# Liability Valuation and Optimal Asset Allocation

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Abstract:

Current approaches to asset-liability management employ a sequence of distinct procedures to value liabilities and determine the asset allocation. First, a discount rate that is usually dictated by accounting standards is used to value liabilities. Second, the asset allocation is determined by maximizing some objective function in the surplus of assets over liabilities, taken as given the valuation of liabilities. We introduce a model that allows for the joint valuation of liabilities and the determination of the optimal asset allocation using discount rates that appropriately reflect default risk. We focus on the case of a defined benefit pension plan.

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## 1. Introduction

This paper deals with liability valuation and optimal asset allocation and the key word is the conjunctive ‘and’. Both in practice and in the theoretical literature, liability valuation and asset allocation are typically treated as completely separate issues, despite lip service to the contrary. A classic example is defined benefit pension fund liabilities and assets.

What we tend to observe is the projected future liability cash flows being discounted using a set of discount rates that fail to reflect the true risk attached to that liability stream. A whole range of discount rates are used in practice: high quality (AA) corporate bonds (as required by various accounting standards: FAS87 (US), FRS17 (UK) and IAS19 (international)); the weighted average return on a notional portfolio of statutory reference assets (as required by the UK Minimum Funding Requirement); and the weighted average expected return on the actual portfolio of assets supporting the liabilities (as used in most actuarial valuations); for more details, see, e.g., Blake (2001).

At the same time, the asset allocation is generally chosen quite independently of the projected liability stream. Typically, at least in Anglo-Saxon countries, the pension plan trustees choose (or are advised by their investment consultant to choose) a high weighting in equities in order to benefit from the equity risk premium and hence ‘lower’ the cost to the plan sponsor of providing pensions. In other countries, e.g., many in continental Europe, pension funds are encouraged to invest heavily in government bonds in order to help governments finance their national debt. Some would argue (e.g., financial economists such as Bodie (1995) and radical actuaries such as Exley, Mehta and Smith (1997), Gold (2001) and Bader and Gold (2003)) that pension funds should be entirely invested in bonds on the grounds that pension funds should not take risks with the sponsoring company’s shareholders’ funds and that pension payments are bond-like in nature. The Boots pension fund in the UK was sufficiently persuaded by this argument that between April 2000 and July 2001 it switched all its assets into bonds (see Blake (2003a, 462-465)).

There might well be rational explanations for why certain of the above practices emerged. For example, AA corporate bonds were chosen for discounting purposes under FAS87 because this was the asset class that US insurance companies used when taking over the pension obligations of insolvent US companies. The UK accounting standard FRS17 adopted the same discount rate even though AA corporate bonds are not a significant investment category in the UK, accounting for only 7% of UK bonds outstanding in 2000, the year FRS17 was announced. Similarly, the adoption of returns on the assets in the pension fund to discount liabilities was intended to minimize any asset-liability mismatch, but leads to the uncomfortable implication that a pension fund can ‘reduce’ the value of its liabilities by investing in a riskier asset class.

The underlying impression from all of this is that there is a lack of consistency between the way in which liabilities are valued and the way in which the asset allocation is decided. We will argue in this paper that they must be jointly determined, otherwise potential inconsistencies emerge. The valuation of the liabilities depends on a discount rate (more precisely a discount term structure) that depends, in turn, on the asset allocation chosen.

We are aware of only one author who explicitly discusses the relationship between the discount rate and asset allocation decisions, namely Petersen (1996) in an empirical study of US pension plans. He argues that a portfolio shift from low-risk to high-risk asset classes should be accompanied by an increase in the discount rate on pension liabilities. He also points out that the discount rate should increase with a decreasing funding ratio. While the latter argument is confirmed by his empirical analysis, the evidence for the former turns out to be somewhat mixed. On the one hand, firms tend to increase the discount rate with a higher equity allocation relative to cash; on the other hand, they tend to increase the discount rate even more with an increasing bond allocation relative to cash, a finding that is clearly inconsistent with his line of reasoning.

VanDerhei (1990) considers the possibility of defaulting on the pension promise in his derivation of fair-value insurance premiums for US defined benefit pension plans covered by the Pension Benefit Guaranty Corporation (PBGC). While he does not account explicitly for the asset allocation of the pension plan, he includes the funding ratio (of assets to liabilities) as an explanatory variable in the default probability regression and reports the anticipated negative impact. He also calculates insurance premiums as the product of the estimated default probability and the estimated scale of default. In related work, Carroll and Niehaus (1998) investigate the impact of unfunded pension liabilities on corporate debt ratings, with higher ratings usually associated with smaller default spreads. They find that ratings increase with over-funding and decrease even more so with under-funding. Over the last few years, with most pension funds having to deal with under-funding problems caused by a combination of equity market declines and a legacy of sponsor contribution holidays, pension deficits have been repeatedly cited by rating agencies as a reason for downgrading actions.[1]

The outline of the paper is as follows. Section 2 presents a model of a pension plan's liabilities and their dependence on the forward term structure of discount rates. The latter are decomposed into a risk-free component and a spread reflecting the default risk of failing to deliver the promised pensions and this is most likely to be triggered by the insolvency of the sponsoring company. This default risk will be quantified in terms of the default probability and the recovery rate in the event of default (both of which will depend on the ratio of pension plan assets to liabilities and the net worth of the sponsor). Following Sharpe and Tint (1990), the objective function for the pension plan's asset allocation problem is a mean-variance function of the surplus ratio. In contrast with previous work, we show that maximizing this objective function with respect to the asset allocation simultaneously generates an appropriate default spread. Since the default spread depends on the value of the defaultable pension

[1] Standard & Poor's published a watch list of 12 European companies in February 2003 because of concerns about the unfunded pensions. Among these companies were Deutsche Post, Linde, Michelin, Sainsbury, Rolls Royce, and Thyssen-Krupp. The subsequent downgrading of Thyssen-Krupp from BBB to junk bond grade BB+ has led to a controversial discussion between S&P and the German pension industry.

claim itself, it is endogenous in the sense of Duffie and Singleton (1999) and this considerably complicates the optimization problem.

We consider a numerical example in Section 3 under the assumption that returns and yields are independent and identically multivariate normal distributed in order to shed more light on the relationship between default spreads and the optimal asset allocation. The main findings are:

- The appropriate default spread decreases with an increasing funding ratio and increasing sponsoring company net worth.
- The appropriate default spread decreases with increasing risk aversion (or an increasing willingness to pay high expected contributions) and leads to a lower equity weighting.
- The spread on AA corporate bonds, the most common asset class used in the discounting of pension plan liabilities in the US and UK, is unlikely to be an appropriate default spread in the valuation of most pension liabilities.

## 2. Asset-Liability Modeling

We consider the defined benefit pension plan of a sponsoring company whose own financial strength can be summarized in period[2] t by an exogenously given net worth $V(t)$. The set-up that we describe is a typical asset and liability management (ALM) exercise for defined benefit pension plans, one that is conducted on a regular basis, for example, during the preparation of the annual financial statements or the triennial actuarial valuation. ALM involves two issues: valuation and asset allocation. While the assets in the pension fund are valued more or less continuously by the financial markets, the current value of pension liabilities is determined by discounting all future pension payments that the company has promised to its employees. The crucial task in this calculation is the determination of the discount rate that appropriately reflects the risk of failing to deliver the promised pension payments in the future in the absence of a perfect insurance vehicle.

### 2.1 Assets and Liabilities

Current ALM approaches rely on an exogenously given discount rate F for the valuation part of the ALM exercise. For example, F could be the current yield of a AA corporate bond with long maturity. In this case, the actuarial liability of the pension plan with N members and an accrual rate based on the sixtieths scale is as follows (see Blake, 2003b, Cairns, 2003)

$$L(t,h_i)=\sum_{i=1}^{N}Y(t,h_i)\frac{W(0)}{W(h_i)}\frac{e_i}{60}\frac{pr(t,h_i)}{(1+F)^{h_i}}\sum_{\ell=1}^{L}\frac{pr(T_i,\ell)}{(1+F)^{\ell}} \tag{1}$$

where the index i refers to an employee who joined the pension plan $e_i$ years ago and has to work $h_i$ additional years before reaching the retirement age in period $T_i = t+h_i$. In the cur-

[2] The model is written in discrete time, as are most of the relevant pricing (e.g. Das and Sundaram, 2000, Cochrane, 2001) and asset allocation (e.g. Sharpe and Tint, 1990) models to which we refer.

rent period t, the employee has accrued pension benefits equal to the share, $e_i/60$, of his projected final salary, which equals the current salary, $Y(t,h_i)$, scaled up by $W(0)/W(h_i)$, the (age-related) increase in earnings over his remaining working life. F is a real discount rate that accounts for (price) inflation. Note that the last sum in (1) is the unit price of a real annuity that is bought at retirement age. We assume without loss of generality that each employee lives for a maximum of L years in retirement, which means that the probability of surviving from t to $t+s$, $pr(t,s)$, falls to zero for all $s > h_i + L$. From now on we assume again without loss of generality that $N = 1$ to simplify notation. The index i will be dropped accordingly.

Our subsequent analysis is based on two important modifications of (1). First, the assumed time-invariance of the discount rate is dropped. Second, the appropriate default spread will be derived rather than taken as given. In order to achieve these modifications, F will be replaced by forward rates that are decomposed into default-free forward rates and default spreads. Let $F(t,s,n)$ be the n-period forward rate at time t for the financial transaction in period $t+s$. Following, e.g., Das and Sundaram (2000), $F(t,s,n)$ is decomposed as

$$1+F(t,s,n) = (1+G(t,s,n))(1+D(t,s,n)) \tag{2}$$

where $G(t,s,n)$ denotes the default-free forward rate and $D(t,s,n)$ the default spread. Given this definition, (1) can be expressed more generally for $N = 1$ as

$$L(t,h) = Y(t,h)\frac{W(0)}{W(h)}\frac{e}{60}\frac{pr(t,h)}{(1+F(t,0,h))^h}\sum_{\ell=1}^{L}\frac{pr(T,\ell)}{(1+F(t,h,\ell))^\ell}. \tag{3}$$

It will be convenient to work with one-period forward rates, which we denote more compactly as $F(t,m) \equiv F(t,m,1)$. They are related, of course, to n-period forward rates by

$$(1+F(t,s,n))^n = \prod_{m=s}^{s+n-1}(1+F(t,m)). \tag{4}$$

We denote with small letters the logs of variables. Thus, we write $f(t,m) = g(t,m) + d(t,m)$ with $f(t,m) = \log(1+F(t,m))$, $g(t,m) = \log(1+G(t,m))$ and $d(t,m) = \log(1+D(t,m))$. Log prices of zero-coupon default-free and defaultable bonds are denoted $q(t,m)$ and $p(t,m)$, respectively. The corresponding one-period log returns are written as $s(t+1,m)$ and $r(t+1,m)$. In terms of one-period log forward rates, log prices and one-period log returns are defined as

$$q(t,m) = -\sum_{n=0}^{m-1} g(t,n), \quad s(t+1,m) = q(t+1,m-1) - q(t,m) = g(t,0) - \sum_{n=1}^{m-1}(g(t+1,n-1) - g(t,n)) \tag{5}$$

$$p(t,m) = -\sum_{n=0}^{m-1} f(t,n), \quad r(t+1,m) = p(t+1,m-1) - p(t,m) = f(t,0) - \sum_{n=1}^{m-1}(f(t+1,n-1) - f(t,n)). \tag{6}$$

Using these properties the current value of the pension liabilities (3) can be rewritten as

$$L(t,h)=U(t,h,e)\sum_{\ell=1}^{L}pr(T,\ell)\exp(p(t,h+\ell)) \quad \text{with} \quad U(t,h,e)=Y(t,h)\frac{W(0)}{W(h)}\frac{e}{60}pr(t,h). \tag{7}$$

Matching these liabilities in the pension fund are assets with market value $A(t)$. The asset-liability position of a defined benefit pension plan is usually summarized by the surplus, $S(t,h)=A(t)-L(t,h)$, or by the funding ratio, $C(t,h)=A(t)/L(t,h)$.

