Corporate Bond Prices and Co-ordination Failure[*,†]

Max Bruche
Financial Markets Group
London School of Economics
Houghton Street
London WC1A 2AE, U. K.
m.e.bruche@lse.ac.uk

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Abstract

It has been suggested (Morris, Shin 2001) that co-ordination failure between holders of debt can affect the price of debt. In essence, fear of premature foreclosure by other debtors can lead to preemptive action, affecting the value of debt. Using a continuous-time framework related to a Merton (1974)-type structural model, this paper demonstrates how such co-ordination failures can affect the prices of corporate bonds. As it turns out, the resulting model is version of a structural model that allows default before maturity, a model feature that has proven to be popular with practitioners.

1 Introduction

Morris and Shin (2001) (cf. also Morris and Shin 2000) argue that co-ordination failure among creditors can have an effect on the price of debt.

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The problem of co-ordination failure is akin to the problem faced by depositors of a bank which is vulnerable to a run. Even if it is not efficient to foreclose on a loan, e. g. when the debtor is fundamentally viable, fear that other creditors may foreclose can lead to preemptive action and inefficient foreclosure.

In general, co-ordination failures can arise among creditors in a context where it is possible for individual creditors to improve their position vis-a-vis the firm at the cost of other creditors (i.e. in a situation with strategic complementarities between the different lenders). This is the case for example if creditors can foreclose individually, leaving other creditors exposed to a firm with lower liquidity, or for instance if some creditors will be able to grab assets in the case of financial difficulties at the expense of other creditors.

For holders of bonds, it is not in general possible to foreclose on the loan represented by their bond as and when they wish - i. e. it is not possible to sell the bond back to the firm, and also it is not in general possible to grab assets and net out claims. In most jurisdictions, separate deals of individual bondholders with the firm to the detriment of other bondholders are expressly forbidden, and the grabbing of assets is prohibited by bankruptcy codes. In the US, for instance the Trust Indenture Act of 1939 specifies that holders of public debt need to give unanimous consent before a firm can alter the principal, interest or maturity of any part of its public debt, and requires that all holders of the same class of debt receive the same treatment. Under Chapter 11, holders of a particular type of debt will receive the same treatment (cf. e.g. Baird and Jackson 1990, Jackson 1986). The aim of these provisions is of course precisely to mitigate co-ordination failures. The scope for co-ordination failure to arise between holders of bonds themselves is therefore limited, and the co-ordination failure argument is probably not applicable to holders of corporate bonds directly (at least not for most jurisdictions with a well-defined legal framework).

However, bonds are typically not the only form of debt for any particular firm - in particular, almost all companies will depend on bank credit of some form in a crucial way, even if they depend on markets to raise the bulk of their funds. Firms that raise short term cash in the commercial paper market, for instance, will almost always have commercial paper backup lines of credit, to have an emergency supply of liquidity in situations where they cannot raise money on the commercial paper markets (i. e. the kind of situation we are interested in). Bank credit is almost always governed by covenants which are meant to afford creditors a measure of protection, and give them the discretion to stop lending in some situations. This in fact serves to create the strategic complementarities necessary for co-ordination failure. Some recent examples have highlighted the role of bank credit and covenants in financial distress, such as the demise of e. g. Vivendi, Kirch, Enron, Energis, Worldcom and Hutchison 3G to name but a few of the more high-profile cases.

For larger firms, any loan or line of credit is likely to be syndicated as banks try to diversify their credit risks. Co-ordination failures and premature inefficient foreclosure can arise between these banks. These co-ordination failures will of course affect the price of bonds. It is in this setting that this paper will model the effect of co-ordination failure on the price of bonds.

Morris and Shin (2001) model co-ordination failure as a function of a fundamental variable, which can easily be interpreted as the asset value of a firm. This naturally suggests using a structural model to price the bond: The structural, or firm-value based approach pioneered by Merton (1974) explains prices as a function of the process driving the asset value of a company: Bonds are treated as a 'bull spread' on the asset value of a firm, and bankruptcy occurs when the face value of debt exceeds the value of assets at some given date.

Empirically, there is evidence that the simplest structural model (the original (Merton 1974) model) seems to require implausibly high volatilities to generate reasonable bond prices (cf. e.g. Jones, Mason, and Rosenfeld 1984, Anderson and Sundaresan 2000, Eom, Helwege, and Huang 2001). Many extensions have been proposed to make it more realistic: e. g. relating to sub-ordination arrangements and indenture provisions (allowing for default before maturity) (Black and Cox 1976), coupon bearing bonds (Geske 1977), stochastic interest rates (Shimko, Tejima, and van Deventer 1993, Longstaff and Schwartz 1995) or an optimally chosen capital structure (e. g. Leland 1994). Strategic issues have only more recently become the focus of attention. Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) look at a game between creditors and shareholders who optimally choose to default on payments, rather than on games between creditors, such as co-ordination failure.

It is interesting to note that allowing default before maturity seems to have been one of the most interesting extensions for commercial implementations of the Merton model so far, it is used e g. in the KMV EDF™ methodology (Crosbie and Bohn 2002) or in the CreditGrades™ model (Finger et al. 2002). Of course, models with a default barrier make it possible for much lower volatilities to produce lower bond prices, given some suitable assumptions about recovery fractions, overcoming some of the shortcomings of the original simple (Merton 1974) model.

This paper will derive a continuous-time structural model of bond prices, with the simplest possible assumptions, integrating co-ordination failure. It will be shown that once limits are taken, the model resembles the Merton model, but with a default barrier, allowing for default before maturity.

2 The model

2.1 Co-ordination failure among short term creditors

Shorter-term financing for firms that issue bonds can take many forms. As these firms are mostly large and well-established corporations, short-term financing is likely to be raised either in the form of commercial paper, supplemented by a commercial paper back-up line of credit, or by a revolving credit facility or line of credit syndicated or extended by more than one bank, as lenders try to diversify credit risk exposure. Any facility or line of credit will almost certainly be governed by covenants intended to protect the creditors. These covenants can be related to e. g. revenues, cash flow etc, and will allow the creditors to foreclose or stop lending in case they are breached.

Co-ordination failure in periods of financial difficulty for the firm could be modelled as follows: A firm needs a certain amount of short-term liquidity (as e. g. working capital, which is proportional to the total value of assets), which it borrows regularly from a syndicate of banks. Assume that the actual amount borrowed is negligible when compared to the total amount of publicly traded debt (the bond).[1] The borrowing facility has previously been negotiated, and is governed by covenants.

