon_{GC} \quad {}_T \ \sigma \ \kappa_3 \ \kappa_4) \\ &\simeq \sigma\sqrt{\tau} + \lambda_1 \sigma\sqrt{\tau} \ {}'''_3 \left[\varphi(\ )\right]^{-1} + \lambda_2 \sigma\sqrt{\tau} \ {}'''_4 \left[\varphi(\ )\right]^{-1} \end{aligned} \quad (38)$$

[26] Following Backus *et al.* (1997), we refer to the relation $\Psi(.)$ between implied volatility and moneyness as the implied volatility smile function.

[27] Equation (38) is an approximation due to the presence of the Gram-Charlier series expansion and of a linear approximation of European call prices in terms of volatility (see Appendix 7).

with (using previous notation):

$$\begin{cases} {}_3''' = (\ _t\ {}^{r\tau})^2 \left({}^{\sigma^2\tau} - 1 \right)^{3/2}\ {}_1(\ )\frac{l(K)}{3!K\,\sigma^2\tau} \\ {}_4''' = (\ _t\ {}^{r\tau})^3 \left({}^{\sigma^2\tau} - 1 \right)^2 [\ {}_2(\ ) + 2\ {}_1(\ )\,\sigma\sqrt{\tau} - \sigma^2\tau]\,\frac{l(K)}{4!K\,{}^2\sigma^3\tau^{3/2}} \end{cases}$$

where the implied volatility function $_{JR}$ *corresponds to the value of* Ψ *that equates the Jarrow and Rudd (1982) price to the value of the Black and Scholes (1973)* formula*, given the values of other parameters fixed and* $\varphi(\ )$ *is the standard normal density function.*

Proof: *see Appendix 7.*

The implied volatility function is then a linear function of the excess skewness and excess of excess kurtosis of the underlying asset log-return risk-neutral density.

Theorem 8 (see Backus *et al.*, 1997). *When the European call market price is given by the Corrado and Su (1996-b and 1997-b)* formula*, the implied volatility smile function can be written*[28] *such as:*[29]

$$\begin{aligned} {}_{CS} &= \Psi(\ _t\quad \tau\quad \upsilon_{GC}\quad \varphi\quad \sigma\ \kappa_3\ \kappa_4) \\ &\simeq \sigma\sqrt{\tau} + \frac{1}{_t}(\ {}^*_{BS} - \ _{BS})\,[\varphi(\ )]^{-1} \\ &\qquad + \gamma_1(\ )\ \sigma\sqrt{\tau}\ \ {}_3''''\ [\varphi(\ )]^{-1} + \gamma_2(\ )\ \sigma\sqrt{\tau}\ \ {}_4''''\ [\varphi(\ )]^{-1} \end{aligned} \tag{39}$$

with (using previous notation):

$$\begin{cases} {}_3'''' = [3!\,(1+\omega)]^{-1}\ \ {}_1(\ ^*)\,\varphi(\ ^*) \\ {}_4'''' = [4!\,(1+\omega)]^{-1}\ \ {}_2(\ ^*)\,\varphi(\ ^*) \end{cases}$$

[28] Equation (39) is also an approximation for several reasons: the Gram-Charlier series expansion, a linear approximation of European call prices in terms of volatility and the elimination of terms involving $\sigma^3\tau^{3/2}$ and $\sigma^4\tau^2$ (see Appendix 4 and 7).

[29] Using the same approach, Baschi *et al.* (2002) derive a similar relation between the implied volatility and the skewness and kurtosis of the risk-neutral distribution. The only difference is that they do not explicitly identify the coefficients of implied volatility function.

where the implied volatility $_{CS}$ *corresponds to the value of* Ψ *that equates the Corrado and Su (1996-b and 1997-b) price to the value of the Black and Scholes (1973)* formula, *given the values of other parameters fixed.*

Proof: *see Appendix 7.*

The implied volatility function can again be expressed as a linear function of the skewness and excess kurtosis of the underlying asset log-return risk-neutral density.

Theorem 9. *When the European call market price is given by the Rubinstein (1998)* formula, *the implied volatility smile function reads:*

$$
\begin{aligned}
{}_{R} &= \Psi(\ {}_{t} \quad \tau \quad v_E \quad \varphi \quad \sigma\ \kappa_3\ \kappa_4) \\
&\simeq \sigma\sqrt{\tau} + \frac{1}{{}_t}(\ {}^{**}_{BS} - \ {}_{BS})\,[\varphi(\ )]^{-1} \\
&\qquad + \gamma_1(\ )\ \sigma\sqrt{\tau}\ \ {}^{\prime\prime\prime\prime\prime}_{3}\,[\varphi(\ )]^{-1} + \gamma_2(\ )\ \sigma\sqrt{\tau}\ \ {}^{\prime\prime\prime\prime\prime}_{4}\,[\varphi(\ )]^{-1} \\
&\qquad\qquad + [\gamma_1(\ )]^2\,\sigma\sqrt{\tau}\ \ {}^{\prime\prime\prime\prime\prime}_{5}\,[\varphi(\ )]^{-1}
\end{aligned}
\tag{40}
$$

with (using previous notation):

$$
\begin{cases}
{}^{\prime\prime\prime\prime\prime}_{3} = [3!\,(1+\ \ )]^{\ -1}\ \ {}_{1}(\ {}^{**})\,\varphi(\ {}^{**}) \\
{}^{\prime\prime\prime\prime\prime}_{4} = \ [4!\,(1+\ \ )]^{-1}\ \ {}_{2}(\ {}^{**})\,\varphi(\ {}^{**}) \\
{}^{\prime\prime\prime\prime\prime}_{5} = 10\,[6!\,(1+\ \ )]^{-1}\ \ {}_{4}(\ {}^{**})\,\varphi(\ {}^{**})
\end{cases}
$$

where the implied volatility $_{R}$ *corresponds to the value of* Ψ *that equates the Rubinstein price (1998) to the value of the Black and Scholes (1973)* formula, *given the values of other parameters fixed.*

Proof: *see Appendix 7.*

As with the Gram-Charlier Type A series expansion, the implied volatility smile function can here be expressed as a function of the skewness and excess kurtosis of the underlying asset log-return risk-neutral density. However, the relation is no longer linear but quadratic as in the previous case.

Figure 6 illustrates the comparison of Jarrow-Rudd, Corrado-Su and Rubinstein's implied volatility smile functions when the risk-neutral density is skewed and leptokurtic. Figures 7 and 8 are dedicated to the comparison of the specific effect of the skewness and of the kurtosis on the shape of the implied volatility smile functions.

- Please insert Figures 6 to 8 somewhere here -

2.4.3 The Greeks

The Greek parameters are of interest since they can be used for testing and hedging purposes. In particular, Delta states the sensitivity of the option price to underlying asset price movements. By definition, it is the first partial derivative of the option price with respect to the underlying asset price. Gamma measures the sensitivity of Delta-hedged strategies to the underlying asset price changes and is defined by the second partial derivative of the option price with respect to the underlying asset price. Vega, Khi and Psi measure the sensitivities of the option price with respect to changes in the volatility, skewness and kurtosis and are defined by the first partial derivatives of the option price. By taking the appropriate derivatives of the Jarrow-Rudd (1982), the Corrado-Su (1996-b and 1997-b) and the Rubinstein (1998) European call option pricing models, we get the following expressions for the Delta, Gamma, Khi and Psi.

Theorem 10 (see Corrado and Su, 1996-b and 1997-b). *When the European call market price is given by the Jarrow and Rudd (1982)* formula*, the Delta, Gamma, Vega, Khi and Psi of the call can be written respectively such as:*

$$\begin{aligned} \Delta_{JR}^{C} &= \frac{\partial \quad _{JR}}{\partial \quad _{t}} \\ &\simeq \Phi(\;) + \lambda_1 \quad _3^{''''} \left[- \quad _2(\;) - 3 \quad _1(\;)\,\sigma\sqrt{\tau} + \sigma^2\tau\right] \\ &\quad +\lambda_2 \quad _4^{''''} \left[- \quad _3(\;) + 6 \quad _2(\;)\,\sigma\sqrt{\tau} + 7 \quad _1(\;)\,\sigma^2\tau - 5\sigma^3\tau^{3/2}\right] \end{aligned} \tag{41}$$

$$
\begin{aligned}
\Gamma_{JR}^{C} &= \frac{\partial^2 \quad _{JR}}{\partial \quad _t^2} \\
&\simeq \left(\quad _t\sigma\sqrt{\tau}\right)^{-1}\Big\{\varphi(\ )+\lambda_1 \quad _3^{''''}\left[\quad _3(\ )-5 \quad _2(\ )\sigma\sqrt{\tau}+5 \quad \sigma^2\tau-6\sigma^3\tau^{3/2}\right] \\
&\quad +\lambda_2 \quad _4^{''''}\left[\quad _4(\ )-9 \quad _3(\ )\sigma\sqrt{\tau}+25 \quad _2(\ )\sigma^2\tau-43\sigma^3\tau^{3/2}+18\sigma^4\tau^2-12\sigma^2\tau\right]\Big\}
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
\upsilon_{JR}^{C} &= \frac{\partial \quad _{JR}}{\partial\sigma} \\
&\simeq \quad _t\sqrt{\tau}\Big\{\varphi(\ )+\lambda_1 \quad _3^{''''}\left[\quad _3(\ )+ \quad _2(\ )\sigma\sqrt{\tau}- \quad \sigma^2\tau-6\sigma^3\tau^{3/2}\right] \\
&\quad +\lambda_2 \quad _4^{''''}\left[\quad _4(\ )- \quad _3(\ )\sigma\sqrt{\tau}-3 \quad _2(\ )\sigma^2\tau+ \quad \sigma^3\tau^{3/2}-18 \quad \sigma\sqrt{\tau}+11\sigma^4\tau^2\right] \\
&\quad +3\sigma^2\tau \quad ^{\sigma^2\tau} \quad _3^{'''}\left[\lambda_1\left(\quad ^{\sigma^2\tau}-1\right)^{-1}-\left(\quad ^{\sigma^2\tau}-1\right)^{-1/2}-\left(\quad ^{\sigma^2\tau}-1\right)^{-1/2}\right] \\
&\quad +4\sigma^2\tau \quad ^{\sigma^2\tau} \quad _4^{'''}\Big[\lambda_2\left(\quad ^{\sigma^2\tau}-1\right)^{-1}-8-15\left(\quad ^{\sigma^2\tau}-1\right)-9\left(\quad ^{\sigma^2\tau}-1\right)^2 \\
&\quad -2\left(\quad ^{\sigma^2\tau}-1\right)^3\Big]\Big\}
\end{aligned}
\tag{43}
$$

$$
\chi_{JR}^{C} = \frac{\partial \quad _{JR}}{\partial\gamma_1(\ )} \simeq \quad _3 \tag{44}
$$

and:

$$
\Psi_{JR}^{C} = \frac{\partial \quad _{JR}}{\partial\gamma_2(\ )} \simeq \quad _4 \tag{45}
$$

with (using previous notation):

$$
\begin{cases}
\quad _3^{''''} = -\left(\quad _t \quad ^{r\tau}\right)^2\left(\quad ^{\sigma^2\tau}-1\right)^{3/2}\frac{l(K)}{3!K\,\sigma^2\tau} \\
\quad _4^{''''} = \left(\quad _t \quad ^{r\tau}\right)^3\left(\quad ^{\sigma^2\tau}-1\right)^{2}\frac{l(K)}{4!K\,^2\sigma^3\tau^{3/2}} \\
\quad _3(\ ) = \quad ^3-4 \quad ^2\sigma\sqrt{\tau}-3 \quad +6 \quad \sigma^2\tau-3\sigma^3\tau^{3/2}+4\sigma\sqrt{\tau}
\end{cases}
$$

where $\Phi(\ )$ $\varphi(\ )$ *and* $(\ )$ *are, respectively, the cumulative density function of the standard Gaussian distribution, the density function of the standard Gaussian distribution and the density function of the lognormal distribution;* $_3(\ )$ *is a polynomial of third order and* λ_1, λ_2, $_1(\ )$ $_2(\ )$, $_3$, $_4$, $_3^{'''}$ *and* $_4^{'''}$ *are defined in equation (25) and (38).*

Proof: *see Appendix 8.*

Theorem 11 (see Backus *et al.*, 1997). *When the European call market price is given by the Corrado and Su (1996-b and 1997-b)* formula*, the Delta, Gamma, Vega, Khi and Psi of the call can be written respectively such as:*

$$\begin{aligned}
\Delta_{CS}^{C} &= \frac{\partial \ _{CS}}{\partial \ _{t}} \\
&\simeq \Phi(\ ^{*}) + \frac{\varphi(\ ^{*})}{\sigma\sqrt{\tau}} + \frac{\varphi(\ ^{*})}{(1+\omega)}\left\{-\frac{1}{\sigma\sqrt{\tau}} + \frac{\gamma_1(\ )}{3!}\left[\ _2(\ ^{*}) - \sigma^2\tau\right]\right. \\
&\quad \left. -\frac{\gamma_2(\ )}{4!}\ _3(\ ^{*})\right\}
\end{aligned} \tag{46}$$

$$\begin{aligned}
\Gamma_{CS}^{C} &= \frac{\partial^2 \ _{CS}}{\partial \ _{t}^{2}} \\
&\simeq \frac{\varphi(\ ^{*})}{_t\sigma\sqrt{\tau}} - \frac{^{*}\varphi(\ ^{*})}{_t\sigma^2\tau} + \frac{\varphi(\ ^{*})}{_t\sigma\sqrt{\tau}\,(1+\omega)}\left\{\frac{^{*}}{\sigma\sqrt{\tau}} + \frac{\gamma_1(\ )}{3!}\left[-\ _3(\ ^{*})\right.\right. \\
&\quad \left. -\ _2(\ ^{*})\,\sigma\sqrt{\tau} + \ ^{*}\sigma^2\tau + 6\sigma^3\tau^{3/2}\right] + \frac{\gamma_2(\ )}{4!}\left[\ _4(\ ^{*}) - \ _3(\ ^{*})\,\sigma\sqrt{\tau}\right. \\
&\quad \left.\left. -2\ ^{*3}\sigma\sqrt{\tau} + \ ^{*}\sigma^3\tau^{3/2} + 8\sigma^4\tau^2\right]\right\}
\end{aligned} \tag{47}$$

$$\begin{aligned}
v_{CS}^{C} &= \frac{\partial \ _{CS}}{\partial\sigma} \\
&\simeq \ _t\sqrt{\tau}\varphi(\ ^{*})\left(1 - \frac{^{*}}{\sigma\sqrt{\tau}}\right) + \frac{_t\sqrt{\tau}\varphi(\ ^{*})}{(1+\omega)}\left\{\frac{^{*}}{\sigma\sqrt{\tau}} + \frac{\gamma_1(\ )}{3!}\left[-\ ^{*}\right.\right. \\
&\quad \left. \times\ _2(\ ^{*}) + \ ^{*}\sigma^2\tau + \ ^{*} + 3\sigma\sqrt{\tau}\right] + \frac{\gamma_2(\ )}{4!}\left[\ ^{*}\ _3(\ ^{*}) + \ _2(\ ^{*})\right. \\
&\quad \left.\left. -7\ ^{*}\sigma\sqrt{\tau} - 3\sigma^2\tau\right]\right\} + \frac{_t\sqrt{\tau}\left(\frac{\gamma_1(f)}{2!}\sigma^2\tau + \frac{\gamma_2(f)}{3!}\sigma^3\tau^{3/2}\right)\varphi(\ ^{*})}{(1+\omega)^2}\left\{\frac{1}{\sigma\sqrt{\tau}}\right. \\
&\quad \left. +\frac{\gamma_1(\ )}{3!}\left[-\ _2(\ ^{*}) + \sigma^2\tau\right] + \frac{\gamma_2(\ )}{4!}\ _3(\ ^{*})\right\}
\end{aligned} \tag{48}$$

$$\begin{aligned}
\chi_{CS}^{C} &= \frac{\partial \ _{CS}}{\partial\gamma_1(\ )} \\
&\simeq \frac{_t\sigma\sqrt{\tau}\ \ _1(\ ^{*})\,\varphi(\ ^{*})}{3!\,(1+\omega)} + \frac{_t\sigma^3\tau^{3/2}\varphi(\ ^{*})}{3!\,(1+\omega)^2}\left\{\frac{1}{\sigma\sqrt{\tau}}\right. \\
&\quad \left. +\frac{\gamma_1(\ )}{3!}\left[-\ _2(\ ^{*}) + \sigma^2\tau\right] + \frac{\gamma_2(\ )}{4!}\ _3(\ ^{*})\right\}
\end{aligned} \tag{49}$$

and:

$$
\begin{aligned}
\Psi^C_{CS} &= \frac{\partial \ _{GC}}{\partial \gamma_2\left(\ \right)} \qquad (50) \\
&\simeq \frac{_t\,\sigma\sqrt{\tau}\left[\ _2\left(\ ^*\right)-\sigma\sqrt{\tau}\right]\varphi\left(\ ^*\right)}{4!\,(1+\omega)} + \frac{_t\,\sigma^4\tau^2\varphi\left(\ ^*\right)}{4!\,(1+\omega)^2}\left\{\frac{1}{\sigma\sqrt{\tau}}\right. \\
&\quad \left. + \frac{\gamma_1\left(\ \right)}{3!}\left[-\ _2\left(\ ^*\right)+\sigma\sqrt{\tau}\right] + \frac{\gamma_2\left(\ \right)}{4!}\ _3\left(\ ^*\right)\right\}
\end{aligned}
$$

where $\Phi\left(\ \right)$ *and* $\varphi\left(\ \right)$ *are the cumulative density function and the density function of the standard Gaussian distribution and* $_1\left(\ \right)$, $_2\left(\ \right)$, *, ω *and* $_3\left(\ \right)$ *are defined respectively in equation (25), (31) and (41).*