## 2.2 Pension Plan Default

We are interested in the determination of the forward spread term structure that is relevant for the computation of (7). In line with the literature on the pricing of corporate bonds (see e.g. Duffie and Singleton, 1999), we derive default spreads for the valuation of the pension claims that appropriately take into account the possibility of default. We define default for a corporate defined benefit pension plan as the event in which the combined value of the plan assets and the net worth of the sponsoring company is insufficient to cover the value of the pension liabilities. We assume that the plan has not defaulted by period t. Thus we focus on the likelihood of the future event

$$L(t+1,h-1)>A(t+1)+V(t+1) \quad \Leftrightarrow \quad S(t+1,h-1)<-V(t+1) \tag{8}$$

We make the assumption that any recovery value in the event of default in period $t+1$ will be related to the period t value of pension liabilities. Thus, we assume, in the event of default, that pension rights are frozen at their default date (i.e. discontinuance date) value, although assets continue to grow. In order to reflect this assumption, the future net worth of the sponsoring company is related to the current liabilities by $V(t+1)=\tau(t+1)\cdot L(t,h)$. Default condition (8) then becomes

$$\frac{S(t+1,h-1)}{L(t,h)}<-\tau(t+1). \tag{9}$$

We will refer to the variable on the left-hand-side as the discontinuance surplus ratio (DSR). It can be derived from the current funding ratio $C(t,h)$ of pension plan assets and liabilities as

$$\frac{S(t+1,h-1)}{L(t,h)}=(1+c)\sum_{k=1}^{K}w(t,k)\exp(x(t+1,k))\cdot C(t,h)-(1+u)\sum_{\ell=1}^{L}v(t,h,\ell)\exp(r(t+1,h+\ell)) \tag{10}$$

where c refers to contributions[3] to the plan assets as a percentage of $A(t)$, $w(t,k)$ denotes the pension fund's proportionate allocation to asset class k and $x(t+1,k)=\log(1+X(t+1,k))$ is the one-period log return for asset class $k=1,\dots,K$. The weights $v(t,h,\ell)$ follow from (7) as

$$v(t,h,\ell)=\frac{pr(T,\ell)}{(1+F(t,h,\ell))^{\ell}}\left(\sum_{l=1}^{L}\frac{pr(T,l)}{(1+F(t,h,l))^{l}}\right)^{-1} \quad \text{with} \quad \sum_{\ell=1}^{L}v(t,h,\ell)=1. \tag{11}$$

[3] These could become negative in the case where the plan member dies before retirement.

Equation (10) shows that the growth rate of the actuarial liability is proportional to the one-period log return of a value-weighted portfolio of defaultable bonds with maturities between $h+1$ and $h+L$. The factor $u = U(t+1, h-1, e+1)/U(t, h, e) - 1$ in the liability growth formula incorporates time- and age-specific wage inflation as well as changes in survival probabilities and benefit accrual. Since this paper focuses on the relationship between discount rates and asset allocation, we simplify matters by assuming that these changes can be summarized by some constant and known growth factor, therefore any sub-indices are omitted for u. This restriction will simplify the notation and helps to focus on the core problem.

We are interested in the probability of default as defined in (9). We also note that the ratio of the future asset value and the company's net worth to the current liability value is a natural measure of the recovery rate in the sense of relating available assets to frozen pension liabilities. Thus, the random variables

$$D(t+1) = 1[S(t+1, h-1)/L(t, h) < -\tau(t+1)] \tag{12}$$
$$R(t+1) = (A(t+1) + V(t+1))/L(t, h) \tag{13}$$
$$K(t+1) = 1 - D(t+1) + D(t+1) \cdot R(t+1) \tag{14}$$

are of particular interest ($1[\cdot]$ in (12) is the indicator function that takes a value of unity if the argument in parentheses is true and zero else). Before deriving the expectations of these variables we have to discuss the available conditioning information. We assume that the stochastic discount factor that prices all traded assets in the economy is a function of random variables $Z_{t+1}$ (that might, for example, approximate consumption growth as in a consumption-based asset pricing model): $M(t+1) \equiv M(Z_{t+1})$. The stochastic discount factor (or pricing kernel) is the same for all traded assets. For example, the prices of one-period default-free and defaultable bonds are given by the fundamental pricing equations

$$\exp(q(t,1)) = E[M(t+1) \mid Z_t] \tag{15}$$
$$\exp(p(t,1)) = E[M(t+1) \cdot K(t+1) \mid Z_t] \tag{16}$$

that relate time $t+1$ payoffs to time t prices by an application of the pricing kernel. We assume that additional information $Z_t^c$ is available concerning the financial state of the sponsoring company. The net worth of the sponsoring company depends both on this information and on the variables describing the stochastic discount factor. Thus, the relevant conditional expectation of $\tau(t+1) \equiv \tau(Z_{t+1}, Z_{t+1}^c)$ is $\bar{\tau}(t,1) = E[\tau(t+1) \mid Z_t, Z_t^c]$ and the corresponding conditional expectations of the random variables (12)-(14) are

$$\pi(t,1) = E[D(t+1) \mid Z_t, Z_t^c] \tag{17}$$
$$\rho(t,1) = E[R(t+1) \mid D(t+1) = 1, Z_t, Z_t^c] \tag{18}$$
$$\Pi(t,1) = E[K(t+1) \mid Z_t, Z_t^c] = 1 - \pi(t,1) + \pi(t,1) \cdot \rho(t,1) \tag{19}$$

where in (18) we also condition on the default event.[4] Equation (17) defines the conditional default probability. Equation (18) defines the conditional expected recovery rate in the event of default. Equation (19) defines the expected DFR as the sum of the expected conditional recovery rates in the events of survival and default, weighted by their respective probability.

We denote m period ahead expected values of $D(t+m)$, $R(t+m)$, $K(t+m)$, $\tau(t+m)$ conditional on current information $Z_t, Z_t^c$ as $\pi(t,m)$, $\rho(t,m)$, $\Pi(t,m)$ and $\bar{\tau}(t,m)$.

### 2.3 Default Spreads

For the derivation of the default spread term structure we employ a conditional mean independence (CMI) assumption for $M(t+1)$ and $K(t+1)$ where the conditioning information includes $Z_t$ and the corporate-specific variables $Z_t^c$:

$$E\left[M(t+1) \mid K(t+1), Z_t, Z_t^c\right] = E\left[M(t+1) \mid Z_t, Z_t^c\right] \ . \qquad \text{(Assumption CMI)}$$

Using CMI, we have the following important result for the ratio of (16) to (19):

$$\begin{aligned}
\frac{\exp(p(t,1))}{\Pi(t,1)} &= E\left[M(t+1)\cdot\frac{K(t+1)}{\Pi(t,1)} \mid Z_t\right] = E\left[E\left[M(t+1)\cdot\frac{K(t+1)}{\Pi(t,1)} \mid Z_t, Z_t^c\right] \mid Z_t\right] \\
&= E\left[E\left[\frac{M(t+1)}{\Pi(t,1)} \mid Z_t, Z_t^c, K(t+1)\right]\cdot E\left[K(t+1) \mid Z_t, Z_t^c\right] \mid Z_t\right] \\
&= E\left[E\left[\frac{M(t+1)}{\Pi(t,1)} \mid Z_t, Z_t^c\right]\cdot \Pi(t,1) \mid Z_t\right] = E\left[M(t+1) \mid Z_t\right] = \exp(q(t,1)).
\end{aligned} \qquad (20)$$

CMI is used to obtain the first expression in the last row. The implications of CMI are similar to those from using risk-neutral probabilities instead of objective probabilities for pricing purposes. Under the objective probability measure expression, (16) can be decomposed into $E[M(t+1)K(t+1) \mid Z_t] = E[M(t+1) \mid Z_t]\cdot E[K(t+1) \mid Z_t] + \operatorname{cov}(M(t+1), K(t+1) \mid Z_t)$. This reduces to $E^*[M(t+1)K(t+1) \mid Z_t] = E^*[M(t+1) \mid Z_t]\cdot E^*[K(t+1) \mid Z_t]$ under the risk-neutral probability measure, which implies $\exp(p(t,1))/E^*[K(t+1) \mid Z_t] = E^*[M(t+1) \mid Z_t] = \exp(q(t,1))$. While this expression is similar to (20), the latter does not require a transition from the risk-neutral to the objective probability measure, which is a huge advantage in practical implementations.[5]

One might ask why $Z_t^c$ is not part of $Z_t$. This exclusion restriction is, of course, crucial for CMI. In the present context, we can justify the exclusion because the defaultable pension claims we are discussing are not traded. They simply serve as a vehicle for determining the appropriate default spreads for the valuation of pension liabilities. The default spreads $d(t,m) = f(t,m) - g(t,m)$ follow immediately from a straightforward generalization of (20) to all

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[4] Equation (17) can be rewritten as $\pi(t+1) = \Pr[R(t+1)<1|Z]$ which is a definition of the (conditional on information Z) value-at-risk (VaR), while $\rho(t+1) = E[R(t+1)|\ R(t+1)<1,Z]$ in (18) defines the complement of the corresponding expected shortfall (ES).

[5] Das and Sundaram (2000), for example, assume that the default probability (17) will be larger under the risk-neutral measure but at the same time treat the recovery rate in the event of default (18) as invariant against the applied measure. But (18) will change when (17) changes because it conditions on default.

maturities m: $\exp(p(t,m)) = \exp(q(t,m))\Pi(t,m)$. Starting from $m = 0$, spreads are calculated recursively using $g(t,m) = -(q(t,m+1) - q(t,m))$ and $f(t,m) = -(p(t,m+1) - p(t,m))$ as

$$
\begin{aligned}
d(t,0) &= f(t,0) - g(t,0) = -p(t,1) + p(t,0) + q(t,1) - q(t,0) = [p(t,1) - \ln \Pi(t,1)] - p(t,1) = -\ln \Pi(t,1) \\
d(t,1) &= f(t,1) - g(t,1) = -p(t,2) + p(t,1) + q(t,2) - q(t,1) = -\ln \Pi(t,2) - d(t,0) \\
d(t,2) &= f(t,2) - g(t,2) = -p(t,3) + p(t,2) + q(t,3) - q(t,2) = -\ln \Pi(t,3) - d(t,0) - d(t,1) \quad \cdots
\end{aligned}
$$

where the first line makes use of the terminal conditions $p(t,0) = 0$ and $q(t,0) = 0$. More generally, the following two equations completely describe the default spread term structure

$$d(t,0) = -\log(\Pi(t,1)) \tag{21}$$

$$d(t,m) = -\log(\Pi(t,m+1)) - \sum_{n=0}^{m-1} d(t,n) \quad \text{for} \quad m > 0 \tag{22}$$

These spreads are endogenous (Duffie and Singleton, 1999) in the sense that they depend on the value of the defaultable claim itself. The spreads depend on the discontinuance surplus ratio which itself depends on the spread. Combining and rearranging equations (21) and (22), the default spread term structure can be expressed as follows (for $m > 0$)

$$\Pi(t,m) = \exp\left(-\sum_{n=0}^{m-1} d(t,n)\right) \tag{23}$$

where $\Pi(t,m)$ is decreasing in m and $\Pi(t,m) = 1$ if and only if $d(t,n) = 0$, for all $n < m$.