Assume that the covenants specify a cash-flow or revenue target, and that revenues or cash-flows are proportional to the total asset value of the firm. Alternatively, assume that the covenant specifies a value of assets (a net worth covenant). In all cases, we can model a breach of the covenant as a fall of the asset value below a pre-specified level. If the covenants are breached, it is up to individual banks to decide whether or not to extend further credit. Assume that if a large enough proportion of the banks decide

[1] For a non-negligible amount of bank debt, the resulting pricing equations would have to be slightly adjusted. The solution and pricing methodology would remain the same, however.

to stop extending credit, the firm has to default on payments and is pushed into bankruptcy.

In this situation, banks will consider the likely actions of other banks, and co-ordination failures (i.e. failure to extend credit when it would be optimal for the banks as a group to do so) can arise.

2.2 The setup

As in the Merton (1974) model, the bond will be priced as a function of the asset value process. We will first set up a discrete time game, where in every period in which covenants are breached, lenders have to decide whether or not to stop lending. We will then be able to derive a critical point for the asset value for which the firm will just be forced into bankruptcy (the 'trigger point'). The asset value changes between periods, such that when we take the continuous time limit, the process will turn out to be a geometric Brownian motion. This will then allow us to price the bond as a combination of barrier options on the asset value using standard techniques, where the barrier is given by the trigger point.

Assume that bank debt is junior to all other debt.[2]

2.2.1 Payoffs

Let C be the critical level of assets stipulated directly or indirectly in the covenant. If the asset value V falls below C, the covenant is breached. Bankruptcy will take place immediately before a time t only when the fraction of banks who stop lending l is larger than or equal to $\frac{V_t}{C}$, such that it will be impossible for the firm to be liquidated because banks stop lending in the case where $V_t > C$ (i.e. when covenants are not breached), even if some banks would prefer to call loans. As V_t falls, it becomes easier for banks to push the firm into bankruptcy.

Assume that if a bank decides to stop lending, it receives a payoff of zero. If it continues lending and the firm is not pushed into bankruptcy, it earns the spread s^* (negotiated at the time of origination of the loan and fixed throughout its life for simplicity) on the employed principal. If it continues lending and the firm is pushed into bankruptcy, it will loose its share of the principal P. Assume that banks are myopic in the sense that they only care about present payoffs. This considerably simplifies the solution of the game.

[2]

The table below illustrates the 'per unit of principal' payoffs that banks need to take into account when making the decision whether to continue or stop lending.

	firm goes bankrupt	firm does not go bankrupt
stop lending	0	0
continue lending	$-P$	s^*P

2.2.2 Information content of prices

A necessary ingredient for co-ordination failure to arise is uncertainty about the actions of other agents. Without common knowledge of the fundamentals (the asset value in our case) of an issuer, agents will not be completely sure of how other agents will act. Suppose there is private as well as public information, then provided that private information is sufficiently precise in relation to public information, i. e. there is sufficient uncertainty about the actions of others, this will create co-ordination failure.

In a comment on Morris and Shin's (2000) paper, Atkeson (2000) doubts that the co-ordination failure idea is applicable to pricing debt. He argues that if agents can see prices, there will be no co-ordination failure, because all information will be revealed in the prices - there is no role for private information, and hence uncertainty about the information of other agents. In our case, banks do not trade and hence their private information will not necessarily be revealed. But even if it were, co-ordination failure could arise, depending on the exact timing assumptions.

Suppose that banks have to make a decision as to whether or not to stop lending after they have received a signal, but before the signals are revealed to all. Subsequently, signals are revealed to all, then trading occurs and information is integrated into prices. Then there might still be co-ordination failure, because the private information has not been made public at the time when the banks need to act.

2.2.3 Timing

Time increments are of size Δ. At time t, identical agents (the banks) know the asset value of this period, V_t. We will later let the number of agents tend to infinity, and will subsequently index them by the unit interval.[3] Relative

[3] Note that similar results could be obtained by starting with n banks and then taking the continuous time limit. Since this would require a more involved argument, it is not done here. The interested reader is referred to Morris and Shin (2002b) for a game-theoretic exposition of the argument.

changes in the asset value are normally distributed. The bonds trade at a price B_t which incorporates the information V_t. Let q denote a time increment that is smaller than Δ ($0 < q < \Delta$). At $t+q$, banks receive a signal X_i about the increase in the asset value - subscript i indexes the different banks, we omit the time subscript to simplify notation. They form a posterior given their information. Given their posterior, they make a decision as to whether or not to stop lending or not.

After it has been determined whether the firm will fail in this period or not, we proceed to the next period: Signals are revealed, the asset value is revealed and the price $B_{t+\Delta}$ incorporating all the information $V_{t+\Delta}$ is formed. We see that as a consequence of these timing assumptions, only public information will be incorporated into prices. This is important as it allows pricing by standard martingale techniques.

In the case of bankruptcy before maturity, the banks receive their payoffs as described above, and the bondholders receive a (recovery) fraction $R < 1$ of the current asset value.

At the maturity of the bond, the holders receive the minimum of their share of the face value of the bond or the asset value of the firm. At this point, there is no cost to reorganisation (the firm will be wound up in any case).

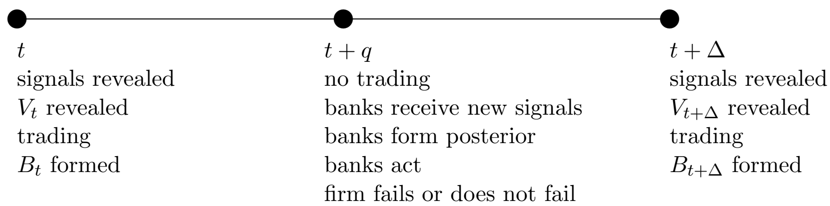

Figure 1: Timing assumptions

2.2.4 Information

The relative increase in the asset value is normally distributed around a drift.

$$V_{t+\Delta} - V_t = \mu_V V_t \Delta + V_t \eta_t, \quad \eta_t \sim NID\left(0, \frac{1}{\alpha}\right)$$

At $t+q$, banks receive a signal X_i (subscript i indexes the different agents) about the impending change in V, with the distribution of the signal, conditional on the asset value V_t given by

$$X_i = V_{t+\Delta} + V_t \varepsilon_i, \quad \varepsilon_i \sim NID\left(0, \frac{1}{\beta}\right),$$

where $Cov(\eta_t, \varepsilon_i) = 0$, i. e. the noise is orthogonal to the innovations in the asset value.

From the signal X_i and the public information V_t, agents form a posterior about the value of the firm in period $t+\Delta$, $V_{t+\Delta}$ (which is also normally distributed).

2.3 The solution

2.3.1 Basic procedure

We follow the same procedure as Morris and Shin (2001) to solve the model. Suppose that agents follow a switching strategy around a certain posterior belief. Given the posterior belief around which agents switch, we can work out how many of them will stop lending, given the asset value in the next period (posterior beliefs will be centered around this asset value in the next period). We can therefore work out what the critical next-period asset value is for which the firm will fail, given the belief in this period around which agents switch. This is the trigger point.

