

Simulated Nonparametric Estimation of Dynamic Models with Applications to Finance*

Unabridged version

Filippo Altissimo
European Central Bank, and CEPR

Antonio Mele
London School of Economics

First draft: April 11, 2001

This version: May 19, 2005

Abstract

This paper introduces a new class of parameter estimators for dynamic models, called Simulated Nonparametric Estimators (SNE). The SNE minimizes appropriate distances between nonparametric joint (or conditional) densities estimated from sample data and nonparametric joint (or conditional) densities estimated from data simulated out of the model of interest. Sample data and model-simulated data are smoothed with the same kernel. This makes the SNE: 1) consistent independently of the amount of smoothing (up to identifiability); and 2) asymptotically root-T normal when the smoothing parameter goes to zero at a reasonably mild rate. Furthermore, the estimator displays the same asymptotic efficiency properties as the maximum-likelihood estimator as soon as the model is Markov in the observable variables. The methods are flexible, simple to implement, and fairly fast; furthermore, they possess finite sample properties that are well approximated by the asymptotic theory. These features are illustrated within the typical estimation problems arising in financial economics.

JEL: C14, C15, C32, G12

Keywords: nonparametric estimation, asset pricing, continuum of moments, simulations

Corresponding author: Antonio Mele, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom. Email: a.mele@lse.ac.uk.

*We wish to thank Yacine Aït-Sahalia, Torben Andersen, Marine Carrasco, Mikhail Chernov, Frank Diebold, Cristian Huse, Dennis Kristensen, Oliver Linton, Nour Meddahi, Angelo Melino, Alex Michaelides, Eric Renault, Christopher Sims, Sam Thompson, seminar participants at CORE, the LSE, the Norwegian School of Economics and Business, Princeton University, the University of Pennsylvania and, especially, Valentina Corradi for her valuable comments and Fabio Fornari who worked with us in the initial phase of this project. We remain, however, solely responsible for any omission and mistakes, as well as for the views expressed in the paper, which do not reflect those of the European Central Bank.

1 Introduction

This paper introduces a new class of parameter estimators for dynamic models with possibly unobserved components, called Simulated Nonparametric Estimators (hereafter SNE). The SNE works by making the finite dimensional distributions of the model's observables as close as possible to their empirical counterparts estimated through standard nonparametric techniques. Since the distribution of the model's observables is in general analytically intractable, we recover it through two steps. In the first step, we simulate the system of interest. In the second step, we obtain model's density estimates through the application of the same nonparametric devices used to smooth the sample data. The result is a consistent and root-T asymptotically normal estimator displaying a number of attractive properties. First, our estimator is based on simulations; consequently, it can be implemented in a straightforward manner to cope with a variety of estimation problems. Second, the SNE is purposely designed to minimize distances of densities smoothed with the same kernel; therefore, up to identifiability, it is consistent regardless of the smoothing parameter behavior. Third, if the SNE is taken to match conditional densities and the model is Markov in the observables, it achieves the same asymptotic efficiency as the maximum-likelihood estimator (MLE). Finally, Monte Carlo experiments reveal that our estimator does exhibit a proper finite sample behavior.

Systems with unobserved components arise naturally in many areas of economics. Examples in macroeconomics include models of stochastic growth with human capital and/or sunspots, job duration models, or models of investment-specific technological changes. Examples arising in finance include latent factor models, processes with jumps, continuous-time Markov chains, and even scalar diffusions. While the general theory we develop in this article is well suited to address estimation issues in all such areas, the specific applications we choose to illustrate our methods cover the typical models arising in financial economics (latent factor models and diffusion models).

As is well-known, the major difficulty arising from the estimation of dynamic models with unobserved components is related to the complexity of evaluating the criterion functions. A natural remedy to this difficulty is to make use of simulation-based methods. The simulated method of moments (McFadden (1989), Pakes and Pollard (1989), Lee and Ingram (1991) and Duffie and Singleton (1993)), the simulated pseudo-maximum likelihood method of Laroque and Salanié (1989, 1993, 1994), the indirect inference approach of Gouriéroux, Monfort and Renault (1993) and Smith (1993), and the efficient method of moments (EMM) of Gallant and Tauchen (1996) represent the first attempts at addressing this problem through extensions of the generalized method of moments. The main characteristic of these approaches is that they are general-purpose. Their drawback is that they lead to inefficient estimators even in the case of fully observed systems. The only exception is the EMM, which becomes indeed efficient as the (parameter) dimension of the auxiliary score gets larger and larger - a condition known as

“smooth embedding”. There exist alternative simulation-based econometric methods, which directly approximate the likelihood function through simulations (e.g., Lee (1995) or Hajivassiliou and McFadden (1998)). These methods do lead to asymptotic efficiency. Yet all the estimators arising within this class of methods are designed to address very specific estimation problems.