Proof: *see Appendix 9.*

Theorem 12. *When the European call market price is given by the Rubinstein (1998)* formula*, the Delta, Gamma, Khi and Psi of the call can be written respectively such as:*

$$
\begin{aligned}
\Delta^C_R &= \frac{\partial \ _R}{\partial \ _t} \qquad (51) \\
&\simeq \Phi\left(\ ^*\right) + \frac{\varphi\left(\ ^{**}\right)}{\sigma\sqrt{\tau}} + \frac{\varphi\left(\ ^{**}\right)}{(1+\ \ )}\left\{-\frac{1}{\sigma\sqrt{\tau}} + \frac{\gamma_1\left(\ \right)}{3!}\left[\ _2\left(\ ^{**}\right)-\sigma^2\tau\right]\right. \\
&\quad \left. -\frac{\gamma_2\left(\ \right)}{4!}\ _3\left(\ ^{**}\right) - \frac{10\left[\gamma_1\left(\ \right)\right]^2}{6!}\ _5\left(\ ^{**}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^C_R &= \frac{\partial^2 \ _R}{\partial \ ^2_t} \qquad (52) \\
&\simeq \frac{\varphi\left(\ ^{**}\right)}{_t\sigma\sqrt{\tau}} - \frac{^{**}\varphi\left(\ ^{**}\right)}{_t\sigma^2\tau} + \frac{\varphi\left(\ ^{**}\right)}{_t\sigma\sqrt{\tau}\,(1+\ \ )}\left\{\frac{^{**}}{\sigma\sqrt{\tau}} + \frac{\gamma_1\left(\ \right)}{3!}\left[-\ _3\left(\ ^{**}\right)\right.\right. \\
&\quad \left. -\ _2\left(\ ^{**}\right)\sigma\sqrt{\tau} + \ ^{**}\sigma^2\tau + 6\sigma^3\tau^{3/2}\right] + \frac{\gamma_2\left(\ \right)}{4!}\left[\ _4\left(\ ^{**}\right) - \ _3\left(\ ^{**}\right)\sigma\sqrt{\tau}\right. \\
&\quad \left.\left. -2\ ^{**3}\sigma\sqrt{\tau} + \ ^{**}\sigma^3\tau^{3/2} + 8\sigma^4\tau^2\right] + \frac{10\left[\gamma_1\left(\ \right)\right]^2}{6!}\left[\ _6\left(\ ^{**}\right) - 5\ _4\left(\ ^{**}\right) - 15\ ^{**^2}\sigma^2\tau\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
v_R^C &= \frac{\partial \;\, _R}{\partial \sigma} &(53)\\
&\simeq \;\, _t\sqrt{\tau}\varphi(\;\,^{**})\left(1-\frac{^{**}}{\sigma\sqrt{\tau}}\right)+\frac{_t\sqrt{\tau}\varphi(\;\,^{**})}{(1+\;\;)}\left\{\frac{^{**}}{\sigma\sqrt{\tau}}+\frac{\gamma_1(\;)}{3!}\left[-\;^{**}\;\,_2(\;^{**})\right.\right.\\
&\quad \left.+\;^{**}\sigma^2\tau+\;\;^{**}+3\sigma\sqrt{\tau}\right]+\frac{\gamma_2(\;)}{4!}\left[\;^{**}\;\,_3(\;^{**})+\;\,_2(\;^{**})-7\;^{**}\sigma\sqrt{\tau}-3\sigma^2\tau\right]\\
&\quad \left.+10\frac{\left[\gamma_1(\;)\;^2\right]}{6!}\;\,_6(\;^{**})\right\}+\frac{_t\sqrt{\tau}\,\varphi(\;\,^{**})\left(\frac{\gamma_1(f)}{2!}\sigma^2\tau^2+\frac{\gamma_2(f)}{3!}\sigma^3\tau^{3/2}+10\frac{\gamma_1(f)^2}{5!}\sigma^5\tau^{5/2}\right)}{(1+\;\;)^2}\\
&\quad \left\{\frac{1}{\sigma\sqrt{\tau}}+\frac{\gamma_1(\;)}{3!}\left[-\;\,_2(\;^{**})+\sigma^2\tau\right]+\frac{\gamma_2(\;)}{4!}\;\,_3(\;^{**})+10\frac{\left[\gamma_1(\;)\;^2\right]}{6!}\;\,_5(\;^{**})\right\}
\end{aligned}
$$

$$
\begin{aligned}
\chi_R^C &= \frac{\partial \;\, _R}{\partial \gamma_1(\;)} &(54)\\
&\simeq -\frac{_t\,\sigma\sqrt{\tau}\left[\;\,_1(\;^{**})+\frac{\gamma_1(f)}{6}\;\,_4(\;^{**})+\sigma^4\tau^2+\sigma\sqrt{\tau}\right]\varphi(\;^{**})}{3!\,(1+\;\;)}\\
&\quad +\frac{_t\,\sigma^3\tau^{3/2}\left(1+\frac{\gamma_1(f)}{6}\sigma^3\tau^{3/2}\right)\varphi(\;^{**})}{3!\,(1+\;\;)^2}\left\{\frac{1}{\sigma\sqrt{\tau}}+\frac{\gamma_1(\;)}{3!}\left[-\;\,_2(\;^{**})+\sigma^2\tau\right]\right.\\
&\quad \left.+\frac{\gamma_2(\;)}{(4!)}\;\,_3(\;^{**})+10\frac{\left[\gamma_1(\;)\right]^2}{6!}\;\,_5(\;^{**})\right\}
\end{aligned}
$$

and:

$$
\begin{aligned}
\Psi_R^C &= \frac{\partial \;\, _R}{\partial \gamma_2(\;)} &(55)\\
&\simeq \frac{_t\,\sigma\sqrt{\tau}\sqrt{\tau}\left[\;\,_2(\;^{**})-\sigma\sqrt{\tau}\right]\varphi(\;^{**})}{4!\,(1+\;\;)}+\frac{_t\,\sigma^4\tau^2\varphi(\;^{**})}{4!\,(1+\;\;)^2}\left\{\frac{1}{\sigma\sqrt{\tau}}\right.\\
&\quad \left.+\frac{\gamma_1(\;)}{3!}\left[-\;\,_2(\;^{**})+\sigma\sqrt{\tau}\right]+\frac{\gamma_2(\;)}{4!}\;\,_3(\;^{**})+\frac{10\left[\gamma_1(\;)\right]^2}{6!}\;\,_5(\;^{**})\right\}
\end{aligned}
$$

with (using previous notation):

$$
\left\{
\begin{aligned}
_5(\;) &= \;^5-6\;^4\sigma\sqrt{\tau}+15\;^3\sigma^2\tau-10\;^3-20\;^2\sigma^3\tau^{3/2}+36\;^2\sigma\sqrt{\tau}\\
&\quad +15\;\sigma^4\tau^2-45\;\sigma^2\tau+15\;-5\sigma^5\tau^{5/2}+20\sigma^3\tau^{3/2}-18\sigma\sqrt{\tau}\\
_6(\;) &= \;^6-6\;^5\sigma\sqrt{\tau}+15\;^4\sigma^2\tau-9\;^4-20\;^3\sigma^3\tau^{3/2}+30\;^3\sigma\sqrt{\tau}+15\;^2\sigma^4\tau\\
&\quad -15\;^2\sigma^2\tau+9\;^2-5\;\sigma^5\tau^{5/2}+15\sigma^4\tau^2-15\sigma^2\tau+3
\end{aligned}
\right.
$$

where $\Phi(\ )$ *and* $\varphi(\ )$ *are, respectively, the cumulative density function and the density function of the standard Gaussian distribution,* $_5(\ )$ *and* $_6(\ )$ *are respectively polynomials of fourth and fifth order and* $_1(\ )$, $_2(\ )$, **, , $_4(\ )$ *and* $_3(\ )$ *are defined respectively in equation (25), (34) and (41).*

Proof: *see Appendix 10.*

The first terms on the right-hand sides of equations (41), (46) and (51); (42), (47) and (52); and (43), (48) and (53) are respectively the Delta, Gamma and Vega of the Black-Scholes (1973) model whilst the other terms adjust the Delta, Gamma and Vega for the presence of skewness and kurtosis in the return distribution. In Figures 9 to 11, we illustrate respectively the differences in Deltas, Gammas and Vegas for the approximate models. Figure 12 displays the Khi of the Corrado-Su and Rubinstein models, and Figure 13 represents the comparison between related Psi.

- Please insert Figures 9 to 13 somewhere here -

3 Conclusion

This article focuses on a way of solving drawbacks of Black and Scholes (1973) model using statistical series expansions to correct the implied density departures from Gaussianity. We investigate several different multi-moment approximate option pricing models in an unified framework, highlighting the difference between Jarrow and Rudd (1982), Corrado and Su (1996-b and 1997-b) and Rubinstein (1998) models. We present the conditions that ensure the respect of the martingale restriction and establish the link between these approximate models and alternative option pricing models such as Black and Scholes (1973) and Hermite polynomial models (see Madan and Milne, 1994, Abken *et al.*, 1996). We also provide analytical *formulae* for related implied densities and implicit volatility smile functions, and illustrate their properties with simulated data. The final contribution of this paper concerns hedging parameters in this setting: we extend the traditional Greeks to deal with higher moment changes.

Our next work will consist in investigating the relative pricing power and hedging performances of all these different fourth-moment options pricing models with market data.

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Appendix 1

It is possible to express - under mild conditions, see below - a continuous density $( )$ as a function of an arbitrary continuous density $( )$ and its cumulants $\kappa_j ( )$ $= [1 \quad -1]$ as follows:

$$( ) = ( ) + \sum_{i=1}^{+\infty} \frac{1}{!} \left\{ \sum_{j=1}^{N-1} (-1)^j \frac{[\kappa_j ( ) - \kappa_j ( )] \quad ^j}{!} \right\}^i ( ) + \varepsilon ( ) \qquad (8)$$

Proof. Let $( )$ and $( )$ be respectively the "true" cumulative density function and the approximating one. We moreover assume that $( ) \quad = ( )$ and $( ) \quad = ( )$ exist, as well as the first N non-central moments of the distribution function . Formally, the first cumulants $\kappa_j ( )$, $= [1 \quad - 1]$ are given by the following equality (see Kendall and Stuart, 1977, p.73):

$$\ln \phi (\quad) = \left[\sum_{j=1}^{N-1} \kappa_j ( ) \frac{( )^j}{!} \right] + \left(^{N-1} \right) \qquad \text{(A.1.1.)}$$

where $\phi (\quad)$ is the characteristic function of $( )$ and $^2 = -1$.

Taking exponential of equation $(\quad 1\,1)$ and using the definition of the characteristic function of $( )$ we obtain:

$$\phi (\quad) = \exp \left\{ \sum_{j=1}^{N-1} [\kappa_j ( ) - \kappa_j ( )] \frac{( )^j}{!} \right\} \phi (\quad) + \left(^{N-1} \right) \qquad \text{(A.1.2.)}$$

Taking the inverse Fourier transform of $(\quad 1\,2)$, yields (see Johnson *et al.*, 1994, p.26):

$$( ) = \exp \left\{ \sum_{j=1}^{N-1} (-1)^j \frac{_j ( )^j}{!} \right\} ( ) + \varepsilon ( ) \qquad \text{(A.1.3.)}$$

with:

$$\begin{cases} ( ) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \quad ^{-itx} \phi (\quad) \\ ( ) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \quad ^{-itx} \phi (\quad) \\ \exp \left[(-1)^j \quad ^j \right] ( ) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \quad ^{-itx} \exp \left[_j \frac{(it)^j}{j!} \right] \phi (\quad) \\ \varepsilon ( ) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \quad ^{-itx} \left(^{N-1} \right) \end{cases}$$

where $_j = [\kappa_j(\ ) - \kappa_j(\ )]$ and is the differentiation operator such as $^j(\ ) = {}^j(\ ) \quad {}^j$.

Expanding the equation $(\quad 1\ 3\,)$ as an infinite polynomial leads to the desired result, that is:

$$(\ ) = \quad (\ ) + \sum_{i=0}^{\infty} \frac{1}{!} \left[\sum_{j=1}^{N-1} (-1)^j \frac{_j(\ )^j}{!} \right]^i \quad (\ ) + \varepsilon(\ ) \tag{A.1.4.}$$

■

Appendix 2

The $j\text{-}th$ cumulant for a normalized sum of n independent and identically standardized random variables $_i$ such as:

$$= \quad {}^{-1/2}\sigma^{-1}\sum_{i=1}^{n}(\ _i - \ ) \tag{11}$$

is proportional to $^{1-j/2}(\quad \geq 2)$.

Proof. By construction, the characteristic function of must verified the following equality:

$$\phi(\ ) = \quad \left[\ ^{(itx)}\right] = \left[\phi_i(\qquad ^{1/2})\right]^n \tag{A.2.1.}$$

where $-\infty \qquad +\infty \quad {}^2 = -1$ and $\phi_i(\ )$ is the characteristic function of $\sigma^{-1}(\ _i - \ )$.

Recalling the definition of cumulants $(\quad 1\ 1\,)$, we must also have, for :

$$\phi(\ ) = \quad \left\{ \frac{1}{2}(\quad)^2 + \frac{1}{3!}\kappa_3(\quad)^3 + \frac{1}{4!}\kappa_4(\quad)^4 \right\} + \quad (\ ^4) \tag{A.2.2.}$$

where $\kappa_j \quad = [1 \qquad 4]$ refers to the - cumulant of , with $\kappa_1 = 0$ and $\kappa_2 = 1$

Following the same approach for $\sigma^{-1}(\ _i - \ )$, we get:

$$\phi_i(\ ) = \quad \left\{ \frac{1}{2}(\quad)^2 + \frac{1}{3!}\kappa_{i,3}(\quad)^3 + \frac{1}{4!}\kappa_{i,4}(\quad)^4 \right\} + \quad (\ ^4) \tag{A.2.3.}$$

where $\kappa_{i,j}$, $(\ \times\ ) = (1 \qquad) \times (1 \qquad 4)$ refers now to the - cumulant of the standardized random variable $\sigma^{-1}(\ _i - \ )$ with $\kappa_{i,1} = 0$ and $\kappa_{i,2} = 1$

Using equation ($\quad 2\ 1$) and equation ($\quad 2\ 3$), we have:

$$\phi(\ ) = \left\{-\frac{1}{2}\ ^2 + \ ^{-1/2}\frac{1}{3!}\kappa_{i,3}\ (\ )^3 + \ ^{-1}\frac{1}{4!}\kappa_{i,4}\ (\ )\right\} + \ (\ ^4) \qquad \text{(A.2.4.)}$$

Identifying terms in ($\quad 2\ 2$) and in ($\quad 2\ 4$) leads to the desired property, that is:

$$\kappa_j = \ ^{1-j/2}\kappa_{i,j} \qquad \text{(A.2.5.)}$$

with κ_j the $j\text{-}th$ cumulant of $\quad$, $\quad \geq 2$ and $\kappa_{i,j}$ the $j\text{-}th$ cumulant of $\sigma^{-1}(\ _i - \ )$. ■

Appendix 3

Under the hypotheses of existence of the five first non-central moments of the underlying asset terminal price density, the choice of the lognormal as the approximate density of the underlying asset terminal price density and perfection and completeness of financial markets, the fair price of an European call $\ _{JR}$ can be expressed as:

$$\ _{JR} \simeq \ _{BS} + \lambda_1\ \ _3 + \lambda_2\ \ _4 \qquad (25)$$

with:

$$\begin{cases} \ _3 = (\ _t\ \ ^{r\tau})^3\left(\ ^{\sigma^2\tau} - 1\right)^{3/2}\frac{e^{-r\tau}}{3!}\ \ _1(\ )\frac{l(K)}{K\sigma\sqrt{\tau}} \\ \ _4 = (\ _t\ \ ^{r\tau})^4\left(\ ^{\sigma^2\tau} - 1\right)^2\frac{e^{-r\tau}}{4!}\left[\ \ _2(\ ) + 2\ \ _1(\ )\sigma\sqrt{\tau} - \sigma^2\tau\right]\frac{l(K)}{K^2\sigma^2\tau} \\ \ _1(\ ) = 2\sigma\sqrt{\tau} - \\ \ _2(\ ) = \ ^2 - 3\ \ \sigma\sqrt{\tau} + 3\sigma^2\tau - 1 \end{cases}$$

and:

$$\lambda_1 = [\gamma_1(\ ) - \gamma_1(\ )]$$

$$\lambda_2 = [\gamma_1(\ ) - \gamma_1(\ )]$$

Proof. Under a lognormal Gram-Charlier series expansion, the risk-neutral price of an

European call written on a stock $_t$ with strike price is:

$$
\begin{aligned}
{}_{JR} &= [\ _t \quad \tau \quad \upsilon_{GC} \quad {}_T\ \sigma\ \kappa_3\ \kappa_4] \\
&= {}^{-r\tau} \int_{S_T=K}^{+\infty} (\ _T - \ ) \left[(\ _T) - \frac{3}{3!} \frac{^3 (\ _T)}{^3_T} \right. \\
&\qquad \left. + \frac{4}{4!} \frac{^4 (\ _T)}{^4_T} \right] \ _T
\end{aligned}
\quad \text{(A.3.1.)}
$$

where the Gram-Charlier series expansion residual is dropped.

In order to evaluate expression $(\ 3\ 1\ )$

we can derive the first moment of the underlying asset price under the risk-neutral density:

$$\begin{aligned}
{}_t &= {}^{-r\tau}\int_0^{+\infty} {}_T \left({}_T \right) {}_T \\
&= {}^{-r\tau}\int_0^{+\infty} {}_T \left[\left({}_T \right) - \frac{{}_3}{3!}\frac{{}^3\left({}_T \right)}{{}_T^3} + \frac{{}_4}{4!}\frac{{}^4\left({}_T \right)}{{}_T^4} \right] {}_T \\
&= {}^{-r\tau}\int_0^{+\infty} {}_T \left({}_T \right) {}_T \\
&\quad - \frac{{}_3}{3!} {}^{-r\tau}\int_0^{+\infty} {}_T \frac{{}^3\left({}_T \right)}{{}_T^3} {}_T \\
&\quad + \frac{{}_4}{4!} {}^{-r\tau}\int_0^{+\infty} {}_T \frac{{}^4\left({}_T \right)}{{}_T^4} {}_T
\end{aligned} \qquad \text{(A.3.7.)}$$

In order to evaluate expression (3 7), we need to calculate the following integral for $= [3\ 4]$:

$${}_j^* = \int_0^{+\infty} {}_T \frac{{}^j\left({}_T \right)}{{}_T^j} {}_T \qquad \text{(A.3.8.)}$$

Integrating by parts this expression, yields:

$$\begin{aligned}
{}_j^* &= \left[{}_T \frac{{}^j\left({}_T \right)}{{}_T^j} \right]_0^{+\infty} - \left[\frac{{}^{j-1}\left({}_T \right)}{{}_T^{j-1}} \right]_0^{+\infty} \\
&= \lim_{S_T\to+\infty} {}_T \frac{{}^{j-1}\left({}_T \right)}{{}_T^{j-1}} - \lim_{S_T\to 0} {}_T \frac{{}^{j-1}\left({}_T \right)}{{}_T^{j-1}} \\
&\quad - \lim_{S_T\to+\infty} \frac{{}^{j-1}\left({}_T \right)}{{}_T^{j-1}} + \lim_{S_T\to 0} \frac{{}^{j-1}\left({}_T \right)}{{}_T^{j-1}}
\end{aligned} \qquad \text{(A.3.9.)}$$

For the lognormal distribution, we also have for ≥ 1 (see Kendall, 1977, p.180):

$$\lim_{S_T\to+\infty} \frac{{}^{j-1}\left({}_T \right)}{{}_T^{j-1}} = \lim_{S_T\to 0} \frac{{}^{j-1}\left({}_T \right)}{{}_T^{j-1}} = 0 \qquad \text{(A.3.10.)}$$

So, using the above expression for $= [3\ 4]$ in (3 7) and dividing it by ${}_t$ we get the following expression:

$$\begin{aligned}
1 &= {}^{-r\tau}\int_0^{+\infty} \frac{{}_T}{{}_t} \left({}_T \right) {}_T \\
&= {}^{-r\tau}\int_{-\infty}^{+\infty} \left({}^{\mu\tau+\sigma\sqrt{\tau}z} \right) \varphi\left(\ \right) \\
&= {}^{\left(-r\tau+\mu\tau+\frac{1}{2}\sigma^2\tau\right)}
\end{aligned} \qquad \text{(A.3.11.)}$$

where the change of variable $= [\ln(\ _T \ \ _t) - \ \tau] \ \sigma\sqrt{\tau}$ have been performed on $_T$.