2.3.2 The discrete time trigger point

In the appendix, section 5.1, the following solution is derived (equation 4):

$$V_t^* = C\Phi\left\{\frac{\alpha}{\sqrt{\beta}}\left(\frac{V_t^*}{V_{t-\Delta}} - 1 - \mu_V \Delta\right) + \frac{\sqrt{\alpha+\beta}}{\sqrt{\beta}}\Phi^{-1}\left\{\frac{1}{1+s^*}\right\}\right\}$$

The trigger point V_t^* is unique if:

$$C\frac{1}{\sqrt{2\pi}}\frac{\alpha}{\sqrt{\beta}}\frac{1}{V_t} < 1$$

(cf. appendix, section 5.1.8, proposition 1, condition A).

2.3.3 Continuous time limit

Now take the continuous time limit. If we want the asset value process to tend to a geometric Brownian motion, we need

$$\lim_{\Delta \to dt} \frac{1}{\alpha} = \sigma_V^2 dt,$$

i.e. the variance of public information about the innovation in the asset value to be proportional to time. So the variance of the innovation is $O(\Delta)$, or the precision is $O(\frac{1}{\Delta})$.

Now a sufficient condition for the uniqueness of the equilibrium described in equation (4) in continuous time, *regardless of the asset value, the parameter C and the face value of debt*, is that

$$\frac{1}{\beta} = o\left(\Delta^2\right),$$

i. e. that private information becomes more precise at a rate faster than Δ^2, because this ensures that condition (A) (s.a.) is always satisfied. This is just to say that we need the quality of private information to be sufficiently high in relation to the quality of public information in order for agents to be sufficiently uncertain about the actions of others to obtain co-ordination failure. As $\Delta \to dt$, $\Delta^2 \to 0$, and hence β grows at a faster rate than α. Consequently, $\frac{\alpha}{\sqrt{\beta}}$ tends to zero, so condition (A) will be satisfied for any permissible V_t. Also, $\frac{\sqrt{\alpha+\beta}}{\sqrt{\beta}} \to 1$. The resulting trigger point equation then reduces to

$$V^* = C\left(\frac{1}{1+s^*}\right). \tag{1}$$

Note that the solution is constant. If we let the intermediate time period $(t+q)$ tend to the period immediately following it $(t+dt)$, the firm fails at t whenever $V(t)$ hits $C\left(\{1/(1+s^*\}\right)$, i. e. when the asset value is a *fraction* $(\{1/(1+s^*\})$ of the covenant (given that $s^* > 0$). So here failure actually occurs at a later point than the first time the covenant is breached. The boundary or trigger point is a decreasing function of the opportunity cost of stopping to lend (loosing the spread s^*) - agents are reluctant to stop lending if it is costly for them to do so.

Due to the special assumptions about payoffs, this function turns out to be quite simple here - it is constant. Of course, one could allow for more general types of costs, but these do not in general produce a closed form solution.

2.3.4 The actions of banks in the continuous time limit

Conditional on the asset value in the next period, the probability that a bank receives a signal which prompts it to stop lending is $\Phi\left\{\frac{1}{V_t}\sqrt{\beta}(X_t^* - V_{t+\Delta})\right\}$. We see that as β tends to infinity, this probability tends either to 1 or to 0. What this means is that because all banks essentially receive the same information (as the signal becomes infinitely precise), the banks will either all stop lending, or will all refrain from doing so. So for any bank, the ex-ante probability of stopping to lend when the other agents do not do so tends to zero. Also, the probability of continuing to lend if all other agents are stopping to lend tends to zero. This is essentially because in the limit, agents receive the same signals, and there is no uncertainty about the asset value. However, strategic uncertainty remains.

2.3.5 Strategic uncertainty in the continuous time limit

Strategic uncertainty remains in the sense that for the marginal agent, the fraction of bondholders that forces reorganisation is still a random variable in the limit, which actually turns out to be uniformly distributed. This implies that the solution of the game is preserved in the limit. This type of result has been discussed at length elsewhere (Morris and Shin 2002a). A formal proof is in the appendix, section 5.2.

To illustrate the point, suppose that we instead start with the assumption that there is no private information, and hence no co-ordination failure. All banks now have the same information, and (in the absence of asymmetries) will therefore either all stop lending, or all continue lending. Suppose there is a strategy that specifies a path for the switching point. At every node of the game, it does not pay to deviate from the strategy when it specifies stopping to lend (because in this case deviating *always* implies loosing the principal P - continuing to lend when everyone else forecloses is not a good idea), and it does not pay to deviate from a strategy when it specifies continuing to lend (deviating *always* implies loosing the spread s^*). It follows that all paths of a trigger point below C can be supported. The important difference to the co-ordination failure case is that there is no strategic uncertainty. Taking the limit of our discrete time co-ordination failure game has allowed us to eliminate all equilibria but one, even though in the continuous-time co-ordination failure case, agents also all have the same information.

2.4 Pricing

Prices will depend on the expected payoff to bondholders under an equivalent martingale measure. Prices are only determined in periods in which the asset value for the period is known, implying that expectations can be taken under the natural filtration of the asset value process.

2.4.1 Payoffs to bondholders

Letting $t+q$ tend to the time period following it, default at time t will occur when $V(t)$ hits the trigger point V^*. Suppose τ is the stopping time at which the asset value process hits the boundary and T denotes maturity. Suppose that bondholders receive a (recovery) fraction R of the asset value in the case of default before maturity (in this case, the asset value will be equal to V^*. Then the payoffs are

1. $\Pi_1(T) = D - \max(D - V_T, 0)$, if $\tau > T$ (no default before maturity)
2. $\Pi_2(\tau) = RV^*$, if $\tau < T$ (default before maturity)

With these payoffs, the model looks very similar to the standard Black and Cox (1976) case. The difference here is that the absorbing boundary is given by our trigger point, which is a fraction of the covenant (in their case it is the covenant), and that the payoff upon hitting this boundary is not equal to the asset value, but to a fraction R of the asset value. Pricing will consequently look very similar, and is relatively straightforward under standard assumptions (e. g. the assumptions that there exists a money market account with a fixed risk free rate and that the asset value is traded[4]).

2.4.2 Bond price

The bond price will be equal to the discounted expected value of the payoffs under the equivalent martingale measure (Q) defined by the money market account as a numeraire.

$$F_B(V_t, t, T) = E_t^Q \left[e^{-r(T-t)} \Pi_1 I\left(\tau > T\right) + e^{-r(T-\tau)} \Pi_2 I\left(\tau \leq T\right) \right]$$

(Here, r denotes the constant risk free interest rate, t denotes the present and I is the indicator function)

[4] In the author's opinion the latter assumption is unpalatable but unfortunately standard in the literature

At this stage pricing is straightforward. It was first explored by Black and Cox (1976). For a good recent treatment, cf. e. g. Ericsson and Reneby (1998).

