# Consistent Testing for Stochastic Dominance:

## A Subsampling Appraoch

**By**

**Olivier Linton**
**and**
**Esfandiar Maasoumi**

**DISCUSSION PAPER 407**

**February 2002**

FINANCIAL MARKETS GROUP

AN ESRC RESEARCH CENTRE

LONDON SCHOOL OF ECONOMICS

E·S·R·C
ECONOMIC & SOCIAL RESEARCH COUNCIL



ISSN 0956-8549-407

# Consistent Testing for Stochastic Dominance: A Subsampling Approach

Oliver Linton*
London School of Economics

Esfandiar Maasoumi†
Southern Methodist University

Yoon-Jae Whang‡
Ewha University

February 13, 2002

**Abstract**

We propose a procedure for estimating the critical values of the Klecan, McFadden, and McFadden (1990) test for first and second order stochastic dominance in the general k-prospect case. Our method is based on subsampling bootstrap. We show that the resulting test is consistent. We allow for correlation amongst the prospects and for the observations to be auto-correlated over time. Importantly, the prospects may be the residuals from certain conditional models.

---

*Department of Economics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom. E-mail address: `lintono@lse.ac.uk`. I would like to thank the Economic and Social Research Council of the United Kingdom and the Cowles Foundation for financial support.

†Department of Economics, Southern Methodist University, Dallas, Texas, 75275-0496. Email: `maasoumi@mail.smu.edu`

‡Department of Economics, Ewha University, Seoul 120-750, Korea. Email address: `whang@ewha.ac.kr` I would like to thank the hospitality of the Cowles Foundation during my visits. This work was supported by the Korea Research Foundation via grant number KRF-2001-041-C00198.

# 1 Introduction

There is a considerable interest in uniform weak ordering of investment strategies, welfare outcomes (income distributions, poverty levels), and in program evaluation exercises. Partial strong orders are more commonly used on the basis of specific utility (loss) functions. This is done when one employs indices of inequality or poverty in welfare, mean-variance (return-volatility) analysis in finance, or performance indices in program evaluation.

The most popular uniform order relations are the Stochastic Dominance (SD) relations of various orders, based on the expected utility paradigm and its mathematical regularity conditions. The following brief definitions will be useful:

Let $X_1$ and $X_2$ be two variables (incomes, returns/prospects) at either two different points in time, or for different regions or countries, or with or without a program (treatment). Let $X_{ki}$ , $i = 1, \ldots, N$; $k = 1, \ldots, K$ denote the not necessarily i.i.d. observations. Let $\mathcal{U}_1$ denote the class of all von Neumann-Morgenstern type utility functions, $u$, such that $u' \geq 0$, (increasing). Also, let $\mathcal{U}_2$ denote the class of all utility functions in $\mathcal{U}_1$ for which $u'' \leq 0$ (strict concavity), and $\mathcal{U}_3$ denote a subset of $\mathcal{U}_2$ for which $u''' \geq 0$. Let $X_{(1p)}$ and $X_{(2p)}$ denote the $p$-th quantiles, and $F_1(x)$ and $F_2(x)$ denote the cumulative distribution functions, respectively.

**Definition 1** *$X_1$ First Order Stochastic Dominates $X_2$, denoted $X_1 \succeq_f X_2$, if and only if:*
*(1) $E[u(X_1)] \geq E[u(X_2)]$ for all $u \in \mathcal{U}_1$, with strict inequality for some $u$; Or*
*(2) $F_1(x) \leq F_2(x)$ for all $x \in \mathcal{X}$ , the support of $X_k$, with strict inequality for some $x$; Or*
*(3) $X_{(1p)} \geq X_{(2p)}$ for all $0 \leq p \leq 1$, with strict inequality for some $p$.*

**Definition 2** *$X_1$ Second Order Stochastic Dominates $X_2$, denoted $X_1 \succeq_s X_2$, if and only if one of the following equivalent conditions holds:*

*(1) $E[u(X_1)] \geq E[u(X_2)]$ for all $u \in \mathcal{U}_2$, with strict inequality for some $u$; Or:*

*(2) $\int_{-\infty}^{x} F_1(t)dt \leq \int_{-\infty}^{x} F_2(t)dt$ for all $x \in \mathcal{X}$, with strict inequality for some $x$;Or:*

*(3) $\Phi_1(p) = \int_0^p X_{(1t)}dt \geq \Phi_2(p) = \int_0^p X_{(2t)}dt$ for all $0 \leq p \leq 1$, with strict inequality for some value(s) $p$.*

Weak orders of SD obtain by eliminating the requirement of strict inequality at some point. When these conditions are not met, as when either Lorenz or Generalized Lorenz Curves of two distributions cross, unambiguous First and Second order SD is not possible. Any partial ordering by specific *indices* that correspond to the utility functions in $\mathcal{U}_1$ and $\mathcal{U}_2$ classes, will not enjoy general consensus. Whitmore introduced the concept of third order stochastic dominance (TSD) in finance, see (e.g.) Whitmore and Findley (1978). Shorrocks and Foster (1987) showed that the addition of a "transfer sensitivity" requirement leads to TSD ranking of income distributions. This requirement is stronger than the Pigou-Dalton principle of transfers since it makes regressive transfers less desirable at lower income levels. Higher order SD relations correspond to increasingly smaller subsets of $\mathcal{U}_2$. Davidson and Duclos (2000) offer a very useful characterization of these relations and their tests.

Kahneman and Tversky (1979) introduced the concept of Prospect Stochastic Dominance (PSD); a refinement of the usual dominance criteria that may better reflect individual behaviour. Let $\mathcal{U}_P$ be the set of all $S$-shaped utility functions in $\mathcal{U}_1$ for which $u''(x) \leq 0$ for all $x > 0$ but $u''(x) \geq 0$ for all $x < 0$. These properties represent risk seeking for losses but risk aversion for gains.[1]

**Definition 3** *$X_1$ Prospect Stochastic Dominates $X_2$, denoted $X_1 \succeq_{PSD} X_2$, if and only if one of the following equivalent conditions holds:*

*(1) $E[u(X_1)] \geq E[u(X_2)]$ for all $u \in \mathcal{U}_P$, with strict inequality for some $u$; Or:*

*(2) $\int_y^x F_1(t)dt \leq \int_y^x F_2(t)dt$ for all pairs $(x, y)$ with $x > 0$ and $y < 0$ with strict inequality for some $(x, y)$; Or:*

*(3) $\int_{p_1}^{p_2} X_{(1t)}dt \geq \int_{p_1}^{p_2} X_{(2t)}dt$ for all $0 \leq p_1 \leq F_1(0) \leq F_2(0) \leq p_2 \leq 1$, with strict inequality for some value(s) $p$.*

Econometric tests for the existence of SD orders involve composite hypotheses on inequality restrictions. These restrictions may be equivalently formulated in terms of distribution functions, their quantiles, or other conditional moments. Different test procedures may also differ in terms of their accommodation of the inequality nature (information) of the SD hypotheses. A recent survey is given in Maasoumi (2001).

[1] Kahneman and Tversky (1979) argue that in making decisions under uncertainty, individuals act as if they had applied monotonic transformations to the underlying probabilities before making payoff comparisons. In Tversky and Kahneman (1992) this idea is refined to make the cumulative distribution function of payoffs the subject of the transformation. Thus, individuals would compare the distributions $F_k^* = T(F_k)$, where $T$ is a monotonic decreasing transformation that can be interpreted as a subjective revision of probabilities that varies across investors. One question is whether stochastic dominance [of first, second, or higher order] is preserved under transformation, or rather what is the set of transformations under which an ordering is preserved. As Levy and Wiener (1998) show, any monotonic transformation preserves first order stochastic dominance, while any monotonic and concave transformation preserves second order dominance relations. The PSD property is preserved under the class of monotonic transformations that are concave for gains and convex for losses.

McFadden (1989) proposed a generalization of the Kolmogorov-Smirnov test of First and Second order SD among K prospects (distributions) based on i.i.d. observations *and independent* prospects. Klecan, Mcfadden, and Mcfadden (1991) extended these tests allowing for dependence in observations, and replacing independence with a general exchangeability amongst the competing prospects. Since the asymptotic null distribution of these tests depends on the unknown distributions and parameters, they proposed a Monte Carlo procedure for the computation of critical values. Maasoumi and Heshmati (2000) and Maasoumi et al. (1997) proposed simple bootstrap versions of the same tests which they employed in empirical applications.

Alternative approaches for testing SD are discussed in Anderson (1996), Davidson and Duclos (2000), Kaur et al. (1994), Dardanoni and Forcina (2000), Bishop et al. (1998), and Xu, Fisher, and Wilson (1995), to name but a few recent contributions. The Xu et al. (1995) paper is an example of the use of $\overline{\chi}^2$ distribution for testing the joint inequality amongst the quantiles. The Davidson and Duclos (2000) is the most general account of the tests for any order SD, based on conditional moments of distributions and, as with most of these alternative approaches, requires control of its size by Studentized maximum modulus or similar techniques. Maasoumi (2001) contains an extensive discussion of these alternatives. Tse and Zhang (2000) provide some Monte Carlo evidence on the power of these alternative tests.

We propose a "subsampling" procedure for estimating the critical values of the Klecan, McFadden, and McFadden (1991) test for first, second order, and prospect stochastic dominance in the general k-prospect case. Our method is based on subsampling bootstrap. We prove that the resulting test is consistent. Our sampling scheme is quite general: for the first time in this literature, we allow for general dependence amongst the prospects, and for the observations to be autocorrelated over time. This is especially necessary in substantive empirical settings where income distributions, say, are compared before and after taxes or some other policy decision, or returns on different funds are compared in the same or interconnected markets. We also allow the prospects themselves to be residuals from some estimated model. This latter generality can be important if one first wants to adjust for systematic effects before comparing distributions. We investigate the finite sample performance of our method on simulated data.

## 2 [redacted]st t t st cs

We shall suppose that there are $K$ prospects $X_1, \ldots, X_k$ and let $\mathcal{A} = \{X_k : k = 1, \ldots, K\}$. Let $\{X_{ki} : i = 1, \ldots, N\}$ be realizations of $X_k$ for $k = 1, \ldots, K$. Suppose now that $\{X_{ki} : i = 1, \ldots, N\}$ are unobserved errors in the linear regression model:

$$Y_{ki} = Z_{ki}'\theta_{k0} + X_{ki},$$

for $i = 1, \ldots, N$ and $k = 1, \ldots, K$, where, $Y_{ki} \in \mathbb{R}$, $Z_{ki} \in \mathbb{R}^L$ and $\theta_{k0} \in \Theta_k \subset \mathbb{R}^L$. We shall suppose that $E(X_{ki}|Z_{ki}) = 0$ a.s. as well as other conditions on the random variables $X_k, Y_k$. We allow for serial dependence of the realizations and for mutual correlation across prospects. Let $X_{ki}(\theta) = Y_{ki} - Z_{ki}'\theta$, $X_{ki} = X_{ki}(\theta_{k0})$, and $\widehat{X}_{ki} = X_{ki}(\widehat{\theta}_k)$, where $\widehat{\theta}_k$ is some sensible estimator of $\theta_{k0}$ whose properties we detail below, i.e., the prospects can be estimated from the data. Since we have a linear model, there are many possible ways of obtaining consistent estimates of the unknown parameters. The motivation for considering estimated prospects is that when data is limited one may want to use a model to adjust for systematic differences. Common practice is to group the data into subsets, say of families with different sizes, or by educational attainment, or subgroups of funds by investment goals, and then make comparisons across homogenous populations. When data are limited this can be difficult.[2] In addition, the preliminary regressions may identify "causes" of different outcomes which may be of substantive interest and useful to control for.

For $k = 1, \ldots, K$, define

$$\begin{aligned} F_k(x, \theta) &= P\left(X_{ki}(\theta) \le x\right) \text{ and} \\ F_{kN}(x, \theta) &= \frac{1}{N}\sum_{i=1}^{N} 1\left(X_{ki}(\theta) \le x\right). \end{aligned}$$

We denote $F_k(x) = F_k(x, \theta_{k0})$ and $F_{kN}(x) = F_{kN}(x, \theta_{k0})$, and let $F(x)$ be the joint c.d.f. of $(X_1, \ldots, X_k)'$. Now define the following functionals of the joint distribution

$$d^* = \min_{k \neq l} \sup_{x \in \mathcal{X}} \left[F_k(x) - F_l(x)\right] \tag{1}$$

$$s^* = \min_{k \neq l} \sup_{x \in \mathcal{X}} \int_{-\infty}^{x} \left[F_k(t) - F_l(t)\right] dt \tag{2}$$

$$p^* = \min_{k \neq l} \sup_{x, -y \in \mathcal{X}_+} \int_{y}^{x} \left[F_k(t) - F_l(t)\right] dt, \tag{3}$$

[2] Another way of controlling for systematic differences is to test a hypothesis about the conditional c.d.f.'s of $Y_k$ given $Z_k$. Similar results can be established in this case.

where $\mathcal{X}$ denotes the support of $X_{ki}$ and $\mathcal{X}_+ = \{x \in \mathcal{X},\, x > 0\}$. We assume that $\mathcal{X}$ is a bounded set, as in Klecan et al. (1990). The hypotheses of interest can now be stated as:

$$H_0^d : d^* \leq 0 \ vs.\ H_1^d : d^* > 0 \tag{4}$$

$$H_0^s : s^* \leq 0 \ vs.\ H_1^s : s^* > 0 \tag{5}$$

$$H_0^p : p^* \leq 0 \ vs.\ H_1^p : p^* > 0. \tag{6}$$

The null hypothesis $H_0^d$ implies that the prospects in $\mathcal{A}$ are not first-degree stochastically maximal, i.e., there exists at least one prospect in $\mathcal{A}$ which first-degree dominates the others. Likewise for the second order and prospect stochastic dominance test.