Taking the logarithm of expression (3 11) and rearranging terms, yields:

$$\tau = \ \tau - \frac{1}{2}\sigma^2\tau \qquad \text{(A.3.12.)}$$

Using this expression, the risk-neutral expected value and variance of the terminal price can be written such as:

$$\begin{cases} _1(\ ) = \ _1(\ ) = \ _t \ ^{\mu\tau+\frac{1}{2}\sigma^2\tau} = \ _t \ ^{r\tau} \\ _2(\ ) = \ _2(\ ) = [\ _1(\ )]^2 \left[\ ^{\sigma^2\tau} - 1\right] = (\ _t \ ^{r\tau})^2 \left[\ ^{\sigma^2\tau} - 1\right] \end{cases} \qquad \text{(A.3.13.)}$$

where the definition of the moments of the lognormal and the equality of the two first cumulants between the true and the approximating distribution have been used.

Substituting (3 5) and (3 13) in equation (3 1) and using the cumulants and the Fisher parameters definitions $(\kappa_3(\ ) = \ _3(\ ) \ \ \kappa_4(\ ) = \ _4(\ ) - 3 \ _2(\ )^2 \ \ \gamma_1(\ ) = \ _3(\ ) \ \ _2^{3/2}(\ )$ and $\gamma_2(\ ) = \ _4(\ ) \ \ _2^2(\ ) - 3)$ the value of an European call becomes:

$$\begin{aligned} _{JR} \ \simeq \ & _{BS} + [\gamma_1(\ ) - \gamma_1(\ )] \ (\ _t \ ^{r\tau})^3 \qquad \text{(A.3.14.)} \\ & \times \left(\ ^{\sigma^2\tau} - 1\right)^{3/2} \frac{^{-r\tau}}{3!} \left(2\,\sigma\sqrt{\tau} - \ \right) \frac{(\ )}{\sigma\sqrt{\tau}} \\ & + [\gamma_2(\ ) - \gamma_2(\ )] \ (\ _t \ ^{r\tau})^4 \left(\ ^{\sigma^2\tau} - 1\right)^2 \frac{^{-r\tau}}{4!} \\ & \times \left(\ ^2 - 5 \ \ \sigma\sqrt{\tau} + 6\,\sigma^2\tau - 1\right) \frac{(\ )}{^2\sigma^2\tau} \end{aligned}$$

where is defined as in Black and Scholes (1973) $formula$.

Using $_1(\ )$ and $_2(\ )$ expressions leads to the desired result. ■

Appendix 4

Under the hypotheses of existence of the five first non-central moments of the underlying asset log-return density, the choice of a normal as the approximate density of the continuous compound return density and perfection and completeness of financial markets, the fair price of an European call $_{CS}$ can be expressed as:

$$_{CS} \simeq \, ^{*}_{BS} + \gamma_1 (\ ) \quad ^{\prime}_{3} + \gamma_2 (\ ) \quad ^{\prime}_{4} \tag{31}$$

with:

$$\begin{cases} ^{\prime}_{3} = [3! \, (1+\omega)]^{-1} \quad _{t} \, \sigma\sqrt{\tau} \ \ _{1} (\ ^{*}) \, \varphi (\ ^{*}) \\ ^{\prime}_{4} = \ [4! \, (1+\omega)]^{-1} \quad _{t} \, \sigma\sqrt{\tau} \ \ _{2} (\ ^{*}) \, \varphi (\ ^{*}) \\ _{1} (\ ) = 2\sigma\sqrt{\tau} - \\ _{2} (\ ) = \ ^{2} - 3 \quad \sigma\sqrt{\tau} + 3\sigma^2\tau - 1 \end{cases}$$

and:

$$\begin{cases} ^{*} = (\sigma\sqrt{\tau})^{-1} \left[\ln (\ _{t} \qquad ^{-r\tau}) + \frac{1}{2}\sigma^2\tau - \ln [(1+\omega)]\right] \\ \omega = \frac{\gamma_1(f)}{3!} \, \sigma^3\tau^{3/2} + \frac{\gamma_2(f)}{4!} \, \sigma^4\tau^2 \end{cases}$$

Proof. Under a Gram-Charlier Type A series expansion, the risk-neutral price of an European call written on a stock $_t$ with strike price is:

$$_{CS} \quad = \quad ^{-r\tau} \int_{z=}^{+\infty}$$

Using the definition of Hermite polynomials, we get:

$$
\begin{aligned}
{}^{**}_{j} &= \int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \left({}_{t} \ {}^{\mu\tau+\sigma\sqrt{\tau}z} - \ \right) (-1)^{j} \ \frac{{}^{j}\varphi(\ )}{{}_{j}} \qquad \text{(A.4.3.)} \\
&= -\int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \left({}_{t} \ {}^{\mu\tau+\sigma\sqrt{\tau}z} - \ \right) \text{—} \left[(-1)^{j-1} \ \frac{{}^{j-1}\varphi(\ )}{{}_{j-1}} \right] \\
&= -\int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \left({}_{t} \ {}^{\mu\tau+\sigma\sqrt{\tau}z} - \ \right) \text{—} \ \ {}_{j-1}(\ )\ \varphi(\ )
\end{aligned}
$$

and an integration by parts yields:

$$
\begin{aligned}
{}^{**}_{j} &= -\left[\left({}_{t} \ {}^{\mu\tau+\sigma\sqrt{\tau}z} - \ \right) \ {}_{j-1}(\ )\ \varphi(\ ) \right]_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \qquad \text{(A.4.4.)} \\
&\quad + \sigma\sqrt{\tau} \ {}_{t} \int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} {}^{\mu\tau+\sigma\sqrt{\tau}z} \ \ {}_{j-1}(\ )\ \varphi(\ )
\end{aligned}
$$

It is readily verified that the first term in the above expression equals zero. Noting also that ${}_{z\to\infty}\varphi(\ ) = 0$, this leaves the expression:

$$
{}^{**}_{j} = \sigma\sqrt{\tau} \ {}_{t} \int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} {}^{\mu\tau+\sigma\sqrt{\tau}z} \ \ {}_{j-1}(\ )\ \varphi(\ ) \qquad \text{(A.4.5.)}
$$

Using once again the definition of Hermite polynomials, we have:

$$
\begin{aligned}
{}^{**}_{j} &= \sigma\sqrt{\tau} \ {}_{t} \int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} {}^{\mu\tau+\sigma\sqrt{\tau}z} \ (-1)^{j-1} \ \frac{{}^{j-1}\varphi(\ )}{{}_{j-1}} \qquad \text{(A.4.6.)} \\
&= -\sigma\sqrt{\tau} \ {}_{t} \int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} {}^{\mu\tau+\sigma\sqrt{\tau}z} \text{—} \left[(-1)^{j-2} \ \frac{{}^{j-2}\varphi(\ )}{{}_{j-2}} \right] \\
&= -\sigma\sqrt{\tau} \ {}_{t} \int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} {}^{\mu\tau+\sigma\sqrt{\tau}z} \text{—} \ \ {}_{j-2}(\ )\ \varphi(\ )
\end{aligned}
$$

and integrating by parts, we get:

$$
\begin{aligned}
{}^{**}_{j} &= -\sigma\sqrt{\tau} \ {}_{t} \left[{}^{\mu\tau+\sigma\sqrt{\tau}z} \ \ {}_{j-2}(\ )\ \varphi(\ ) \right]_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \qquad \text{(A.4.7.)} \\
&\quad + \left(\sigma\sqrt{\tau}\right)^{2} \ {}_{t} \int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} {}^{\mu\tau+\sigma\sqrt{\tau}z} \ \ {}_{j-2}(\ )\ \varphi(\ ) \\
&= \sigma\sqrt{\tau} \ \ {}_{j-2}\left(\frac{\ln(\ \ {}_{t}) - \ \tau}{\sigma\sqrt{\tau}} \right) \varphi\left(\frac{\ln(\ \ {}_{t}) - \ \tau}{\sigma\sqrt{\tau}} \right) \\
&\quad + \left(\sigma\sqrt{\tau}\right)^{2} \ {}_{t} \int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} {}^{\mu\tau+\sigma\sqrt{\tau}z} \ \ {}_{j-2}(\ )\ \varphi(\ )
\end{aligned}
$$

Then, by induction, we obtain:

$$
\begin{aligned}
{}_j^{**} &= \sigma\sqrt{\tau} \quad {}_{j-2}\left(\frac{\ln(\quad {}_t) - \quad \tau}{\sigma\sqrt{\tau}}\right)\varphi\left(\frac{\ln(\quad {}_t) - \quad \tau}{\sigma\sqrt{\tau}}\right) \qquad \text{(A.4.8.)}\\
&\quad + \sigma\sqrt{\tau}\left[\sigma\sqrt{\tau} \quad {}_{j-3}\left(\frac{\ln(\quad {}_t) - \quad \tau}{\sigma\sqrt{\tau}}\right)\varphi\left(\frac{\ln(\quad {}_t) - \quad \tau}{\sigma\sqrt{\tau}}\right)\right.\\
&\quad \left. + \left(\sigma\sqrt{\tau}\right)^2 \quad {}_t\int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \quad {}^{\mu\tau+\sigma\sqrt{\tau}z} \quad {}_{j-3}(\ )\,\varphi(\ ) \quad \right]\\
&= \quad \varphi\left(\frac{\ln(\ {}_t \quad) + \quad \tau}{\sigma\sqrt{\tau}}\right)\sum_{k=1}^{j-1}\left(\sigma\sqrt{\tau}\right)^k \quad {}_{j-1-k}\left(\frac{\ln(\quad {}_t) - \quad \tau}{\sigma\sqrt{\tau}}\right)\\
&\quad + \left(\sigma\sqrt{\tau}\right)^j \quad {}_t\int_{\frac{\ln(K/S_t)-\mu\tau}{\sigma\sqrt{\tau}}}^{+\infty} \quad {}^{\mu\tau+\sigma\sqrt{\tau}z}\,\varphi(\ )\\
&= \quad \varphi\left(\frac{\ln(\ {}_t \quad) + \quad \tau}{\sigma\sqrt{\tau}}\right)\sum_{k=1}^{j-1}\left(\sigma\sqrt{\tau}\right)^k \quad {}_{j-1-k}\left(\frac{\ln(\quad {}_t) - \quad \tau}{\sigma\sqrt{\tau}}\right)\\
&\quad + \left(\sigma\sqrt{\tau}\right)^j \quad {}_t \quad {}^{\mu\tau+\sigma^2\tau/2}\Phi\left(\frac{\ln(\ {}_t \quad) + \quad \tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right)
\end{aligned}
$$

Using the following equality (see Appendix 5):

$$\varphi\left(\frac{\ln(\ {}_t \quad) + \quad \tau}{\sigma\sqrt{\tau}}\right) = \quad {}_t \quad {}^{\mu\tau+\sigma^2\tau/2}\varphi\left(\frac{\ln(\ {}_t \quad) + \quad \tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right) \qquad \text{(A.4.9.)}$$

leads to the following expression for ${}_j^{**}$:

$$
\begin{aligned}
{}_j^{**} &= \quad {}_t \quad {}^{\mu\tau+\sigma^2\tau/2}\left[\sum_{k=1}^{j-1}\left(\sigma\sqrt{\tau}\right)^k \quad {}_{j-1-k}\left(\frac{\ln(\quad {}_t) - \quad \tau}{\sigma\sqrt{\tau}}\right)\varphi\left(\frac{\ln(\ {}_t \quad) + \quad \tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right)\right.\\
&\quad \left. + \left(\sigma\sqrt{\tau}\right)^j\,\Phi\left(\frac{\ln(\ {}_t \quad) + \quad \tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right)\right] \qquad \text{(A.4.10.)}
\end{aligned}
$$

From the martingale restriction, that is:

$$ {}_t = \quad {}^{-r\tau} \quad {}_Q[\ \ {}_T \mid \ \ {}_t] \qquad \text{(A.4.11.)}$$

we can derive the first moment of the underlying asset log-return under the risk-neutral density. Indeed, equation (4 11) implies that:

$$\begin{aligned} 1 &= {}^{-r\tau} \; {}_{Q}\left[\frac{T}{t} \Big| \; {}_{t} \right] && \text{(A.4.12.)} \\ &= {}^{-r\tau} \; {}_{Q}\left[{}^{\mu\tau+\sigma\sqrt{\tau}z} \Big| \; {}_{t} \right] \\ &= {}^{-r\tau} \int_{-\infty}^{+\infty} \left({}^{\mu\tau+\sigma\sqrt{\tau}z} \right) \left[1 + \frac{\gamma_1(\;)}{3!} \; {}_{3}(\;) + \frac{\gamma_2(\;)}{4!} \; {}_{4}(\;) \right] \varphi(\;) \end{aligned}$$

In order to evaluate expression (A.4.12), we need to compute the following integral:

$$ {}_{j}^{***} = \int_{-\infty}^{+\infty} {}^{\mu\tau+\sigma\sqrt{\tau}z} \; {}_{j}(\;)\,\varphi(\;) \qquad \text{(A.4.13.)}$$

for $= [3 \;\; 4]$.

Note that if the current underlying asset price is unitary, the exercise price is equal to zero and the limits of integration are taken between minus and plus infinity, then integral ${}_{j}^{***}$ is equivalent to the integral ${}_{j}^{**}$. Thus, integrating expression (4 13) by parts yields for $= [3 \;\; 4]$:

$$ {}_{j}^{***} = \left(\sigma\sqrt{\tau}\right)^{j} \int_{-\infty}^{+\infty} {}^{\mu\tau+\sigma\sqrt{\tau}z}\,\varphi(\;) \qquad \text{(A.4.14.)}$$

Equation (4 12) then becomes:

$$\begin{aligned} 1 &= {}^{-r\tau} \int_{-\infty}^{+\infty} \left({}^{\mu\tau+\sigma\sqrt{\tau}z} \right) \varphi(\;) && \text{(A.4.15.)} \\ &\quad + \frac{\gamma_1(\;)}{3!} \sigma^3 \tau^{3/2} \; {}^{-r\tau} \int_{-\infty}^{+\infty} \left({}^{\mu\tau+\sigma\sqrt{\tau}z} \right) \varphi(\;) \\ &\quad + \frac{\gamma_2(\;)}{4!} \sigma^4 \tau^2 \; {}^{-r\tau} \int_{-\infty}^{+\infty} \left({}^{\mu\tau+\sigma\sqrt{\tau}z} \right) \varphi(\;) \\ &= {}^{-r\tau+\mu\tau+\frac{1}{2}\sigma^2\tau} \left(1 + \frac{\gamma_1(\;)}{3!} \sigma^3 \tau^{3/2} + \frac{\gamma_2(\;)}{4!} \sigma^4 \tau^2 \right) \end{aligned}$$

Taking the logarithm of expression (4 15) and rearranging terms, yields:

$$ \tau = \; \tau - \frac{1}{2}\sigma^2\tau - \ln\left[1 + \frac{\gamma_1(\;)}{3!} \sigma^3 \tau^{3/2} + \frac{\gamma_2(\;)}{4!} \sigma^4 \tau^2 \right] \qquad \text{(A.4.16.)}$$

Substituting this expression into equation ($4\ 10$) and using Hermite polynomials such as $_0(\ )=1$ $_1(\ )=$ $_2(\ )=\ ^2-1$, the value of an European call becomes:

$$
\begin{aligned}
_{CS} \;=\;& \frac{_t\Phi(\ ^*)}{\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2\right)} - \ ^{-r\tau}\ \ \Phi\left(\ ^* - \sigma\sqrt{\tau}\right) && \text{(A.4.17.)}\\
&+\frac{\gamma_1(\ )\ \ _t}{3!\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2\right)}\left[\left(2\sigma^2\tau-\ ^*\sigma\sqrt{\tau}\right)\varphi(\ ^*)\right.\\
&\left.+\sigma^3\tau^{3/2}\Phi(\ ^*)\right]+\frac{\gamma_2(\ )\ \ _t\varphi(\ ^*)}{4!\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2\right)}\\
&\left[\left(3\sigma^3\tau^{3/2}-3\ ^*\sigma^2\tau+\ ^{*2}\sigma\sqrt{\tau}-\sigma\sqrt{\tau}\right)\varphi(\ ^*)+\sigma^4\tau^2\Phi(\ ^*)\right]
\end{aligned}
$$

that is:

$$
\begin{aligned}
_{CS} \;=\;& _t\Phi(\ ^*) - \ ^{-r\tau}\ \ \Phi\left(\ ^* - \sigma\sqrt{\tau}\right) && \text{(A.4.18.)}\\
&+\frac{\gamma_1(\ )\ \ _t}{3!\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2\right)}\left(-\ ^*\sigma\sqrt{\tau}+2\sigma^2\tau\right)\varphi(\ ^*)\\
&+\frac{\gamma_2(\ )\ \ _t}{4!\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2\right)}\left(\ ^{*2}\sigma\sqrt{\tau}-3\ ^*\sigma^2\tau\right.\\
&\left.+3\sigma^3\tau^{3/2}-\sigma\sqrt{\tau}\right)\varphi(\ ^*)
\end{aligned}
$$

with:

$$
^* = \frac{\log(\ _t\ \ \ ^{-r\tau})+\frac{1}{2}\sigma^2\tau-\ln\left[1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2\right]}{\sigma\sqrt{\tau}}
$$

Using $_1(\ )$, $_2(\ )$ and ω expressions leads to the desired result. ■

Appendix 5

For any European call, the following equality is verified:

$$\varphi\left(\frac{\ln(\ _t\ \ )+\ \ \tau}{\sigma\sqrt{\tau}}\right) = \ _t\ \ ^{\mu\tau+\sigma^2\tau/2}\varphi\left(\frac{\ln(\ _t\ \ )+\ \ \tau+\sigma^2\tau}{\sigma\sqrt{\tau}}\right) \quad \text{(A.4.9.)}$$

Proof. Following Stoll and Whaley (1993, p.245), factoring out $\sigma\sqrt{\tau}$ in $[\ln(\ _t\ \ )+\ \ \tau](\sigma\sqrt{\tau})^{-1}$ and taking the square gives:

$$\begin{aligned}
\left[\frac{\ln(\ _t\ \ )+\ \ \tau}{\sigma\sqrt{\tau}}\right]^2 &= \left[\frac{\ln(\ _t\ \ )+\ \ \tau+\sigma^2\tau}{\sigma\sqrt{\tau}}-\sigma\sqrt{\tau}\right]^2 \qquad \text{(A.5.1.)}\\
&= \left[\frac{\ln(\ _t\ \ )+\ \ \tau+\sigma^2\tau}{\sigma\sqrt{\tau}}\right]^2 - 2\left[\frac{\ln(\ _t\ \ )+\ \ \tau+\sigma^2\tau}{\sigma\sqrt{\tau}}\right]\sigma\sqrt{\tau}+\sigma^2\tau\\
&= \left[\frac{\ln(\ _t\ \ )+\ \ \tau+\sigma^2\tau}{\sigma\sqrt{\tau}}\right]^2 - 2\left[\ln(\ _t\ \ )+\ \ \tau+\sigma^2\tau\right]+\sigma^2\tau\\
&= \left[\frac{\ln(\ _t\ \ )+\ \ \tau+\sigma^2\tau}{\sigma\sqrt{\tau}}\right]^2 - 2\left[\ln(\ _t\ \ )+\ \ \tau+\frac{\sigma^2\tau}{2}\right]\\
&= \left[\frac{\ln(\ _t\ \ )+\ \ \tau+\sigma^2\tau}{\sigma\sqrt{\tau}}\right]^2 - 2\ln\left(\ _t\ \ ^{\mu\tau+\sigma^2\tau/2}\ \ \right)
\end{aligned}$$