The bond can be viewed as a portfolio of barrier options. We can view Π_1 as a combination of a long position in a down-and-out call with strike price 0 (which is of course just equivalent to a down-and-out position in the underlying asset value), and a short position in a down-and-out call in with a strike price of D. This captures the fact that a bond can be viewed as a bull spread on the asset value. We can view Π_2 as a long position in RV^* units of a dollar-in-boundary claim (a claim that pays one dollar in the case the boundary is hit before maturity). The price is the sum of prices of these positions:

$$\begin{aligned} F_B(V(t), t, T, V^*) = F_{C,DO}(V(t), 0, t, T) - F_{C,DO}(V(t), D, t, T) \\ + RV^* F_{DIB}(V(t), t, T) \end{aligned}$$

where $F_{C,DO}(V(t), Z, t, T)$ denotes the price of a down-and-out call with strike price Z on the underlying V at t with maturity T. Similarly, $F_{DIB}(V(t), t, T)$ denotes the price of a dollar-in-boundary claim. The interested reader is referred to the appendix, section 5.3 for details of how the components are priced.

2.4.3 Equity price

[5]

Similarly, we can interpret equity as a down-and-out call option on the asset value. Hence the price of equity is given by

$$F_E(V(t), t, T, V^*) = F_{C,DO}(V(t), D, t, T).$$

The sum of the value of equity and the value of debt equals the market value of the firm:

$$F_{MV}(V(t), t, T, V^*) = F_{C,DO}(V(t), 0, t, T) + RV^* F_{DIB}(V(t), t, T)$$

This is in essence the sum of the price of the down-and-out asset value of the firm, plus the price of the down-and-in claim representing the recovery

[5]Note that the equity price presented here depends on the assumption that the amount borrowed by banks is negligible when compared to the total amount of publicly traded debt. For a non-negligible amount, the equations would have to be adjusted slightly.

value in the case of default. Note that the market value is not equal to the asset value of the firm unless $R = 1$, i. e. there is no cost associated with bankruptcy.

2.4.4 Extension to coupon-paying debt

An extension to a coupon-paying bond is simple, using methods described by Ericsson and Reneby (1998). If we assume that coupon i is paid in case the boundary is not reached prior to its maturity t_i, and not paid if the boundary is reached and reorganisation takes place at or before maturity, then we can think of the coupon essentially as a down-and-out binary cash call with the strike price and barrier equal to the trigger point. If the coupon rate is c of face value D, then the value of the bond will be increased by

$$B_c = cD \sum_i F_{BCC,DO} \left(V(t), t, t_i, V^* \right)$$

The value of equity will be decreased by

$$E_c = - \left(1 - \varkappa \right) cD \sum_i F_{BCC,DO} \left(V(t), t, t_i, V^* \right),$$

where $F_{BCC,DO}$ denotes the pricing function of a down-and-out binary cash call, and $\varkappa$ is the tax rate - this captures the value of the tax shield to equity.

3 Discussion

Note that the present model is a special case of a very generic bond pricing approach (Ericsson and Reneby 1998) that views a bond as a portfolio of simple and barrier claims. This also goes for other bond pricing models integrating strategic interaction (Mella-Barral and Perraudin 1997). In essence, incorporating elements of strategic interaction into bond pricing produces arguments as to what the simple and barrier claims should be that make up the bond. Strategic interaction produces different payoffs in different states of the world which determine the price. Of course, any other model that produces the same payoffs will also produce the same price.

3.1 Comparison to Black-Cox and Merton

In the Black and Cox (1976) case, the amount recovered if bondholders liquidate the firm is simply $V(t)$. It is trivial to show that this is always more than $B(t)$ (intuitively, this is the case since $B(t)$ represents a claim to $V(T)$ in some states of the world, and a claim to a value less than $V(T)$ in others, whereas $V(t)$ represents a claim to $V(T)$ in all states of the world). Liquidation or bankruptcy will therefore always occur when the covenant is breached, as it is a dominant strategy for bondholders to liquidate.

Note that in the co-ordination failure model, if the opportunity cost of stopping to lend is set to zero ($s^* = 0$), and there are no costs to bankruptcy ($R = 1$), the trigger point is C, which is simply the value of the covenant, and the amount recovered is the same as in the Black and Cox (1976) model. The prices will coincide. Essentially, if the cost of stopping lending is zero, then it becomes a dominant strategy to do so as soon as possible, if there are no bankruptcy costs.

If we let $s^* \to \infty$, it is always preferable to continue lending, and the barrier drops to zero. In this case, bankruptcy before maturity does not occur, and the price will coincide with the Merton (1974) price.

The effect producing the price that differs from Merton price is twofold: Firstly, the possibility of early default - i.e. receiving money before the maturity of the bond - increases the value of the bond, as in the Black-Cox case. But secondly, since there is a cost of reorganisation $R < 1$, this decreases the value of the bond, ceteris paribus. Because these two effects conflict, the co-ordination failure model does not produce a discount vis-a-vis the Merton case in all situations. However, it will make it possible for realistic volatilities to generate low bond prices, as opposed to the Merton model.

3.2 Comparison to Morris-Shin

Morris and Shin (2001) refer to a version of equation (4), and argue that if one assumes the trigger point to be fixed, one would underprice debt, as the trigger point is actually a decreasing function of the asset value. So as the asset value decreases, the trigger point moves up. Ignoring this effect would cause overpricing. The effect mentioned by Morris and Shin (2001) does not cause the difference in the price to the Merton (1974) model here, because the continuous time limit of the trigger point (equation 1) is not a function of the asset value - it is constant.

3.3 Comparison to commercial implementations

In their implementation of a Merton (1974) model with a barrier, the KMV corporation calculates the default barrier as a function of the book value of long-term debt, convertible debt, preferred equity and common equity. In general KMV finds that the

> the default point [...] generally lies somewhere between total liabilities and current, or short-term liabilities

(Crosbie and Bohn 2002, p. 3). We can view this as an ad-hoc attempt at calculating the trigger point V^*, which is of course likely to be related to all these balance sheet items.

Another commercial implementation, the CreditGrades$^{\text{TM}}$ model (Finger et al. 2002), acknowledges the difficulty in determining the appropriate default barrier in a pricing exercise and treats it as random. The mean of the default barrier is calculated from balance sheet items, in a manner similar to the KMV calculations, and from average recovery rates, and their standard deviation. The main effect of introducing a random barrier is to produce higher spreads at low volatilities for shorter-dated instruments.

Ultimately, both models look very similar to the model presented in this paper, the difference being only in assumptions about the exact location of the default barrier and recovery fractions.