The test statistics we consider are based on the empirical analogues of (1)-(3). They are defined to be:

$$\begin{aligned} D_N &= \min_{k \neq l} \sup_{x \in \mathcal{X}} \sqrt{N} \left[ F_{kN}(x, \widehat{\theta}_k) - F_{lN}(x, \widehat{\theta}_l) \right] \\ S_N &= \min_{k \neq l} \sup_{x \in \mathcal{X}} \sqrt{N} \int_{-\infty}^{x} \left[ F_{kN}(t, \widehat{\theta}_k) - F_{lN}(t, \widehat{\theta}_l) \right] dt \\ P_N &= \min_{k \neq l} \sup_{x, -y \in \mathcal{X}_+} \sqrt{N} \int_{y}^{x} \left[ F_{kN}(t, \widehat{\theta}_k) - F_{lN}(t, \widehat{\theta}_l) \right] dt. \end{aligned}$$

These are precisely the Klecan et al. (1990) test statistics except that we have allowed the prospects to have been estimated from the data. The computation of these statistics is discussed in Klecan et al. (1990) and in our empirical section below.

In computing the test statistics there is an issue of how to approximate the supremum in $D_N, S_N$ and $P_N$, and the integrals in $S_N$ and $P_N$. One approach is to compute the maximum over the data points, as in Andrews (1997). We take this issue up later in the numerical section.

# 3 Asymptotic Null Distributions

## 3.1 Regularity Conditions

We need the following assumptions to analyze the asymptotic behavior of $D_N$:

**Assumption 1:** (i) $\{(X_{ki}\ , Z_{ki}) : i = 1, \ldots, n\}$ is a strictly stationary and $\alpha$- mixing sequence with $\alpha(m) = O(m^{-A})$ for some $A > \max\{(Q-1)(1+Q/2),\, 1+2/\delta\}$ for all $1 \leq k \leq K$, where $Q$ is an even integer that satisfies $Q > 2(L+1)$ and $\delta$ is a positive constant that also appears in

Assumption 2(ii) below. (ii) $E\|Z_{ki}\|^2 < \infty$ for all $1 \le k \le K$, for all $i \ge 1$. (iii) The conditional distribution $H_k(\cdot|Z_{ki})$ of $X_{ki}$ given $Z_{ki}$ has bounded density with respect to Lebesgue measure a.s. for all $1 \le k \le K$, for all $i \ge 1$.

**Assumption 2:** (i) The parameter estimator satisfies $\sqrt{N}(\widehat{\theta}_k - \theta_{k0}) = (1/\sqrt{N})\sum_{i=1}^N \Gamma_{k0}\psi_k(X_{ki}, Z_{ki}, \theta_{k0}) + o_p(1)$, where $\Gamma_{k0}$ is a non-stochastic matrix for all $1 \le k \le K$; (ii) The function $\psi_k(y, z, \theta) : \mathbb{R} \times \mathbb{R}^L \times \Theta \to \mathbb{R}^L$ is measurable and satisfies (a) $E\psi_k(Y_{ki}, Z_{ki}, \theta_{k0}) = 0$ and (b) $E\|\psi_k(Y_{ki}, Z_{ki}, \theta_{k0})\|^{2+\delta} < \infty$ for some $\delta > 0$ for all $1 \le k \le K$, for all $i \ge 1$.

**Assumption 3:** (i) The function $F_k(x, \theta)$ is differentiable in $\theta$ on a neighborhood $\Theta_0$ of $\theta_0$ for all $1 \le k \le K$; (ii) For all sequence of positive constants $\{\xi_N : N \ge 1\}$ such that $\xi_N \to 0$, $\sup_{x\in\mathcal{X}}\sup_{\theta:\|\theta-\theta_0\|\le\xi_N} \|\partial F_k(x,\theta)/\partial\theta - \Delta_{k0}(x)\| \to 0$ for all $1 \le k \le K$, where $\Delta_{k0}(x) = \partial F_k(x, \theta_{k0})/\partial\theta$; (iii) $\sup_{x\in\mathcal{X}} \|\Delta_{k0}(x)\| < \infty$ for all $1 \le k \le K$.

For the tests $S_N$ and $P_N$ we need the following modifications of Assumption 3 respectively:

**Assumption 3**$^*$ : (i) Assumption 3(i) holds; (ii) For all $1 \le k \le K$ for all sequence of positive constants $\{\xi_N : N \ge 1\}$ such that $\xi_N \to 0$, $\sup_{x\in\mathcal{X}}\sup_{\theta:\|\theta-\theta_0\|\le\xi_N} \left\|(\partial/\partial\theta)\int_{-\infty}^x F_k(t,\theta)dt - \Lambda_{k0}(x)\right\| \to 0$, where $\Lambda_{k0}(x) = (\partial/\partial\theta)\int F_k(y, \theta_{k0})dy$; (iii) $\sup_{x\in\mathcal{X}}\|\Lambda_{k0}(x)\| < \infty$ for all $1 \le k \le K$.

**Assumption 3**$^{**}$ : (i) Assumption 3(i) holds; (ii) For all $1 \le k \le K$ for all sequence of positive constants $\{\xi_N : N \ge 1\}$ such that $\xi_N \to 0$, $\sup_{x,-y\in\mathcal{X}_+}\sup_{\theta:\|\theta-\theta_0\|\le\xi_N} \left\|(\partial/\partial\theta)\int_y^x F_k(t,\theta)dt - \Xi_{k0}(x,y)\right\| \to 0$, where $\Xi_{k0}(x,y) = (\partial/\partial\theta)\int_y^x F_k(t,\theta_{k0})dt$; (iii) $\sup_{x,-y\in\mathcal{X}_+}\|\Xi_{k0}(x,y)\| < \infty$ for all $1 \le k \le K$.

Assumptions 3 and 3* (or 3$^{**}$) differ in the amount of smoothness required. For second order (or prospect) stochastic dominance, less smoothness is required.

## 3.2 The Null Distributions

Define the empirical processes in $x, \theta$

$$
\begin{aligned}
\nu_{kN}^d(x,\theta) &= \frac{1}{\sqrt{N}}\sum_{i=1}^N \left[1\left(X_{ki}(\theta) \le x\right) - F_k(x,\theta)\right] \\
\nu_{kN}^s(x,\theta) &= \frac{1}{\sqrt{N}}\sum_{i=1}^N \left[\int_{-\infty}^x 1\left(X_{ki}(\theta) \le t\right)dt - \int_{-\infty}^x F_k(t,\theta)dt\right] \\
\nu_{kN}^p(x,y,\theta) &= \frac{1}{\sqrt{N}}\sum_{i=1}^N \left[\int_y^x 1\left(X_{ki}(\theta) \le t\right)dt - \int_y^x F_k(t,\theta)dt\right] \qquad (7)
\end{aligned}
$$

Let $( \ \widetilde{d}_{kl}(\cdot) \quad \nu_{k0}' \quad \nu_{l0}' \ )'$ be a mean zero Gaussian process with covariance functions given by

$$C^d(x_1, x_2) = \lim_{N\to\infty} E \begin{pmatrix} v_{kN}^d(x_1, \theta_{k0}) - v_{lN}^d(x_1, \theta_{l0}) \\ \sqrt{N}\psi_{kN}(\theta_{k0}) \\ \sqrt{N}\psi_{lN}(\theta_{l0}) \end{pmatrix} \begin{pmatrix} v_{kN}^d(x_2, \theta_{k0}) - v_{lN}^d(x_2, \theta_{l0}) \\ \sqrt{N}\psi_{kN}(\theta_{k0}) \\ \sqrt{N}\psi_{lN}(\theta_{l0}) \end{pmatrix}'. \tag{8}$$

We analogously define $( \ \widetilde{s}_{kl}(\cdot) \quad \nu_{k0}' \quad \nu_{l0}' \ )'$ and $( \ \widetilde{p}_{kl}(\cdot,\cdot) \quad \nu_{k0}' \quad \nu_{l0}' \ )'$ to be mean zero Gaussian processes with covariance functions given by $C^s(x_1, x_2)$ and $C^p(x_1, y_1, x_2, y_2)$ respectively.

The limiting null distributions of our test statistics are given in the following theorem.

**Theorem 1.** (a) *Suppose Assumptions 1-3 hold. Then, under the null $H_0^d$, we have*

$$D_N \Rightarrow \begin{cases} \min_{k\neq l} \sup_{x\in\mathcal{B}_{kl}^d} \left[\widetilde{d}_{kl}(x) + \Delta_{k0}(x)'\Gamma_{k0}\nu_{k0} - \Delta_{l0}(x)'\Gamma_{l0}\nu_{l0}\right] & \text{if } d = 0 \\ -\infty & \text{if } d < 0, \end{cases}$$

*where* $\mathcal{B}_{kl}^d = \{x \in \mathcal{X} : F_k(x) = F_l(x)\}$.

*(b) Suppose Assumptions 1(i), 1(ii), 2 and 3* hold. Then, under the null $H_0^s$, we have*

$$S_N \Rightarrow \begin{cases} \min_{k\neq l} \sup_{x\in\mathcal{B}_{kl}^s} \left[\widetilde{s}_{kl}(x) + \Lambda_{k0}(x)'\Gamma_{k0}\nu_{k0} - \Lambda_{l0}(x)'\Gamma_{l0}\nu_{l0}\right] & \text{if } s = 0 \\ -\infty & \text{if } s < 0, \end{cases}$$

*where* $\mathcal{B}_{kl}^s = \{x \in \mathcal{X} : \int_{-\infty}^x F_k(t)dt = \int_{-\infty}^x F_l(t)dt\}$.

*(c) Suppose Assumptions 1(i), 1(ii), 2 and 3** hold. Then, under the null $H_0^p$, we have*

$$P_N \Rightarrow \begin{cases} \min_{k\neq l} \sup_{(x,y)\in\mathcal{B}_{kl}^p} \left[\widetilde{p}_{kl}(x,y) + \Xi_{k0}(x)'\Gamma_{k0}\nu_{k0} - \Xi_{l0}(x)'\Gamma_{l0}\nu_{l0}\right] & \text{if } p = 0 \\ -\infty & \text{if } p < 0, \end{cases}$$

*where* $\mathcal{B}_{kl}^p = \{(x, y) : x \in \mathcal{X}_+, -y \in \mathcal{X}_+ \text{ and } \int_y^x F_k(t)dt = \int_y^x F_l(t)dt\}$.

The asymptotic null distributions of $D_N$, $S_N$ and $P_N$ depend on the "true" parameters $\{\theta_{k0} : k = 1, \ldots, K\}$ and distribution functions $\{F_k(\cdot) : k = 1, \ldots, K\}$. This implies that the asymptotic critical values for $D_N$, $S_N$, $P_N$ can not be tabulated once and for all. However, a subsampling procedure can be used to approximate the null distributions.

# 4 Critical Values based on Subsampling and Bootstrap

In this section, we consider the use of subsampling to approximate the null distributions of our test statistics. In contrast to the simulation approach of Klecan et. al. (1990), our procedure does not require the assumption of generalized exchangeability of the underlying random variables. Indeed, we require no additional assumptions beyond those that have already been made.