Evaluating the standard normal density at $[\ln(\ _t\ \ )+\ \ \tau](\sigma\sqrt{\tau})^{-1}$ we get:

$$\begin{aligned}
\varphi\left(\frac{\ln(\ _t\ \ )+\ \ \tau}{\sigma\sqrt{\tau}}\right) &= \frac{1}{\sqrt{2\pi}}\ \ ^{-\left\{\left[\ln(S_t/K)+\mu\tau+\sigma^2\tau\right]\left(\sigma\sqrt{\tau}\right)^{-1}-\sigma\sqrt{\tau}\right\}^2/2} \qquad \text{(A.5.2.)}\\
&= \frac{1}{\sqrt{2\pi}}\ \ ^{-\left[\ln(S_t/K)+\mu\tau+\sigma^2\tau\right]\left(\sigma\sqrt{\tau}\right)^{-1}/2+\ln\left(S_t e^{\mu\tau+\sigma^2\tau/2}/K\right)}\\
&= \frac{1}{\sqrt{2\pi}}\ \ ^{-\left[\ln(S_t/K)+\mu\tau+\sigma^2\tau\right]\left(\sigma\sqrt{\tau}\right)^{-1}/2}\ \ ^{\ln\left(S_t\, e^{\mu\tau+\sigma^2\tau/2}/K\right)}\\
&= \varphi\left(\frac{\ln(\ _t\ \ )+\ \ \tau+\sigma^2\tau}{\sigma\sqrt{\tau}}\right)\ \ _t\ \ ^{\mu\tau+\sigma^2\tau/2}
\end{aligned}$$

Rearranging equation (A.5.2.), we obtain the following identity:

$$\varphi\left(\frac{\ln(\ _t\ \ )+\ \ \tau}{\sigma\sqrt{\tau}}\right) = \ _t\ \ ^{\mu\tau+\sigma^2\tau/2}\varphi\left(\frac{\ln(\ _t\ \ )+\ \ \tau+\sigma^2\tau}{\sigma\sqrt{\tau}}\right) \quad \text{(A.5.3.)}$$

In the particular case where $\tau = \quad \tau - \sigma^2\tau \quad 2$ (see Black and Scholes, 1973), this expression becomes:

$$\varphi\left(\quad - \sigma\sqrt{\tau}\right) = \quad _t \quad ^{r\tau}\varphi\left(\quad \right) \qquad \text{(A.5.4.)}$$

where corresponds to the Black Scholes standardized moneyness. ■

Appendix 6

Under the hypotheses of existence of the five first non-central moments of the underlying asset log-return density, the choice of the normal as the approximate density of the continuous compound return density, and perfection and completeness of financial markets, the fair price of an European call $_R$ written on a stock $_t$ with strike price can be written as:

$$_R = \quad ^{**}_{BS} + \gamma_1\left(\quad \right) \quad ''_3 + \gamma_2\left(\quad \right) \quad ''_4 + \gamma_1\left(\quad \right)^2 \quad ''_5 \qquad (34)$$

with:

$$\begin{cases} ''_3 = [3!\,(1 + \quad)]^{-1} \quad _t\, \sigma\sqrt{\tau} \quad _1\left(\quad ^{**}\right)\varphi\left(\quad ^{**}\right) \\ ''_4 = [4!\,(1 + \quad)]^{-1} \quad _t\, \sigma\sqrt{\tau} \quad _2\left(\quad ^{**}\right)\varphi\left(\quad ^{**}\right) \\ ''_5 = 10\,[6!\,(1 + \quad)]^{-1} \quad _t\, \sigma\sqrt{\tau} \quad _4\left(\quad ^{**}\right)\varphi\left(\quad ^{**}\right) \\ _1\left(\quad \right) = 2\sigma\sqrt{\tau} - \\ _2\left(\quad \right) = \quad ^2 - 3 \quad \sigma\sqrt{\tau} + 3\sigma^2\tau - 1 \\ _4\left(\quad \right) = \quad ^4 - 5 \quad ^3\sigma\sqrt{\tau} + 10 \quad ^2\sigma^2\tau - 6 \quad ^2 - 10 \quad \sigma^3\tau^{3/2} + 15 \quad \sigma\sqrt{\tau} + 5\sigma^4\tau^2 \\ -10\sigma^2\tau + 3 \end{cases}$$

and:

$$\begin{cases} ^{**} = \left(\sigma\sqrt{\tau}\right)^{-1}\left[\ln\left(\quad _t \quad ^{-r\tau}\right) + \frac{1}{2}\sigma^2\tau - \ln\left[(1 + \quad)\right]\right] \\ \quad = \frac{\gamma_1(f)}{3!}\,\sigma^3\tau^{3/2} + \frac{\gamma_2(f)}{4!}\,\sigma^4\tau^2 + 10\,\frac{\gamma_1(f)^2}{6!}\sigma^6\tau^3 \end{cases}$$

Proof. Following the same approach as previously, but using now a normal Edgeworth series expansion for the risk-neutral density of the underlying asset log-return, with $_6\left(\quad \right) =$

$^{6}-15\ ^{4}+45\ ^{2}-15$, yields:

$$
\begin{aligned}
{}_{R} &= {}_{t}\Phi\left(\ ^{**}\right)- \ ^{-r\tau}\ \Phi\left(\ ^{**}-\sigma\sqrt{\tau}\right) \qquad \text{(A.6.1.)}\\
&+\frac{\gamma_1(\ )\ \ {}_{t}\varphi\left(\ ^{**}\right)}{3!\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2+10\,\frac{\gamma_1(f)^2}{6!}\sigma^6\tau^3\right)}\left(2\sigma^2\tau-\ ^{**}\sigma\sqrt{\tau}\right)\\
&+\frac{\gamma_2(\ )\ \ {}_{t}\varphi\left(\ ^{**}\right)}{4!\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2+10\,\frac{\gamma_1(f)^2}{6!}\sigma^6\tau^3\right)}\left(3\,\sigma^3\tau^{3/2}\right.\\
&\left.-3\ ^{**}\sigma^2\tau+\ ^{**2}\,\sigma\sqrt{\tau}-\sigma\sqrt{\tau}\right)\\
&+\frac{10\,\gamma_1(\ )^2\ \ {}_{t}\varphi\left(\ ^{**}\right)}{6!\left(1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2+10\,\frac{\gamma_1(f)^2}{6!}\sigma^6\tau^3\right)}\left[\sigma\sqrt{\tau}\left(\ ^{4}-6\ ^{2}\right.\right.\\
&\left.\left.+3\right)+\sigma^2\tau\left(-5\ ^{3}+15\ \right)+\sigma^3\tau^{3/2}\left(10\ ^{2}-10\right)-10\ \sigma^4\tau^2+5\sigma^5\tau^{5/2}\right]
\end{aligned}
$$

with:

$$
^{**}=\frac{\log\left(\ {}_{t}\ \ \ ^{-r\tau}\right)+\frac{1}{2}\sigma^2\tau-\ln\left[1+\frac{\gamma_1(f)}{3!}\sigma^3\tau^{3/2}+\frac{\gamma_2(f)}{4!}\sigma^4\tau^2+10\,\frac{\gamma_1(f)^2}{6!}\sigma^6\tau^3\right]}{\sigma\sqrt{\tau}}
$$

Using $_{1}(\ )$, $_{2}(\ )$, $_{3}(\ )$ and expressions leads to the desired result. ■

Appendix 7

When the European call market price is given respectively by the Jarrow-Rudd (1982), the Corrado-Su (1996) or the Rubinstein (1998) *formula*, the implied volatility smile function can be written such as equation (38), (39) or (40):

Proof. The implied volatility smile function corresponds to the volatility Ψ that equates the market price of the option to the value of the Black and Scholes (1973) *formula*, given values of other parameters fixed, that is:

$$
\begin{aligned}
&= \ ^{-r\tau}\int_{S_T=K}^{+\infty}\left(\ {}_{T}-\ \right)\ \left(\ {}_{T}\right)\ \ {}_{T} \qquad \text{(A.7.1.)}\\
&= {}_{t}\,\Phi\left[\ (\Psi)\right]-\ \ ^{-r\tau}\,\Phi\left[\ (\Psi)-\Psi\right]
\end{aligned}
$$

where $(\Psi) = [\log(\ _t \quad ^{-r\tau}) + 0\ 5\Psi^2]\,\Psi^{-1}$ and $\Phi(\ )$ represent respectively the Black and Scholes' measure of moneyness and the standard normal distribution evaluated at the implied volatility level.

A linear approximation of this expression around the "true" volatility of the underlying asset $\sigma\sqrt{\tau}$ gives:

$$\begin{aligned} \cong \quad & _t\,\Phi\left[\ (\sigma\sqrt{\tau})\right] - \quad ^{-r\tau}\Phi\left[\ (\sigma\sqrt{\tau}) - \sigma\sqrt{\tau}\right] \\ & + \ _t\,\varphi\left[\ (\sigma\sqrt{\tau})\right]\left(-\frac{\ln \ _t \quad ^{-r\tau}}{\sigma^2\tau} + \frac{1}{2}\right)(\Psi - \sigma\sqrt{\tau}) \\ & - \quad ^{-r\tau}\varphi\left[\ (\sigma\sqrt{\tau}) - \sigma\sqrt{\tau}\right]\left(-\frac{\ln \ _t \quad ^{-r\tau}}{\sigma^2\tau} - \frac{1}{2}\right)(\Psi - \sigma\sqrt{\tau}) \end{aligned} \quad \text{(A.7.2.)}$$

with $\varphi(\ )$ the standard normal density function.

Using the following equality (see Appendix 5):

$$\varphi\left[\ (\sigma\sqrt{\tau}) - \sigma\sqrt{\tau}\right] = \ _t \quad ^{r\tau}\varphi\left[\ (\sigma\sqrt{\tau})\right] \quad \text{(A.7.3.)}$$

equation $(\quad 7\ 2\ )$ simplifies to:

$$\begin{aligned} \cong \quad & _t\,\Phi\left[\ (\sigma\sqrt{\tau})\right] - \quad ^{-r\tau}\Phi\left[\ (\sigma\sqrt{\tau}) - \sigma\sqrt{\tau}\right] \\ & + \ _t\,\varphi\left[\ (\sigma\sqrt{\tau})\right](\Psi - \sigma\sqrt{\tau}) \end{aligned} \quad \text{(A.7.4.)}$$

Rearranging equation (A.7.1.) leads to the following general expression for the implied volatility smile function:

$$\Psi = \sigma\sqrt{\tau} + \frac{1}{\ _t}(\quad - \quad _{BS})\,\varphi\left[\ (\sigma\sqrt{\tau})\right]^{-1} \quad \text{(A.7.5.)}$$

with:

$$_{BS} = \ _t\Phi\left[\ (\sigma\sqrt{\tau})\right] - \quad ^{-r\tau}\Phi\left[\ (\sigma\sqrt{\tau}) - \sigma\sqrt{\tau}\right]$$

Depending on the approximate option pricing model considered in equation (A.7.5.) leads to expression (38), (39) or (40). ■

Appendix 8

When the European call market price is given by the Jarrow-Rudd (1982) *formula*, the Greek parameters of the call can be written respectively as equation (41), (42), (43), (44) and (45).

Proof. Consider the Jarrow-Rudd (1982) *formula* of an European call option:

$$_{JR} \simeq \ _{BS} + \lambda_1 \ _3 + \lambda_2 \ _4 \qquad \text{(A.8.1.)}$$

with:

$$\begin{cases} \lambda_1 = [\gamma_1 (\ ) - \gamma_1 ( )] \\ \lambda_2 = [\gamma_2 (\ ) - \gamma_2 ( )] \end{cases}$$

and:

$$\begin{cases} _3 = (\ _t \ ^{r\tau})^3 \left(\ ^{\sigma^2\tau} - 1\right)^{3/2} \frac{e^{-r\tau}}{3!} \ _1 (\ ) \frac{l(K)}{K\sigma\sqrt{\tau}} \\ _4 = (\ _t \ ^{r\tau})^4 \left(\ ^{\sigma^2\tau} - 1\right)^2 \frac{e^{-r\tau}}{4!} [\ _2 (\ ) + 2 \ _1 (\ ) \sigma\sqrt{\tau} - \sigma^2\tau] \frac{l(K)}{K^2\sigma^2\tau} \\ _1 (\ ) = 2\sigma\sqrt{\tau} - \\ _2 (\ ) = \ ^2 - 3 \ \sigma\sqrt{\tau} + 3\sigma^2\tau - 1 \end{cases}$$

Differentiating the Jarrow-Rudd *formula* (A.8.1.) with respect to the underlying price, we get:

$$\frac{\partial \ _{JR}}{\partial \ _t} = \frac{\partial \ _{BS}}{\partial \ _t} + \lambda_1 \frac{\partial \ _3}{\partial \ _t} + \lambda_2 \frac{\partial \ _4}{\partial \ _t} \qquad \text{(A.8.2.)}$$

with:

$$\begin{aligned} \frac{\partial \ _3}{\partial \ _t} &= -\frac{(\ ^{r\tau})^2}{3! \ \sigma\sqrt{\tau}} \left(\ ^{\sigma^2\tau} - 1\right)^{3/2} \Big[-3 (\ _t)^2 \ _3 (\ ) \ (\ ) \\ &\quad + (\ _t)^3 \frac{\partial}{\partial \ _t} (\ ) - (\ _t)^3 \ _3 (\ ) \frac{\partial (\ )}{\partial \ _t}\Big] \\ &= -(\ _t \ ^{r\tau})^2 \left(\ ^{\sigma^2\tau} - 1\right)^{3/2} \frac{(\ )}{3! \ \sigma^2\tau} \left[- \ _2(\ ) - 3 \ _1(\ )\sigma\sqrt{\tau} + \sigma^2\tau\right] \end{aligned} \qquad \text{(A.8.3.)}$$

and:

$$\begin{aligned}\frac{\partial\ \ _4}{\partial\ \ _t} &= \frac{(\ \ ^{r\tau})^3}{4!\ \ ^2\sigma^2\tau}\left(\ \ ^{\sigma^2\tau}-1\right)^2\Big[4(\ \ _t)^3\left[\ \ _2(\ \ )+2\ \ _1(\ \ )\sigma\sqrt{\tau}-\sigma^2\tau\right] \\ &\quad (\ \ )+(\ \ _t)^4\left(2\ \frac{\partial}{\partial\ \ _t}-5\frac{\partial}{\partial\ \ _t}\sigma\sqrt{\tau}\right)\ (\ \ ) \\ &\quad +(\ \ _t)^4\left[\ \ _2(\ \ )+2\ \ _1(\ \ )\sigma\sqrt{\tau}-\sigma^2\tau\right]\frac{\partial\ (\ \ )}{\partial\ \ _t}\Big] \\ &= (\ \ _t\ \ ^{r\tau})^3\left(\ \ ^{\sigma^2\tau}-1\right)^2\frac{(\ \ )}{4!\ \ ^2\sigma^3\tau^{3/2}}\Big[-\ \ _3(\ \ ) \\ &\quad +6\ \ _2(\ \ )\sigma\sqrt{\tau}+7\ \ _1(\ \ )\sigma^2\tau-5\sigma^3\tau^{3/2}\Big]\end{aligned} \quad \text{(A.8.4.)}$$

where:

$$\begin{cases}\frac{\partial d}{\partial S_t}=\frac{1}{S_t\sigma\sqrt{\tau}} \\ \frac{\partial l(K)}{\partial S_t}=\frac{\left(d-\sigma\sqrt{\tau}\right)}{S_t\sigma\sqrt{\tau}}\ (\ \ )\end{cases} \quad \text{(A.8.5.)}$$

Substituting expression (8 3) and (8 4) in equation (8 1) leads to the Delta *formula* (41) for the Jarrow and Rudd (1982) model.

Differentiating once again expression (8 1) with respect to the underlying asset price, we have:

$$\begin{aligned}\frac{\partial\Delta^C_{JR}}{\partial\ \ _t} &= \frac{\partial\Delta_{BS}}{\partial\ \ _t}+\lambda_1\Bigg[\frac{\partial\ \ ^{''''}_3}{\partial\ \ _t}\left(-\ \ _4(\ \ )-3\ \ _3(\ \ )\sigma\sqrt{\tau}+\sigma^2\tau\right) \\ &\quad +\ \ ^{''''}_3\left(-2\ \frac{\partial}{\partial\ \ _t}+6\frac{\partial}{\partial\ \ _t}\sigma\sqrt{\tau}\right)\Bigg] \\ &\quad +\lambda_2\Bigg[\frac{\partial\ \ ^{''''}_4}{\partial\ \ _t}\left[-\ \ _3(\ \ )+6\ \ _2(\ \ )\sigma\sqrt{\tau}+7\ \ _1(\ \ )\sigma^2\tau-5\sigma^3\tau^{3/2}\right] \\ &\quad +\ \ ^{''''}_4\left(-3\ \ ^2\frac{\partial}{\partial\ \ _t}+20\ \frac{\partial}{\partial\ \ _t}-31\frac{\partial}{\partial\ \ _t}\sigma^2\tau+3\frac{\partial}{\partial\ \ _t}\right)\Bigg]\end{aligned} \quad \text{(A.8.6.)}$$

with:

$$\begin{aligned}\frac{\partial\ \ ^{''''}_3}{\partial\ \ _t} &= -\frac{(\ \ ^{r\tau})^2}{3!\ \ \sigma^2\tau}\left(\ \ ^{\sigma^2\tau}-1\right)^{3/2}\left[2\ \ _t\ (\ \ )+(\ \ _t)^2\frac{\partial\ (\ \ )}{\partial\ \ _t}\right] \\ &= (\ \ _t\sigma\sqrt{\tau})^{-1}\left\{-(\ \ _t\ \ ^{r\tau})^2\left(\ \ ^{\sigma^2\tau}-1\right)^{3/2}\frac{(\ \ )}{3!\ \ \sigma^2\tau}\left[\ \ _1(\ \ )+\sigma\sqrt{\tau}\right]\right\}\end{aligned} \quad \text{(A.8.7.)}$$

and:

$$\frac{\partial \ {}_4^{''''}}{\partial \ {}_t} = -\frac{(\ {}^{r\tau})^3}{4! \ {}^2\sigma^3\tau^{3/2}}\left(\ {}^{\sigma^2\tau} - 1\right)^2\left[3 \ {}_t^2 \ (\ \ ) + (\ {}_t)^3 \frac{\partial \ (\ \ )}{\partial \ {}_t}\right] \tag{A.8.8.}$$

$$= \left(\ {}_t\sigma\sqrt{\tau}\right)^{-1}\left\{-(\ {}_t \ {}^{r\tau})^3\left(\ {}^{\sigma^2\tau} - 1\right)^2 \frac{(\ \ )}{4! \ {}^2\sigma^3\tau^{3/2}}\left[\ {}_1(\ \ ) + \sigma\sqrt{\tau}\right]\right\}$$

Substituting expression (8 7) and (8 8) in equation (8 6), factoring out $\left(\ {}_t\sigma\sqrt{\tau}\right)^{-1}$ leads to the Gamma $formula$ (42) for the Jarrow and Rudd (1982) model.