Additionaly, argument presented in this paper suggests that when calculating an estimate of the default barrier from balance sheet items, it might be worthwile looking into the structure of short-term debt, short-term liquidity needs and any covenants that might be in place. The co-ordination failure approach as well as recent experience demonstrates that these are potentially a lot more important than longer term debt in determining the location of a default barrier.

4 Concluding Remarks

This paper argues that co-ordination failure is likely to arise between banks that have jointly lent to a large firm. These co-ordination failures are very likely to have an effect on the price of any bonds issued by the firm. Using a game-theoretic argument based on Morris and Shin's (2001) model, a continuous-time structural model of a bond price in the presence of such co-ordination failure is derived. The resulting pricing equation is based on an endogenously derived default barrier that allows default before maturity. Although many extensions to the Merton (1974) model have been suggested since the initial introduction of a default barrier (Black and Cox 1976), this enhancement has been one of the most popular ones with practitioners, and it has been implemented by e. g. the KMV Corporation (now Moody's KMV) (Crosbie and Bohn 2002), and CreditGrades^{TM} (Finger et al. 2002). In view of the analysis presented here as well as some recent cases of financial distress (e. g. Vivendi, Kirch, Enron, Energis, Worldcom or Hutchison 3G) it might worthwhile to look at the role of covenants, short-term bank debt and short-term liquidity needs in the estimation of default barriers.

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5 Appendix

5.1 Solution of the discrete time model

5.1.1 Basic procedure

We follow the same procedure as Morris and Shin (2001) to solve the model. Suppose that agents follow a switching strategy around a certain posterior belief. Given the posterior belief around which agents switch, it is possible to derive the fraction of them that will foreclose, given the asset value in the next period (posterior beliefs will be centered around this asset value in the next period). We can therefore work out what the critical next-period asset value is for which the firm will fail, given the belief in this period around which agents switch.

Also, we can use the fact that agents will switch if they believe that they will obtain a higher utility from doing so. Once we have defined utilities, this allows us to derive the critical posterior belief, given a critical next period asset value for which the firm fails.

So we have two equations in two unknowns, which can then be solved for the critical asset value for which the firm fails - the trigger point.

5.1.2 Information

For convenience, the assumptions about information are restated here. The relative increase in the asset value is normally distributed around a drift.

$$V_{t+\Delta} - V_t = \mu_V V_t \Delta + V_t \eta_t, \quad \eta_t \sim NID\left(0, \frac{1}{\alpha}\right)$$

Agents receive a signal X_i (subscript i indexes the different agents) about this increase with a distribution conditional on the asset value V_t given by

$$X_i = V_{t+\Delta} + V_t \varepsilon_i, \quad \varepsilon_i \sim NID\left(0, \frac{1}{\beta}\right),$$

with $Cov(\eta_t, \varepsilon_i) = 0$, i. e. the noise is orthogonal to the innovations in the fundamental.

5.1.3 Posteriors

From the signal X_i and the public information V_t, agents form a posterior about the value of the firm in period $t + \Delta$, $V_{t+\Delta}$ which is normal with mean and variance given by

$$\rho_i = E(V_{t+\Delta}|X_i) = \frac{\alpha(1+\mu_V\Delta)V_t + \beta X_i}{\alpha+\beta}$$

and

$$Var(V_{t+\Delta}|X_i) = \frac{(V_t)^2}{\alpha+\beta}.$$

5.1.4 Critical value of $V_{t+\Delta}$ for which the firm fails

Given the posterior belief around which agents switch, we work out how many of them will foreclose, given the asset value in the next period (posterior beliefs will be centered around this asset value in the next period). We then work out what the critical next-period asset value is for which the firm fails, given the belief in this period around which agents switch.

Suppose agents follow a switching strategy around ρ^*, i. e. agents foreclose when their posterior is below ρ^*. Then an agent will not foreclose if and only if the private signal is bigger than

$$X^* = \frac{\alpha+\beta}{\beta}\rho^* - \frac{\alpha}{\beta}(1+\mu_V\Delta)V_t.$$

Conditional on state $V_{t+\Delta}$, the distribution of X_i is normal with mean $V_{t+\Delta}$ and precision $\frac{\beta}{V_t^2}$. So the ex-ante probability for any agent of foreclosing is equal to

$$\Phi\left\{\frac{1}{V_t}\sqrt{\beta}(X^* - V_{t+\Delta})\right\}.$$

As the number of agents tends to infinity, the fraction of agents that foreclose will be equal to this ex ante probability for any individual agent by the law of large numbers.

Since the firm fails if the fraction that forecloses is $l \geq \frac{V_{t+\Delta}}{C}$, the critical value of $V_{t+\Delta}$ (denoted by $V_{t+\Delta}^*$) for which the firm fails at t is given by $V_{t+\Delta}^* = C\Phi\left\{\frac{1}{V_t}\sqrt{\beta}(X^* - V_{t+\Delta}^*)\right\}$ or

$$V_{t+\Delta}^* = C\Phi\left\{\frac{1}{V_t}\left(\frac{\alpha}{\sqrt{\beta}}\left(\rho^* - (1+\mu_V\Delta)V_t\right) + \sqrt{\beta}\left(\rho^* - V_{t+\Delta}^*\right)\right)\right\}. \qquad (2)$$

5.1.5 Utility

Banks are myopic and only care about present utility u which is a function of present consumption c_{t+q}. This assumption makes it possible to solve the game stage by stage, and simplifies the solution considerably. Consumption in any intermediate period $t+q$ is as described in the payoff matrix in the main text, reproduced here for convenience:

	firm fails (F)	firm does not fail (C)
foreclose (f)	0	0
do not foreclose (n)	$-P$	s^*P

This is consumption per fraction of the loan that the bank holds. Let ξ denote the fraction the bank holds.

Then consumption in the case the agent decides to foreclose (f) and the firm fails (F) is

$$c_{t+q}[fF] = 0,$$

consumption in the case the agent decides to foreclose (f) but the firm does not fail, i. e. it continues (C) is

$$c_{t+q}[fC] = 0,$$

consumption in the case the agent decides not to foreclose (n) and the firm fails (F) is

$$c_{t+q}[nF] = -P\xi_t,$$

and consumption in case the agent decides not to foreclose (n) and the firm does not fail, i. e. it continues C is

$$c_{t+q}[nC] = s^*P\xi_t.$$

5.1.6 Critical value of ρ

We now use the fact that agents will switch if they believe that they will obtain a higher utility from doing so. We then derive the critical posterior belief, given a critical next period asset value for which the firm is reorganised.

Now the marginal agent (one that is indifferent between forcing reorganisation or not) has a posterior over the asset value which has its mean just at the switching point (i.e. ρ for this agent is equal to ρ^*). For her the expected utility of not foreclosing should just equal the expected utility of foreclosing.