### 2.4 Optimal Asset Allocation

Having introduced an asset-liability modeling framework that provides an interdependency of valuation and asset allocation, an optimality criterion for choosing the asset allocation $w(t,k)$, for $k = 1,\ldots,K$, and the contribution rate c in (10) needs to be discussed. At the same time, the equilibrium default spread (which appropriately reflects the likelihood that the promised pension payments cannot be delivered) is determined.

The determination of the optimal values of these variables will be carried out in two stages. First, the selected objective function is optimized with respect to the optimal asset allocation. Second, the contribution rate is determined to achieve certain funding targets for the pension fund over a given time horizon, usually known as a control period. For example, c may be chosen to achieve 100% funding over a three-year control period. The following analysis focuses on the first optimization problem that is solved for some given c. Other studies solve these problems simultaneously. For example, Haberman and Sung (1994) focus on the derivation of an optimal contribution strategy that simultaneously minimizes ‘contribution rate risk’, unexpected deviations from a targeted contribution rate, and ‘solvency risk’, unexpected deviations from a targeted funding level. Nevertheless, a two-stage process is common in practical implementations of ALM.

Sharpe and Tint (1990) optimize a mean-variance function in the ratio of future surplus to current assets. Similarly, we propose a mean-variance objective function in the DSR (10), which is the ratio of future surplus to current liabilities. A mean-variance objective function in the surplus with an exogenously given default spread is widely used for plan asset allocation decisions and therefore provides a natural starting point for the present paper[6]. Although this objective function is written in terms of the first and second moments of the scaled surplus distribution, higher moments are usually part of the objective function in the present context because they affect the default spread as is clear from (12) and (13). We therefore assume that the pension fund minimizes the conditional variance of the DSR, given that the conditional mean of the DSR equals some given $E[DSR]$ and portfolio weights sum to unity

$$\min_{w(t)} w'(t)V_t[y(t+1)]w(t) \quad \text{s.t.} \quad w'(t)E_t[y(t+1)] = E[DSR] \text{ and } w'(t)1_K = 1 \quad \text{where} \tag{24}$$

$$y(t+1) = \exp(x(t+1))(1+c)C(t,h) - (1+u)v'(t,h)\exp(r(t+1,h)),\ x'(t+1) = (x(t+1,1),\ldots,x(t+1,K)),$$
$$w'(t) = (w(t,1),\ldots,w'(t,K)),\ v'(t,h) = (v(t,h,1),\ldots,v(t,h,L)),\ r'(t+1,h) = (r(t+1,h+1),\ldots,r(t+1,h+L))$$

$1_K$ denotes a Kx1 vector of ones and $E_t$ and $V_t$ are the expectation and variance operators conditional on time t information $Z_t, Z_t^c$. A smaller $E[DSR]$ corresponds to a more risk averse behavior and a higher willingness on the part of the sponsor to pay contributions.

Solving (24) for the optimal asset allocation is complicated by the fact that the default spreads entering the surplus equation themselves depend on the allocation $w(t)$ as well. As a consequence of this dependency, any change in the asset allocation will immediately affect the current value of liabilities, which enters the objective function in the denominator of $C(t,h)$. This is the main difference from asset allocation problems in the tradition of Markowitz (1952) for the case without liabilities and Sharpe and Tint (1990) for the case with liabilities. The dependency becomes clear by noting that the first-order conditions $dJ(t)/dw(t,k)$ for a minimum of the objective function $J(t) = w'(t)V_t[y(t+1)]w(t)$ with respect to the portfolio weight $w(t,k)$ involve the sum of derivatives of the form $\partial J(t)/\partial w(t,k) + \partial J(t)/\partial d(t,m) \cdot d(d(t,m))/dw(t,k)$ for maturities $m = 0,\ldots,h+L-1$. Determining the signs of the two components of the derivatives requires assumptions regarding the distribution of asset returns and forward rates. Deriving the sign of the expression $d(d(t,m))/dw(t,k) = \exp(-d(t,m)) \cdot [(1-\rho(t,m))d\Pi(t,m)/dw(t,k) - \Pi(t,m)d\rho(t,m)/dw(t,k)]$ remains a difficult analytical task even for given distributional assumptions.

## 3. A Numerical Example

In order to shed more light on the relationship between optimal asset allocation and default spreads, we construct a simple example involving an individual who lives for two periods,

[6] Although we use a mean-variance objective function, there are many types of objective functions involving (10), including those for multi-period allocation decisions, that are compatible with the idea of deriving simultaneously the optimal asset allocation and the appropriate default spread.

one period of work and one period of retirement, so that $h = L = 1$. We also assume that there are only two asset classes, so that $K = 2$. Finally we impose the simplifying assumption of a time- and maturity-invariant forward rate curve for the numerical example, i.e. $f(t,0) = f(t,1) = f(t+1,0)$. With this assumption we need to calibrate just one single default-free forward rate, $g(t,0)$, and obtain just one single default spread, $d(t,0)$, that can be conveniently compared with the historical spread of the yield of a AA corporate bond with long maturity over $g(t,0)$. The DSR (10) then becomes

$$
\begin{aligned}
&\frac{S(t+1,0)}{L(t,1)} = w'(t)y(t+1) = w'(t)\left[\exp(x(t+1))(1+c)C(t,1) - (1+u)\exp(r(t+1,2))\right] \\
&= w'(t)\left[\exp(x(t+1))(1+c)\left(\frac{A(t)}{U(t,1,e)}\exp(f(t,0)+f(t,1))\right) - (1+u)\exp(f(t,0)+f(t,1)-f(t+1,0))\right] \\
&= w'(t)\left[\exp(x(t+1))(1+c)(C_0(t,1)\exp(2d(t,0))) - (1+u)\exp(g(t,0)+d(t,0))\right]
\end{aligned}
\tag{25}
$$

where the funding ratio has been replaced by the term in large round parentheses in the second line and by $C_0(t,1)\exp(2d(t,0))$ in the third line, making use of the definition of the funding ratio for a risk-free pension plan with zero default spread, $C_0(t,1) = A(t)/U(t,1,e)\exp(2g(t,0))$. We assume that the vector $(g(t,0), x(t+1,1), x(t+1,2))'$ is independent and identically multivariate normal distributed as

$$
\begin{pmatrix} g(t,0) \\ x(t+1,1) \\ x(t+1,2) \end{pmatrix} \sim N\left(\begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_{00} & & \\ \sigma_{10} & \sigma_{11} & \\ \sigma_{20} & \sigma_{21} & \sigma_{22} \end{bmatrix}\right)
\tag{26}
$$

which allows us to look at unconditional moment functions and to drop the time index. The mean and variance components of the objective function (24) simplify to

$$
E[y] = m_A - m_L \quad \text{and} \quad V[y] = v_A^2 + v_L^2 1_2 1_2' - 2 \cdot 1_2' \otimes v_{AL} \quad \text{with}
\tag{27}
$$

$$
\begin{aligned}
m_A &= m_X(1+c)\overline{C}(1)\exp(2d(0)), \quad m_L = (1+u)m_Y\exp(d(0)) \\
v_A^2 &= v_X^2\left((1+c)\overline{C}(1)\right)^2\exp(4d(0)), \quad v_L^2 = v_Y^2(1+u)^2\exp(2d(0)), \\
v_{AL} &= v_{XY}(1+c)(1+u)\overline{C}(1)\exp(3d(0))
\end{aligned}
$$

where $\otimes$ denotes the Kronecker product, $\overline{C}(1) = C_0(1)E[\exp(-2g(0))]$, and using the following moments derived from the properties of the log normal distribution

$$
\begin{aligned}
m_X &= E[\exp(x+2g(0))] = \{m_{X,i}\}_{i=1,2} = \{\exp(\mu_i + 2\mu_0 + 0.5\sigma_{ii} + 2\sigma_{00} + 2\sigma_{i0})\}_{i=1,2} \\
m_Y &= E[\exp(g(0))] = \exp(\mu_0 + 0.5\sigma_{00}), \quad v_Y^2 = V[\exp(g(0))] = m_Y^2(\exp(\sigma_{00}) - 1) \\
v_X^2 &= V[\exp(x+2g(0))] = \{m_{X,i}m_{X,j}(\exp(\sigma_{ij} + 4\sigma_{00} + 2\sigma_{i0} + 2\sigma_{j0}) - 1)\}_{i,j=1,2} \\
v_{XY} &= COV[\exp(x+2g(0)), \exp(g(0))] = \{m_{X,i}m_Y(\exp(\sigma_{i0} + 2\sigma_{00}) - 1)\}_{i=1,2}.
\end{aligned}
$$

Most importantly, assumption (26) allows us to derive explicit expressions for the two components of the default spread $\exp(-d(0)) = 1-\pi(1)+\pi(1)\rho(1)$, namely

$$\pi(1) = \Phi\left(-\frac{E[DSR]+\bar{\tau}(1)}{\sqrt{w'V[y]w}}\right) \quad \text{and} \quad \rho(1) = \bar{\tau}(1) + m_A + \frac{v_{AL} - v_A^2}{\sqrt{w'V[y]w}\;\pi(1)}\phi\left(-\frac{E[DSR]+\bar{\tau}(1)}{\sqrt{w'V[y]w}}\right) \tag{28}$$

using properties of the standard normal distribution (see, e.g., Gourieroux and Monfort, 1995, ch. B.3.4.b) with p.d.f. $\phi(\cdot)$ and c.d.f. $\Phi(\cdot)$.

We now optimize (24) with (27) using an iterative procedure based on the following idea. We can generate an explicit solution for the optimal asset allocation in (24) if we condition on a fixed spread $d_i$ in iteration i. Call this $w^* \mid d_i$, which is derived e.g. by Cochrane (2001, p. 85) using the terms $\alpha = E[y \mid d_i]'[y \mid d_i]E[y \mid d_i]$, $\beta = E[y \mid d_i]'V[y \mid d_i]1_2$ and $\gamma = 1_2'V[y \mid d_i]1_2$ as

$$w^* \mid d_i = V[y \mid d_i]^{-1}\left[\frac{(\gamma E[DSR]-\beta)E[y \mid d_i] + (\alpha - \beta E[DSR])1_2}{\alpha\gamma - \beta^2}\right]. \tag{29}$$

Thus, we initialize the iterations by computing (29) for a starting value of $d_1 = 0$, plug $w^* \mid d_1$ into (28), compute $d_2 = -\log(1-\pi(1)+\pi(1)\rho(1))$ and $w^* \mid d_2$ in the next iteration and continue with this iterative procedure until both the asset allocation and the default spread converge in the sense that their last update is smaller than 0.00001. Of course, this iterative process will yield the same solution as the direct optimization of (24) by means of numerical optimization methods, but, in the present context, is much less complicated.