We now discuss the asymptotic validity of the subsampling procedure for the test $D_N$ (The argument for the tests $S_N$ and $P_N$ is similar and hence is omitted). Let $W_i = \{(Y_{ki}, Z_{ki}) : k = 1, \ldots, K\}$ for $i = 1, \ldots, N$. With some abuse of notation, the test statistic $D_N$ can be re-written as a function of the data $\{W_i : i = 1, \ldots, N\}$ :

$$D_N = \sqrt{N} d_N(W_1, \ldots, W_N),$$

where

$$d_N(W_1, \ldots, W_N) = \min_{k \neq l} \sup_{x \in \mathcal{X}} \left[ F_{kN}(x, \widehat{\theta}_k) - F_{lN}(x, \widehat{\theta}_l) \right]. \tag{9}$$

Let

$$G_N(w) = \Pr\left( \sqrt{N} d_N(W_1, \ldots, W_N) \leq w \right) \tag{10}$$

denote the distribution function of $D_N$. Let $d_{N,b,i}$ be equal to the statistic $d_b$ evaluated at the subsample $\{W_i, \ldots, W_{i+b-1}\}$ of size $b$, i.e.,

$$d_{N,b,i} = d_b(W_i, W_{i+1}, \ldots, W_{i+b-1}) \text{ for } i = 1, \ldots, N - b + 1.$$

This means that we have to recompute $\widehat{\theta}_l(W_i, W_{i+1}, \ldots, W_{i+b-1})$ using just the subsample as well. We note that each subsample of size $b$ (taken without replacement from the original data) is indeed a sample of size $b$ from the true sampling distribution of the original data. Hence, it is clear that one can approximate the sampling distribution of $D_N$ using the distribution of the values of $d_{N,b,i}$ computed over $N - b + 1$ different subsamples of size $b$. That is, we approximate the sampling distribution $G_N$ of $D_N$ by

$$\widehat{G}_{N,b}(w) = \frac{1}{N - b + 1} \sum_{i=1}^{N-b+1} 1\left( \sqrt{b} d_{N,b,i} \leq w \right).$$

Let $g_{N,b}(1 - \alpha)$ denote the $(1 - \alpha)$-th sample quantile of $\widehat{G}_{N,b}(\cdot)$, i.e.,

$$g_{N,b}(1 - \alpha) = \inf\{w : \widehat{G}_{N,b}(w) \geq 1 - \alpha\}.$$

We call it the *subsample critical value* of significance level $\alpha$. Thus, we reject the null hypothesis at the significance level $\alpha$ if $D_N > g_{N,b}(1-\alpha)$. The computation of this critical value is not particularly onerous, although it depends on how big $b$ is. The subsampling bootstrap has been proposed in Politis and Romano (1994) and is thoroughly reviewed in Politis, Romano, and Wolf (1999). It is well known to be a universal method that can 'solve' any problem. In particular, it works in heavy tailed distributions, in unit root cases, in non-standard asymptotics, etc.

We now justify the above subsampling procedure. Let $g(1-\alpha)$ denote the $(1-\alpha)$-th quantile of the asymptotic null distribution of $D_N$ (given in Theorem 1(a)).

**Theorem 2**. *Suppose Assumptions 1-3 hold. Assume $b/N \to 0$ and $b \to 0$ as $N \to \infty$. Then, under the null hypothesis $H_0^d$, we have when $d = 0$ that*

$$(a)\ g_{N,b}(1-\alpha) \xrightarrow{p} g(1-\alpha)$$

$$(b)\ \Pr[D_N > g_{N,b}(1-\alpha)] \to \alpha$$

*as $N \to \infty$.*

Since $d = 0$ is the least favorable case, we have that

$$\sup_{d \in H_0^d} \Pr[D_N > g_{N,b}(1-\alpha)] \leq \alpha + o(1).$$

The following theorem establishes the consistency of our test:

**Theorem 3**. *Suppose Assumptions 1-3 hold. Assume $b/N \to 0$ and $b \to 0$ as $N \to \infty$. Then, under the alternative hypothesis $H_1^d$, we have*

$$\Pr\left[D_N > g_{N,b}(1-\alpha)\right] \to 1 \text{ as } N \to \infty.$$

Remark. Results analogous to Theorems 2 and 3 hold for the test $S_N$ ($P_N$) under Assumptions 1(i)-(ii), 2 and $3^*$($3^{**}$). The proof is similar to those of the latter theorems.

In practice, the choice of $b$ is important and rather difficult. Delgado, Rodriguez-Poo, and Wolf (2001) propose a method for selecting $b$ in the context of hypothesis testing within the maximum score estimator, although no optimality properties of this method were proven.

## 5 Numerical Results

Tse and Zhang (2000) have provided some Monte Carlo evidence on the power of the alternative tests proposed by Davidson and Duclos (2000), the "DD test", and Anderson (1996). They also shed light on the convergence to the Gaussian limiting distribution of these tests. The evidence on the latter issue is not very encouraging except for very large sample sizes, and they conclude that the DD test has better power than the Anderson test for the cases they considered. In this section we report some numerical results on the performance of the test statistics and the subsample critical values.

In the income distribution field, an often empirically plausible candidate is the Burr Type XII distribution, $B(\alpha, \beta)$. This is a two parameter family defined by:

$$F(x) = 1 - (1 + x^{\alpha})^{-\beta}, \quad x \geq 0$$

where $\beta > 1/\alpha > 0$. This distribution has a convenient inverse:

$$X = F^{-1}(v) = [(1 - v)^{-\frac{1}{\beta}} - 1]^{\frac{1}{\alpha}}, \quad 0 \leq v < 1$$

We can generate uniformly distributed random numbers, $v$, in order to generate variables $X$ with the Burr distribution for different values of the two parameters. Manipulation of the two parameters will produce unrankable prospects, as well as dominant ones. We also considered normal distributions.

We investigated five different $(k = 2)$ designs:

D1. $X_j \sim B(4.7, 0.55)$ [$d = s = 0$ least favorable case]

D2. $X_1 \sim B(4.7, 0.55)$ and $X_2 \sim B(0.55, 4.7)$ [$d = -0.168, s = 0$ ]

D3. $X_1 \sim B(1, 1)$ and $X_2 \sim B(0.5, 1)$ [$d = 0.150, s = 0.077$]

D4. $X_1 \sim N(1, 16)$ and $X_2(0, 1)$ with mutual correlation of 0.5 [$d = 0.207, s = 0.341$]

D5. $X_1 \sim N(0, 4)$ and $X_2 \sim N(0, 1)$ with mutual correlation of 0.5 [$d = 0.162, s = 0$]

Design D1 was used by Tse and Zhang (2000), while D4 was used in McFadden et al. (1991). In each case we did $1,000$ replications.

We chose a total of nine different subsampling rules

$$b(n) = c_j n^{a_j}$$

with $c_j \in \{2, 2.5, 3\}$ and $a \in \{0.33, 0.5, 0.75\}$, where $n \in \{250, 500, 1000\}$.

Regarding the computation of $\sup_{x \in \mathcal{X}}[F_{1N}(x) - F_{2N}(x)]$ and $\sup_{x \in \mathcal{X}} \int_{-\infty}^{x}[F_{1N}(y) - F_{2N}(y)]dy$, we considered several approaches. One method is to compute

$$\max_{1 \leq j \leq q(n)} [F_{1N}(X_j^*) - F_{2N}(X_j^*)],$$

where $X_j^*$, $j = 1, \ldots, q(n)$ were bootstrap draws from the pooled marginal and $q = q(n) \leq n$. A second approach is just to use an equally spaced grid on the range of the pooled distribution. In practice, both these methods work satisfactorily, but we have found better performance when we also 'trim' out extreme observations. This is especially likely in the Burr case. In the experiments we report here we chose an equally spaced grid of points on the 5% symmetrically trimmed pooled sample. We chose $q = n$.

The results are shown in Tables 1-5, Table $j$ corresponding to design $j$. In Design 1, size distortion is larger for FOSD than SOSD, but improves with sample size. In D2, the rejection frequencies for FOSD are essentially zero, reflecting the negativity of $d$ here. The rejection frequency for SOSD is close to one for small subsamples, but the larger subsamples show much smaller rejection frequency; presumably, somewhere in between there is a subsample that has the right rejection frequency. In D3, the power is better for FOSD than for SOSD, but in both cases power increases with sample size. In the mutually dependent designs D4-D4 the results are very similar. It is clear that when prospects are "second order maximal" one needs very large samples (subsamples) to detect it. This makes perfect sense since, especially for weaker levels of SSD, as in D2-D3, the "crossings " occur very late in the right tails.

Figures 1-3 show the effect of subsample size on rejection frequency in Design 1 for the case that $\alpha = 0.05$: the SOSD test always has a subsample size that make rejection frequency exactly 0.05. The FOSD test is persistently oversized, but this improves with sample size.

Finally, we applied our tests to a dataset of daily returns on the Dow Jones Industrials and the S&P500 stock returns from 8/24/88 to 8/22/00, a total of 3131 observations. Figures 4 and 5 plot the c.d.f.'s and s.d.f.'s of the two series. Not surprisingly, our test rejects the null hypothesis that $d \leq 0$ at the 5% level for any of the nine subsample sizes in our grid, while we accept the null hypothesis that $s \leq 0$ at the 5% level for all but the very largest subsample sizes. Figure 6 compares the s.d.f.'s of the globally standardized return series [i.e., $(r_t - \overline{r})/s_r$, where $\overline{r}$ is the sample mean and $s_r^2$ is the sample variance]; now as expected we can't reject the null of no second order domination at the 5% level, but there does appear to be some second order domination in the right tail, which suggests that mean and variance alone are not sufficient statistics for the investor.

## 6 Concluding Remarks

The subsample bootstrap appears to be an effective way of computing critical values in this test of stochastic dominance, delivering good performance for sample sizes as low as 250. Much work remains to be done. First, we intend to evaluate our test on more demanding situations of testing for prospect dominance with time series data and many prospects. This certainly is a challenging situation.

Second, there are a number of modifications to the test statistic and bootstrap algorithm that may be beneficial. For example, we can consider the weighted criterion $\min_{k \neq l} \sup_{x \in \mathcal{X}} w_{kl} \cdot [F_k(x) - F_l(x)]$ for any weighting sequence $w_{kl}$ that is positive with probability one. Such weighting might be important when there are many prospects and considerable variation in the values of $F_k(x) - F_l(x)$. Also, some methodology for choosing $b$ is desirable, although difficult.

## A A [illegible]

We let $C_j$ for some integer $j \geq 1$

**Pr[illegible] L[illegible] 1.** The result follows from Theorem 2.2 of Andrews and Pollard (1994) if we verify the mixing and bracketing conditions in the theorem. We first verify the conditions for part (a) of Lemma 1. The mixing condition is implied by Assumption 1(i). The bracketing condition also holds by the following argument: Let

$$\mathcal{F}_d = \{1\left(X_{ki}(\theta) \leq x\right) : (x,\theta) \in \mathcal{X} \times \Theta\}. \tag{A.7}$$

Then, $\mathcal{F}_d$ is a class of uniformly bounded functions satisfying the $L^2$-continuity condition, because we have

$$\begin{aligned}
&\sup_{i\geq 1} E \sup_{\substack{(x',\theta')\in\mathcal{X}\times\Theta: \\ |x'-x|\leq r_1, \|\theta'-\theta\|\leq r_2, \sqrt{r_1^2+r_2^2}\leq r}} \left|1\left(X_{ki}(\theta') \leq x'\right) - 1\left(X_{ki}(\theta)\leq x\right)\right|^2 \\
= \;& E \sup_{\substack{(x',\theta')\in\mathcal{X}\times\Theta: \\ |x'-x|\leq r_1, \|\theta'-\theta\|\leq r_2, \sqrt{r_1^2+r_2^2}\leq r}} \left|1\left(X_{ki} \leq Z_{ki}'(\theta'-\theta_0)+x'\right) - 1\left(X_{ki} \leq Z_{ki}'(\theta-\theta_0)+x\right)\right|^2 \\
\leq \;& E1\left(\left|X_{ki} - Z_{ki}'(\theta-\theta_0) - x\right| \leq \|Z_{ki}\| r_1 + r_2\right) \\
\leq \;& C_1\left(E\|Z_{ki}\| r_1 + r_2\right) \\
\leq \;& C_2 r,
\end{aligned}$$

where the second inequality holds by Assumption 1(iii) and $C_2 = \sqrt{2}C_1\left(E\|Z_{ki}\| \vee 1\right)$ is finite by Assumption 1(ii). Now the desired bracketing condition holds because the $L^2$-continuity condition implies that the bracketing number satisfies

$$N(\varepsilon, \mathcal{F}_d) \leq C_3\left(\frac{1}{\varepsilon}\right)^{L+1}, \tag{A.8}$$

see Andrews and Pollard (1994, p.121).

We next verify part (b). Since the mixing condition holds by Assumption 1(i), we need to establish the bracketing condition. Let

$$\mathcal{F}_s = \left\{\int_{-\infty}^{x} 1\left(X_{ki}(\theta) \leq t\right) dt : (x,\theta) \in \mathcal{X}\times\Theta\right\}. \tag{A.9}$$

where the third line follows from the inequality $|\max\{a,0\}-\max\{b,0\}| \leq |a-b|$ and Cauchy-Schwarz inequality. Therefore, since $\sup_{k,i} E\left\|Z_{ki}\right\|^2 < \infty$, the bracketing condition (A.8) holds for the class $\mathcal{F}_s$ as desired.