Differentiating the Jarrow and Rudd equation (8 1) with respect to the volatility gives:

$$\frac{\partial \ {}_{JR}}{\partial\sigma} = \frac{\partial \ {}_{BS}}{\partial\sigma} + \lambda_1\frac{\partial \ {}_3}{\partial\sigma} + \lambda_2\frac{\partial \ {}_4}{\partial\sigma} - \frac{\partial\gamma_1(\ )}{\partial\sigma} \ {}_3 - \partial\gamma$$

$$
\begin{aligned}
= \quad & -\left({}_t \; {}^{r\tau}\right)^4 \frac{{}^{-r\tau}}{4!} \frac{(\quad)}{{}^2\sigma^3\tau} \Big[4\sigma^2\tau \; {}^{\sigma^2\tau} \Big({}^{\sigma^2\tau} - 1\Big) \\
& \times \big[\; {}_4(\;) + 2 \; {}_3(\;)\sigma\sqrt{\tau} - \sigma^2\tau\big] + \Big({}^{\sigma^2\tau} - 1\Big)^2 \\
& \times \left(-2 \; {}^2 - 2 \; \sigma\sqrt{\tau} + 7\sigma^2\tau\right) + \Big({}^{\sigma^2\tau} - 1\Big)^2 \\
& \big[\; {}_2(\;) + 2 \quad {}_1(\;)\sigma\sqrt{\tau} - \sigma^2\tau\big] \big[- \quad {}_1(\;) + \; \sigma\sqrt{\tau} - 3\big]\Big]
\end{aligned}
$$

and:

$$
\begin{cases}
\frac{\partial\gamma_1(l)}{\partial\sigma} = 3\sigma\tau \; {}^{\sigma^2\tau}\left[\Big({}^{\sigma^2\tau} - 1\Big)^{-\frac{1}{2}} + \Big({}^{\sigma^2\tau} - 1\Big)^{\frac{1}{2}}\right] \\
\frac{\partial\gamma_2(l)}{\partial\sigma} = \Big[4\sigma\tau \; {}^{\sigma^2\tau} + 15\Big({}^{\sigma^2\tau} - 1\Big) \\
\qquad\qquad +9\Big({}^{\sigma^2\tau} - 1\Big)^2 + 2\Big({}^{\sigma^2\tau} - 1\Big)^3\Big]
\end{cases}
\tag{A.8.12.}
$$

where:

$$
\begin{cases}
\frac{\partial d}{\partial\sigma} = -\frac{\left(d - \sigma\sqrt{\tau}\right)}{\sigma} \\
\frac{\partial l(K)}{\partial\sigma} = \frac{\left(d^2 - d\sigma\sqrt{\tau} - 1\right)}{\sigma} \; (\quad)
\end{cases}
\tag{A.8.13.}
$$

Substituting expression $(\quad 8\ 10)$ and $(\quad 8\ 11)$ in equation $(\quad 8\ 9)$ and factoring out ${}_t\sqrt{\tau}$ leads to the Vega $formula$ (43) for the Jarrow-Rudd (1982) model.

Differentiating expression $(\quad 8\ 1)$ with respect to the excess of skewness and to the excess of the excess kurtosis leads directly to the equation (44) and (45) of the Jarrow-Rudd Khi and Psi, that is:

$$
\begin{cases}
\frac{\partial C_{JR}}{\partial\gamma_1(f)} = \chi^C_{JR} = \quad {}_3 \\
\frac{\partial C_{JR}}{\partial\gamma_2(f)} = \Psi^C_{JR} \quad {}_4
\end{cases}
\tag{A.8.14.}
$$

■

Appendix 9

When the European call market price is given by the Corrado and Su (1996-b and 1997-b) $formula$, the Greeks of a call can be written respectively such as equations (46), (47), (48), (49) and (50).

Proof. Consider the Corrado and Su (1996-b and 1997-b) *formula* of an European call option:

$$ {}_{CS} = {}^{*}_{BS} + \gamma_1 (\ ) \ {}'_3 + \gamma_2 (\ ) \ {}'_4 \qquad \text{(A.9.1.)} $$

with:

$$ \begin{cases} {}'_3 = [3!\,(1+\omega)]^{-1} \ {}_t\, \sigma\sqrt{\tau} \ {}_1(\ ^{*})\,\varphi(\ ^{*}) \\ {}'_4 = [4!\,(1+\omega)]^{-1} \ {}_t\, \sigma\sqrt{\tau} \ {}_2(\ ^{*})\,\varphi(\ ^{*}) \end{cases} $$

Differentiating the Corrado and Su *formula* (9 1) with respect to the underlying asset price, we get:

$$ \frac{\partial \ {}_{CS}}{\partial \ {}_t} = \frac{\partial \ {}^{*}_{BS}}{\partial \ {}_t} + \gamma_1(\ )\,\frac{\partial \ {}'_3}{\partial \ {}_t} + \gamma_2(\ )\,\frac{\partial \ {}'_4}{\partial \ {}_t} \qquad \text{(A.9.2.)} $$

with:

$$ \begin{aligned} \frac{\partial \ {}^{*}_{BS}}{\partial \ {}_t} &= \Phi(\ ^{*}) + \ {}_t \frac{\partial\Phi(\ ^{*})}{\partial \ ^{*}}\frac{\partial \ ^{*}}{\partial \ {}_t} - \ {}^{-r\tau} \ \frac{\partial\Phi(\ ^{*}-\sigma\sqrt{\tau})}{\partial(\ ^{*}-\sigma\sqrt{\tau})}\frac{\partial(\ ^{*}-\sigma\sqrt{\tau})}{\partial \ {}_t} \qquad \text{(A.9.3.)} \\ &= \Phi(\ ^{*}) + \ {}_t\varphi(\ ^{*})\frac{\partial \ ^{*}}{\partial \ {}_t} - \ {}^{-r\tau} \ \varphi\left(\ ^{*}-\sigma\sqrt{\tau}\right)\frac{\partial(\ ^{*}-\sigma\sqrt{\tau})}{\partial \ {}_t} \\ &= \Phi(\ ^{*}) + \ {}_t\varphi(\ ^{*})\frac{\partial \ ^{*}}{\partial \ {}_t} - \frac{\ {}_t\,\varphi(\ ^{*})}{(1+\omega)}\frac{\partial \ ^{*}}{\partial \ {}_t} \\ &= \Phi(\ ^{*}) + \frac{\varphi(\ ^{*})}{\sigma\sqrt{\tau}} - \frac{\varphi(\ ^{*})}{\sigma\sqrt{\tau}\,(1+\omega)} \end{aligned} $$

$$ \begin{aligned} \frac{\partial \ {}'_3}{\partial \ {}_t} &= \frac{\sigma\sqrt{\tau}}{3!\,(1+\omega)}\Big[\left(2\sigma\sqrt{\tau} - \ ^{*}\right)\varphi(\ ^{*}) \qquad \text{(A.9.4.)} \\ &\quad - \ {}_t\frac{\partial \ ^{*}}{\partial \ {}_t}\varphi(\ ^{*}) + \ {}_t\left(2\sigma\sqrt{\tau} - \ ^{*}\right)\frac{\partial\varphi(\ ^{*})}{\partial \ ^{*}}\frac{\partial \ ^{*}}{\partial \ {}_t}\Big] \\ &= \frac{\varphi(\ ^{*})\left[\ {}_2(\ ^{*}) - \sigma^2\tau\right]}{3!\,(1+\omega)} \end{aligned} $$

and:

$$ \begin{aligned} \frac{\partial \ {}'_4}{\partial \ {}_t} &= \frac{\sigma\sqrt{\tau}}{4!\,(1+\omega)}\Big[\ {}_2(\ ^{*})\,\varphi(\ ^{*}) + \ {}_t\left(2\ ^{*}\frac{\partial \ ^{*}}{\partial \ {}_t} - 3\frac{\partial \ ^{*}}{\partial \ {}_t}\sigma\sqrt{\tau}\right)\varphi(\ ^{*}) \qquad \text{(A.9.5.)} \\ &\quad + \ {}_t \ {}_2(\ ^{*})\frac{\partial\varphi(\ ^{*})}{\partial \ ^{*}}\frac{\partial \ ^{*}}{\partial \ {}_t}\Big] \\ &= -\frac{\varphi(\ ^{*}) \ {}_3(\ ^{*})}{4!\,(1+\omega)} \end{aligned} $$

where:

$$\begin{cases} \frac{\partial d^*}{\partial S_t} = (\ {}_t\sigma\sqrt{\tau})^{-1} \\ \frac{\partial \varphi(d^*)}{\partial d^*} = - \ {}^*\varphi(\ {}^*) \end{cases}$$

Substituting expression (9 3) (9 4) and (9 5) in equation (9 2), factoring out $\varphi(\ {}^*)(1+\omega)^{-1}$ leads to the Delta $formula$ (46) for the Corrado-Su (1996-b and 1997-b) model.

Differentiating once again expression (9 1) with respect to the underlying asset price, we have:

$$\frac{\partial \Delta_{CS}}{\partial \ {}_t} = \frac{\partial \Delta^*_{BS}}{\partial \ {}_t} + \gamma_1 (\ ) \frac{\partial^2 \ {}'_3}{\partial \ {}^2_t} + \gamma_2 (\ ) \frac{\partial^2 \ {}'_4}{\partial \ {}^2_t} \qquad \text{(A.9.6.)}$$

where:

$$\begin{aligned} \frac{\partial \Delta^*_{BS}}{\partial \ {}_t} &= \frac{\partial \Phi(\ {}^*)}{\partial \ {}^*}\frac{\partial \ {}^*}{\partial \ {}_t} + \frac{1}{\sigma\sqrt{\tau}}\frac{\partial\varphi(\ {}^*)}{\partial \ {}^*}\frac{\partial \ {}^*}{\partial \ {}_t} - \frac{1}{\sigma\sqrt{\tau}(1+\omega)}\frac{\partial\varphi(\ {}^*)}{\partial \ {}^*}\frac{\partial \ {}^*}{\partial \ {}_t} \qquad \text{(A.9.7.)} \\ &= \frac{\varphi(\ {}^*)}{{}_t\sigma\sqrt{\tau}} - \frac{{}^*\varphi(\ {}^*)}{{}_t\sigma^2\tau} + \frac{{}^*\varphi(\ {}^*)}{{}_t\sigma^2\tau(1+\omega)} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \ {}'_3}{\partial \ {}^2_t} &= \frac{1}{3!(1+\omega)}\frac{\partial\varphi(\ {}^*)}{\partial \ {}^*}\frac{\partial \ {}^*}{\partial \ {}_t}\left[\ {}_2(\ {}^*) - \sigma^2\tau\right] + \varphi(\ {}^*)\left(2 \ {}^*\frac{\partial \ {}^*}{\partial \ {}_t} - 3\frac{\partial \ {}^*}{\partial \ {}_t}\sigma\sqrt{\tau}\right) \\ &= \frac{\varphi(\ {}^*)\left[- \ {}_3(\ {}^*) - \ {}_2(\ {}^*)\sigma\sqrt{\tau} + \ {}^*\sigma^2\tau + 6\sigma^3\tau^{3/2}\right]}{3! \ {}_t\sigma\sqrt{\tau}(1+\omega)} \qquad \text{(A.9.8.)} \end{aligned}$$

and:

$$\begin{aligned} \frac{\partial^2 \ {}'_4}{\partial \ {}^2_t} &= \frac{1}{4!(1+\omega)}\left[\frac{\partial\varphi(\ {}^*)}{\partial \ {}^*}\frac{\partial \ {}^*}{\partial \ {}_t} \ {}_3(\ {}^*)\right. \qquad \text{(A.9.9.)} \\ &\quad \left. +\varphi(\ {}^*)\left(-3 \ {}^{*2}\frac{\partial \ {}^*}{\partial \ {}_t} + 8 \ {}^*\frac{\partial \ {}^*}{\partial \ {}_t}\sigma\sqrt{\tau} + 3\frac{\partial \ {}^*}{\partial \ {}_t} - 6\frac{\partial \ {}^*}{\partial \ {}_t}\sigma^2\tau\right)\right] \\ &= \frac{\varphi(\ {}^*)\left[\ {}_4(\ {}^*) - \ {}_3(\ {}^*)\sigma\sqrt{\tau} - 2 \ {}^{*3}\sigma\sqrt{\tau} + \ {}^*\sigma^3\tau^{3/2} + 8\sigma^4\tau^2\right]}{4! \ {}_t\sigma\sqrt{\tau}(1+\omega)} \end{aligned}$$

Substituting expression (9 7) (9 8) and (9 9) in equation (9 6) and factoring out $\varphi(\ {}^*)\left[\ {}_t\sigma\sqrt{\tau}(1+\omega)\right]^{-1}$ leads to the Gamma $formula$ (47) for the Corrado-Su (1996-b and 1997-b) model.

Differentiating the Corrado-Su equation (9 1) with respect to the volatility, we get:

$$\frac{\partial \ _{CS}}{\partial\sigma} = \frac{\partial \ ^{*}_{BS}}{\partial\sigma} + \gamma_1 (\ ) \frac{\partial \ '_{3}}{\partial\sigma} + \gamma_2 (\ ) \frac{\partial \ '_{4}}{\partial\sigma} \qquad \text{(A.9.10.)}$$

where:

$$\begin{aligned} \frac{\partial \ ^{*}_{BS}}{\partial\sigma} &= \ _t\varphi(\ ^*)\sqrt{\tau} + \sigma^{-1} \ _t\varphi(\ ^*)\left[\ ^* + \frac{\left(\frac{\gamma_1(f)}{2!}\sigma^2\tau + \frac{\gamma_2(f)}{3!}\sigma^3\tau^{3/2}\right)}{(1+\omega)}\right] \qquad \text{(A.9.11.)} \\ & \times\left[-1 + \frac{1}{(1+\omega)}\right] \end{aligned}$$

$$\begin{aligned} \frac{\partial \ '_{3}}{\partial\sigma} &= \frac{\ _t\sqrt{\tau}}{3!\,(1+\omega)}\left\{\left[4\sigma\sqrt{\tau}\varphi(\ ^*) + 2\sigma^2\sqrt{\tau}\frac{\partial\varphi(\ ^*)}{\partial \ ^*}\frac{\partial \ ^*}{\partial\sigma}\right.\right. \qquad \text{(A.9.12.)} \\ & \left. - \ ^*\varphi(\ ^*) - \sigma\frac{\partial \ ^*}{\partial\sigma}\varphi(\ ^*) - \ ^*\sigma\frac{\partial\varphi(\ ^*)}{\partial \ ^*}\frac{\partial \ ^*}{\partial\sigma}\right] \\ & \left. - \frac{\sigma \ _1(\ ^*)\,\varphi(\ ^*)\left(\frac{\gamma_1(f)}{2!}\sigma^2\tau^{3/2} + \frac{\gamma_2(f)}{3!}\sigma^3\tau^2\right)}{(1+\omega)}\right\} \\ &= \frac{\ _t\sqrt{\tau}\,\varphi(\ ^*)}{3!\,(1+\omega)}\left\{\left(- \ ^* \ _2(\ ^*) + \ ^*\sigma^2\tau + \ ^* + 3\sigma\sqrt{\tau}\right)\right. \\ & \left. + \frac{\left(\frac{\gamma_1(f)}{2!}\sigma^2\tau^2 + \frac{\gamma_2(f)}{3!}\sigma^3\tau^{3/2}\right)}{(1+\omega)}\left[- \ _2(\ ^*) + \sigma^2\tau\right]\right\} \end{aligned}$$

$$\begin{aligned} \frac{\partial \ '_{4}}{\partial\sigma} &= \frac{\ _t\sqrt{\tau}}{4!\,(1+\omega)}\left\{\left[\ ^{*2}\varphi(\ ^*) + 2 \ ^*\frac{\partial \ ^*}{\partial\sigma}\sigma\varphi(\ ^*)\right.\right. \qquad \text{(A.9.13.)} \\ & + \sigma \ ^{*2}\frac{\partial\varphi(\ ^*)}{\partial \ ^*}\frac{\partial \ ^*}{\partial\sigma} - 6 \ ^*\varphi(\ ^*)\,\sigma\sqrt{\tau} - 3\frac{\partial \ ^*}{\partial\sigma}\sigma^2\sqrt{\tau}\varphi(\ ^*) \\ & -3 \ ^*\sigma^2\sqrt{\tau}\frac{\partial\varphi(\ ^*)}{\partial \ ^*}\frac{\partial \ ^*}{\partial\sigma} + 9\sigma^2\tau\varphi(\ ^*) + 3\sigma^3\tau + \frac{\partial\varphi(\ ^*)}{\partial \ ^*}\frac{\partial \ ^*}{\partial\sigma} \\ & \left. -\varphi(\ ^*) - \sigma\frac{\partial\varphi(\ ^*)}{\partial \ ^*}\frac{\partial \ ^*}{\partial\sigma}\right] \\ & \left. - \frac{\sigma \ _2(\ ^*)\,\varphi(\ ^*)\left(\frac{\gamma_1(f)}{2!}\sigma^2\tau^{3/2} + \frac{\gamma_2(f)}{3!}\sigma^3\tau^2\right)}{(1+\omega)}\right\} \\ &= \frac{\ _t\sqrt{\tau}\,\varphi(\ ^*)}{4!\,(1+\omega)}\{[\end{aligned}$$

where:

$$\frac{\partial \ ^{*}}{\partial \sigma} = -\frac{1}{\sigma}\left[\ ^{*} - \sigma\sqrt{\tau} + \frac{\left(\frac{\gamma_1(f)}{2!}\,\sigma^2\tau + \frac{\gamma_2(f)}{3!}\,\sigma^3\tau^{3/2}\right)}{(1+\omega)}\right] \qquad \text{(A.9.14.)}$$

Substituting expressions $(\ \ 9\ 11\ )$, $(\ \ 9\ 12\ )$ and $(\ \ 9\ 13\ )$ in equation $(\ \ 9\ 10\ )$, factoring out $\ _t\sqrt{\tau}\varphi(\ ^{*})\,(1+\omega)^{-1}$ leads to the Vega $formula$ (48) for the Corrado-Su (1996-b and 1997-b) model.

Differentiating the Corrado-Su equation $(\ \ 9\ 1\ )$ with respect to the skewness we obtain:

$$\frac{\partial \ _{CS}}{\partial \gamma_1(\ )} = \frac{\partial \ ^{*}_{BS}}{\partial \gamma_1(\ )} + \ _3' + \gamma_1(\ )$$

where:

$$\frac{\partial \ ^{*}}{\partial \gamma_1 (\ )} = -\frac{\sigma^2 \tau}{3!\,(1+\omega)}$$

Substituting expression (9 16), (9 17) and (9 18) in equation (9 15), factoring out $\ _t\, \sigma^3 \tau^{3/2} \varphi(\ ^{*}) \left[3!\,(1+\omega)^2\right]^{-1}$ leads to the Khi *formula* (49) for the Corrado-Su (1996-b and 1997-b) model.