We conduct two experiments for pension plans with an assumed funding ratio (using a zero default spread) of $C_0(1) = 0.930$ and $C_0(1) = 0.945$. These particular choices are motivated by our desire to present a set of interior solutions to the optimization problem. It will become clear below that a funding ratio much below $C_0(1) = 0.930$ will imply bond short selling while a funding ratio much above $C_0(1) = 0.945$ will generate zero default spreads given our choices for the other parameters. In both experiments we consider a range of possible default thresholds $\bar{\tau}(1)$ from 0.00 to 0.10. Recall that $\bar{\tau}(1)$ denotes the conditional expectation of the company's future net worth per unit of current pension liabilities. Thus, we focus on companies with a comparably large burden of pension liabilities. We do this because it is precisely these companies which face a significant default risk in the sense of having a high default probability and a low recovery rate for their pension plans in the case of default.

Both experiments are based on time- and age-specific wage inflation $u = 0.02$ and contributions $c = 0.00$. We match the moments of the asset returns and the default-free yield to sample moments computed from time series data of US market indices. We use the real yield of a Treasury bond with 30 years maturity for $g(0)$ and real total return indices for US Treasury bonds of all maturities (JPM index) and US equities (MSCI index) for the two assets. The first choice implies that one period in our example has a length that equals the typical average maturity of corporate pension liabilities.

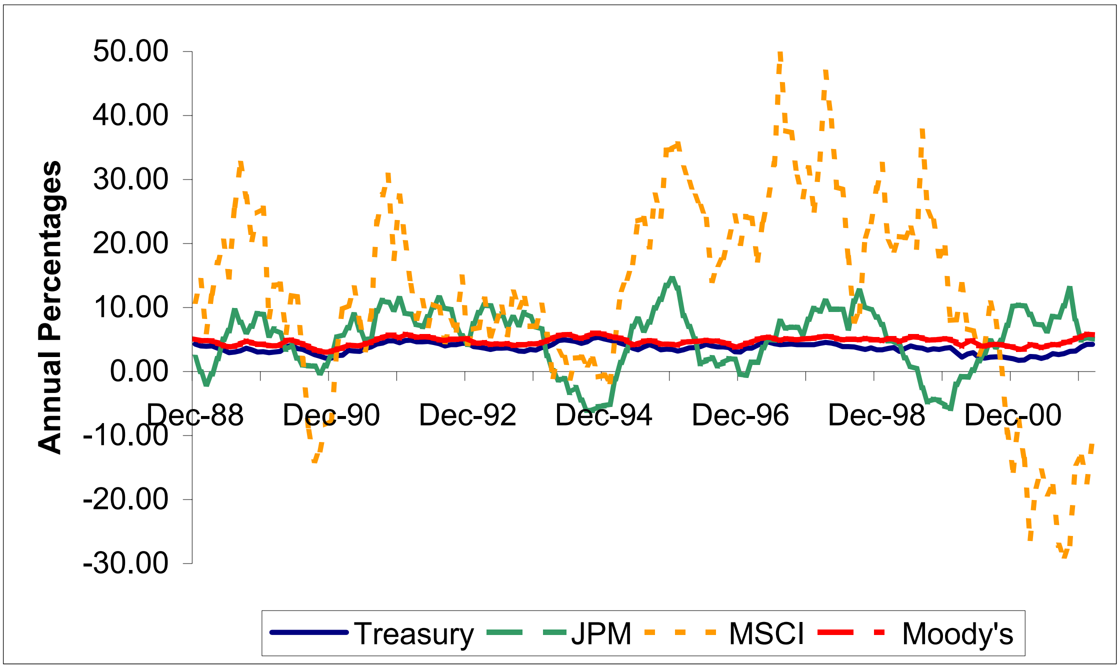
Table 1 contains descriptive statistics for all the variables we need, together with Moody's (real) yield index for AA rated corporate bonds with maturities 20 years and above. This index is frequently used for determining the discount rate for the calculation of corporate pension liabilities in the USA. The average yield on this index exceeds the average yield on 30-year Treasury bonds by 1.05 percentage points.[7] We will use this spread as a benchmark against which we compare the endogenously derived default spread of our model. The sample period is December 1988 (the month the first annual total return could be computed from the JPM index which started in December 1987) to February 2002 (the month the US Treasury stopped publishing yields on bonds with 30 years maturity). Figure 1 presents the yield and return data.

Table 1: Descriptive Statistics of the Data

| Index | Mean | Volatility | Correlation Matrix | | | |
|---|---|---|---|---|---|---|
| Treasury | **3.65%** | **0.78%** | **1.0000** | | | |
| JPM | **4.96%** | **4.76%** | **-0.1466** | **1.0000** | | |
| MSCI | **11.23%** | **15.23%** | **0.3619** | **0.1448** | **1.0000** | |
| Moody's | **4.70%** | **0.62%** | **0.8691** | **-0.1915** | **0.2485** | **1.0000** |

Notes: Sample period is December 1988 – February 2002. Treasury: annual real yields on US Treasury securities with 30 years maturity. JPM: total annual real return on JP Morgan US Treasury index for all maturities. MSCI: total annual real return on MSCI USA equity index. Moody's: annual real yields on AA corporate bonds with maturities 20 years and above. The means are annualized geometric means. The volatilities are annualized standard deviations. Data source: Datastream.

Figure 1: Annual Total Real Returns and Yields




Notes: The graph displays annualized means of the real yield of a US Treasury bond with 30 years maturity (Treasury), the real yield of the Moody's index of AA rated corporate bonds with maturities 20 years and above (Moody's), the real return of the JP Morgan US Treasury bond index for all maturities (JPM) and the real return of the MSCI US equity index (MSCI) over the period December 1988 – February 2002.

[7] Ideally, we would have liked to use the yield on 20-year Treasury bonds as a match for the Moody's index, but the definition of the 20-year index changed over the sample period. We were therefore forced to use 30-year Treasury bonds as the next best alternative.

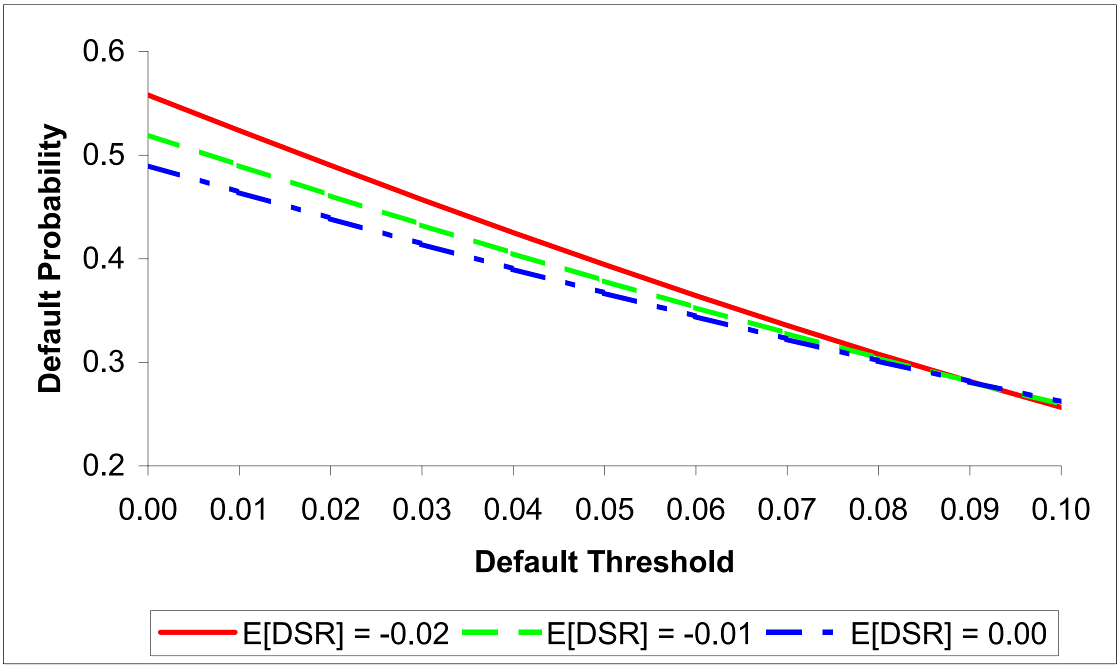
The results of the two experiments are depicted in Figures 2-11. Tables in the Appendix contain the data underlying these Figures. We first discuss movements along the curves (i.e. for a given $E[DSR]$) within each graph (i.e. for a given funding ratio $C_0(1)$), then discuss movements between curves (i.e. for a different $E[DSR]$) within each graph and finally compare the results between graphs (i.e. for different funding ratios $C_0(1)$).

For a given pension funding ratio ($C_0(1)$) and expected discontinuance surplus ratio (e.g. $E[DSR]=0$), higher company net worth (corresponding to a higher default threshold) has two effects. First, it lowers the default probability (since insolvency is less likely): this is shown by the downward sloping curves in Figures 2 and 3. Second, it increases the recovery rate (since we assume the pension fund can claim up to 100% of the net worth of the company): this is shown by the upward sloping curves in Figures 4 and 5. The combined effect is that the default spread is lower the higher the company's net worth as shown by the downward sloping curves in Figures 6 and 7. Since the default spread is increasing in the default probability and decreasing in the recovery rate (see (19)), this result follows mechanically from Figures 2-5. As the spread falls, the expected value of the liabilities rises and gets closer to its promised value, i.e. the value of the liabilities using a zero default spread. The optimal equity weighting increases with the company's net worth because a higher allocation to equities (with their higher expected returns) increases the expected surplus and this is needed to match the higher level of liabilities in the denominator of the DSR, $S(t+1,0)/L(t,1)$, implied by the lower default spread (see (25)). This explains the upward sloping curves in Figures 8 and 9.

The positive relationship that we find between company net worth and pension fund equity weighting is not, however, consistent with the analyses of Black (1980) and Tepper (1981) or the empirical findings of Bodie et al. (1985) which show that profitable taxpaying companies will attempt to reduce their tax liabilities by investing bonds. This difference in results is explained by the absence in our model of the distorting effect of taxes.