The proof of part (c) is similar to that of part (b) except that we now take

$$\mathcal{F}_p = \left\{ \int_y^x 1\left(X_{ki}(\theta) \leq t\right) dt : (x,-y,\theta) \in \mathcal{X}_+ \times \mathcal{X}_+ \times \Theta \right\} \tag{A.11}$$

and verify the Lipschitz condition using (A.10) and triangle inequality. ■

**Lemma 2** *(a) Suppose Assumptions 1-3 hold. Then, we have* $\forall k = 1,\ldots,K$,

$$\sup_{x\in\mathcal{X}} \left|\nu_{kN}^d(x,\widehat{\theta}_k) - \nu_{kN}^d(x,\theta_{k0})\right| \xrightarrow{p} 0. \tag{A.12}$$

*(b) Suppose Assumptions 1(i)(ii), 2 and 3* hold. Then, we have* $\forall k = 1,\ldots,K$,

$$\sup_{x\in\mathcal{X}} \left|\nu_{kN}^s(x,\widehat{\theta}_k) - \nu_{kN}^s(x,\theta_{k0})\right| \xrightarrow{p} 0. \tag{A.13}$$

*(c) Suppose Assumptions 1(i)(ii), 2 and 3** hold. Then, we have* $\forall k = 1,\ldots,K$,

$$\sup_{x,-y\in\mathcal{X}_+} \left|\nu_{kN}^p(x,y,\widehat{\theta}_k) - \nu_{kN}^p(x,y,\theta_{k0})\right| \xrightarrow{p} 0. \tag{A.14}$$

**Pr[illegible] Lemma 2.** We first verify part (a). Consider the pseudometric (A.2). We have

$$\begin{aligned}
& \sup_{x\in\mathcal{X}} \rho_d^*\left(\left(x,\widehat{\theta}_k\right),(x,\theta_{k0})\right)^2 \\
= \ & \sup_{x\in\mathcal{X}} E\left[1(X_{ki}(\theta)\leq x) - 1(X_{ki}(\theta_{k0})\leq x)\right]^2\Big|_{\theta=\widehat{\theta}_k} \\
= \ & \sup_{x\in\mathcal{X}} \iint \left[1\left(\widetilde{x} \leq x + z'(\widehat{\theta}_k-\theta_{k0})\right) - 1\left(\widetilde{x}\leq x\right)\right]^2 dH_k(\widetilde{x}|z)dP_k(z) \qquad \text{(A.15)} \\
\leq \ & \sup_{x\in\mathcal{X}} \iint 1\left(x - \left\|z'(\widehat{\theta}_k-\theta_{k0})\right\| \leq \widetilde{x} \leq x + \left\|z'(\widehat{\theta}_k-\theta_{k0})\right\|\right) dH_k(\widetilde{x}|z)dP_k(z) \\
\leq \ & C_1 \int \left\|z'(\widehat{\theta}_k-\theta_{k0})\right\| dP_k(z) \\
\leq \ & C_1 \left\|\widehat{\theta}_k-\theta_{k0}\right\| E\left\|Z_{ki}\right\| \xrightarrow{p} 0,
\end{aligned}$$

where $P_k(\cdot)$ denotes the distribution function of $Z_{ki}$ and the inequality in the 5th line holds by Assumption 1(iii) and a one-term Taylor expansion, and the last convergence to zero holds by Assumptions 1(ii) and 2. Now, (A.12) holds since we have: $\forall\varepsilon>0$, $\forall\eta>0$, $\exists\delta>0$ such that

$$\overline{\lim_{N\to\infty}} P\left(\sup_{x\in\mathcal{X}} \left|\nu_{kN}^d(x,\widehat{\theta}_k) - \nu_{kN}^d(x,\theta_{k0})\right| > \eta\right)$$

$$
\begin{aligned}
&\leq \overline{\lim_{N\to\infty}} P\left(\sup_{x\in\mathcal{X}}\left|\nu^d_{kN}(x,\widehat{\theta}_k)-\nu_{kN}(x,\theta_{k0})\right|>\eta,\ \sup_{x\in\mathcal{X}}\rho^*_d\left(\left(x,\widehat{\theta}_k\right),(x,\theta_{k0})\right)<\delta\right) \\
&\quad +\overline{\lim_{N\to\infty}} P\left(\sup_{x\in\mathcal{X}}\rho^*_d\left(\left(x,\widehat{\theta}_k\right),(x,\theta_{k0})\right)\geq\delta\right) \qquad\qquad (A.16)\\
&\leq \overline{\lim_{N\to\infty}} P^*\left(\sup_{\rho^*_d((x_1,\theta_1),(x_2,\theta_2))<\delta}\left|\nu^d_{kN}(x_1,\theta_1)-\nu^d_{kN}(x_2,\theta_2)\right|>\eta\right)\\
&< \frac{\varepsilon}{\eta},
\end{aligned}
$$

where the last term on the right hand side of the first inequality is zero by (A.15) and the last inequality holds by the stochastic equicontinuity result (A.1) Since $\varepsilon/\eta > 0$ is arbitrary, (A.12) follows.

We next establish part (b). We have

$$
\begin{aligned}
&\sup_{x\in\mathcal{X}}\rho^*_s\left(\left(x,\widehat{\theta}_k\right),(x,\theta_{k0})\right)^2\\
&= \sup_{x\in\mathcal{X}} E\left[\int_{-\infty}^{x}\left(1(X_{ki}(\theta)\leq t)-1(X_{ki}(\theta_{k0})\leq t)\right)dt\right]^2\Bigg|_{\theta=\widehat{\theta}_k}\\
&\leq \left\|\widehat{\theta}_k-\theta_{k0}\right\|^2 E\left\|Z_{ki}\right\|^2 \xrightarrow{p} 0
\end{aligned}
$$

by Assumptions 1(ii) and 2. Now part (b) holds using an argument similar to the one used to verify part (a). The proof of part (c) is similar. ■

**L[illegible] 3** *(a) Suppose Assumptions 1-3 hold. Then, we have* $\forall k=1,\ldots,K$,

$$
\sqrt{N}\sup_{x\in\mathcal{X}}\left\|F_k(x,\widehat{\theta}_k)-F_k(x,\theta_{k0})-\Delta'_{k0}(x)\Gamma_{k0}\overline{\psi}_{kN}(\theta_{k0})\right\|=o_p(1).
$$

*(b) Suppose Assumptions 1(i)-(ii), 2 and 3* hold. Then, we have* $\forall k=1,\ldots,K$,

$$
\sqrt{N}\sup_{x\in\mathcal{X}}\left\|\int_{-\infty}^{x}F_k(t,\widehat{\theta}_k)dt-\int_{-\infty}^{x}F_k(t,\theta_{k0})dt-\Lambda'_{k0}(x)\Gamma_{k0}\overline{\psi}_{kN}(\theta_{k0})\right\|=o_p(1).
$$

*(c) Suppose Assumptions 1(i)-(ii), 2 and 3** hold. Then, we have* $\forall k=1,\ldots,K$,

$$
\sqrt{N}\sup_{x,-y\in\mathcal{X}_+}\left\|\int_{y}^{x}F_k(t,\widehat{\theta}_k)dt-\int_{y}^{x}F_k(t,\theta_{k0})dt-\Xi'_{k0}(x,y)\Gamma_{k0}\overline{\psi}_{kN}(\theta_{k0})\right\|=o_p(1).
$$

**Pr[illegible] L[illegible] 3.** We verify part (a). Proof of parts (b) and (c) is similar. A mean value expansion gives

$$
F_k(x,\widehat{\theta}_k)=F_k(x,\theta_{k0})+\frac{\partial F_k(x,\theta^*_k(x))}{\partial\theta'}(\widehat{\theta}_k-\theta_{k0}),
$$

where $\theta_k^*(x)$ lies between $\widehat{\theta}_k$ and $\theta_{k0}$. By Assumption 2, we have $\sqrt{N}(\widehat{\theta}_k - \theta_{k0}) = O_p(1)$. This implies that there exists a sequence of constants $\{\xi_N : N \geq 1\}$ such that $\xi_N \to 0$ and $P\left(\left\|\widehat{\theta}_k - \theta_{k0}\right\| \leq \xi_N\right) \to 1$. The latter implies that $P\left(\sup_{x\in\mathcal{X}} \|\theta_k^*(x) - \theta_{k0}\| \leq \xi_N\right) \to 1$. Let

$$\begin{aligned} A_N &= \sup_{x\in\mathcal{X}} \left\|\frac{\partial F_k(x, \theta_k^*(x))}{\partial \theta} - \Delta_{k0}(x)\right\| \text{ and} \\ B_N &= \sup_{x\in\mathcal{X}} \sup_{\theta:\|\theta-\theta_{k0}\|\leq\xi_N} \left\|\frac{\partial F_k(x, \theta)}{\partial \theta} - \Delta_{k0}(x)\right\|. \end{aligned}$$

Then, we have $A_N = o_p(1)$ since $P(A_N \leq B_N) \to 1$ by construction and $B_N = o(1)$ by Assumption 3(ii). Now we have the desired result:

$$\begin{aligned} & \sqrt{N} \sup_{x\in\mathcal{X}} \left\|F_k(x, \widehat{\theta}_k) - F_k(x, \theta_{k0}) - \Delta_{k0}'(x)\Gamma_{k0}\overline{\psi}_{kN}(\theta_{k0})\right\| \\ = \;& \sqrt{N} \sup_{x\in\mathcal{X}} \left\|\frac{\partial F_k(x, \theta_k^*(x))}{\partial \theta'}(\widehat{\theta}_k - \theta_{k0}) - \Delta_{k0}'(x)\Gamma_{k0}\overline{\psi}_{kN}(\theta_{k0})\right\| \\ \leq \;& A_N\sqrt{N}\left\|\widehat{\theta}_k - \theta_{k0}\right\| + \sup_{x\in\mathcal{X}} \|\Delta_{k0}(x)\| \left\|\sqrt{N}(\widehat{\theta}_k - \theta_{k0}) - \Gamma_{k0}\sqrt{N}\overline{\psi}_{kN}(\theta_{k0})\right\| \\ = \;& o_p(1), \end{aligned}$$

where the inequality holds by the triangle inequality and the last equality holds by Assumptions 2 and 3(iii). ■

**Lemma 4** *(a) Suppose Assumptions 1-3 hold. Then, we have*

$$\begin{pmatrix} v_{kN}^d(\cdot, \theta_{k0}) - v_{lN}^d(\cdot, \theta_{l0}) \\ \sqrt{N}\overline{\psi}_{kN}(\theta_{k0}) \\ \sqrt{N}\overline{\psi}_{lN}(\theta_{l0}) \end{pmatrix} \Rightarrow \begin{pmatrix} \widetilde{d}_{kl}(\cdot) \\ \nu_{k0} \\ \nu_{l0} \end{pmatrix}$$

*$\forall\ k, l = 1, \ldots, K$ and the sample paths of $\widetilde{d}_{kl}(\cdot)$ are uniformly continuous with respect to pseudometric $\rho_d$ on $\mathcal{X}$ with probability one, where*

$$\rho_d(x_1, x_2) = \left\{E\left[(1(X_{ki} \leq x_1) - 1(X_{li} \leq x_1)) - (1(X_{ki} \leq x_2) - 1(X_{li} \leq x_2))\right]^2\right\}^{1/2}.$$

*(b) Suppose Assumptions 1(i)–(ii), 2 and 3* hold. Then, we have*

$$\begin{pmatrix} v_{kN}^s(\cdot, \theta_{k0}) - v_{lN}^s(\cdot, \theta_{l0}) \\ \sqrt{N}\overline{\psi}_{kN}(\theta_{k0}) \\ \sqrt{N}\overline{\psi}_{lN}(\theta_{l0}) \end{pmatrix} \Rightarrow \begin{pmatrix} \widetilde{s}_{kl}(\cdot) \\ \nu_{k0} \\ \nu_{l0} \end{pmatrix}$$

$\forall\ k,l=1,\ldots,K$ *and the sample paths of* $\widetilde{s}_{kl}(\cdot)$ *are uniformly continuous with respect to pseudometric* $\rho_s$ *on* $\mathcal{X}$ *with probability one, where*

$$\rho_s\left(x_1,x_2\right)=\left\{E\left[\int_{-\infty}^{x_1}\left(1(X_{ki}\leq t)-1(X_{li}\leq t)\right)dt-\int_{-\infty}^{x_2}\left(1(X_{ki}\leq t)-1(X_{li}\leq t)\right)dt\right]^2\right\}^{1/2}.$$

*(c) Suppose Assumptions 1(i)–(ii), 2 and 3** hold. Then, we have*

$$\begin{pmatrix} v_{kN}^p(\cdot,\cdot,\theta_{k0})-v_{lN}^p(\cdot,\cdot,\theta_{l0}) \\ \sqrt{N}\psi_{kN}(\theta_{k0}) \\ \sqrt{N}\psi_{lN}(\theta_{l0}) \end{pmatrix} \Rightarrow \begin{pmatrix} \widetilde{p}_{kl}(\cdot,\cdot) \\ \nu_{k0} \\ \nu_{l0} \end{pmatrix}$$