Differentiating the Corrado-Su equation (9 1) with respect to the kurtosis leads to:

$$\frac{\partial \ _{CS}}{\partial \gamma_2 (\ )} = \frac{\partial \ ^{*}_{BS}}{\partial \gamma_2 (\ )} + \gamma_1 (\ )\, \frac{\partial \ '_3}{\partial \gamma_2 (\ )} + \ '_4 + \gamma_2 (\ )\, \frac{\partial \ '_4}{\partial \gamma_2 (\ )} \qquad \text{(A.9.19.)}$$

with:

$$\begin{aligned} \frac{\partial \ ^{*}_{BS}}{\partial \gamma_2 (\ )} &= \ _t \frac{\partial \Phi(\ ^{*})}{\partial \ ^{*}} \frac{\partial \ ^{*}}{\partial \gamma_2 (\ )} - \ ^{-r\tau} \ \frac{\partial \Phi(\ ^{*} - \sigma\sqrt{\tau})}{\partial (\ ^{*} - \sigma\sqrt{\tau})} \frac{\partial (\ ^{*} - \sigma\sqrt{\tau})}{\partial \gamma_2 (\ )} \qquad \text{(A.9.20.)} \\ &= \frac{\ _t\, \sigma^3 \tau^{3/2} \varphi(\ ^{*})}{4!\,(1+\omega)} \left[-1 + \frac{1}{(1+\omega)}\right] \end{aligned}$$

$$\begin{aligned} \frac{\partial \ '_3}{\partial \gamma_2 (\ )} &= \frac{\ _t \sigma\sqrt{\tau}}{3!\,(1+\omega)^2} \left[\left(2\sigma\sqrt{\tau} \frac{\partial \varphi(\ ^{*})}{\partial \ ^{*}} \frac{\partial \ ^{*}}{\partial \gamma_2 (\ )} - \frac{\partial \ ^{*}}{\partial \gamma_2 (\ )} \varphi(\ ^{*}) \right.\right. \qquad \text{(A.9.21.)} \\ &\quad \left.\left. - \ ^{*} \frac{\partial \varphi(\ ^{*})}{\partial \ ^{*}} \frac{\partial \ ^{*}}{\partial \gamma_2 (\ )}\right)(1+\omega) - \frac{\sigma^4 \tau^2 \ _1(\ ^{*})\, \varphi(\ ^{*})}{4!}\right] \\ &= \frac{\ _t \sigma^4 \tau^2 \varphi(\ ^{*}) \left[- \ _2(\ ^{*}) + \sigma^2 \tau\right]}{(3!)\,(4!)\,(1+\omega)^2} \end{aligned}$$

and:

$$\begin{aligned} \frac{\partial \ '_4}{\partial \gamma_2 (\ )} &= \frac{\ _t \sigma\sqrt{\tau}}{4!\,(1+\omega)^2} \left[\left(2 \ ^{*} \frac{\partial \ ^{*}}{\partial \gamma_2 (\ )} \varphi(\ ^{*}) + \ ^{*2} \frac{\partial \varphi(\ ^{*})}{\partial \ ^{*}} \frac{\partial \ ^{*}}{\partial \gamma_2 (\ )} \right.\right. \qquad \text{(A.9.22.)} \\ &\quad -3 \frac{\partial \ ^{*}}{\partial \gamma_2 (\ )} \sigma\sqrt{\tau} \varphi(\ ^{*}) - 3 \ ^{*} \sigma\sqrt{\tau} \frac{\partial \varphi(\ ^{*})}{\partial \ ^{*}} \frac{\partial \ ^{*}}{\partial \gamma_2 (\ )} + 3\sigma^2 \tau \frac{\partial \varphi(\ ^{*})}{\partial \ ^{*}} \frac{\partial \ ^{*}}{\partial \gamma_2 (\ )} \\ &\quad \left.\left. - \frac{\partial \varphi(\ ^{*})}{\partial \ ^{*}} \frac{\partial \ ^{*}}{\partial \gamma_2 (\ )}\right)(1+\omega) - \frac{\sigma^4 \tau^2 \ _2(\ ^{*}) \varphi(\ ^{*})}{4!}\right] \\ &= \frac{\ _t \sigma^4 \tau^2 \varphi(\ ^{*}) \ _3(\ ^{*})}{(4!)^2\,(1+\omega)^2} \end{aligned}$$

where:

$$\frac{\partial \ ^{*}}{\partial \gamma_2 (\ )} = -\frac{\sigma^3 \tau^{3/2}}{4!\,(1+\omega)}$$

Substituting expression ($9\ 20$), ($9\ 21$) and ($9\ 22$) in equation ($9\ 19$), factoring out $_t\,\sigma^4\tau^2\varphi(\ ^*)\left[4!\,(1+\omega)^2\right]^{-1}$ /

and:

$$
\begin{aligned}
\frac{\partial \ {}''_5}{\partial \ _t} &= \frac{10\,\sigma\sqrt{\tau}}{6!\,(1+\quad)}\Big[\ _4(\ ^{**})\,\varphi(\ ^{**}) + \ _t\Big(4\ ^{**3}\frac{\partial\ ^{**}}{\partial\ _t} - 15\ ^{**2}\frac{\partial\ ^{**}}{\partial\ _t}\sigma\sqrt{\tau} \\
&\quad +20\ ^{**}\frac{\partial\ ^{**}}{\partial\ _t}\sigma^2\tau - 12\ ^{**}\frac{\partial\ ^{**}}{\partial\ _t} - 10\frac{\partial\ ^{**}}{\partial\ _t}\sigma^3\tau^{3/2} \\
&\quad +15\frac{\partial\ ^{**}}{\partial\ _t}\sigma\sqrt{\tau}\Big)\times\varphi(\ ^{**}) + \ _t\ _4(\ ^{**})\,\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\ _t}\Big] \\
&= -\frac{10\,\varphi(\ ^{**})}{6!\,(1+\quad)}\ _5(\ ^{**})
\end{aligned}
\qquad \text{(A.10.6.)}
$$

where:

$$
\begin{cases}
\frac{\partial d^{**}}{\partial S_t} = (\ _t\sigma\sqrt{\tau})^{-1} \\
\frac{\partial\varphi(d^{**})}{\partial d^{**}} = -\ ^{**}\varphi(\ ^{**})
\end{cases}
$$

Substituting expressions $(\ \ 10\ 3\ )$ $(\ \ 10\ 4\ )$ $(\ \ 10\ 5\ )$ and $(\ \ 10\ 6\ )$ in equation $(\ \ 10\ 2\ )$ and factoring out $\varphi(\ ^{**})\,(1+\quad)^{-1}$ leads to the Delta $formula$ (51) for the Rubinstein (1998) model.

Differentiating expression $(\ \ 10\ 1\ )$ with respect to the underlying asset price, we have:

$$
\frac{\partial\Delta_{CS}}{\partial\ _t} = \frac{\partial\Delta^{**}_{BS}}{\partial\ _t} + \gamma_1(\ )\frac{\partial^2\ {}''_3}{\partial\ ^2_t} + \gamma_2(\ )\frac{\partial^2\ {}''_4}{\partial\ ^2_t} + \gamma_1(\ )^2\frac{\partial^2\ {}''_5}{\partial\ ^2_t} \qquad \text{(A.10.7.)}
$$

with:

$$
\frac{\partial\Delta^{**}_{BS}}{\partial\ _t} = \frac{\varphi(\ ^{**})}{_t\sigma\sqrt{\tau}} - \frac{^{**}\varphi(\ ^{**})}{_t\sigma^2\tau} + \frac{^{**}\varphi(\ ^{**})}{_t\sigma^2\tau\,(1+\quad)} \qquad \text{(A.10.8.)}
$$

and:

$$
\frac{\partial^2\ {}''_3}{\partial\ ^2_t} = \frac{\varphi(\ ^{**})\left[-\ _3(\ ^{**}) - \ _2(\ ^{**})\,\sigma\sqrt{\tau} + \ ^{**}\sigma^2\tau + 6\sigma^3\tau^{3/2}\right]}{3!\ \ _t\sigma\sqrt{\tau}\,(1+\quad)} \qquad \text{(A.10.9.)}
$$

$$
\frac{\partial^2\ {}''_4}{\partial\ ^2_t} = \frac{\varphi(\ ^{*})\left[\ _4(\ ^{**}) + \ _3(\ ^{**})\,\sigma\sqrt{\tau} - 2\ ^{**^3}\sigma\sqrt{\tau} + \ ^{**}\sigma^3\tau^{3/2} + 8\sigma^4\tau^2\right]}{4!\ \ _t\sigma\sqrt{\tau}\,(1+\quad)} \qquad \text{(A.10.10.)}
$$

$$\frac{\partial^2 \;\; ''_5}{\partial \;\; ^2_t} = \frac{10}{6!\,(1+\quad)}\left[-\frac{\partial\varphi(\;\;^{**})}{\partial\;\;^{**}}\frac{\partial\;\;^{**}}{\partial\;\;_t}\;\;_5(\;\;^{**})\right. \tag{A.10.11.}$$

$$+\varphi(\;\;^{**})\left(-5\;\;^4\frac{\partial\;\;^{**}}{\partial\;\;_t}+24\;\;^{**3}\frac{\partial\;\;^{**}}{\partial\;\;_t}\sigma\sqrt{\tau}-45\;\;^{**2}\frac{\partial\;\;^{**}}{\partial\;\;_t}\sigma^2\tau\right.$$

$$+40\;\;^{**2}\frac{\partial\;\;^{**}}{\partial\;\;_t}\sigma^3\tau^{3/2}+30\;\;^{**2}\frac{\partial\;\;^{**}}{\partial\;\;_t}-72\;\;^{**}\frac{\partial\;\;^{**}}{\partial\;\;_t}\sigma\sqrt{\tau}-15\frac{\partial\;\;^{**}}{\partial\;\;_t}\sigma^4\tau^2$$

$$\left.\left.+45\frac{\partial\;\;^{**}}{\partial\;\;_t}\sigma^2\tau-15\frac{\partial\;\;^{**}}{\partial\;\;_t}\right)\right]$$

$$= \frac{10\varphi(\;\;^{**})\left[\;\;_6(\;\;^{**})-5\;\;_4(\;\;^{**})-15\;\;^{**2}\sigma^2\tau\right]}{6!\;\;_t\sigma\sqrt{\tau}\,(1+\quad)}$$

Substituting expressions (10 8), (10 9), (10 10) and (10 11) in equation (10 7) and factoring out $\varphi(\;\;^{**})\,[\;\;_t\sigma\sqrt{\tau}\,(1+\quad)]^{-1}$ leads to the Gamma *formula* (52) for the Rubinstein (1998) model.

Differentiating the Rubinstein equation (10 1) with respect to the volatility, we get:

$$\frac{\partial\;\;_R}{\partial\sigma}=\frac{\partial\;\;^{**}_{BS}}{\partial\sigma}+\gamma_1(\;\;)\frac{\partial\;\;''_3}{\partial\sigma}+\gamma_2(\;\;)\frac{\partial\;\;''_4}{\partial\sigma}+\gamma_1(\;\;)^2\frac{\partial\;\;''_5}{\partial\sigma} \tag{A.10.12.}$$

where:

$$\frac{\partial\;\;^{**}_{BS}}{\partial\sigma} = \;\;_t\varphi(\;\;^{**})\sqrt{\tau} \tag{A.10.13.}$$

$$+\sigma^{-1}\;\;_t\varphi(\;\;^{**})\left[\;\;^{**}+\frac{\left(\frac{\gamma_1(f)}{2!}\sigma^2\tau+\frac{\gamma_2(f)}{3!}\sigma^3\tau^{3/2}+10\frac{\gamma_1(f)^2}{5!}\sigma^5\tau^{5/2}\right)}{(1+\quad)}\right]$$

$$\times\left[-1+\frac{1}{(1+\quad)}\right]$$

$$\frac{\partial \;{}''_3}{\partial\sigma} = \frac{t\sqrt{\tau}}{3!\,(1+\;\;)}\Big\{\big[4\sigma\sqrt{\tau}\varphi(\;^{**}) \qquad \text{(A.10.14.)}$$

$$+2\sigma^2\sqrt{\tau}\frac{\partial\varphi(\;^{**})}{\partial\;^{**}}\frac{\partial\;^{**}}{\partial\sigma} - \;^{**}\varphi(\;^{**}) - \sigma\frac{\partial\;^{**}}{\partial\sigma}\varphi(\;^{**}) - \;^{*}\sigma\frac{\partial\varphi(\;^{**})}{\partial\;^{**}}\frac{\partial\;^{**}}{\partial\sigma}\Big]$$

$$-\frac{\sigma\;\;_1(\;^{**})\,\varphi(\;^{**})\left(\frac{\gamma_1(f)}{2!}\,\sigma^2\tau^{3/2} + \frac{\gamma_2(f)}{3!}\,\sigma^3\tau^2 + 10\frac{\gamma_1(f)^2}{5!}\,\sigma^5\tau^3\right)}{(1+\;\;)}\Bigg\}$$

$$= \frac{t\sqrt{\tau}\,\varphi(\;^{**})}{3!\,(1+\;\;)}\Big\{\big[-\;^{**}\;\;_2(\;^{**}) + \;^{**} + \;^{**}\sigma^2\tau + 3\sigma\sqrt{\tau}\big]$$

$$+\frac{\left(\frac{\gamma_1(f)}{2!}\,\sigma^2\tau^2 + \frac{\gamma_2(f)}{3!}\,\sigma^3\tau^{3/2} + 10\frac{\gamma_1(f)^2}{5!}\,\sigma^5\tau^{5/2}\right)}{(1+\;\;)}\big[-\;\;_2(\;^{**}) + \sigma^2\tau\big]\Bigg\}$$

$$\frac{\partial \;{}''_4}{\partial\sigma} = \frac{t\sqrt{\tau}}{4!\,(1+\;\;)}\Big\{\big[\;^{**2}\varphi(\;^{**}) \qquad \text{(A.10.15.)}$$

$$+2\;^{**}\frac{\partial\;^{**}}{\partial\sigma}\sigma\varphi(\;^{**}) + \sigma\;^{**2}\frac{\partial\varphi(\;^{**})}{\partial\;^{**}}\frac{\partial\;^{**}}{\partial\sigma} - 6\;^{*}\varphi(\;^{**})\,\sigma\sqrt{\tau}$$

$$-3\frac{\partial\;^{**}}{\partial\sigma}\sigma^2\sqrt{\tau}\varphi(\;^{**}) - 3\;^{**}\sigma^2\sqrt{\tau}\frac{\partial\varphi(\;^{**})}{\partial\;^{**}}\frac{\partial\;^{**}}{\partial\sigma} + 9\sigma^2\tau\varphi(\;^{**})$$

$$+3\sigma^3\tau + \frac{\partial\varphi(\;^{**})}{\partial\;^{**}}\frac{\partial\;^{**}}{\partial\sigma} - \varphi(\;^{**}) - \sigma\frac{\partial\varphi(\;^{**})}{\partial\;^{**}}\frac{\partial\;^{**}}{\partial\sigma}\Big]$$

$$-\sigma\;\;_2(\;^{**})\,\varphi(\;^{**})$$

$$\times\frac{\left(\frac{\gamma_1(f)}{2!}\,\sigma^2\tau^{3/2} + \frac{\gamma_2(f)}{3!}\,\sigma^3\tau^2 + 10\frac{\gamma_1(f)^2}{5!}\,\sigma^5\tau^3\right)}{(1+\;\;)}\Bigg\}$$

$$= \frac{t\sqrt{\tau}\,\varphi(\;^{**})}{4!\,(1+\;\;)}\Big\{\big[\;^{**}\;\;_3(\;^{**}) + \;\;_2(\;^{**}) - 7\;^{**}\sigma\sqrt{\tau} - 3\sigma^2\tau\big]$$

$$+\frac{\left(\frac{\gamma_1(f)}{2!}\,\sigma^2\tau^2 + \frac{\gamma_2(f)}{3!}\,\sigma^3\tau^{3/2} + 10\frac{\gamma_1(f)^2}{5!}\,\sigma^5\tau^{5/2}\right)}{(1+\;\;)}\;\;_3(\;^{**})\Bigg\}$$

and:

$$\frac{\partial \ {}''_5}{\partial\sigma} = \frac{10 \ {}_t\sqrt{\tau}}{6!\,(1+ \quad)}\Big\{\Big[\ {}^{**4}\varphi(\ {}^{**}) \qquad \text{(A.10.16.)}$$

$$+4\ {}^{**3}\frac{\partial \ {}^{**}}{\partial\sigma}\sigma\varphi(\ {}^{**}) + \ {}^{**4}\sigma\frac{\partial\varphi(\ {}^{**})}{\partial \ {}^{**}}\frac{\partial \ {}^{**}}{\partial\sigma} - 10\ {}^{**3}\sigma\sqrt{\tau}\,\varphi(\ {}^{**})$$

$$-15\ {}^{**2}\frac{\partial \ {}^{**}}{\partial\sigma}\sigma^2\sqrt{\tau}\varphi(\ {}^{**}) - 5\ {}^{**3}\sigma^2\sqrt{\tau}\frac{\partial\varphi(\ {}^{**})}{\partial \ {}^{**}}\frac{\partial \ {}^{**}}{\partial\sigma} + 30\ {}^{**2}\sigma^2\tau\varphi(\ {}^{**})$$

$$+20\ {}^{**}\sigma^3\tau\frac{\partial \ {}^{**}}{\partial\sigma}\varphi(\ {}^{**}) + 10\ {}^{**2}\sigma^3\tau\frac{\partial\varphi(\ {}^{**})}{\partial \ {}^{**}}\frac{\partial \ {}^{**}}{\partial\sigma} - 6\ {}^{**2}\varphi(\ {}^{**})$$

$$-12\ {}^{**}\frac{\partial \ {}^{**}}{\partial\sigma}\sigma\varphi(\ {}^{**}) - 6\ {}^{**2}\sigma\frac{\partial\varphi(\ {}^{**})}{\partial \ {}^{**}}\frac{\partial \ {}^{**}}{\partial\sigma} - 40\ {}^{**}\sigma^3\tau^{3/2}\varphi(\ {}^{**})$$

$$-10\frac{\partial \ {}^{**}}{\partial\sigma}\sigma^4\tau^{3/2}\varphi(\ {}^{**}) - 10\ {}^{**}\sigma^4\tau^{3/2}\frac{\partial\varphi(\ {}^{**})}{\partial \ {}^{**}}\frac{\partial \ {}^{**}}{\partial\sigma} + 30\ {}^{**}\sigma\sqrt{\tau}\varphi(\ {}^{**})$$

$$+15\frac{\partial \ {}^{**}}{\partial\sigma}\sigma^2\sqrt{\tau}\varphi(\ {}^{**}) + 15\ {}^{**}\sigma^2\sqrt{\tau}\frac{\partial\varphi(\ {}^{**})}{\partial \ {}^{**}}\frac{\partial \ {}^{**}}{\partial\sigma} + 25\sigma^4\tau^2\varphi(\ {}^{**})$$

$$+5\sigma^5\tau^2\frac{\partial\varphi(\ {}^{**})}{\partial \ {}^{**}}\frac{\partial \ {}^{**}}{\partial\sigma} - 30\sigma^2\tau\varphi(\ {}^{**}) - 10\sigma^3\tau\frac{\partial\varphi(\ {}^{**})}{\partial \ {}^{**}}\frac{\partial \ {}^{**}}{\partial\sigma} + 3\varphi(\ {}^{**})$$

$$+3\sigma\frac{\partial\varphi(\ {}^{**})}{\partial \ {}^{**}}\frac{\partial \ {}^{**}}{\partial\sigma}\Big] - \sigma\ {}_4(\ {}^{**})\,\varphi(\ {}^{**})$$

$$\times\frac{\left(\frac{\gamma_1(f)}{2!}\,\sigma^2\tau^{3/2} + \frac{\gamma_2(f)}{3!}\,\sigma^3\tau^2 + 10\frac{\gamma_1(f)^2}{5!}\,\sigma^5\tau^3\right)}{(1+ \quad)}\Big\}$$

$$= \frac{10\ {}_t\sqrt{\tau}\,\varphi(\ {}^{**})}{6!\,(1+ \quad)}\Big[\ {}_6(\ {}^{**})$$

$$-\frac{\left(\frac{\gamma_1(f)}{2!}\,\sigma^2\tau^2 + \frac{\gamma_2(f)}{3!}\,\sigma^3\tau^{3/2} + 10\frac{\gamma_1(f)^2}{5!}\,\sigma^5\tau^{5/2}\right)}{(1+ \quad)}\ {}_5(\ {}^{**})\Big]$$

where:

$$\frac{\partial \ {}^{**}}{\partial\sigma} = -\frac{1}{\sigma}\left[\ {}^{**} - \sigma\sqrt{\tau} + \frac{\frac{\gamma_1(f)}{2!}\,\sigma^2\tau + \frac{\gamma_2(f)}{3!}\,\sigma^3\tau^{3/2} + 10\frac{\gamma_1(f)^2}{5!}\,\sigma^5\tau^{5/2}}{(1+ \quad)}\right]$$

Substituting expression ($10\ 13$), ($10\ 14$), ($10\ 15$), ($10\ 16$) in equation ($10\ 12$), factoring out ${}_t\sqrt{\tau}\varphi(\ {}^{**})\,(1+ \quad)^{-1}$ leads to the Vega $formula$ (53) for the Rubinstein (1998) model.