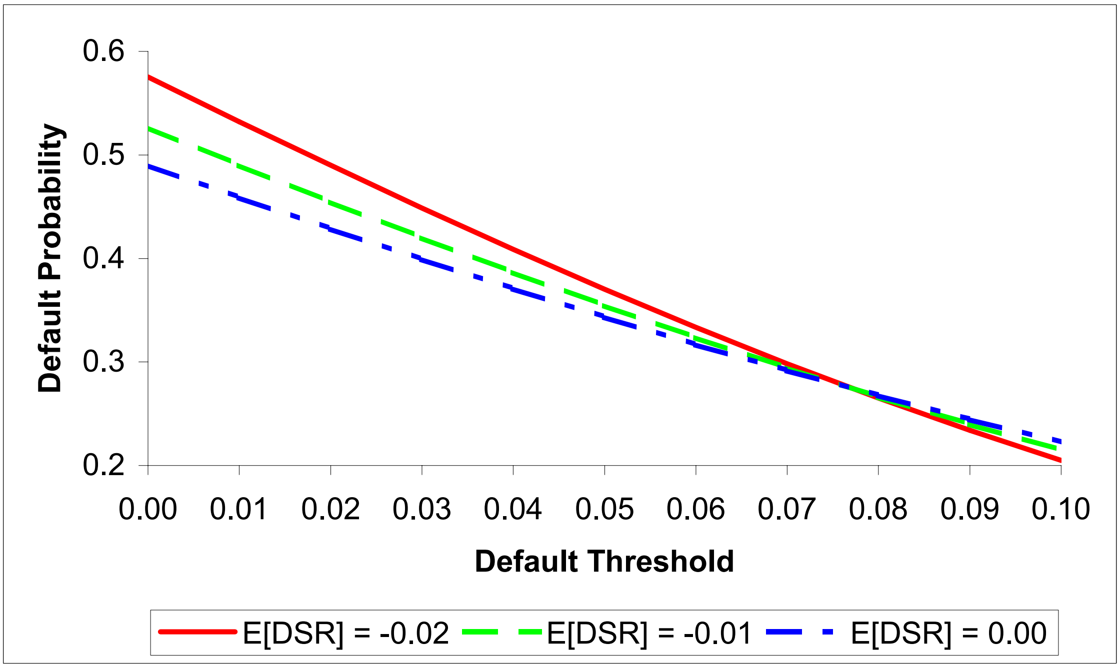
For a given pension funding ratio and default threshold, an increase in the $E[DSR]$ raises the default probability when the default threshold is low and lowers the default probability when the default threshold is high: the curves in Figures 2 and 3 intersect with the $E[DSR]=-0.02$ curve highest to the left of the intersection and the $E[DSR]=0$ curve highest to the right of the intersection. This is explained as follows. An increase in $E[DSR]$ increases both the mean and variance of the distribution (see equation (24)). This has the effect of flattening the density function around the mean (which itself shifts to the right) and fattening the density function in the tails (compare the dashed and solid curves in the upper panel of Figure B1 in Appendix B). There must exist a default threshold for which the area under each curve to the left of this threshold (which measures the default probability) is the same (this is given by the intersection point in the lower panel of Figure B1). For lower default thresholds, the default probability increases when $E[DSR]$ increases; the opposite holds for higher default thresholds. This explains why the downward sloping lines in Figures 2 and 3 intersect: they correspond to different cumulative distribution functions with different means and variances.

Figure 2: Default Probabilities $\pi(1)$ for Different $E[DSR]$ and $\overline{\tau}(1)$; $C_0(1) = 0.930$



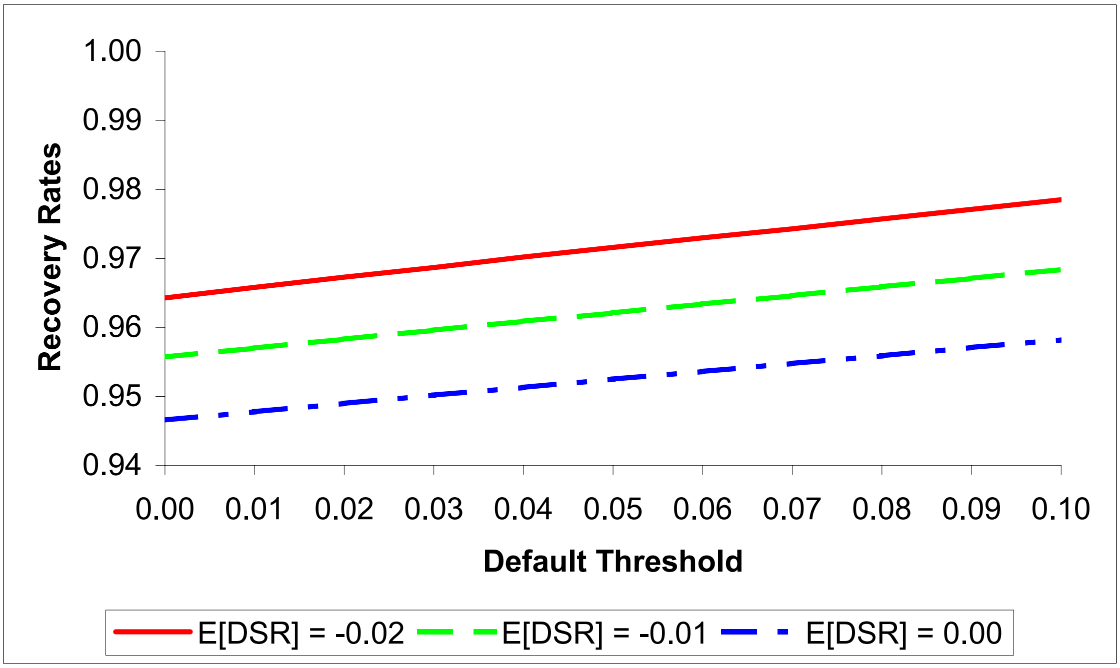

Notes: $C_0(1)$ is the funding ratio of the pension plan assuming a zero default spread. E[DSR] denotes the targeted mean discontinuance surplus ratio. The default threshold is the ratio of the sponsoring company's future net worth to the current value of liabilities.

Figure 3: Default Probabilities $\pi(1)$ for Different $E[DSR]$ and $\overline{\tau}(1)$; $C_0(1) = 0.945$



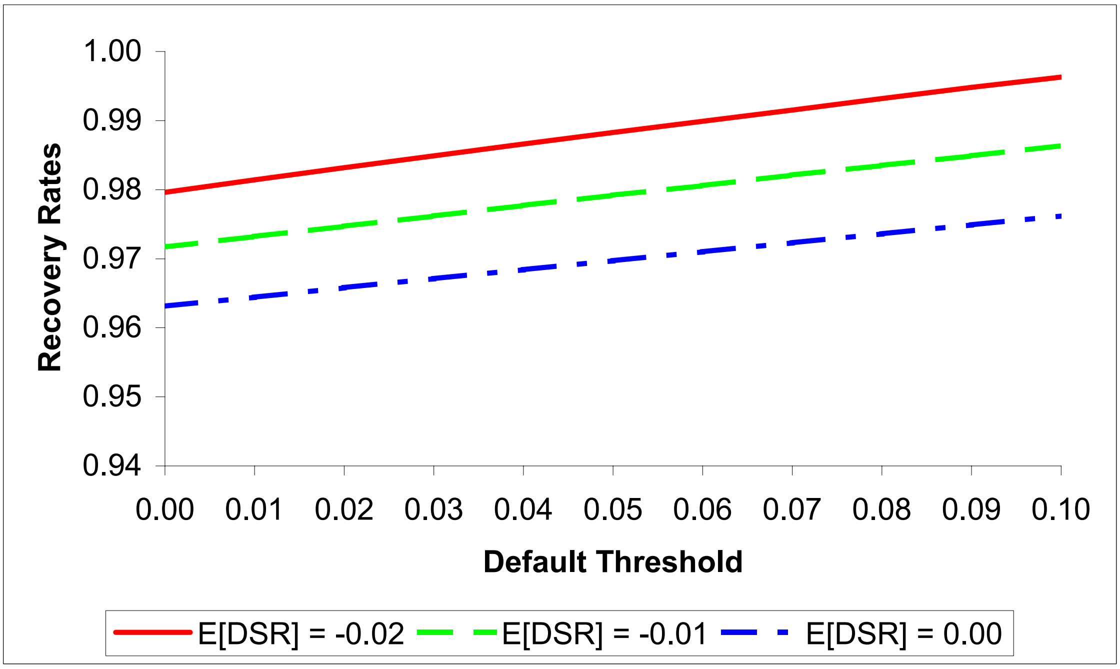

Notes: Cf. Figure 2.

Figure 4: Recovery Rates $\rho(1)$ for Different $E[DSR]$ and $\overline{\tau}(1)$; $C_0(1) = 0.930$



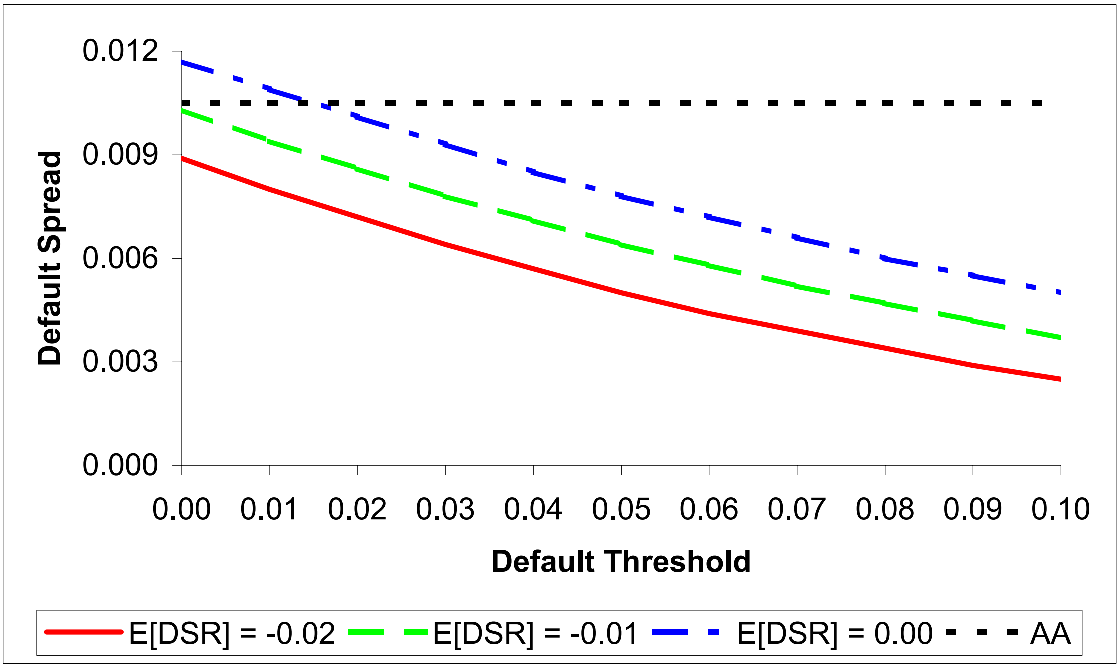

Notes: Cf. Figure 2.