$\forall\ k,l=1,\ldots,K$ *and the sample paths of* $\widetilde{p}_{kl}(\cdot,\cdot)$ *are uniformly continuous with respect to pseudometric* $\rho_p$ *on* $\mathcal{X}_+\times\ \mathcal{X}_-$ *with probability one, where*

$$\rho_s\left((x_1,y_1),(x_2,y_2)\right)=\left\{E\left[\int_{y_1}^{x_1}\left(1(X_{ki}\leq t)-1(X_{li}\leq t)\right)dt-\int_{y_2}^{x_2}\left(1(X_{ki}\leq t)-1(X_{li}\leq t)\right)dt\right]^2\right\}^{1/2}.$$

**Pr L[redacted] 4.** Consider part (a) first. By Theorem 10.2 of Pollard (1990), the result of Lemma 4 holds if we have (i) total boundedness of pseudometric space $(\mathcal{X},\rho_d)$ (ii) stochastic equicontinuity of $\{v_{kN}^d(\cdot,\theta_{k0})-v_{lN}^d(\cdot,\theta_{l0}):N\geq 1\}$ and (iii) finite dimensional (fidi) convergence. Conditions (i) and (ii) follow from Lemma 1. We now verify condition (iii). We need to show that $v_{kN}^d(x_1,\theta_{k0})-v_{lN}^d(x_1,\theta_{l0}),\ldots,v_{kN}^d(x_J,\theta_{k0})-v_{lN}^d(x_J,\theta_{l0}),\sqrt{N}\psi_{kN}(\theta_{k0})',\ \sqrt{N}\psi_{lN}(\theta_{l0})'$ converges in distribution to $\left(\widetilde{d}_{kl}(x_1),\ldots,\widetilde{d}_{kl}(x_J),\nu_{k0}',\ \nu_{l0}'\right)'$ $\forall x_j\in\mathcal{X}$, $\forall j\leq J,\forall J\geq 1$. This result holds by the Cramer-Wold device and a CLT for bounded random variables (e.g., Hall and Heyde (1980, Corollary 5.1, p.132)) because the underlying random sequence$\{X_{ki}:i=1,\ldots,n\}$ is strictly stationary and $\alpha$- mixing with the mixing coefficients satisfying $\sum_{m=1}^{\infty}\alpha(m)\ <\infty$ by Assumption 1 and we have $|1(X_{ki}\leq x)-1(X_{li}\leq x)|\leq 2<\infty$. This establishes part (a).

Next, for part (b), we need to verify the fidi convergence (ii) again. Note that the moment condition of Hall and Heyde (1980, Corollary 5.1) holds since we have

$$E\left|\int_{-\infty}^{x}\left(1(X_{ki}\leq t)-1(X_{li}\leq t)\right)dt\right|^{2+\delta}\leq E\left|X_{ki}-X_{li}\right|^{2+\delta}<\infty.$$

The mixing condition also holds since we have $\sum\alpha(m)^{-A}\leq C\sum m^{-A\delta/(2+\delta)}<\infty$ by Assumption 1(i) as desired. Proof of part (c) is similar. ■

**Pr** [redacted] **1.** We only verify part (a). Proof of parts (b) and (c) is analogous. Consider first the case when $d = 0$. In this case, we verify that, if $F_k(x) \leq F_l(x)$ with equality holding for $x \in \mathcal{B}^d_{kl}$, then

$$\begin{aligned} \widehat{D}_{kl} &\equiv \sup_{x \in \mathcal{X}} \sqrt{N} \left[ F_{kN}(x, \widehat{\theta}_k) - F_{lN}(x, \widehat{\theta}_l) \right] \\ &\Rightarrow \sup_{x \in \mathcal{B}^d_{kl}} \left[ \widetilde{d}_{kl}(\cdot) + \Delta_{k0}(\cdot)' \Gamma_{k0} \nu_{k0} - \Delta_{l0}(\cdot)' \Gamma_{l0} \nu_{l0} \right]. \end{aligned} \tag{A.17}$$

Then, the result of Theorem 1 (a) follows immediately from continuous mapping theorem.

We now establish (A.17). Lemmas 2 and 3 imply

$$\begin{aligned} \widehat{D}_{kl}(x) &\equiv \sqrt{N} \left[ F_{kN}(x, \widehat{\theta}_k) - F_{lN}(x, \widehat{\theta}_l) \right] \\ &= \nu^d_{kN}(x, \widehat{\theta}_k) - \nu^d_{lN}(x, \widehat{\theta}_l) + \sqrt{N} \left[ F_k(x, \widehat{\theta}_k) - F_l(x, \widehat{\theta}_l) \right] \\ &= \overline{D}_{kl}(x) + o_p(1) \text{ uniformly in } x \in \mathcal{X}, \end{aligned}$$

where

$$\overline{D}_{kl}(x) = D^0_{kl}(x) + D^1_{kl}(x) \tag{A.18}$$

$$\begin{aligned} D^0_{kl}(x) &= \nu^d_{kN}(x, \theta_{k0}) - \nu^d_{lN}(x, \theta_{l0}) \\ &\quad + \Delta_{k0}(x) \Gamma_{k0} \sqrt{N} \overline{\psi}_{kN}(\theta_{k0}) - \Delta_{l0}(x) \Gamma_{l0} \sqrt{N} \overline{\psi}_{lN}(\theta_{l0}) \\ D^1_{kl}(x) &= \sqrt{N} \left[ F_k(x) - F_l(x) \right]. \end{aligned} \tag{A.19}$$

We need to verify

$$\sup_{x \in \mathcal{X}} \overline{D}_{kl}(x) \Rightarrow \sup_{x \in \mathcal{B}^d_{kl}} d_{kl}(x). \tag{A.20}$$

Note that

$$\sup_{x \in \mathcal{B}^d_{kl}} D^0_{kl}(x) \Rightarrow \sup_{x \in \mathcal{B}^d_{kl}} d_{kl}(x) \tag{A.21}$$

by Lemma 4 and continuous mapping theorem. Note also that $\overline{D}_{kl}(x) = D^0_{kl}(x)$ for $x \in \mathcal{B}^d_{kl}$. Given $\varepsilon > 0$, this implies that

$$P \left( \sup_{x \in \mathcal{X}} \overline{D}_{kl}(x) \leq \varepsilon \right) \leq P \left( \sup_{x \in \mathcal{B}^d_{kl}} D^0_{kl}(x) \leq \varepsilon \right). \tag{A.22}$$

On the other hand, Lemma 4 and Assumptions 1(i), 2(ii) and 3(iii) imply that given $\lambda$ and $\gamma > 0$, there exists $\delta > 0$ such that

$$P \left( \sup_{\substack{\rho(x,y) < \delta \\ y \in \mathcal{B}^d_{kl}}} \left| D^0_{kl}(x) - D^0_{kl}(y) \right| > \lambda \right) < \gamma \tag{A.23}$$

and

$$\sup_{x\in\mathcal{X}} \left|D_{kl}^{0}(x)\right| = O_p(1). \tag{A.24}$$

The results (A.23) and (A.24) imply that we have

$$P\left(\sup_{x\in\mathcal{B}_{kl}^{d}} D_{kl}^{0}(x) \le \varepsilon\right) \le P\left(\sup_{x\in\mathcal{X}} \overline{D}_{kl}(x) \le \varepsilon + \lambda\right) + 2\gamma \tag{A.25}$$

for $N$ sufficiently large, which follows from arguments similar to those in the proof of Theorem 6 of Klecan et. al. (1990, p.15). Taking $\lambda$ and $\gamma$ small and using (A.21), (A.22) and (A.25) now establish the desired result (A.20).

Next suppose $d < 0$. In this case, the set $\mathcal{B}_{kl}^{d}$ is an empty set and hence $F_k(x) < F_l(x)\ \forall x \in \mathcal{X}$ for some $k, l$. Then, $\sup_{x\in\mathcal{X}} \overline{D}_{kl}(x)$ defined in (A.18) will be dominated by the term $D_{kl}^{1}(x)$ which diverges to minus infinity for any $x \in \mathcal{X}$ as required. ■

**Pr** [redacted] **2.** Let

$$d_{\infty}^{*} = \min_{k\neq l} \sup_{x\in\mathcal{B}_{kl}^{d}} \left[\widetilde{d}_{kl}(x) + \Delta_{k0}(x)'\Gamma_{k0}\nu_{k0} - \Delta_{l0}(x)'\Gamma_{l0}\nu_{l0}\right].$$

Let the asymptotic null distribution of $D_N$ be given by $G(w) \equiv P\left(d_{\infty}^{*} \le w\right)$. This distribution is absolutely continuous because it is a functional of a Gaussian process whose covariance function is nonsingular, see Lifshits (1982). Therefore, part (a) of Theorem 2 holds if we establish

$$\widehat{G}_{N,b}(w) \xrightarrow{p} G(w) \quad \forall w \in \mathbb{R}. \tag{A.26}$$

Let

$$\begin{aligned} G_b(w) &= P\left(\sqrt{b}d_{N,b,i} \le w\right) \\ &= P\left(\sqrt{b}d_b(W_i, \ldots, W_{i+b-1}) \le w\right) \\ &= P\left(\sqrt{b}d_b(W_1, \ldots, W_b) \le w\right). \end{aligned}$$

By Theorem 1(a), we have $\lim_{b\to\infty} G_b(w) = G(w)$ , where $w$ is a continuity point of $G(\cdot)$. Therefore, to establish (A.26), it suffices to verify

$$\widehat{G}_{N,b}(w) - G_b(w) \xrightarrow{p} 0 \quad \forall w \in \mathbb{R}. \tag{A.27}$$

We now verify (A.27). Note first that

$$E\widehat{G}_{N,b}(w) = G_b(w). \tag{A.28}$$

Let

$$I_i = 1\left(\sqrt{b} d_b(W_i, \ldots, W_{i+b-1}) \leq w\right)$$

for $i = 1, \ldots, N$. We have

$$\begin{aligned}
\operatorname{var}\left(\widehat{G}_{N,b}(w)\right) &= \operatorname{var}\left(\frac{1}{N-b+1}\sum_{i=1}^{N-b+1} I_i\right) \\
&= \frac{1}{N-b+1}\left[S_{N-b+1,0} + 2\sum_{m=1}^{b-1} S_{N-b+1,m} + 2\sum_{m=b}^{N-b} S_{N-b+1,m}\right] \\
&\equiv A_1 + A_2 + A_3\ ,\ say,
\end{aligned}$$

where

$$S_{N-b+1,m} = \frac{1}{N-b+1}\sum_{i=1}^{N-b+1-m} Cov\left(I_i, I_{i+m}\right).$$

Note that

$$|A_1 + A_2| \leq O(\frac{b}{N}) = o(1). \tag{A.29}$$

Also, we have

$$\begin{aligned}
|A_3| &= \left|\frac{2}{N-b+1}\sum_{m=b}^{N-b}\left\{\frac{1}{N-b+1}\sum_{i=1}^{N-b+1-m} Cov\left(I_i, I_{i+m}\right)\right\}\right| \\
&\leq \frac{8}{(N-b+1)^2}\sum_{m=b}^{N-b}\sum_{i=1}^{N-b+1-m}\alpha_X(m-b+1) \\
&\leq \frac{8}{N-b+1}\sum_{m=1}^{N-2b+1}\alpha_X(m) \\
&\to 0 \text{ as } N \to \infty,
\end{aligned} \tag{A.30}$$

where the first inequality holds by Theorem A.5 of Hall and Heyde (1980) and the last convergence to zero holds by Assumption 1(i). Now the desired result (A.27) follows immediately from (A.28)-(A.30). This establishes part (a) of Theorem 2. Given this result, part (b) of Theorem 2 holds since we have

$$P\left(D_N > g_{N,b}(1-\alpha)\right) = P\left(D_N > g(1-\alpha) + o_p(1)\right) \to \alpha \text{ as } n \to \infty.$$

■

**Pr** [redacted] **3.** By lemmas 2-4, we have

$$d_N(W_1, \ldots, W_N) \xrightarrow{p} d^*,$$

where $d^*$ is as defined in (1). Note that under $H_1^d$, we have $d^* > 0$. Now consider the empirical distribution of $d_{N,b,i} = d_b(W_i, \ldots, W_{i+b-1})$ :

$$\widehat{G}^0_{N,b}(w) = \frac{1}{N-b+1} \sum_{i=1}^{N-b+1} 1\left(d_{N,b,i} \le w\right) = \widehat{G}_{N,b}\left(\sqrt{b}w\right).$$

Let

$$G^0_b(w) = P\left(d_b(W_1, \ldots, W_b) \le w\right).$$

By an argument analogous to those used to verify (A.27), we have

$$\widehat{G}^0_{N,b}(w) - G^0_b(w) \xrightarrow{p} 0 \ .$$

Since $d_b(W_1, \ldots, W_b) \xrightarrow{p} d^*$, $\widehat{G}^0_{N,b}(\cdot)$ converges in distribution to a point mass at $d^*$. It also follows that

$$g^0_{N,b}(1-\alpha) = \inf\left\{w : \widehat{G}^0_{N,b}(w) \ge 1-\alpha\right\} \xrightarrow{p} d^*.$$

Therefore, we have

$$\begin{aligned} P\left(D_N > g_{N,b}(1-\alpha)\right) &= P\left(\sqrt{N}d_N(W_1, \ldots, W_N) > \sqrt{b}g^0_{N,b}(1-\alpha)\right) \\ &= P\left(\sqrt{\frac{N}{b}}d_N(W_1, \ldots, W_N) > g^0_{N,b}(1-\alpha)\right) \\ &= P\left(\sqrt{\frac{N}{b}}d_N(W_1, \ldots, W_N) > d^* + o_p(1)\right) \\ &= P\left(\sqrt{\frac{N}{b}}d_N(W_1, \ldots, W_N) > d^*\right) + o(1) \\ &\to 1, \end{aligned}$$

where the last convergence holds since $\underline{\lim}_{N\to\infty}\left(\frac{N}{b}\right) > 1$ and $d_N(W_1, \ldots, W_N) \xrightarrow{p} d^* > 0$ as desired. ■

## REFERENCE


Anderson, G.J. (1996),"Nonparametric tests of stochastic dominance in income distributions," **Econometrica**, 64, 1183-1193.