This *defines* the switching point. Using F to denote the posterior cumulative distribution (given the belief) over the asset value $V_{t+\Delta}$ we can write:

$$\int_{-\infty}^{V^*_{t+\Delta}} u\left(c_{t+q}\left[fR\right]\right)dF + \int_{V^*_{t+\Delta}}^{\infty} u\left(c_{t+q}\left[fC\right]\right)dF =$$

$$\int_{-\infty}^{V^*_{t+\Delta}} u\left(c_{t+q}\left[nR\right]\right)dF + \int_{V^*_{t+\Delta}}^{\infty} u\left(c_{t+q}\left[nC\right]\right)dF$$

Note that the utility at $t+q$ does not depend on $V_{t+\Delta}$. We can therefore write:

$$u\left(c_{t+q}\left[fR\right]\right)\Pr\left(V_{t+\Delta} < V^*_{t+\Delta}\right) + u\left(c_{t+q}\left[fC\right]\right)\Pr\left(V_{t+\Delta} > V^*_{t+\Delta}\right) =$$

$$u\left(c_{t+q}\left[nR\right]\right)\Pr\left(V_{t+\Delta} < V^*_{t+\Delta}\right) + u\left(c_{t+q}\left[nC\right]\right)\Pr\left(V_{t+\Delta} > V^*_{t+\Delta}\right)$$

We can write this probability as:

$$\Pr\left(V_{t+\Delta} > V^*_{t+\Delta}\right) = \Phi\left\{\frac{\sqrt{\alpha+\beta}}{V_t}\left(\rho^* - V^*_{t+\Delta}\right)\right\}$$
$$\left(= 1 - \Pr\left(V_{t+\Delta} < V^*_{t+\Delta}\right)\right).$$

We insert this and rearrange to obtain

$$\rho^* - V^*_{t+\Delta} = \frac{V_t}{\sqrt{\alpha+\beta}}\Phi^{-1}\left\{\frac{u\left(c_t[fF]\right) - u\left(c_t[nF]\right)}{u\left(c_t[nC]\right) - u\left(c_t[nF]\right) + u\left(c_t[fF]\right) - u\left(c_t[fC]\right)}\right\}$$

Note that by definition, $c_t[fC] = c_t[fF]$, so the last two terms in the denominator drop out.

Now take limits as the number of agents goes to infinity. This will imply that the fraction of the loan held by any individual agent goes to zero, $\xi_t \to 0$. Note that we have a fraction of functions of ξ_t, and can apply l'Hopital's rule. In the limit, all c_{t+q} are equal, so

$$\lim_{\xi_t \to 0} \frac{u\left(c_t[fF]\right) - u\left(c_t[nF]\right)}{u\left(c_t[nC]\right) - u\left(c_t[nF]\right))} =$$

$$\frac{u'(c_{t+q})(0) - u'(c_{t+q})(-P)}{u'(c_{t+q})(s^*P) - u'(c_{t+q})(-P)} = \frac{1}{1+s^*}.$$

So the limit of our equation is

$$\rho^* - V^*_{t+\Delta} = \frac{V_t}{\sqrt{\alpha+\beta}} \Phi^{-1}\left\{\frac{1}{1+s^*}\right\}. \tag{3}$$

This equation together with (2) pins down the critical value of beliefs and the asset value.

5.1.7 Equilibrium forced reorganisation

Combining equations (3) and (2) we can solve for the failure point at which the asset value in the next period causes failure in this period:

$$V^*_{t+\Delta} = C\Phi\left\{\frac{\alpha}{\sqrt{\beta}}\left(\frac{V^*_{t+\Delta}}{V_t} - 1 - \mu_V\Delta\right) + \frac{\sqrt{\alpha+\beta}}{\sqrt{\beta}}\Phi^{-1}\left\{\frac{1}{1+s^*}\right\}\right\} \tag{4}$$

Reorganisation at time $t+q$ will occur when V hits V^* at $t+\Delta$.

5.1.8 Uniqueness

To simplify notation, define

$$Z = \frac{\alpha}{\sqrt{\beta}}\left(\frac{V^*_{t+\Delta}}{V_t} - 1 - \mu_V\Delta\right) + \frac{\sqrt{\alpha+\beta}}{\sqrt{\beta}}\Phi^{-1}\left\{\frac{1}{1+s^*}\right\}$$

and

Condition A $C\frac{1}{\sqrt{2\pi}}\frac{\alpha}{\sqrt{\beta}}\frac{1}{V_t} < 1$

Proposition 1 *The trigger point $V^*_{t+\Delta}$ is unique if condition (A) is satisfied.*

Proof.
This is a version of the proof in Morris and Shin (2001). A sufficient condition for a unique solution is that the slope of

$$C\Phi\{Z\}$$

is less than one everywhere. This slope is equal to

$$C\varphi\{Z\}\frac{\alpha}{\sqrt{\beta}}\frac{1}{V_t}.$$

It reaches a maximum where the argument of the normal density is 0, the maximum there will be $\frac{1}{\sqrt{2\pi}}$. Hence a sufficient condition for a unique solution is that

$$C\frac{1}{\sqrt{2\pi}}\frac{\alpha}{\sqrt{\beta}}\frac{1}{V_t} < 1.$$

■

5.2 Uncertainty in the limit

It can be shown that the marginal or pivotal agent views the fraction of banks that attempt to foreclose as a random variable that is uniformly distributed in the continuous-time limit, and hence that strategic uncertainty remains. Note that these kind of results have been discussed at length elsewhere (Morris and Shin 2002a).

Proposition 2 *The distribution of l given the belief ρ^* of the marginal agent is uniform in the limit*

Proof.
The proportion of people who receive a lower signal X^* is

$$l = \Phi\left(\frac{\sqrt{\beta}}{V_t}\left(X^* - V_{t+\Delta}\right)\right).$$

The question to ask is: What is the probability that a fraction less than z of the other bondholders receive a signal higher than that of the marginal agent, conditional on the marginal agent's belief, or what is $\Pr\left((1-l) < z \mid \rho^*\right)$?