Figure 5: Recovery Rates $\rho(1)$ for Different $E[DSR]$ and $\overline{\tau}(1)$; $C_0(1) = 0.945$



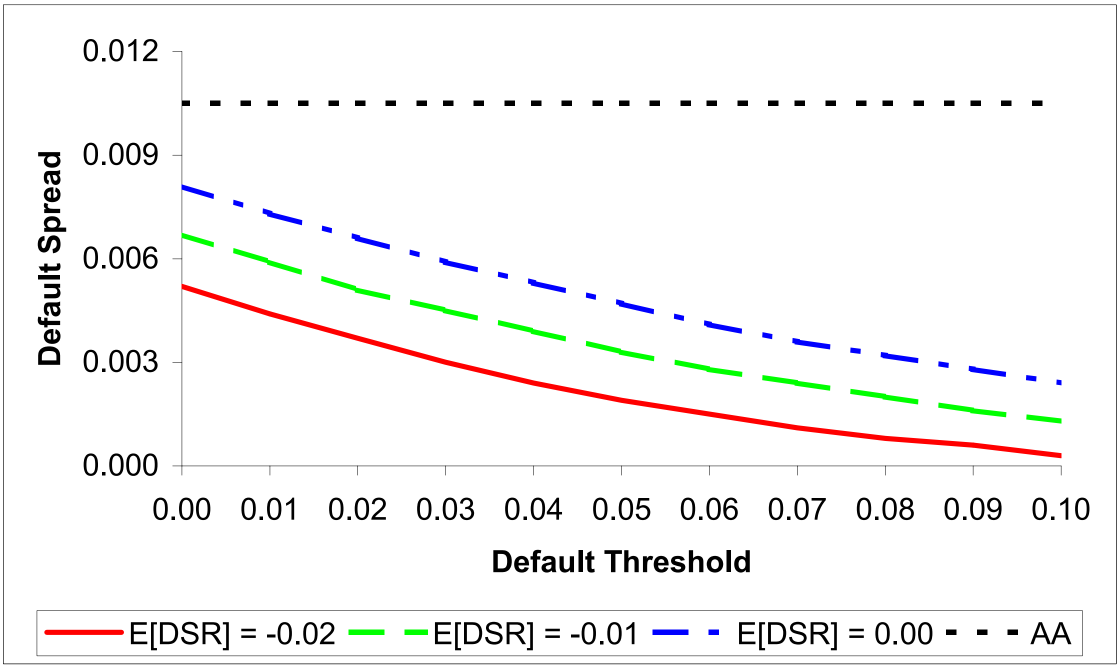

Notes: Cf. Figure 2.

Figure 6: Default Spreads $d(0)$ for Different $E[DSR]$ and $\bar{\tau}(1)$; $C_0(1) = 0.930$



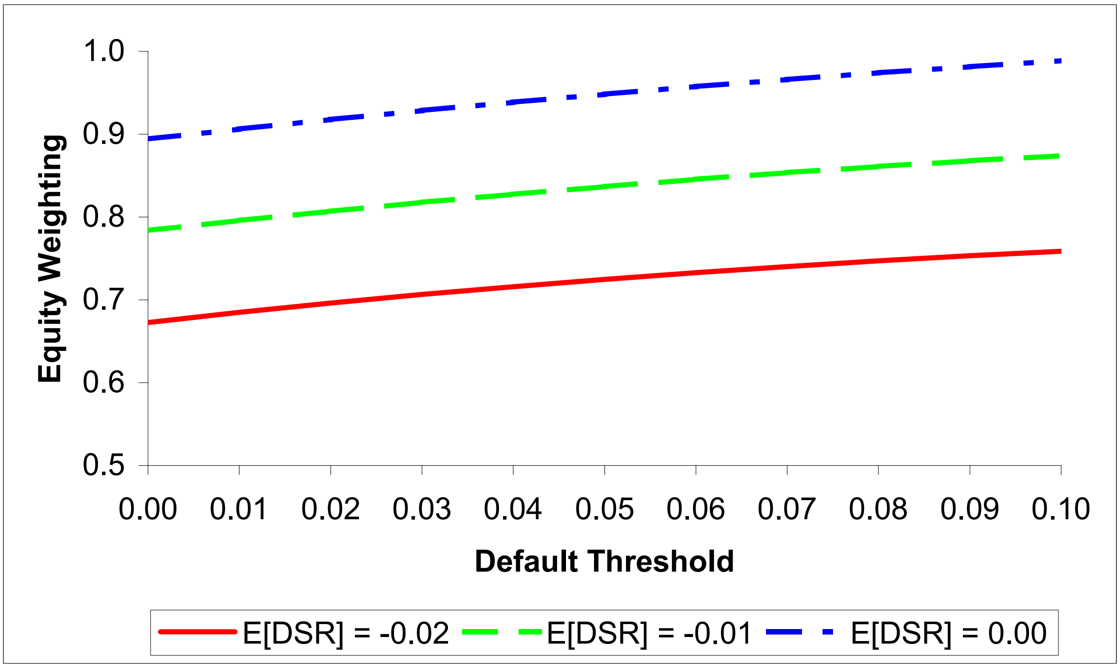

Notes: Cf. Figure 2.

Figure 7: Default Spreads $d(0)$ for Different $E[DSR]$ and $\bar{\tau}(1)$; $C_0(1) = 0.945$



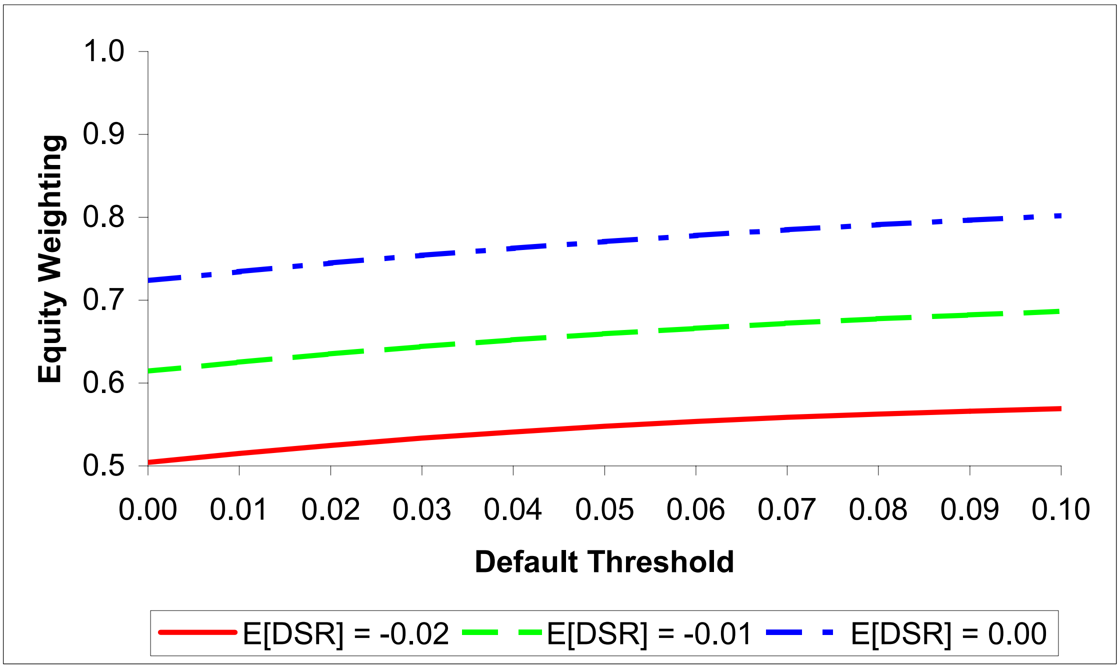

Notes: Cf. Figure 2.

Figure 8: Equity Weightings for Different $E[DSR]$ and $\overline{\tau}(1)$; $C_0(1) = 0.930$



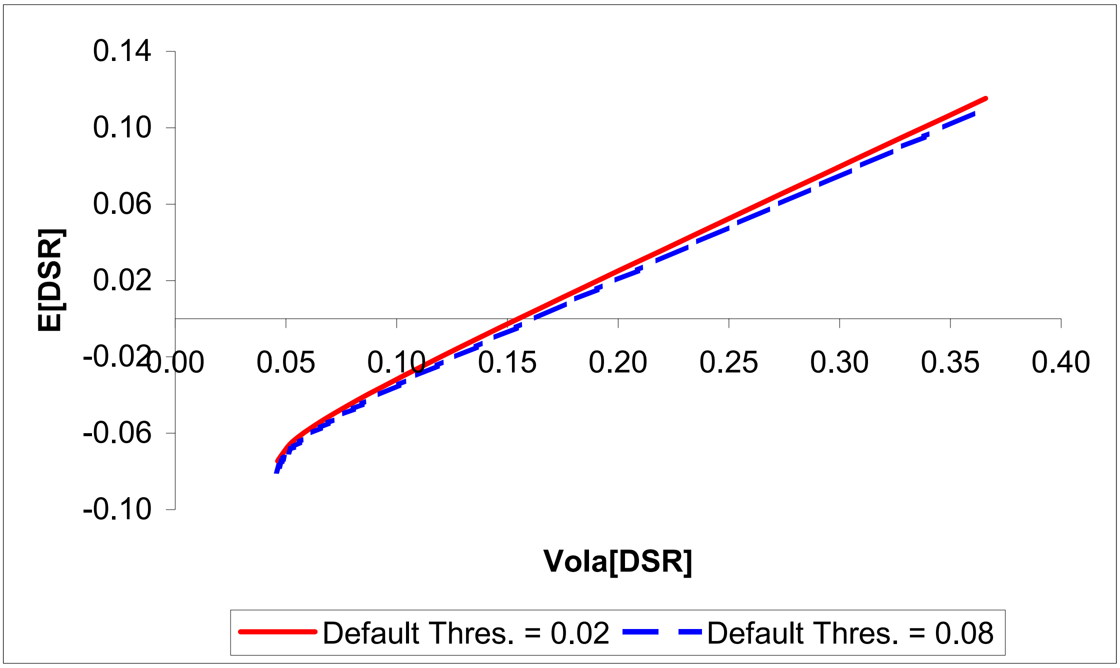

Notes: Cf. Figure 2.

Figure 9: Equity Weightings for Different $E[DSR]$ and $\overline{\tau}(1)$; $C_0(1) = 0.945$



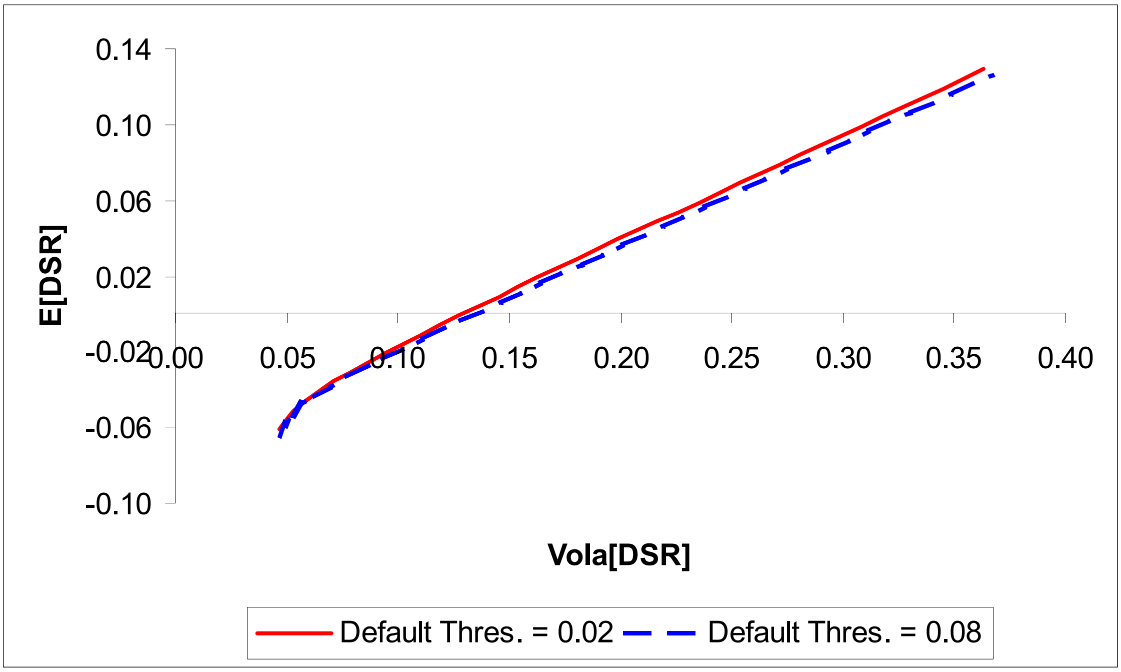

Notes: Cf. Figure 2.