Andrews, D. W. K. (1994). An introduction to functional central limit theorems for dependent stochastic processes, **I t r t t t st c R v** 62, 119-132.

Andrews, D.W.K., (1997): "A Conditional Kolmogorov Test," **E tr c** 65, 1097-1128.

Bartholomew, D.J. (1959a), "A test of homogeneity for ordered alternatives," **B tr**, 46, 36-48.

Bartholomew, D.J. (1959b), "A test of homogeneity for ordered alternatives," **B tr**, 46, 328-35.

Bawa, V.S. (1975), "Optimal rules for ordering uncertain prospects," **J r F c E -  cs**, 2, 95-121.

Bishop, J.A., J.P.Formby, and P.D. Thisle (1992), "Convergence of the South and non-South income distributions, 1969-1979", **A r c E c R v**, 82, 262-272.

Dardanoni, V. and A. Forcina (1999), "Inference for Lorenz curve orderings", **E tr cs J r-** 2, 49-75.

Davidson, R. and J-Y Duclos (1997), "Statistical inference for the measurement of the incidence of taxes and transfers", **E tr c**, 52, 761-76.

Davidson R. and J-Y. Duclos (2000), "Statistical inference for stochastic dominance and for the measurement of poverty and inequality", **E tr c**, 68, 1435-1464.

Delgado, M., J.M. Rodriguez-Poo and M. Wolf (2001): "Subsampling inference in cube root asymptotics with an application to Manski's maximum score estimator," **E cs L tt rs** 73, 241-250.

Durbin, J.(1973), **D str t th r r t sts s th s str t ct**, SIAM, Philadelphia.

Durbin, J. (1985), "The first passage density of a continuous Gaussian process to a general boundary," **J r A Pr t**, 22, 99-122.

Gourieroux, C., Holly, A., and A. Monfort (1982), "Likelihood ratio test, Wald test and Kuhn-Tucker test in linear models with inequality constraints in the regression parameters," **E tr c**, 50, 63-80.

Hadar, J. and W.R. Russell (1969),"Rules for ordering uncertain prospects, **American Economic Review**, 59, 25-34.

Hall, P. and Heyde, C. C. (1980). **Martingale Limit Theory and its Application**. Academic Press, New York.

Hanoch,G. and H. Levy (1969),"The efficiency analysis of choices involving risk," **Review of Economic Studies**, 36, 335-346.

Kahneman, D., and A. Tversky (1979): "Prospect Theory of Decisions Under Risk," **Econometrica** 47, 263-291.

Kaur, A., B.L.S. Prakasa Rao, and H. Singh (1994),"Testing for second-order stochastic dominance of two distributions," **Econometric Theory**, 10, 849-866.

Klecan, L., R. McFadden, and D. McFadden (1991),"A robust test for stochastic dominance," Working paper, Economics Dept., MIT.

Lehmann, E.L. (1959), **Testing Statistical Hypotheses**, J. Wiley and Sons, N.Y.

Levy, H., and Z. Wiener (1998): "Stochastic Dominance and Prospect Dominance with Subjective Weighting Functions"

Perlman, M.D. (1969), "One-sided testing problems in multivariate analysis," [illegible], 40, 549-62.

Pollard, D. (1990). [illegible], CBMS Conference Series in Probability and Statistics, Vol. 2. Institute of Mathematical Statistics, Hayward.

Politis, D. N., and J. P. Romano (1994): "Large sample confidence regions based on subsamples under minimal assumptions." [illegible] 22, 2031-2050.

Politis, D. N., J. P. Romano and M. Wolf (1999). [illegible]. Springer-Verlag, New York.

Porter, R.B., J.R.Wart, and D.L. Ferguson (1973), "Efficient algorithms for conducting stochastic dominance tests on large number of portfolios," [illegible], 8, 71-81.

Porter, R. (1978), "Portfolio applications: Empirical studies," in G. Whitmore and M. Findley (eds.) [illegible], Lexington: Lexington Books, pp. 117-161.

Shorrocks, A.F.(1983), "Ranking income distributions," [illegible], 50, 3-17.

Shorrocks A., and J. Foster (1987), "Transfer sensitive inequality measures," [illegible], 54, 485-497.

Tse, Y.K. and X.B. Zhang (2000), "A Monte carlo Investigation of Some Tests for Stochastic Dominance", National University of Singapore, Mimeo.

Tversky, A. and D. Kahneman (1992): "Advances in Prospect Theory:Cumulative Representation of Uncertainty," [illegible] 5, 297-323.

Whitmore,G.A. and M.C. Findley (1978), [illegible] Heath,Lexington :Mars.

Xu, K., G. Fisher, and D. Wilson (1995), "New distribution-free tests for stochastic dominance," Working paper No. 95-02, February, Dept. of Economics, Dalhousie University, Halifax, Nova Scotia.

| | •( ) | **FO D95** | **FO D90** | **FO D80** | **O D95** | **O D90** | **O D80** |
|---|---|---|---|---|---|---|---|
| | $2n^{0.33}[12]$ | 0.1960 | 0.2670 | 0.2780 | 0.0320 | 0.0660 | 0.1720 |
| | $2.5n^{0.33}[15]$ | 0.1040 | 0.2010 | 0.3430 | 0.0420 | 0.0860 | 0.1880 |
| | $3n^{0.33}[18]$ | 0.1300 | 0.2000 | 0.3980 | 0.0380 | 0.0700 | 0.1800 |
| | $2n^{0.5}[32]$ | 0.1030 | 0.1620 | 0.3040 | 0.0450 | 0.0790 | 0.1820 |
| 250 | $2.5n^{0.5}[40]$ | 0.1000 | 0.1670 | 0.2940 | 0.0480 | 0.0870 | 0.1890 |
| | $3n^{0.5}[48]$ | 0.1000 | 0.1440 | 0.2640 | 0.0540 | 0.0910 | 0.1840 |
| | $2n^{0.75}[126]$ | 0.0890 | 0.1570 | 0.2490 | 0.0810 | 0.1410 | 0.2410 |
| | $2.5n^{0.75}[157]$ | 0.1060 | 0.1600 | 0.2520 | 0.1060 | 0.1540 | 0.2440 |
| | $3n^{0.75}[189]$ | 0.1160 | 0.1570 | 0.2480 | 0.1170 | 0.1640 | 0.2570 |
| | | | | | | | |
| | $2n^{0.33}[16]$ | 0.1270 | 0.1840 | 0.3770 | 0.0380 | 0.0750 | 0.1630 |
| | $2.5n^{0.33}[20]$ | 0.1280 | 0.1960 | 0.3990 | 0.0430 | 0.0800 | 0.1660 |
| | $3n^{0.33}[24]$ | 0.1160 | 0.2230 | 0.2920 | 0.0340 | 0.0770 | 0.1570 |
| | $2n^{0.5}[44]$ | 0.0980 | 0.1700 | 0.2770 | 0.0380 | 0.0790 | 0.1570 |
| 500 | $2.5n^{0.5}[55]$ | 0.0740 | 0.1290 | 0.2420 | 0.0460 | 0.0850 | 0.1700 |
| | $3n^{0.5}[66]$ | 0.0870 | 0.1490 | 0.2540 | 0.0490 | 0.0840 | 0.1700 |
| | $2n^{0.75}[212]$ | 0.0740 | 0.1160 | 0.2190 | 0.0590 | 0.1030 | 0.2140 |
| | $2.5n^{0.75}[265]$ | 0.0780 | 0.1330 | 0.2340 | 0.0780 | 0.1270 | 0.2230 |
| | $3n^{0.75}[318]$ | 0.1030 | 0.1410 | 0.2290 | 0.0860 | 0.1330 | 0.2310 |
| | | | | | | | |
| | $2n^{0.33}[20]$ | 0.1260 | 0.1990 | 0.4400 | 0.0420 | 0.0950 | 0.1890 |
| | $2.5n^{0.33}[25]$ | 0.1080 | 0.2640 | 0.3220 | 0.0450 | 0.0970 | 0.1890 |
| | $3n^{0.33}[30]$ | 0.1160 | 0.1570 | 0.3340 | 0.0460 | 0.0990 | 0.1870 |
| | $2n^{0.5}[64]$ | 0.0810 | 0.1600 | 0.2740 | 0.0420 | 0.0920 | 0.1860 |
| 1000 | $2.5n^{0.5}[80]$ | 0.0760 | 0.1350 | 0.2500 | 0.0520 | 0.0960 | 0.1910 |
| | $3n^{0.5}[96]$ | 0.0690 | 0.1340 | 0.2360 | 0.0460 | 0.0910 | 0.1930 |
| | $2n^{0.75}[356]$ | 0.0710 | 0.1100 | 0.1970 | 0.0660 | 0.1020 | 0.1940 |
| | $2.5n^{0.75}[445]$ | 0.0800 | 0.1280 | 0.2110 | 0.0710 | 0.1080 | 0.2130 |
| | $3n^{0.75}[534]$ | 0.0870 | 0.1280 | 0.2040 | 0.0770 | 0.1190 | 0.2280 |

Table 1:

| | •( ) | **FO D95** | **FO D90** | **FO D80** | **O D95** | **O D90** | **O D80** |
|---|---|---|---|---|---|---|---|
| | $2n^{0.33}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2.5n^{0.33}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $3n^{0.33}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2n^{0.5}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| 250 | $2.5n^{0.5}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $3n^{0.5}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2n^{0.75}$ | 0.0000 | 0.0000 | 0.0000 | 0.4570 | 0.5550 | 0.6820 |
| | $2.5n^{0.75}$ | 0.0000 | 0.0000 | 0.0000 | 0.0010 | 0.0010 | 0.0070 |
| | $3n^{0.75}$ | 0.0000 | 0.0000 | 0.0010 | 0.0010 | 0.0010 | 0.0010 |
| | | | | | | | |
| | $2n^{0.33}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2.5n^{0.33}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $3n^{0.33}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2n^{0.5}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| 500 | $2.5n^{0.5}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $3n^{0.5}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2n^{0.75}$ | 0.0000 | 0.0000 | 0.0000 | 0.0010 | 0.0010 | 0.0020 |
| | $2.5n^{0.75}$ | 0.0000 | 0.0000 | 0.0000 | 0.0010 | 0.0010 | 0.0010 |
| | $3n^{0.75}$ | 0.0000 | 0.0000 | 0.0000 | 0.0010 | 0.0010 | 0.0010 |
| | | | | | | | |
| | $2n^{0.33}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2.5n^{0.33}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $3n^{0.33}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2n^{0.5}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1000 | $2.5n^{0.5}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $3n^{0.5}$ | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2n^{0.75}$ | 0.0000 | 0.0000 | 0.0000 | 0.0010 | 0.0010 | 0.0010 |
| | $2.5n^{0.75}$ | 0.0000 | 0.0000 | 0.0000 | 0.0010 | 0.0010 | 0.0010 |
| | $3n^{0.75}$ | 0.0000 | 0.0000 | 0.0000 | 0.0010 | 0.0010 | 0.0010 |