Now the event

$$1 - l < z$$

is equivalent to

$$1 - \Phi\left(\frac{\sqrt{\beta}}{V_t}\left(X^* - V_{t+\Delta}\right)\right) < z$$

or (rearranging)

$$V_{t+\Delta} < X^* + \frac{V_t}{\sqrt{\beta}}\Phi^{-1}\left(1-z\right).$$

So the probability we are looking for is $\Pr\left(V_{t+\Delta} < X^* + \frac{V_t}{\sqrt{\beta}}\Phi^{-1}\left(1-z\right) \mid \rho^*\right)$. The posterior of the marginal agent over $V_{t+\Delta}$ has mean ρ^* and variance $\frac{V_t^2}{\alpha+\beta}$, hence this probability is

$$\Pr\left((1-l) < z \mid \rho^*\right) = \Phi\left(\frac{\sqrt{\alpha+\beta}}{V_t}\left(X^* + \frac{V_t}{\sqrt{\beta}}\Phi^{-1}(1-z) - \rho^*\right)\right).$$

Now as we take limits, $\rho^* \to X^*$, since private information becomes infinitely more precise than public information (the agent attaches all weight to the signal and none to the mean of the prior), and $\frac{\sqrt{\alpha+\beta}}{\sqrt{\beta}} \to 1$. It follows that

$$\Pr\left((1-l) < z \mid \rho^*\right) = 1 - z,$$

or

$$\Pr\left(l < 1 - z \mid \rho^*\right) = 1 - z.$$

So the cumulative distribution of l is the identity function, which implies that the density will be uniform. ■

5.3 Pricing the bond

Pricing in our context is a standard procedure. For a good recent treatment, cf. e. g. Ericsson and Reneby (1998). These methods were first used by Black and Cox (1976). Essentially, the bond will be viewed as a portfolio of barrier options. Some standard results are restated for convenience.

5.3.1 Default process

Define the default process

$$Y(t) \equiv \frac{1}{\sigma_V}\ln\frac{V(t)}{V^*}. \tag{5}$$

This process will take the value zero in default. It is a Wiener process that can be written as $dY = \mu_Y^M dt + dW^M$, where μ_Y^M is the drift of Y under measure M. Note that it is standard to assume that V is traded.[6] It follows that under standard assumptions, the drift of V under the risk neutral measure Q (with the money market account as numeraire) is r, and the drift of Y under Q will be $\mu_Y^Q = \frac{r-\frac{1}{2}\sigma_V^2}{\sigma_V}$.

[6] In the author's opinion, this is an unsatisfactory assumption, however it is standard in the literature.

5.3.2 Standard distributional results

There are some useful standard results for this default process.

The first passage time density (under M) at time t of the default process to zero is given by (cf. e. g. Øksendal 2000)

$$f^M(Y(t),\tau) \equiv \frac{Y(t)}{\sqrt{2\pi\,(\tau-t)^3}} e^{-\frac{1}{2}\frac{1}{(\tau-t)}\left(Y(t)+\mu_Y^M(\tau-t)\right)^2}. \tag{6}$$

The point mass for the first passage time is given by

$$\begin{aligned}\Pr^{M}(\tau \le T) &= \int_t^T f^M(Y(t),\tau)d\tau = \Phi\left(\frac{-Y(t)-\mu_Y^M(T-t)}{\sqrt{T-t}}\right) \\ &+ e^{-2\mu_Y^M Y(t)}\Phi\left(\frac{-Y(t)+\mu_Y^M(T-t)}{\sqrt{T-t}}\right)\end{aligned} \tag{7}$$

where Φ is the normal cumulative density. The density of $Y(T)$, given that Y has not hit 0 before maturity, is given by

$$\begin{aligned}f^M\left(Y(T);Y(t),t,T\right) &= \frac{1}{\sqrt{2\pi\,(T-t)}} e^{-\frac{1}{2}\left(\frac{Y(T)-Y(t)-\mu_Y^M(T-t)}{\sqrt{T-t}}\right)} \\ &- e^{-2\mu_Y^M Y(t)}\frac{1}{\sqrt{2\pi\,(T-t)}} e^{-\frac{1}{2}\left(\frac{Y(T)+Y(t)-\mu_Y^M(T-t)}{\sqrt{T-t}}\right)}\end{aligned} \tag{8}$$

Using the definition of $Y(t)$, we can change variables and work out the corresponding results under Q, where $\mu_Y^Q = \frac{r-\frac{1}{2}\sigma_V^2}{\sigma}$, or the measure G, where $\mu_Y^G = -\frac{r-\frac{1}{2}\sigma_V^2}{\sigma}$.

Use $\tilde{r}$ to denote $r-\frac{1}{2}\sigma_V^2$, x to denote $\frac{V^*}{V(t)}$ and θ to denote $\frac{r+\frac{1}{2}\sigma_V^2}{\sigma_V^2}$, then:

$$\Pr^{G}(\tau \le T) = \Phi(d_5) + x^{-2\theta}\Phi(d_6) \tag{9}$$

where

$$d_5 = \frac{\ln x + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma_V\sqrt{T-t}} \tag{10}$$

and

$$d_6 = \frac{\ln x - \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma_V\sqrt{T-t}}. \tag{11}$$

Also,

$$f^Q\left(\ln V(T); \ln V(t)t, T\right) \tag{12}$$
$$= \frac{1}{\sqrt{2\pi\left(T-t\right)}} e^{-\frac{1}{2}\left(\frac{\ln V(T) - \ln V(t) - \tilde{r}(T-t)}{\sigma_V \sqrt{T-t}}\right)}$$
$$-\left(\frac{V^*}{V(t)}\right)^{\frac{2\tilde{r}}{\sigma_V^2}} \frac{1}{\sqrt{2\pi\left(T-t\right)}} e^{-\frac{1}{2}\left(\frac{\ln V(T) - \ln\left(V(t)x^2\right) - \tilde{r}(T-t)}{\sigma_V \sqrt{T-t}}\right)}$$

5.3.3 Prices of constituent claims

Dollar-in-boundary claim The price of a claim that pays 1 when the boundary is hit (dollar-in-boundary claim) is

$$F_{DIB}\left(V(t), t\right) = E^Q\left[e^{-r(T-\tau)} I\left(\tau \leq T\right)\right] = \int_t^T e^{-r(\tau - t)} f^Q(Y(t), \tau) d\tau$$

which (completing the squares and integrating) yields (see e.g. Ericsson and Reneby 1998, in the appendix)

$$F_{DIB}\left(V(t), t\right) = e^{-\left(\mu_Y^Q - \mu_Y^G\right)Y(t)} \overset{G}{\Pr}\left(\tau \leq T\right). \tag{13}$$

Noting that

$$\frac{\left(\mu_Y^Q + \sqrt{\left(\mu_Y^Q\right)^2 + 2r}\right)}{\sigma_V} = \frac{2r}{\sigma_V^2}$$

we can write this as

$$F_{DIB}\left(V(t), t\right) = x^{\frac{2r}{\sigma_V^2}} \left(\Phi\left(d_5\right) + x^{-2\theta}\Phi\left(d_6\right)\right)$$

or

$$F_{DIB}\left(V(t), t\right) = \frac{V(t)}{V^*}\left\{x^{2\theta}\Phi\left(d_5\right) + \Phi\left(d_6\right)\right\} \tag{14}$$

Down-and-out claim Define the payoff $\Pi_{Tr}(V(T))$ as the payoff $\Pi(V(T))$ truncated at V^*:

Then we can derive the price of the down-and-out claim in terms of the prices of the truncated down-and-out claims with starting values for the process of $V(t)$ and $V(t)x^2$, using the distribution given in (8):

$$F_{\Pi,DO}\left(V(t),t\right) = F_{\Pi_{Tr}}(V(t),t) - x^{\frac{2\tilde{r}}{\sigma_V^2}} F_{\Pi_{Tr}}\left(V(t)x^2,t\right) \tag{15}$$

For a good exposition, see e. g. Björk (1998).