Figure 10: Efficient Frontier in DSR; $C_0(1) = 0.930$




Notes: Cf. Figure 2. Vola[DSR] denotes the volatility or standard deviation of the DSR.

Figure 11: Efficient Frontier in DSR; $C_0(1) = 0.945$




Notes: Cf. Figure 10.

For a given pension funding ratio and default threshold, an increase in $E[DSR]$ lowers the recovery rate (see Figures 4 and 5). This is because the value of the company's equity and the equity in the pension fund must now fall by larger amounts before the pension fund becomes insolvent and so the pension fund, perversely, has a greater shortfall to recover.[8] Note that there is no crossover of the curves in this case: from the second equation in (28), the size of $\rho(1)$ is dominated by the first term $\bar{\tau}(1)+m_A$ so the impact of changes in $\pi(1)$, which only affect the second term, is small.

For a given pension funding ratio and default threshold, an increase in $E[DSR]$ increases the default spread (see Figures 6 and 7). This indicates that the negative impact of the recovery rate (recovery rates fall with increasing $E[DSR]$) on the default spread dominates the positive impact of the default probability (default probabilities are lower with increasing $E[DSR]$ to the left of the intersection points in Figures 2 and 3)[9] on the default spread.

For a given pension funding ratio and default threshold, an increase in the $E[DSR]$ (which, as mentioned above, corresponds to a decrease in risk aversion) naturally results in a higher equity weighting (see Figures 8 and 9). In the case of $C_0(1)=0.930$ and a default threshold of 0.05, the equity weighting is around 72% when $E[DSR]=-0.02$ and around 95% when $E[DSR]=0$.

Sharpe (1976) and Bodie et al. (1987) show that firms facing financial difficulties or temporary cash flow shortages have an incentive to raise the required discount rate by investing in equities to lower both reported liabilities and the contribution rate to the plan. In our numerical example, a higher discount rate always follows from a more aggressive investment strategy (an increase in E[DSR]) that results in an increase in equity weightings whatever the financial strength of the sponsoring company. The positive relationship between equity weightings and the default spread, anticipated by common sense, holds for movements between curves for a given default threshold (cf. Figures 6 and 8, and Figures 7 and 9). By contrast, movements along a given curve which hold $E[DSR]$ constant and vary the default threshold show a negative relationship between the default spread and equity weightings (cf. the curves labeled $E[DSR]=0$ in Figures 6 and 8, and Figures 7 and 9).

For a given $E[DSR]$ and default threshold, higher funding ratios lead to lower default probabilities so long as default probabilities are smaller than 0.5. For default probabilities larger than 0.5, higher funding ratios increase the default probability (see Tables A1 and A2 in Appendix A). This is because at probability 0.5, $E[DSR]$ is exactly equal to the default threshold regardless of the funding ratio. Thus, the cumulative distribution functions intersect at probability 0.5 as is clear from (28) (and Figure B1 in Appendix B). Probabilities that are relatively higher to the left of the intersection switch to being relatively lower to the right of the intersection. Comparing Figures 2 and 3, the curves have a common fixed point passing through a

[8] The result also follows immediately from the relationship between VaR and ES (cf. footnote 4): If $\pi(t+1)$ = Pr[R(t+1)<1|Z] decreases with increasing E[DSR], $\rho(t+1)$ = E[R(t+1)| R(t+1)<1,Z] has to decrease as well by definition.
[9] To the right of the intersection points the effects of the default probability and the recovery rate on the spread are reinforcing and the spread unambiguously increases with increasing E[DSR].

default probability of 0.5, but the curves in Figure 3 are steeper than those in Figure 2. VanDerhei (1990) obtains a significant negative coefficient on the funding variable in a logit regression for the default probability using a sample of US pension plans terminated between 1981 and 1984. This result is consistent with our findings for default probabilities below 0.5.

For a given $E[DSR]$ and default threshold, a 1.5 percentage point increase in the funding ratio implies approximately a 1.5 percentage point increase in the recovery rate (the curves in Fiure 5 lie approximately 1.5 percentage points above Figure 4). Regardless of the default probability being above or below 0.5, a higher funding ratio unambiguously implies a lower default spread for a given $E[DSR]$ and default threshold (the curves in Figure 7 are lower than the curves in Figure 6) which indicates that the recovery rate has a greater impact on the default spread than on the default probability. This is consistent with empirical results from Petersen (1996) who reports a highly significant negative (albeit small in absolute value) coefficient on the funding variable in a discount rate regression using US data from 1988-1991.

Figures 6 and 7 also plot the average AA corporate bond yield for the sample period. Only under very particular circumstances it is optimal to discount pension liabilities using the AA corporate bond yield: Figure 6 shows these to be $C_0(1) = 0.930$, $E[DSR] = 0$ and $\overline{\tau}(1)$ approximately equal 0.015 in our example. The figures also show that in general the AA corporate bond yield is likely to be an inappropriate discount rate for valuing pension liabilities.

Finally, Figures 10 and 11 display efficient frontiers in mean–volatility graphs for the optimized discontinuance surplus ratio. An increase in the funding ratio implies an upward shift of the efficient frontier. An increase in the net worth of the sponsoring company does not affect the efficient frontier to any significant extent.

## 4. Conclusion

This paper challenges current practice in asset-liability management (ALM) in a fundamental way. We have shown that any exogenously determined discount rate is unlikely in general to be suitable for determining the value of liabilities since it will not reflect the true risk of failing to deliver the promised future liability payments. The appropriate discount rate will depend on a number of factors, the most important of which are: the asset allocation, the funding ratio and the financial strength of the guarantor of the liabilities such as the corporate sponsor of a defined benefit pension plan.

In order to focus on the main relationship between discount rates and asset allocation, we introduced some simplifications, which we would like to abandon in future work:

- We would like to extend the theoretical framework by establishing a relationship between the plan sponsor's core business and the financial strength of the pension plan. Webb (2004) treats the deficit of a corporate pension plan as corporate debt with funding requirements and priority rules in the event of company insolvency and examines the impact of the plan sponsor's financial position on the pension plan's investment policy. One way of considering these dependencies in our framework would be to relate the threshold defining default to the plan sponsor's financial strength in a reduced form approach.

- We would like to examine the asset-liability modeling exercise from the point of view of the different stakeholders in the corporate pension plan, principally the sponsor and the members. Each of these different viewpoints involves different (possibly inter-temporal) objective functions, different risk aversion parameters and hence different optimal asset allocations, equilibrium discount rates and liability valuations.
- Another interesting line of research using our methodology would be to analyze the role of a pension plan insurance scheme, which already exists in the US (Pension Benefit Guaranty Corporation, PBGC[10]) and has recently been introduced in the UK (Pension Protection Fund).

Finally, an empirical application of our model remains an important task. In an empirical analysis we could abandon the kind of distributional assumptions we imposed in the numerical example, i.e. i.i.d. normal returns and yields. To estimate the conditional expectation components of the default spread, one could use nonparametric estimation techniques as proposed by Scaillet (forthcoming) for the estimation of conditional VaR and ES risk measures. The first order conditions for the optimal portfolio weights given conditioning information might be solved along the lines of the nonparametric Kernel-M estimation approach suggested by Brandt (1999) and Brandt and Aït-Sahalia (2001). Data on pension plan defaults and recovery rates in the event of default is not required because the default spread is endogenous and therefore completely described by the evolution of the sum of pension plan assets and the net worth of the sponsoring company relative to the liabilities of the pension scheme.

[10] The PBGC can take up to 30% of the net worth of a company in the event of a pension plan default.

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## Appendix A

Table A1: Simulation Results for the Numerical Example Based on $C_0(1) = 0.930$

| $\bar{\tau}(1)$ | $E[DSR]$ | $\sqrt{V[DSR]}$ | $\pi(1)$ | $\rho(1)$ | $d(0)$ | Bond | Equity |
|---|---|---|---|---|---|---|---|
| AA | -0.02 | 0.1139 | - | - | 0.0105 | 0.3488 | 0.6512 |
| 0.00 | -0.02 | 0.1168 | 0.5680 | 0.9643 | 0.0089 | 0.3273 | 0.6727 |
| 0.01 | -0.02 | 0.1184 | 0.5337 | 0.9658 | 0.0080 | 0.3152 | 0.6848 |
| 0.02 | -0.02 | 0.1199 | 0.5000 | 0.9673 | 0.0072 | 0.3040 | 0.6960 |
| 0.03 | -0.02 | 0.1213 | 0.4671 | 0.9687 | 0.0064 | 0.2936 | 0.7064 |
| 0.04 | -0.02 | 0.1226 | 0.4352 | 0.9702 | 0.0057 | 0.2841 | 0.7159 |
| 0.05 | -0.02 | 0.1237 | 0.4042 | 0.9716 | 0.0050 | 0.2752 | 0.7248 |
| 0.06 | -0.02 | 0.1248 | 0.3743 | 0.9730 | 0.0044 | 0.2671 | 0.7329 |
| 0.07 | -0.02 | 0.1258 | 0.3455 | 0.9743 | 0.0039 | 0.2597 | 0.7403 |
| 0.08 | -0.02 | 0.1267 | 0.3179 | 0.9757 | 0.0034 | 0.2529 | 0.7471 |
| 0.09 | -0.02 | 0.1275 | 0.2915 | 0.9771 | 0.0029 | 0.2468 | 0.7532 |
| 0.10 | -0.02 | 0.1282 | 0.2664 | 0.9785 | 0.0025 | 0.2413 | 0.7587 |
| AA | -0.01 | 0.1338 | - | - | 0.0105 | 0.2187 | 0.7813 |
| 0.00 | -0.01 | 0.1342 | 0.5297 | 0.9557 | 0.0103 | 0.2162 | 0.7838 |
| 0.01 | -0.01 | 0.1358 | 0.5000 | 0.9570 | 0.0094 | 0.2042 | 0.7958 |
| 0.02 | -0.01 | 0.1373 | 0.4710 | 0.9583 | 0.0086 | 0.1929 | 0.8071 |
| 0.03 | -0.01 | 0.1387 | 0.4427 | 0.9596 | 0.0078 | 0.1824 | 0.8176 |
| 0.04 | -0.01 | 0.1401 | 0.4152 | 0.9609 | 0.0071 | 0.1724 | 0.8276 |
| 0.05 | -0.01 | 0.1413 | 0.3886 | 0.9621 | 0.0064 | 0.1632 | 0.8368 |
| 0.06 | -0.01 | 0.1425 | 0.3628 | 0.9634 | 0.0058 | 0.1545 | 0.8455 |
| 0.07 | -0.01 | 0.1435 | 0.3380 | 0.9646 | 0.0052 | 0.1465 | 0.8535 |
| 0.08 | -0.01 | 0.1445 | 0.3141 | 0.9659 | 0.0047 | 0.1390 | 0.8610 |
| 0.09 | -0.01 | 0.1455 | 0.2912 | 0.9671 | 0.0042 | 0.1321 | 0.8679 |
| 0.10 | -0.01 | 0.1463 | 0.2692 | 0.9684 | 0.0037 | 0.1258 | 0.8742 |
| AA | 0.00 | 0.1542 | - | - | 0.0105 | 0.0886 | 0.9114 |
| 0.00 | 0.00 | 0.1518 | 0.5000 | 0.9466 | 0.0117 | 0.1059 | 0.8941 |
| 0.01 | 0.00 | 0.1535 | 0.4740 | 0.9478 | 0.0109 | 0.0939 | 0.9061 |
| 0.02 | 0.00 | 0.1550 | 0.4487 | 0.9490 | 0.0101 | 0.0824 | 0.9176 |
| 0.03 | 0.00 | 0.1565 | 0.4240 | 0.9502 | 0.0093 | 0.0716 | 0.9284 |
| 0.04 | 0.00 | 0.1579 | 0.4000 | 0.9513 | 0.0085 | 0.0613 | 0.9387 |
| 0.05 | 0.00 | 0.1592 | 0.3767 | 0.9525 | 0.0078 | 0.0517 | 0.9483 |
| 0.06 | 0.00 | 0.1604 | 0.3542 | 0.9536 | 0.0072 | 0.0426 | 0.9574 |
| 0.07 | 0.00 | 0.1615 | 0.3324 | 0.9548 | 0.0066 | 0.0340 | 0.9660 |
| 0.08 | 0.00 | 0.1626 | 0.3114 | 0.9559 | 0.0060 | 0.0259 | 0.9741 |
| 0.09 | 0.00 | 0.1636 | 0.2911 | 0.9571 | 0.0055 | 0.0184 | 0.9816 |
| 0.10 | 0.00 | 0.1646 | 0.2717 | 0.9582 | 0.0050 | 0.0113 | 0.9887 |