Table 2:

| | •( ) | FO D95 | FO D90 | FO D80 | O D95 | O D90 | O D80 |
|---|---|---|---|---|---|---|---|
| | $2n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.0120 | 0.0590 | 0.3240 |
| | $2.5n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.0490 | 0.1540 | 0.4730 |
| | $3n^{0.33}$ | 0.9990 | 0.9990 | 1.0000 | 0.0140 | 0.0540 | 0.2370 |
| | $2n^{0.5}$ | 0.9970 | 0.9990 | 0.9990 | 0.0800 | 0.1900 | 0.4760 |
| 250 | $2.5n^{0.5}$ | 0.9960 | 0.9990 | 0.9990 | 0.1140 | 0.2480 | 0.5210 |
| | $3n^{0.5}$ | 0.9920 | 0.9980 | 0.9990 | 0.0860 | 0.1910 | 0.4560 |
| | $2n^{0.75}$ | 0.8840 | 0.9160 | 0.9600 | 0.2500 | 0.3670 | 0.5090 |
| | $2.5n^{0.75}$ | 0.7570 | 0.8080 | 0.8750 | 0.2870 | 0.3730 | 0.4980 |
| | $3n^{0.75}$ | 0.6290 | 0.6930 | 0.7670 | 0.3000 | 0.3740 | 0.4910 |
| | | | | | | | |
| | $2n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.0430 | 0.2320 | 0.7100 |
| | $2.5n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.1100 | 0.3800 | 0.7940 |
| | $3n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.0260 | 0.1600 | 0.5060 |
| | $2n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.2090 | 0.4530 | 0.7960 |
| 500 | $2.5n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.3220 | 0.6080 | 0.8970 |
| | $3n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.3220 | 0.5820 | 0.8890 |
| | $2n^{0.75}$ | 0.9920 | 0.9980 | 0.9990 | 0.4720 | 0.6010 | 0.7650 |
| | $2.5n^{0.75}$ | 0.9680 | 0.9790 | 0.9890 | 0.4900 | 0.5950 | 0.7160 |
| | $3n^{0.75}$ | 0.9260 | 0.9490 | 0.9710 | 0.4760 | 0.5460 | 0.7000 |
| | | | | | | | |
| | $2n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.2880 | 0.7560 | 0.9940 |
| | $2.5n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.4480 | 0.8300 | 0.9950 |
| | $3n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.5370 | 0.8720 | 0.9980 |
| | $2n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.6510 | 0.9350 | 0.9990 |
| 1000 | $2.5n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.7100 | 0.9440 | 0.9990 |
| | $3n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.6910 | 0.9340 | 0.9990 |
| | $2n^{0.75}$ | 1.0000 | 1.0000 | 1.0000 | 0.7570 | 0.8500 | 0.9380 |
| | $2.5n^{0.75}$ | 1.0000 | 1.0000 | 1.0000 | 0.7640 | 0.8270 | 0.9130 |
| | $3n^{0.75}$ | 0.9990 | 0.9990 | 0.9990 | 0.7130 | 0.7830 | 0.8840 |

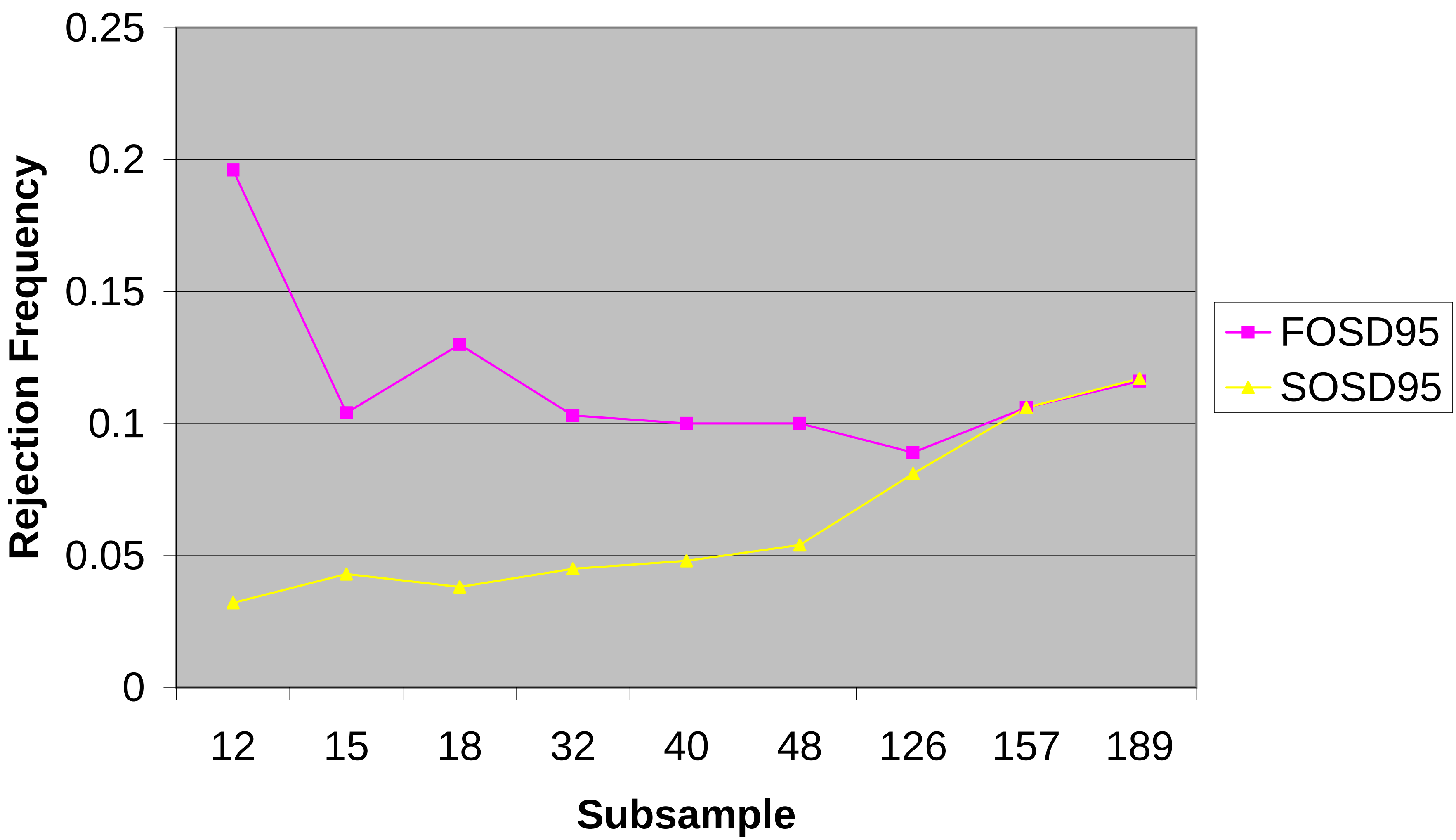
Table 3:

| | •( ) | FO D95 | FO D90 | FO D80 | O D95 | O D90 | O D80 |
|---|---|---|---|---|---|---|---|
| | $2n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.9870 | 0.9960 | 0.9990 |
| | $2.5n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.9870 | 1.0000 | 1.0000 |
| | $3n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.9850 | 0.9980 | 1.0000 |
| | $2n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.9290 | 0.9760 | 0.9940 |
| 250 | $2.5n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.9310 | 0.9700 | 0.9960 |
| | $3n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.9060 | 0.9610 | 0.9930 |
| | $2n^{0.75}$ | 0.9940 | 0.9960 | 0.9990 | 0.6320 | 0.7160 | 0.8210 |
| | $2.5n^{0.75}$ | 0.9610 | 0.9790 | 0.9910 | 0.5350 | 0.6150 | 0.7230 |
| | $3n^{0.75}$ | 0.8800 | 0.9130 | 0.9470 | 0.5090 | 0.5770 | 0.6890 |
| | | | | | | | |
| | $2n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2.5n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $3n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| | $2n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.9990 | 1.0000 | 1.0000 |
| 500 | $2.5n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.9970 | 1.0000 | 1.0000 |
| | $3n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.9940 | 0.9970 | 1.0000 |
| | $2n$ | | | | | | |

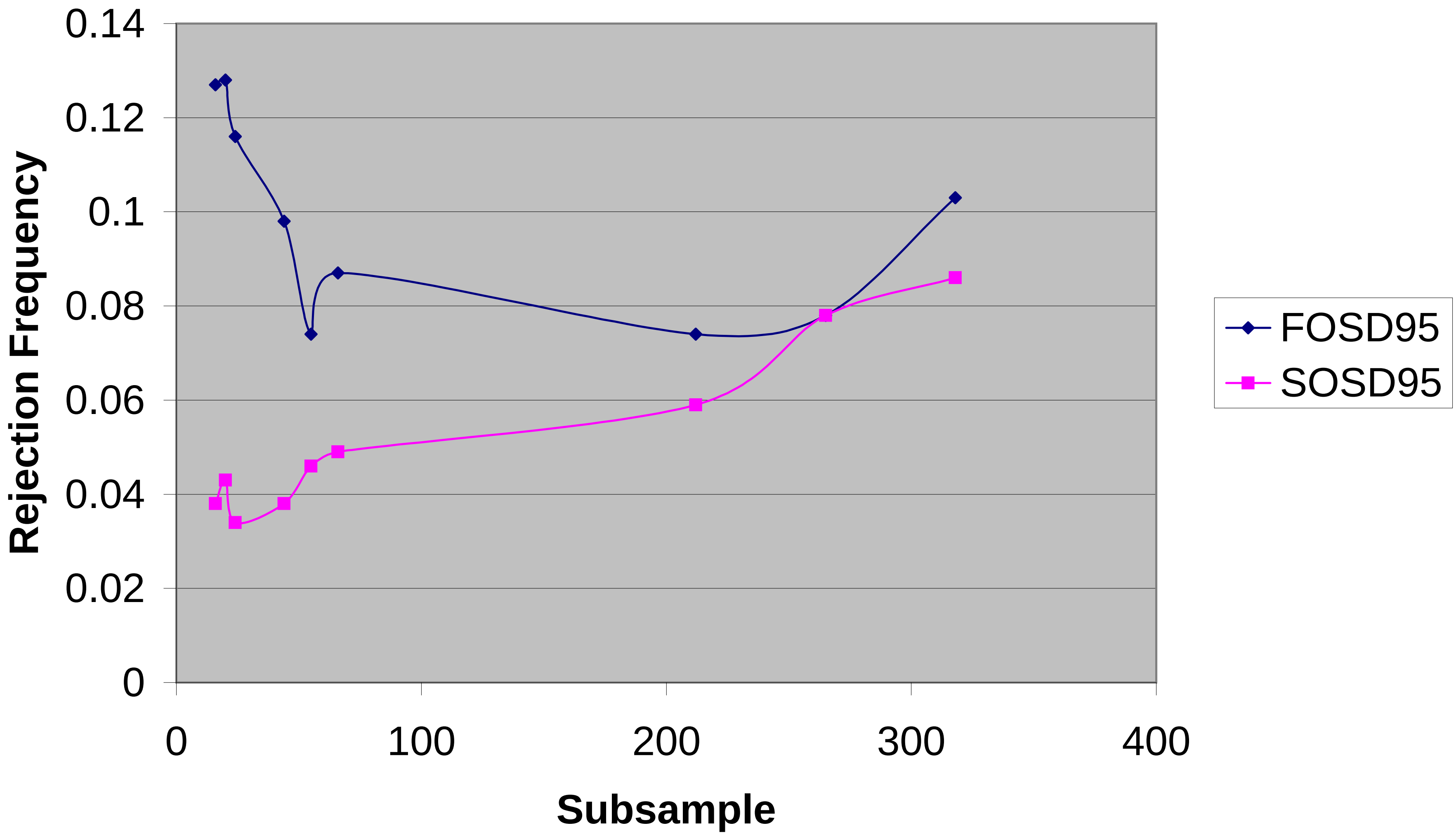
| | •( ) | **FO D95** | **FO D90** | **FO D80** | **O D95** | **O D90** | **O D80** |
|---|---|---|---|---|---|---|---|
| | $2n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.2960 | 0.3550 | 0.4210 |
| | $2.5n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.3090 | 0.3580 | 0.4150 |
| | $3n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.2810 | 0.3300 | 0.3930 |
| | $2n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.2360 | 0.2750 | 0.3660 |
| 250 | $2.5n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.2260 | 0.2690 | 0.3650 |
| | $3n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.2010 | 0.2440 | 0.3380 |
| | $2n^{0.75}$ | 0.9610 | 0.9770 | 0.9890 | 0.1540 | 0.2110 | 0.3250 |
| | $2.5n^{0.75}$ | 0.9010 | 0.9390 | 0.9610 | 0.1400 | 0.1840 | 0.2520 |
| | $3n^{0.75}$ | 0.7870 | 0.8320 | 0.8930 | 0.1070 | 0.1360 | 0.1960 |
| | | | | | | | |
| | $2n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.3060 | 0.3510 | 0.4180 |
| | $2.5n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.2990 | 0.3400 | 0.4070 |
| | $3n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.2750 | 0.3160 | 0.3650 |
| | $2n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.2200 | 0.2610 | 0.3200 |
| 500 | $2.5n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.2060 | 0.2470 | 0.3140 |
| | $3n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.1850 | 0.2210 | 0.2970 |
| | $2n^{0.75}$ | 0.9990 | 0.9990 | 1.0000 | 0.0760 | 0.1050 | 0.1770 |
| | $2.5n^{0.75}$ | 0.9970 | 0.9990 | 0.9990 | 0.0830 | 0.1240 | 0.1960 |
| | $3n^{0.75}$ | 0.9860 | 0.9940 | 0.9960 | 0.0930 | 0.1190 | 0.1670 |
| | | | | | | | |
| | $2n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.2850 | 0.3330 | 0.3880 |
| | $2.5n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.2710 | 0.3160 | 0.3700 |
| | $3n^{0.33}$ | 1.0000 | 1.0000 | 1.0000 | 0.2550 | 0.2960 | 0.3550 |
| | $2n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.1890 | 0.2260 | 0.2640 |
| 1000 | $2.5n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.1740 | 0.2040 | 0.2600 |
| | $3n^{0.5}$ | 1.0000 | 1.0000 | 1.0000 | 0.1540 | 0.1810 | 0.2380 |
| | $2n^{0.75}$ | 1.0000 | 1.0000 | 1.0000 | 0.0630 | 0.0860 | 0.1560 |
| | $2.5n^{0.75}$ | 1.0000 | 1.0000 | 1.0000 | 0.0650 | 0.0940 | 0.1670 |
| | $3n^{0.75}$ | 1.0000 | 1.0000 | 1.0000 | 0.0820 | 0.1040 | 0.1560 |