Down-and-out call For example, consider a down-and-out call. For pricing a down-and-out call with strike price Z, we need to know what the price of a truncated call is. For a call whose price is truncated at V^*, with a starting value of the process equal to S, the price will be the simple Black-Scholes price if $V^* \leq Z$, i. e. if the truncated payoff is just equal to the normal call payoff:

$$F_C\left(S,Z,t\right) = S\Phi\left(d_1\right) - e^{-r(T-t)}Z\Phi\left(d_2\right) \tag{16}$$

where

$$d_2\left(S\right) = \frac{\ln\left(\frac{S}{Z}\right) + \tilde{r}\left(T-t\right)}{\sigma_V\sqrt{T-t}} \tag{17}$$

and

$$d_1\left(S\right) = d_2 + \sigma_V\sqrt{T-t}. \tag{18}$$

If $V^* > Z$, the price will be different. Denote the price of the truncated call as $F_{C,Tr}$. It is given by

$$F_{C_{Tr}}\left(S,Z,t\right) = S\Phi(d_3) - e^{-r(T-t)}Z\Phi(d_4) \tag{19}$$

where now

$$d_4\left(S\right) = \frac{\ln\left(\frac{S}{V^*}\right) + \tilde{r}\left(T-t\right)}{\sigma_V\sqrt{T-t}} \tag{20}$$

and

$$d_3\left(S\right) = d_4 + \sigma_V\sqrt{T-t} \tag{21}$$

We can now insert these pricing functions into the equation for a down-and-out-price to obtain the pricing functions $F_{C,DO}$.

Down-and-out call with strike price 0 By the above formulas, this is given by

$$F_{C,DO}\left(V(t),0,t\right)=V(t)\left[\Phi\left\{d_3(V(t)\right\}-x^{2\theta}\left(\Phi\left\{d_3\left(V(t)x^2\right)\right\}\right)\right] \tag{22}$$

Down-and-out call with strike price D**, price of equity** Suppose we have a call with a strike price of D, which is bigger than V^*, so that the payoff function is not truncated, then its price is

$$\begin{aligned} F_{C,DO}\left(V(t),D,t\right) &= V(t)\left(\Phi\left\{d_1(V(t)\right\}-x^{2\theta}\Phi\left\{d_1\left(V(t)x^2\right)\right\}\right) \\ &\quad + e^{-r(T-t)}D\left(x^{\frac{2\tilde{r}}{\sigma_V^2}}\Phi\left\{d_2\left(V(t)x^2\right)\right\}-\Phi\left\{d_2(V(t)\right\}\right) \end{aligned} \tag{23}$$

This is also the value of equity (F_E).

Note that the derivative of the value of equity with respect to the asset value is

$$\frac{\partial F_E}{\partial V}=\Phi\left\{d_1\left(V\right)\right\}+\frac{2\tilde{r}}{\sigma_V^2}\left(\frac{x}{\left(V^*\right)^2}\right)\left(Vx^2\Phi\left\{d_1\left(Vx^2\right)\right\}-e^{-r(T-t)}D\Phi\left\{d_2\left(Vx^2\right)\right\}\right) \tag{24}$$

So using Itô's lemma, we can now find the volatility of equity as

$$\sigma_{F_E}=\frac{V}{F_E}\frac{\partial F_E}{\partial V}\sigma_V \tag{25}$$

Bond price Recall that the dollar-in-boundary claim is

$$F_{DIB}\left(V(t),t\right)=\frac{V(t)}{V^*}\left[x^{2\theta}\Phi\left\{d_5\right\}+\Phi\left\{d_6\right\}\right]. \tag{26}$$

Supposing that the recovery fraction is R, we will need RV^* of these - this is what we get for holding the bond in case of early default.

Putting it all together, we can calculate the price of the bond as

$$\begin{aligned} B(V(t),t,T,V^*) &= F_{C,DO}(V(t),0,t,T) \\ &\quad - F_{C,DO}(V(t),D,t,T) \\ &\quad + RV^*F_{DIB}\left(V(t),t\right) \end{aligned} \tag{27}$$

$$
\begin{aligned}
B(V(t),t,T,V^*) &= V(t)\left[\Phi\left\{d_3(V(t)\right\} - x^{2\theta}\left(\Phi\left\{d_3\left(V(t)x^2\right)\right\}\right)\right] \qquad (28)\\
&- V(t)\left(\Phi\left\{d_1(V(t)\right\} - x^{2\theta}\Phi\left\{d_1\left(V(t)x^2\right)\right\}\right)\\
&- e^{-r(T-t)}D\left(x^{\frac{2\tilde{r}}{\sigma_V^2}}\Phi\left\{d_2\left(V(t)x^2\right)\right\} - \Phi\left\{d_2\left(V(t)\right)\right\}\right)\\
&+ RV^*\frac{V(t)}{V^*}\left[x^{2\theta}\Phi\left\{d_5\right\} + \Phi\left\{d_6\right\}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
x &= \frac{V^*}{V(t)}\\
\theta &= \frac{r+\frac{1}{2}\sigma_V^2}{\sigma_V^2}\\
\tilde{r} &= r - \frac{1}{2}\sigma^2\\
d_1(V(t)) &= \frac{\ln\left(\frac{V(t)}{D}\right) + \left(r+\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}\\
d_2(V(t)) &= \frac{\ln\left(\frac{V(t)}{D}\right) + \left(r-\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}\\
d_3(V(t)) &= \frac{-\ln x + \left(r+\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}\\
d_1(V(t)x^2) &= \frac{\ln\left(\frac{V(t)}{D}\right) + 2\ln x + \left(r+\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}\\
d_2(V(t)x^2) &= \frac{\ln\left(\frac{V(t)}{D}\right) + 2\ln x + \left(r-\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}\\
d_3(V(t)x^2) &= \frac{\ln x + \left(r+\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}\\
d_5 &= \frac{\ln x + \left(r+\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}(= d_3(V(t)x^2))\\
d_6 &= \frac{\ln x - \left(r+\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}(= -d_3(V(t)))
\end{aligned}
$$

Now note that $-d_3(V(t)) = d_6$, and $d_5 = d_3\left(V(t)x^2\right)$.