Notes: $C_0(1)$ is the funding ratio of the pension plan assuming a zero default spread. The first column contains the default threshold, i.e. the sponsoring company's future net worth per unit of the current value of liabilities. "AA" in the first column refers to the benchmark results obtained from using the exogenous historical default spread of AA bonds. The second and third columns present the mean and volatility of the optimized discontinuance surplus ratio. d(0) is the endogenous default spread with components $\pi(1)$, the default probability, and $\rho(1)$, the recovery rate. "Bond" and "Equity" refer to the optimized asset allocation.

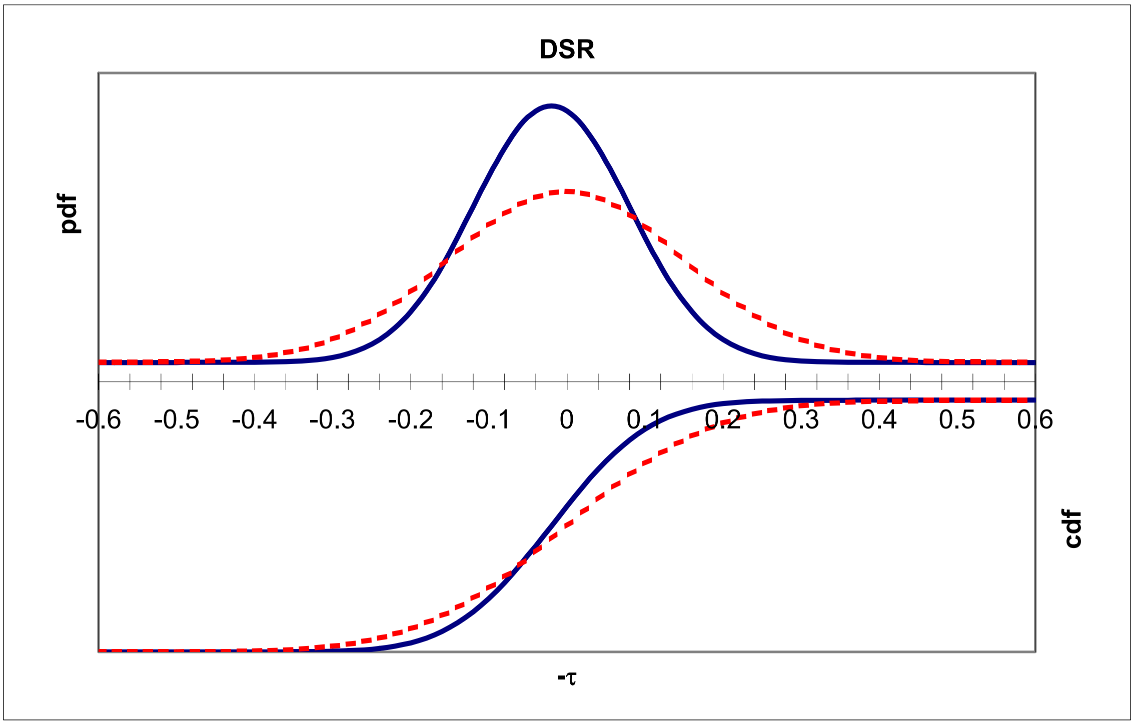
Table A2: Simulation Results for the Numerical Example Based on $C_0(1) = 0.945$

| $\bar{\tau}(1)$ | $E[DSR]$ | $\sqrt{V[DSR]}$ | $\pi(1)$ | $\rho(1)$ | $d(0)$ | Bond | Equity |
|---|---|---|---|---|---|---|---|
| AA | -0.02 | 0.0840 | - | - | 0.0105 | 0.5654 | 0.4346 |
| 0.00 | -0.02 | 0.0929 | 0.5853 | 0.9796 | 0.0052 | 0.4957 | 0.5043 |
| 0.01 | -0.02 | 0.0943 | 0.5422 | 0.9814 | 0.0044 | 0.4848 | 0.5152 |
| 0.02 | -0.02 | 0.0955 | 0.5000 | 0.9832 | 0.0037 | 0.4752 | 0.5248 |
| 0.03 | -0.02 | 0.0966 | 0.4588 | 0.9849 | 0.0030 | 0.4665 | 0.5335 |
| 0.04 | -0.02 | 0.0976 | 0.4188 | 0.9866 | 0.0024 | 0.4589 | 0.5411 |
| 0.05 | -0.02 | 0.0985 | 0.3803 | 0.9883 | 0.0019 | 0.4522 | 0.5478 |
| 0.06 | -0.02 | 0.0992 | 0.3435 | 0.9899 | 0.0015 | 0.4464 | 0.5536 |
| 0.07 | -0.02 | 0.0999 | 0.3083 | 0.9915 | 0.0011 | 0.4415 | 0.5585 |
| 0.08 | -0.02 | 0.1004 | 0.2751 | 0.9932 | 0.0008 | 0.4373 | 0.5627 |
| 0.09 | -0.02 | 0.1009 | 0.2439 | 0.9948 | 0.0006 | 0.4338 | 0.5662 |
| 0.10 | -0.02 | 0.1013 | 0.2148 | 0.9963 | 0.0003 | 0.4309 | 0.5691 |
| AA | -0.01 | 0.1024 | - | - | 0.0105 | 0.4374 | 0.5626 |
| 0.00 | -0.01 | 0.1093 | 0.5364 | 0.9717 | 0.0067 | 0.3856 | 0.6144 |
| 0.01 | -0.01 | 0.1108 | 0.5000 | 0.9732 | 0.0059 | 0.3749 | 0.6251 |
| 0.02 | -0.01 | 0.1121 | 0.4645 | 0.9747 | 0.0051 | 0.3650 | 0.6350 |
| 0.03 | -0.01 | 0.1133 | 0.4299 | 0.9762 | 0.0045 | 0.3561 | 0.6439 |
| 0.04 | -0.01 | 0.1144 | 0.3966 | 0.9777 | 0.0039 | 0.3479 | 0.6521 |
| 0.05 | -0.01 | 0.1154 | 0.3644 | 0.9792 | 0.0033 | 0.3405 | 0.6595 |
| 0.06 | -0.01 | 0.1163 | 0.3336 | 0.9806 | 0.0028 | 0.3338 | 0.6662 |
| 0.07 | -0.01 | 0.1171 | 0.3042 | 0.9821 | 0.0024 | 0.3278 | 0.6722 |
| 0.08 | -0.01 | 0.1178 | 0.2762 | 0.9835 | 0.0020 | 0.3225 | 0.6775 |
| 0.09 | -0.01 | 0.1184 | 0.2497 | 0.9849 | 0.0016 | 0.3178 | 0.6822 |
| 0.10 | -0.01 | 0.1190 | 0.2247 | 0.9864 | 0.0013 | 0.3137 | 0.6863 |
| AA | 0.00 | 0.1219 | - | - | 0.0105 | 0.3094 | 0.6906 |
| 0.00 | 0.00 | 0.1264 | 0.5000 | 0.9631 | 0.0081 | 0.2764 | 0.7236 |
| 0.01 | 0.00 | 0.1279 | 0.4688 | 0.9644 | 0.0073 | 0.2655 | 0.7345 |
| 0.02 | 0.00 | 0.1292 | 0.4385 | 0.9658 | 0.0066 | 0.2554 | 0.7446 |
| 0.03 | 0.00 | 0.1305 | 0.4091 | 0.9671 | 0.0059 | 0.2460 | 0.7540 |
| 0.04 | 0.00 | 0.1317 | 0.3807 | 0.9684 | 0.0053 | 0.2374 | 0.7626 |
| 0.05 | 0.00 | 0.1328 | 0.3533 | 0.9697 | 0.0047 | 0.2293 | 0.7707 |
| 0.06 | 0.00 | 0.1338 | 0.3269 | 0.9710 | 0.0041 | 0.2219 | 0.7781 |
| 0.07 | 0.00 | 0.1347 | 0.3017 | 0.9723 | 0.0036 | 0.2151 | 0.7849 |
| 0.08 | 0.00 | 0.1356 | 0.2775 | 0.9736 | 0.0032 | 0.2089 | 0.7911 |
| 0.09 | 0.00 | 0.1363 | 0.2546 | 0.9749 | 0.0028 | 0.2033 | 0.7967 |
| 0.10 | 0.00 | 0.1370 | 0.2327 | 0.9762 | 0.0024 | 0.1982 | 0.8018 |

Notes: Cf. Table A1.

## Appendix B

Figure B1: Two Different DSR Distributions (pdf and cdf)




Notes: The graph is only used to explain the intersection of default probabilities in Figures 2 and 3. In this example, the critical threshold for which both default probabilities are the same is in the magnitude of –0.07.