Differentiating the Rubinstein equation (10 1) with respect to the skewness we obtain:

$$\frac{\partial \ _R}{\partial\gamma_1(\ )} = \frac{\partial \ _{BS}^{**}}{\partial\gamma_1(\ )} + \ _3'' + \gamma_1(\ )\frac{\partial \ _3''}{\partial\gamma_1(\ )} + \gamma_2(\ )\frac{\partial \ _4''}{\partial\gamma_1(\ )} + 2\gamma_1(\ ) \ _5'' + [\gamma_1(\ )]^2\frac{\partial \ _5''}{\partial\gamma_1(\ )}$$ (A.10.17.)

with:

$$\frac{\partial \ _{BS}^{**}}{\partial\gamma_1(\ )} = \ _t\frac{\partial\Phi(\ ^{**})}{\partial \ ^{**}}\frac{\partial \ ^{**}}{\partial\gamma_1(\ )} - \ ^{-r\tau} \ \frac{\partial\Phi(\ ^{**}-\sigma\sqrt{\tau})}{\partial(\ ^{**}-\sigma\sqrt{\tau})}\frac{\partial(\ ^{**}-\sigma\sqrt{\tau})}{\partial\gamma_1(\ )}$$ (A.10.18.)

$$= \frac{\ _t\sigma^2\tau\left(1+\frac{\gamma_1(f)}{6}\sigma^3\tau^{3/2}\right)\varphi(\ ^{**})}{3!\,(1+\ )} \times \left[-1+\frac{1}{(1+\ )}\right]$$

$$\frac{\partial \ _3''}{\partial\gamma_1(\ )} = \frac{\ _t\sigma\sqrt{\tau}}{3!\,(1+\ )^2}\left[\left(2\sigma\sqrt{\tau}\frac{\partial\varphi(\ ^{**})}{\partial \ ^{**}}\right.\right.$$ (A.10.19.)

$$\left.\times\frac{\partial \ ^{**}}{\partial\gamma_1(\ )} - \frac{\partial \ ^{**}}{\partial\gamma_1(\ )}\varphi(\ ^{**}) - \ ^{**}\frac{\partial\varphi(\ ^{**})}{\partial \ ^{**}}\frac{\partial \ ^{**}}{\partial\gamma_1(\ )}\right)$$

$$\times(1+\ )$$

$$\left.-\frac{\sigma^3\tau^{3/2}\left(1+\frac{\gamma_1(f)}{6}\sigma^3\tau^{3/2}\right) \ _3(\ ^{**})\varphi(\ ^{**})}{3!}\right]$$

$$= \frac{\ _t\sigma^3\tau^{3/2}\left(1+\frac{\gamma_1(f)}{6}\sigma^3\tau^{3/2}\right)\varphi(\ ^{**})\left[-\ _2(\ ^{**})+\sigma^2\tau\right]}{(3!)^2\,(1+\ )^2}$$

$$\frac{\partial \ _4''}{\partial\gamma_1(\ )} = \frac{\ _t\sigma\sqrt{\tau}}{4!\,(1+\ )^2}\left[\left(2\ ^{**}\frac{\partial \ ^{**}}{\partial\gamma_1(\ )}\varphi(\ ^{**}) + \ ^{**2}\frac{\partial\varphi(\ ^{**})}{\partial \ ^{**}}\frac{\partial \ ^{**}}{\partial\gamma_1(\ )}\right.\right.$$ (A.10.20.)

$$-3\frac{\partial \ ^{**}}{\partial\gamma_1(\ )}\sigma\sqrt{\tau}\varphi(\ ^{**}) - 3\ ^{**}\sigma\sqrt{\tau}\frac{\partial\varphi(\ ^{**})}{\partial \ ^{**}}\frac{\partial \ ^{**}}{\partial\gamma_1(\ )}$$

$$\left.+3\sigma^2\tau\frac{\partial\varphi(\ ^{**})}{\partial \ ^{**}}\frac{\partial \ ^{**}}{\partial\gamma_1(\ )} - \frac{\partial\varphi(\ ^{**})}{\partial \ ^{**}}\frac{\partial \ ^{**}}{\partial\gamma_1(\ )}\right)\times(1+\ )$$

$$\left.-\frac{\sigma^3\tau^{3/2}\left(1+\frac{\gamma_1(f)}{6}\sigma^3\tau^{3/2}\right) \ _2(\ ^{**})\varphi(\ ^{**})}{3!}\right]$$

$$= \frac{\ _t\sigma^3\tau^{3/2}\left(1+\frac{\gamma_1(f)}{6}\sigma^3\tau^{3/2}\right)\varphi(\ ^{**}) \ _3(\ ^{**})}{(3!)\,(4!)\,(1+\ )^2}$$

and:

$$\frac{\partial \;\; ''_{5}}{\partial\gamma_1(\;)} = \frac{10 \;\; _t\sigma\sqrt{\tau}}{6!\,(1+\;\;)^2}\left[\left(4 \;\; ^{**3}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)}\varphi(\;\; ^{**}) + \;\; ^{**4}\frac{\partial\varphi(\;\; ^{**})}{\partial \;\; ^{**}}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)}\right.\right. \tag{A.10.21.}$$
$$-15 \;\; ^{**2}\sigma\sqrt{\tau}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)}\varphi(\;\; ^{**}) - 5 \;\; ^{**3}\sigma\sqrt{\tau}\frac{\partial\varphi(\;\; ^{**})}{\partial \;\; ^{**}}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)} - 12 \;\; ^{**}$$
$$+20 \;\; ^{**}\sigma^2\tau\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)}\varphi(\;\; ^{**}) + 10 \;\; ^{**2}\sigma^2\tau\frac{\partial\varphi(\;\; ^{**})}{\partial \;\; ^{**}}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)}\varphi(\;\; ^{**})$$
$$-6 \;\; ^{**2}\frac{\partial\varphi(\;\; ^{**})}{\partial \;\; ^{**}}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)} - 10\sigma^3\tau^{3/2}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)}\varphi(\;\; ^{**}) - 10 \;\; ^{**}\sigma^3\tau^{3/2}$$
$$\times\frac{\partial\varphi(\;\; ^{**})}{\partial \;\; ^{**}}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)} + 15\sigma\sqrt{\tau}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)}\varphi(\;\; ^{**}) + 15 \;\; ^{**}\sigma\sqrt{\tau}\frac{\partial\varphi(\;\; ^{**})}{\partial \;\; ^{**}}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)}$$
$$+5\sigma^4\tau^2\frac{\partial\varphi(\;\; ^{**})}{\partial \;\; ^{**}}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)} - 10\sigma^2\tau\frac{\partial\varphi(\;\; ^{**})}{\partial \;\; ^{**}}\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)} + 3\frac{\partial\varphi(\;\; ^{**})}{\partial \;\; ^{**}}$$
$$\left.\left.\times\frac{\partial \;\; ^{**}}{\partial\gamma_1(\;)}\right)(1+\;\;) - \frac{\sigma^3\tau^{3/2}\left(1+\frac{\gamma_1(f)}{6}\sigma^3\tau^{3/2}\right)\varphi(\;\; ^{**})}{3!} \;\; _4(\;\; ^{**})\right]$$
$$= -\frac{10 \;\; _t\sigma^3\tau^{3/2}\left(1+\frac{\gamma_1(f)}{6}\sigma^3\tau^{3/2}\right)\varphi(\;\; ^{**})}{(3!)\,(6!)\,(1+\;\;)^2} \;\; _5(\;\; ^{**})$$

where:

$$\begin{cases} \frac{\partial\varphi(d^{**})}{\partial d^{**}} = - \;\; ^{**}\varphi(\;\; ^{**}) \\ \frac{\partial d^{**}}{\partial\gamma_1(f)} = -\frac{\sigma^2\tau\left(1+\frac{\gamma_1(f)}{6}\sigma^3\tau^{3/2}\right)}{3!(1+\varpi)} \end{cases}$$

Substituting expression (9 18), (9 19), (9 20) and (9 21) in equation (10 17), factoring out $\left\{ \;\; _t\,\sigma^3\tau^{3/2}\left(1+\frac{\gamma_1(f)}{6}\sigma^3\tau^{3/2}\right)\varphi(\;\; ^{**})\left[3!\,(1+\;\;)^2\right]^{-1}\right\}$ leads to the Khi $formula$ (54) for the Rubinstein (1998) model.

Differentiating the Rubinstein equation (10 1) with respect to the kurtosis leads to:

$$\frac{\partial \;\; _R}{\partial\gamma_2(\;)} = \frac{\partial \;\; ^{**}_{BS}}{\partial\gamma_2(\;)} + \gamma_1(\;)\frac{\partial \;\; ''_{3}}{\partial\gamma_2(\;)} + \;\; ''_{4} + \gamma_2(\;)\frac{\partial \;\; ''_{4}}{\partial\gamma_2(\;)} + [\gamma_1(\;)]^2\frac{\partial \;\; ''_{5}}{\partial\gamma_2(\;)} \tag{A.10.22.}$$

with:

$$\frac{\partial \;\; ^{**}_{BS}}{\partial\gamma_2(\;)} = \;\; _t\frac{\partial\Phi(\;\; ^{**})}{\partial \;\; ^{**}}\frac{\partial \;\; ^{**}}{\partial\gamma_2(\;)} - \;\; ^{-r\tau}\;\;\frac{\partial\Phi(\;\; ^{**}-\sigma\sqrt{\tau})}{\partial(\;\; ^{**}-\sigma\sqrt{\tau})}\frac{\partial(\;\; ^{**}-\sigma\sqrt{\tau})}{\partial\gamma_2(\;)} \tag{A.10.23.}$$
$$= \frac{\;\; _t\,\sigma^3\tau^{3/2}\varphi(\;\; ^{*})}{4!\,(1+\;\;)} \times \left[-1+\frac{1}{(1+\;\;)}\right]$$

$$\frac{\partial\ {}''_3}{\partial\gamma_2(\ )} = \frac{{}_t\sigma\sqrt{\tau}}{3!\,(1+\ )^2}\left[\left(2\sigma\sqrt{\tau}\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\times\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} - \frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\varphi(\ ^{**}) - \ ^{**}\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\right)\times(1+\ ) - \frac{\sigma^4\tau^2\ {}_1(\ ^{**})\,\varphi(\ ^{**})}{4!}\right] = \frac{{}_t\sigma^4\tau^2\left[-\ {}_2(\ ^{**})+\sigma^2\tau\right]\varphi(\ ^{**})}{(3!)\,(4!)\,(1+\ )^2} \qquad \text{(A.10.24.)}$$

$$\frac{\partial\ {}''_4}{\partial\gamma_2(\ )} = \frac{{}_t\sigma\sqrt{\tau}}{4!\,(1+\ )^2}\left[\left(2\ ^{**}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\varphi(\ ^{**}) + \ ^{**2}\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} - 3\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\sigma\sqrt{\tau}\varphi(\ ^{**}) - 3\ ^{**}\sigma\sqrt{\tau}\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} + 3\sigma^2\tau\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} - \frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\right)\times(1+\ ) - \frac{\sigma^4\tau^2\ {}_2(\ ^{**})\varphi(\ ^{**})}{4!}\right] = \frac{{}_t\sigma^4\tau^2\varphi(\ ^{**})\ {}_3(\ ^{**})}{(4!)^2\,(1+\ )^2} \qquad \text{(A.10.25.)}$$

and:

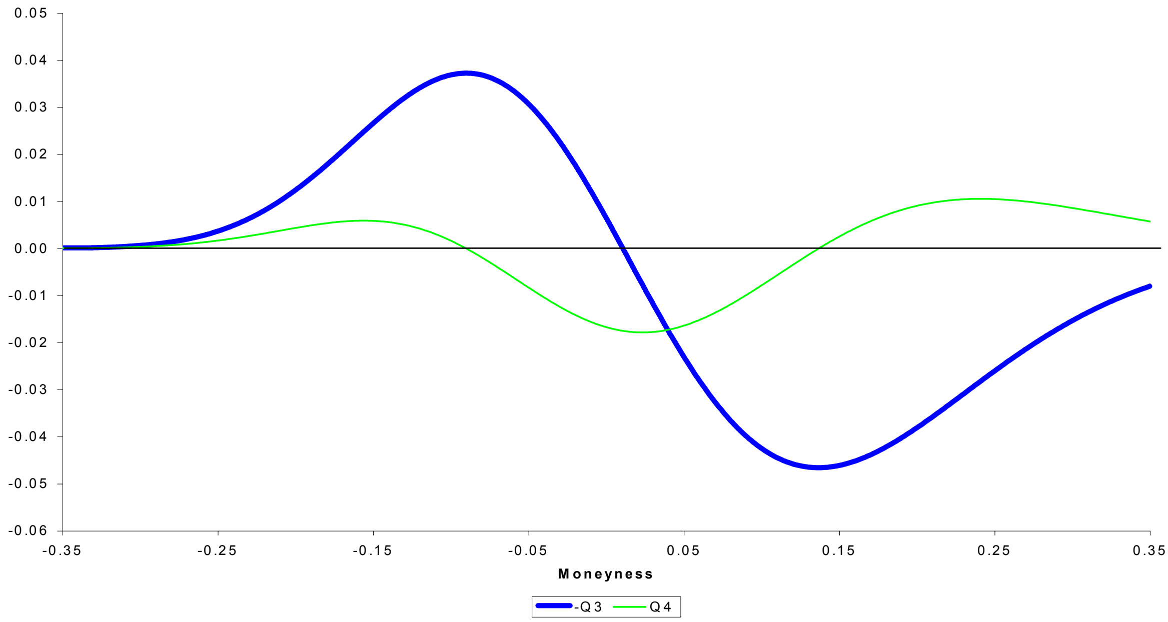
$$\frac{\partial\ {}''_5}{\partial\gamma_2(\ )} = \frac{10\ {}_t\sigma\sqrt{\tau}}{6!\,(1+\ )^2}\left[\left(4\ ^{**3}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\varphi(\ ^{**}) + \ ^{**4}\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} - 15\ ^{**2}\sigma\sqrt{\tau}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\varphi(\ ^{**}) - 5\ ^{**3}\sigma\sqrt{\tau}\times\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} + 20\ ^{**}\sigma^2\tau\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\varphi(\ ^{**}) + 10\ ^{**2}\sigma^2\tau\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} - 12\ ^{**}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\varphi(\ ^{**}) - 6\ ^{**2}\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} - 10\sigma^3\tau^{3/2}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\varphi(\ ^{**}) - 10\ ^{**}\sigma^3\tau^{3/2}\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} + 15\sigma\sqrt{\tau}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\varphi(\ ^{**}) + 15\ ^{**}\sigma\sqrt{\tau} + \frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\times\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} + 5\sigma^4\tau^2\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} - 10\sigma^2\tau\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\frac{\partial\ ^{**}}{\partial\gamma_2(\ )} + 3\frac{\partial\varphi(\ ^{**})}{\partial\ ^{**}}\times\frac{\partial\ ^{**}}{\partial\gamma_2(\ )}\right)(1+\ ) - \frac{\sigma^4\tau^2\varphi(\ ^{**})}{4!}\ {}_4(\ ^{**})\right] = -\frac{10\ {}_t\sigma^4\tau^2\varphi(\ ^{**})}{(4!)\,(6!)\,(1+\ )^2}\ {}_5(\ ^{**}) \qquad \text{(A.10.26.)}$$

where:

$$\frac{\partial\ ^{*}}{\partial\gamma_2(\ )} = -\frac{\sigma^3\tau^{3/2}}{4!\,(1+\ )}$$

Substituting expressions $(\quad 10\,23\,)$, $(\quad 10\,24\,)$, $(\quad 10\,25\,)$ and $(\quad 10\,26\,)$ in equation $(\quad 10\,22\,)$, factoring out $\;_{t}\,\sigma^4\tau^2\varphi(\;\;^{**})\left[4!\,(1+\quad)^2\right]^{-1}$ leads to the Psi $formula$ (55) for the Rubinstein (1998) model.