Table 5:

Figure 1: n=250



# Figure 2: n=500



## Figure 3: n=1000

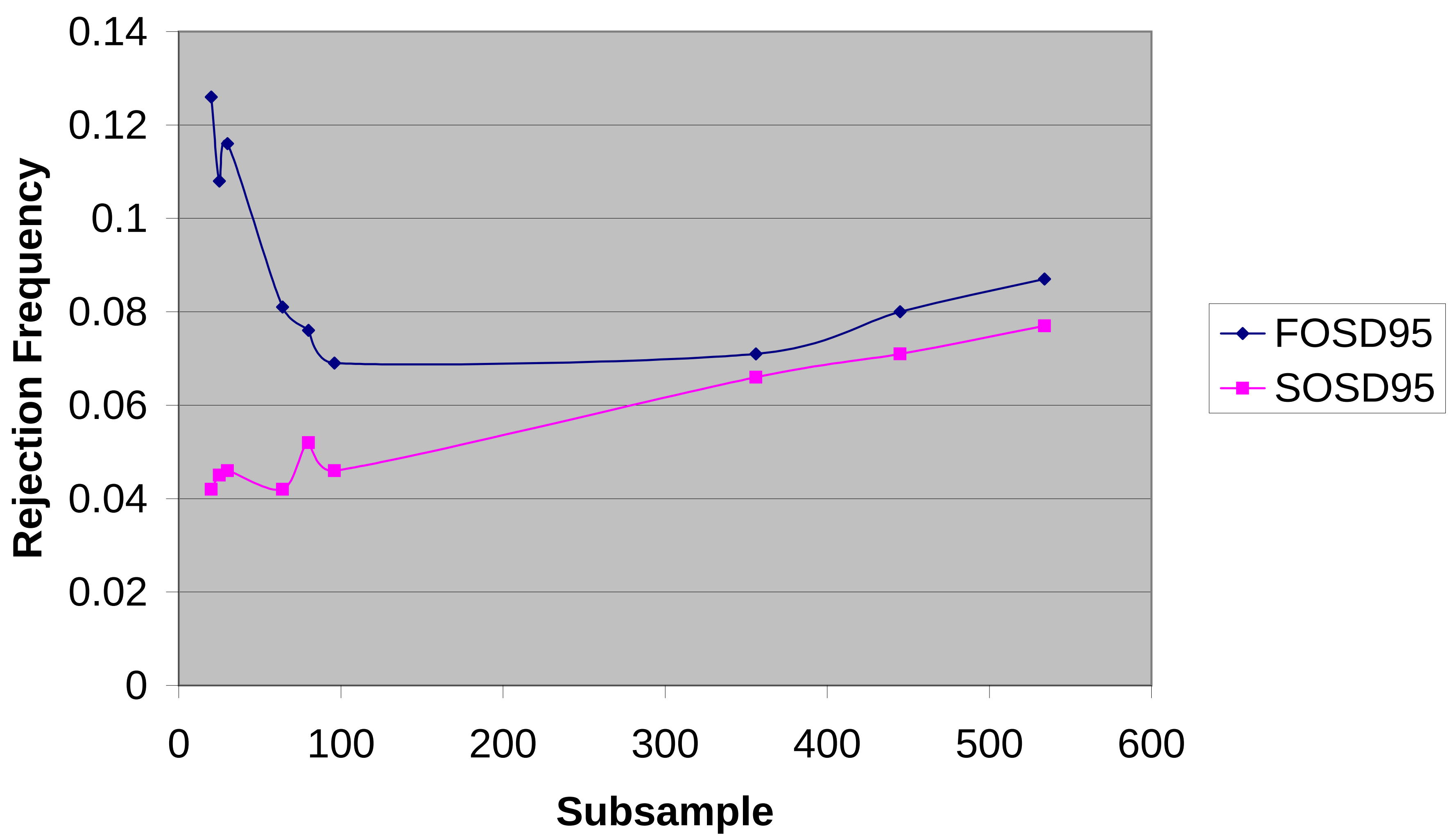
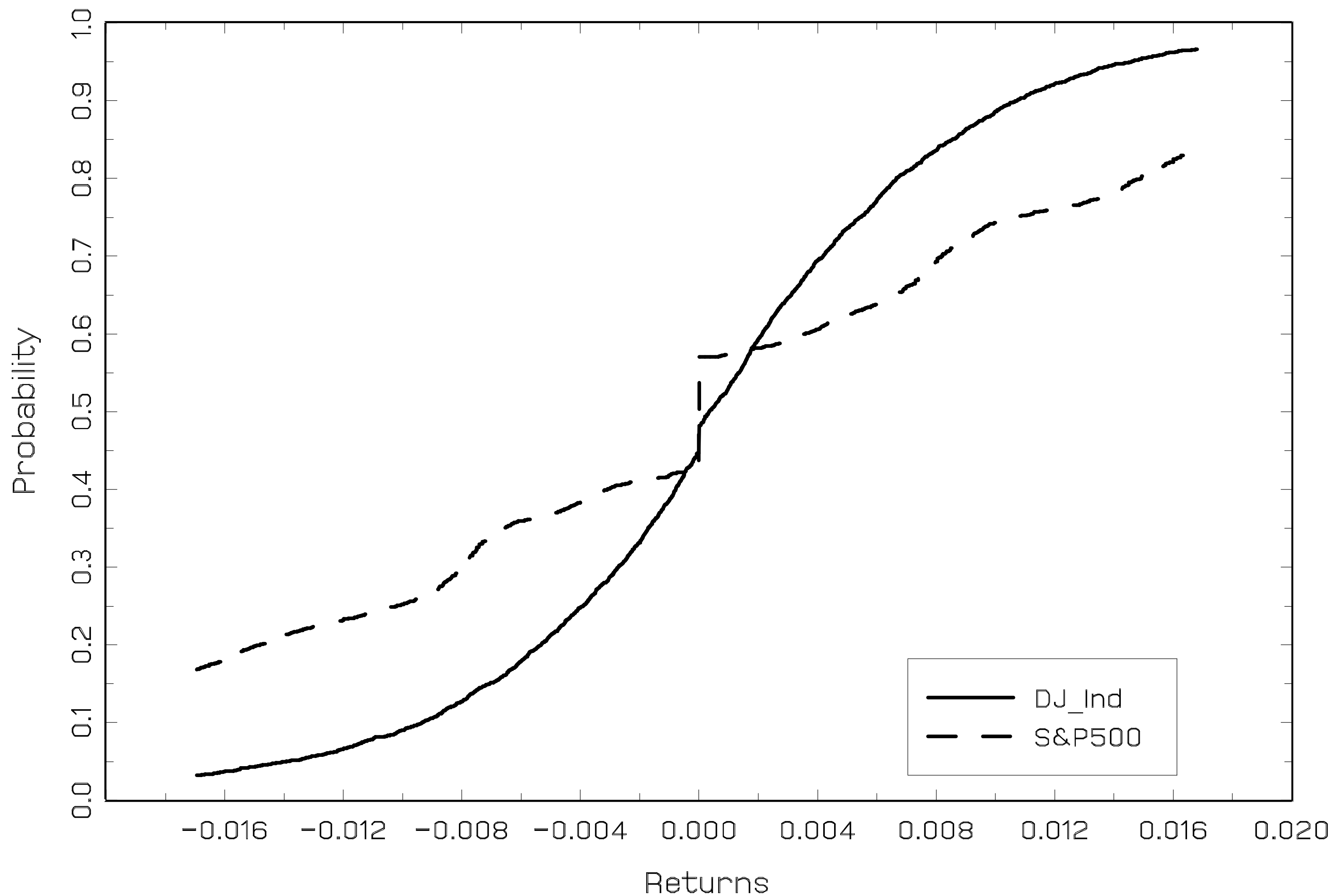


Figure 4: CDF's of DJ_Ind and S&P500



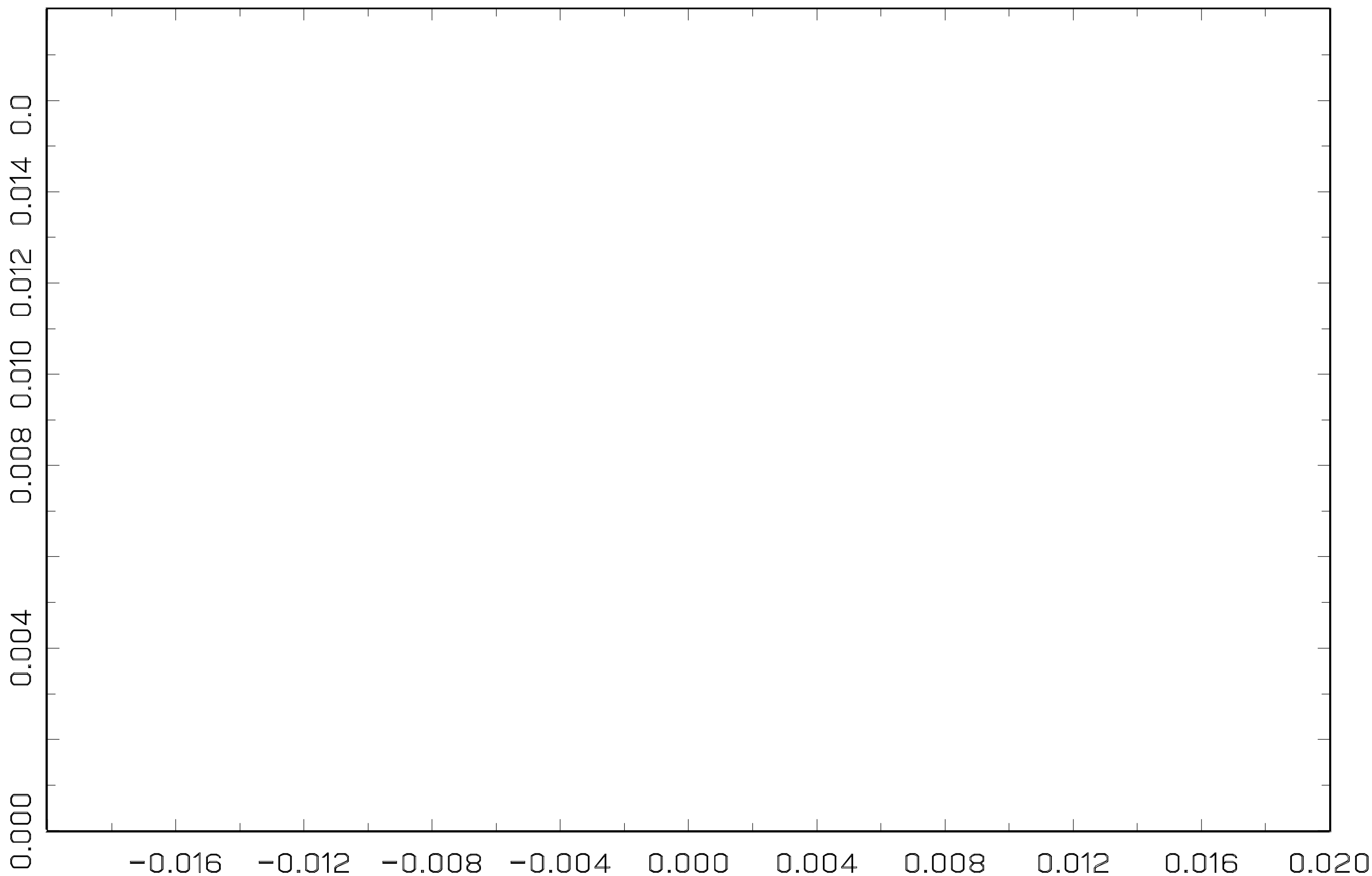

0.000
0.004
0.008
0.010
0.012
0.014
0.0
-0.016
-0.012
-0.008
-0.004
0.000
0.004
0.008
0.012
0.016
0.020

Figure 6: SDF's of Standardized DJ_Ind and Standardized S&P500