■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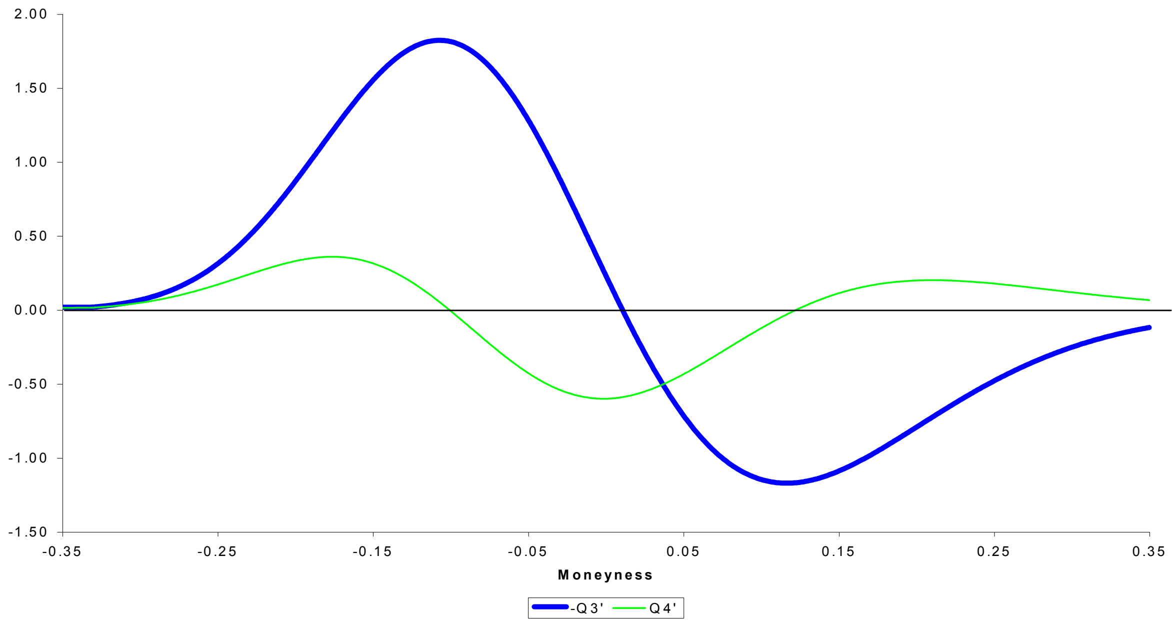
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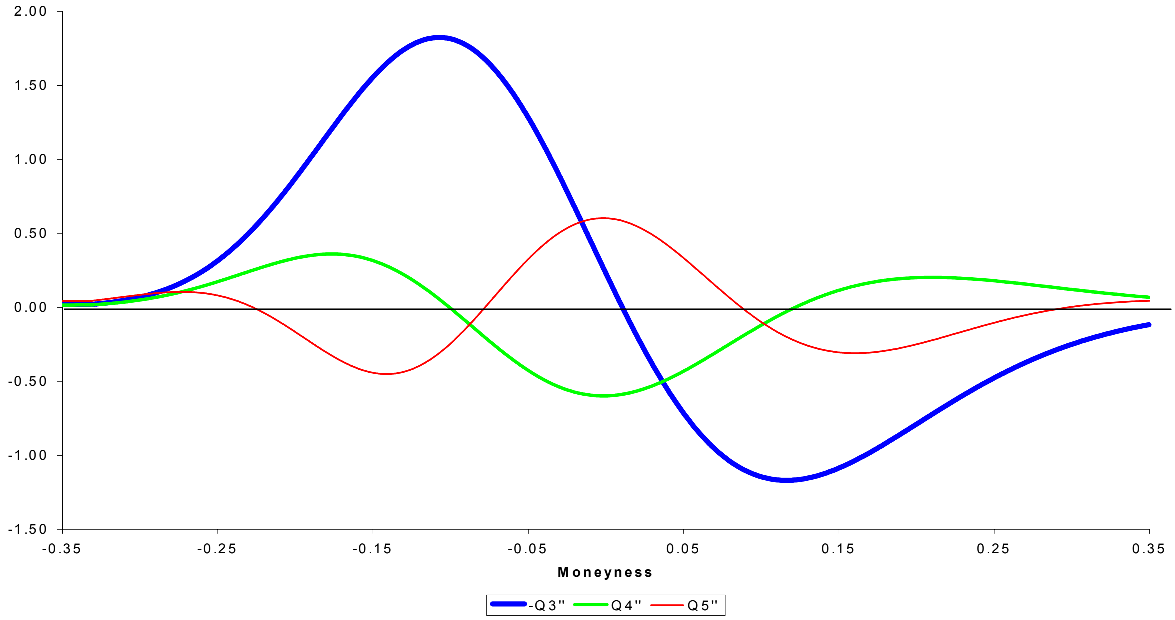
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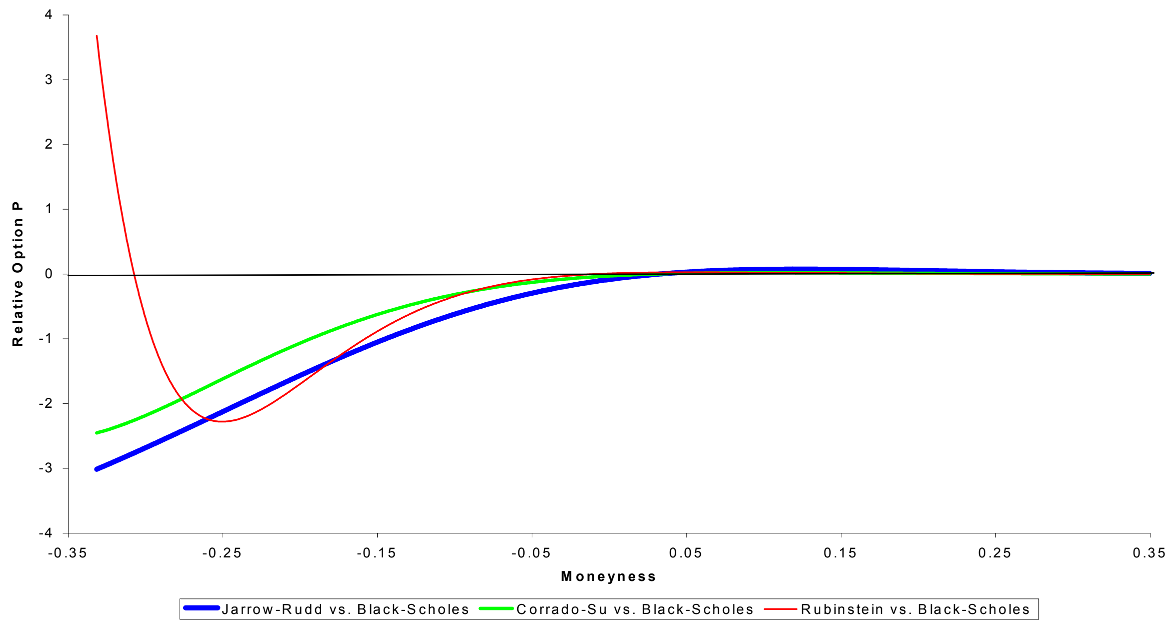
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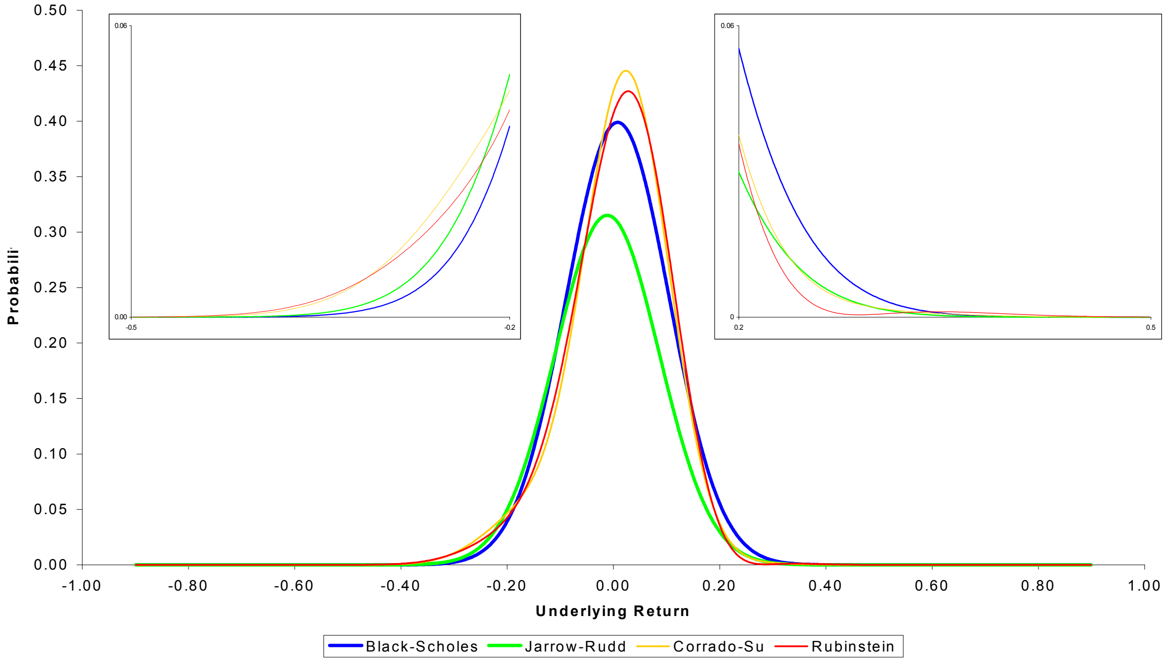
Moneyness
Jarrow-Rudd vs. Black-Scholes
Corrado-Su vs. Black-Scholes
Rubinstein vs. Black-Scholes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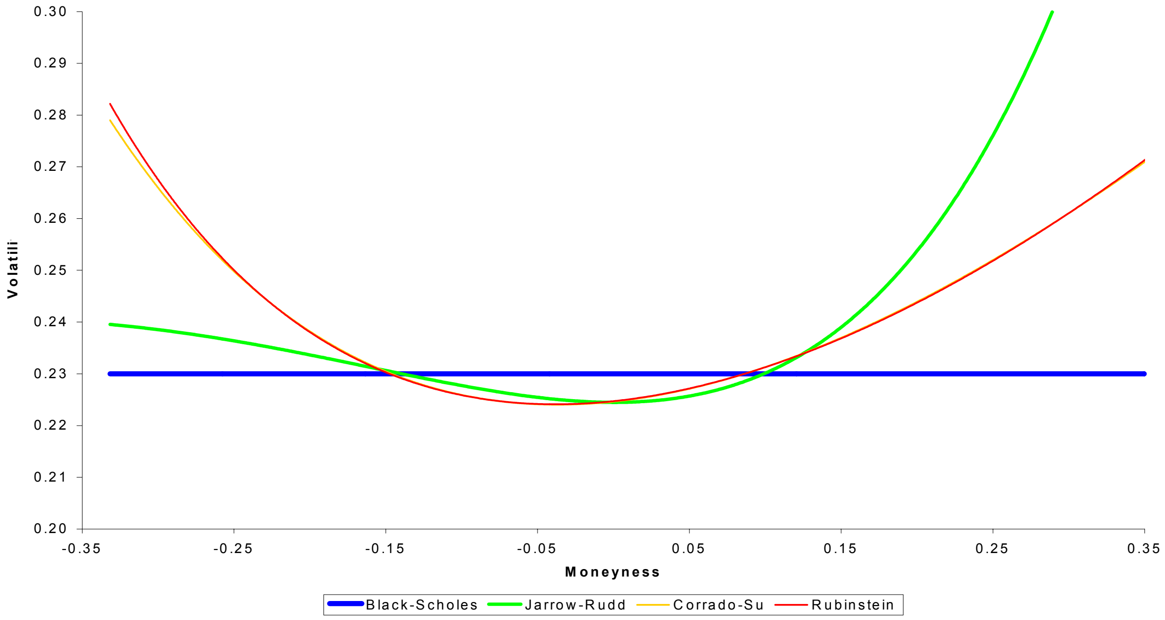
Underlying Return
Black-Scholes
Jarrow-Rudd
Corrado-Su
Rubinstein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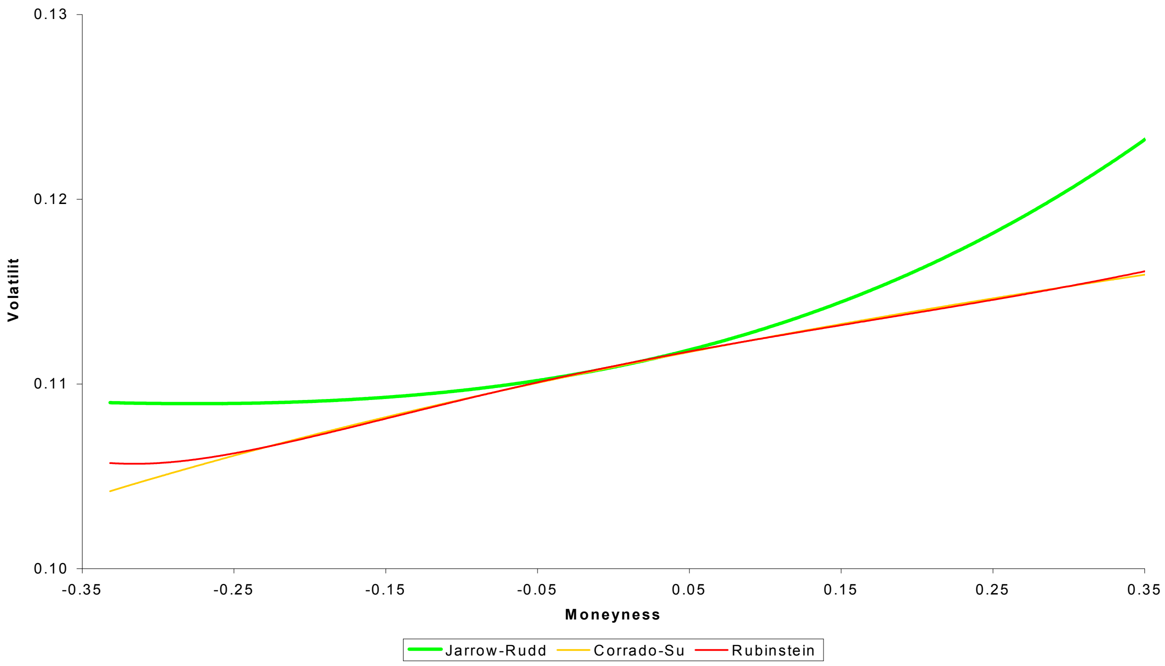
Moneyness
Black-Scholes
Jarrow-Rudd
Corrado-Su
Rubinstein

30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Corrado-Su
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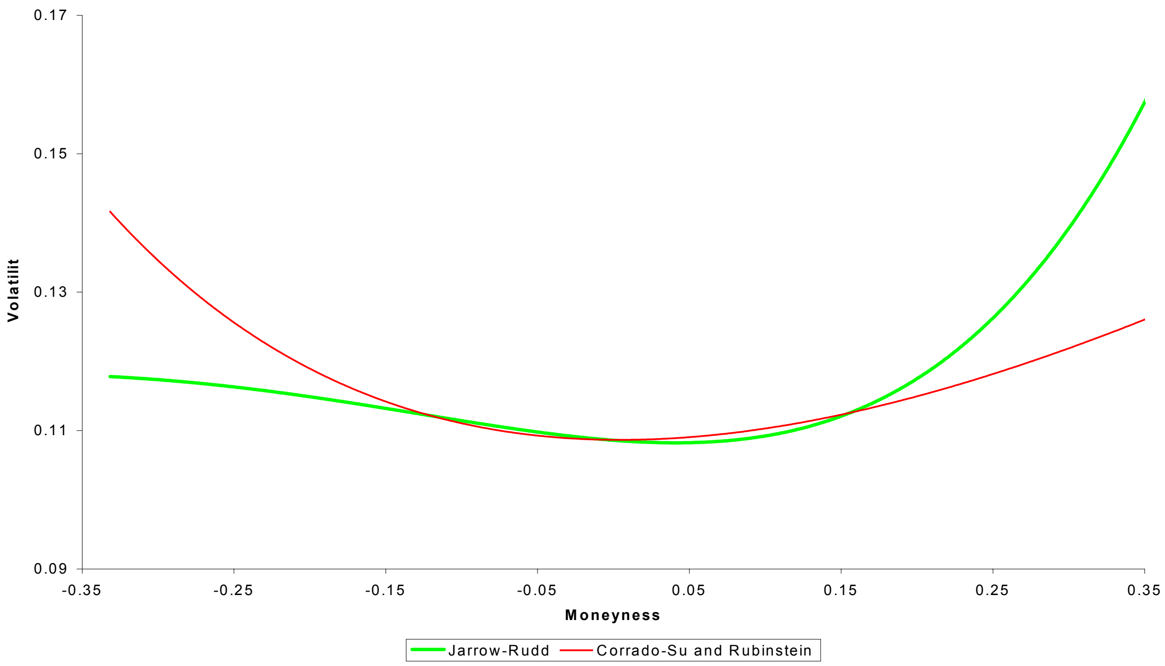
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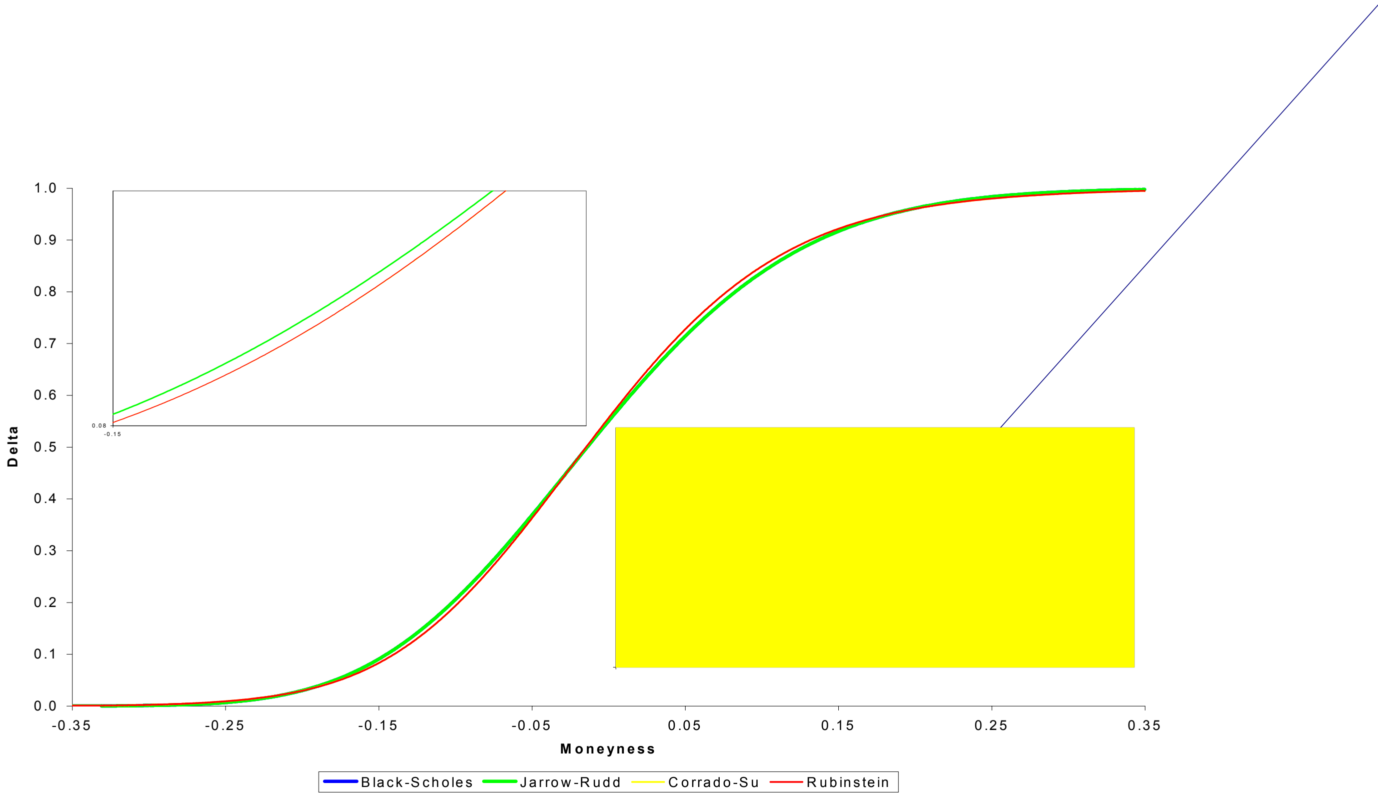
Moneyness
Jarrow-Rudd
Corrado-Su and Rubinstein

31

32



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-0.15
-0.05
0.05
0.15
0.25
0.35
Moneyness
Black-Scholes
Jarrow-Rudd
Corrado-Su
Rubinstein