More recently, the focus of the literature has shifted towards a search for estimators combining the attractive features of both moments generating techniques and ML. In addition to the EMM, two particularly important contributions in this area are Fermanian and Salanié (2004) and Carrasco, Chernov, Florens and Ghysels (2004). Precisely, Fermanian and Salanié (2004) introduced a general-purpose method in which the (intractable) likelihood function is approximated by kernel estimates obtained through simulations of the model of interest. The resulting estimator, called nonparametric simulated ML (NPSML) estimator, is then both consistent and asymptotically efficient as the number of simulations goes to infinity and the smoothing parameter goes to zero at some (typical) convergence rate. Carrasco, Chernov, Florens and Ghysels (2004) developed a general estimation technology which also leads to asymptotic efficiency in the case of fully observed Markov processes. Their method leads to a “continuum of moment conditions” matching model-based (simulated) characteristic functions to data-based characteristic functions.

This article belongs to this new strand of the literature. Our strategy is indeed to construct criterion functions leading to a general estimation approach. And in many cases of interest, these criterion functions are asymptotically equivalent to Neyman’s chi-square measures of distance. It is precisely such an asymptotic equivalence which makes our resulting estimators asymptotically efficient. However, we emphasize that our estimators are quite distinct from any possible approximation to the MLE - they thus work rather differently from the Fermanian and Salanié NPSML estimator. In the language of indirect inference theory, we rely on “auxiliary criterion functions”, which generally give rise to asymptotically inefficient but consistent estimators. But as soon as model and data’s transition densities are estimated with a smoothing parameter converging to zero, these criterion functions converge to Neyman’s chi-squares, and our estimator becomes efficient. In this sense, the role played by the smoothing parameter in our context parallels the role played by the smooth embedding condition within the EMM.¹ One distinctive feature of our method is that we allow the smoothing parameter to go to zero at a reasonably mild rate. Furthermore, we smooth model-generated data and observations with the same kernel. Therefore, the behavior of the smoothing parameter does not affect the *consistency* of the estimator. An asymptotically shrinking smoothing parameter can only affect the *precision* of our estimator.

Our method is also related to the estimators introduced by Carrasco, Chernov, Florens and Ghysels (2004). Indeed, our SNE also relies on a “continuum of moments”, but in a very different manner. First, we do not need an infinite number of simulations to ensure consistency and

¹We are grateful to one anonymous referee and Christopher Sims for bringing this point to our attention.

asymptotic normality of our estimators. Second, we use more classical ideas from the statistical literature, and develop estimating equations leading to match model-based density estimates (not characteristic functions) to their empirical counterparts. As for the NPSML estimator, the SNE is thus both conceptually very simple and fairly easy to implement. Earlier estimators based on ideas similar to ours include the ones introduced by Gallant (2001) and Billio and Monfort (2003). Precisely, Gallant (2001) estimator matches cumulative distributions, but it does not lead to asymptotic efficiency. Billio and Monfort (2003) estimator minimizes distances between observation-based and simulated-based expectations of test functions smoothed with kernel methods. While their estimator is not asymptotically efficient, it is still (up to identifiability) consistent independently of the amount of smoothing. Yet the rate of convergence of their estimator is nonparametric - although the rate of convergence to zero of their smoothing parameter can be made very slow. As noted earlier, the convergence rate of our estimator is the usual parametric one, but this attractive feature of our methods is obtained with one additional computational cost: To match nonparametric density estimates, the evaluation of our objective functions requires the computation of a Riemann integral.

Finally, Aït-Sahalia (1996) is one additional fundamental contribution which this article is clearly related to. Aït-Sahalia developed a minimum distance estimator in which the *measure of distance* is a special case of the general class of measures of distance we consider here. But our estimator is different for three additional important reasons. First, the asymptotic behavior of Aït-Sahalia's estimator critically depends on the smoothing parameter; as we argued earlier, our estimator is designed in a way that the smoothing parameter plays a relatively more marginal role. Second, Aït-Sahalia's estimator only matches marginal densities. Third, Aït-Sahalia's method does not rely on simulations; therefore, it is feasible only when the density implied by the model has a fairly tractable form.

The paper is organized in the following manner. Section 2 introduces basic notation and assumptions for the model of interest. Section 3 provides large sample theory. Section 4 illustrates how our methods can be used to estimate the typical diffusion models arising in financial economics. Section 5 assesses the finite sample properties of our estimators. Section 6 concludes. The appendix gathers proofs and regularity conditions omitted in the main text.

2 The model of interest

Let $\Theta \subset \mathbb{R}^{p_\theta}$ be a compact parameter set, and for a given parameter vector $\theta_0 \in \Theta$, consider the following reduced-form data generating process:

$$y_{t+1} = f(y_t, \epsilon_{t+1}; \theta_0), \quad t = 0, 1, \dots, \quad (1)$$

where $y_t \in \mathbb{R}^d$, f is known and $\{\epsilon_t\}_{t=1, \dots}$ is a sequence of \mathbb{R}^d -valued identically and independently distributed random variables (with known distribution). The purpose of this paper is to provide estimators of the true parameter vector θ_0 . We consider a general situation in which some components of y are not observed. Accordingly, we partition vector y as:

$$y = \begin{pmatrix} y^o \\ \cdots \\ y^u \end{pmatrix},$$

where $y^o \in Y^o \subseteq \mathbb{R}^{q^*}$ is the vector of observable variables and $y^u \in Y^u \subseteq \mathbb{R}^{d-q^*}$ is the vector of unobservable variables. Data are collected in a $q^* \times T$ matrix with elements $\{y_{j,t}^o\}_{j=1, \dots, q^*; t=1, \dots, T}$, where $y_{j,t}^o$ denotes the t -th observation of the j -th component of vector y^o , and T is the sample size. Since our general interest lies in the estimation of partially observed processes, we may wish to recover as much information as possible about the dependence structure of the observables in (1). We thus set $q = q^*(1 + l)$, for some $l \geq 1$, let $y_t^o = (y_{1,t}^o, \dots, y_{q^*,t}^o)$ and

$$x_t \equiv (y_t^o, \dots, y_{t-l}^o), \quad t = t_l \equiv 1 + l, \dots, T, \quad (2)$$

and define $X \subseteq \mathbb{R}^q$ as the domain of x_t . In practice, there is a clear trade-off between increasing the highest lag l and both speed of computations *and* the curse of dimensionality. In Section 3.2, we succinctly present a few practical devices on how to cope with the curse of dimensionality.

Let $\pi(x; \theta)$ denote the joint density induced by (1) on x when the parameter vector is $\theta \in \Theta$. Let $\pi_0(x) \equiv \pi(x; \theta_0)$ and let $|\nabla_\theta \pi(x; \theta)|_2$ denote the outer product of vector $\nabla_\theta \pi(x; \theta)$. We now make assumptions further characterizing the family of processes we are investigating.

Assumption 1 (a) $\pi(x; \theta)$ is continuous and bounded on $X \times \Theta$. **(b)** For all $x \in X$, function $\theta \mapsto \pi(x; \theta)$ is twice differentiable and its derivatives are bounded on Θ . Furthermore, f is continuous and twice differentiable on Θ .

To ensure the feasibility of the asymptotic theory related to our estimation methods, we also need to make the following assumption on the decay of dependence in the observables in (1):

Assumption 2. Vector y is a Markov β -mixing sequence with mixing coefficients β_k satisfying $\lim_{k \rightarrow \infty} k^\mu \beta_k \rightarrow 0$, for some $\mu > 1$.

The mixing condition of assumption 2 is critical for the application of a functional central limit theorem due to Arcones and Yu (1994). Precisely, assumption 2 ensures convergence of suitably rescaled integrals of kernel functions to stochastic integrals involving generalized Brownian Bridges. This kind of convergence is exactly what we need to prove asymptotic normality of our estimators.

3 Theory

3.1 “Twin-smoothing”

Our estimation methodology is related to the classical literature on goodness-of-fit tests initiated by Bickel and Rosenblatt (1973). Let π_T be a nonparametric estimator of π_0 , obtained as $\pi_T(x) \equiv (T\lambda^q)^{-1} \sum_{t=t_l}^T K((x_t - x)/\lambda)$, where $x \in \mathbb{R}^q$, the bandwidth $\lambda > 0$, and K is a symmetric bounded kernel of the r -th order.² Consider the following empirical measure of distance:

$$I_T(\theta) = \int_{\mathbb{R}^q} [\pi(x; \theta) - \pi_T(x)]^2 w_T(x) dx, \quad (3)$$

where $w_T > 0$ is a weighting function possibly depending on data, and θ is a given parameter value. Let $\hat{\theta}$ be some consistent estimator of θ_0 . Typical measures of fit of the parametric model $\{\pi(\cdot; \theta), \theta \in \Theta\}$ to data are based on the empirical distance $I_T(\hat{\theta})$.³ Alternatively, the empirical distance in (3) can be utilized to *estimate* the unknown parameter vector θ_0 . For example, Aït-Sahalia (1996) defined an estimator minimizing (3) (with weighting function $w_T \equiv \pi_T$) in the context of scalar diffusions:

$$\theta_T^I = \arg \min_{\theta \in \Theta} I_T(\theta). \quad (4)$$

An important feature of the empirical measure of distance $I_T(\hat{\theta})$ is that a *parametric* density estimate, $\pi(\cdot; \hat{\theta})$, is matched to a *nonparametric* one, $\pi_T(\cdot)$. Under correct model specification, $\pi_T(x) \xrightarrow{P} K * \pi(x; \theta_0) \equiv \int_{\mathbb{R}^q} \lambda^{-q} K((u - x)/\lambda) \pi(u; \theta_0) du$ (x -pointwise). As is well-known, the result that $\pi_T(\cdot) \xrightarrow{P} \pi(\cdot; \theta_0)$ only holds if the bandwidth satisfies $\lambda \equiv \lambda_T$, $\lim_{T \rightarrow \infty} \lambda_T \rightarrow 0$ and $\lim_{T \rightarrow \infty} T\lambda_T^q \rightarrow \infty$. Therefore, bandwidth choice is critical for (3) and (4) to be really informative in finite samples.

²A symmetric kernel K is a symmetric function around zero that integrates to one. It is said to be of the r -th order if: 1) $\forall \mu \in \mathbb{N}^q : |\mu| \in \{1, \dots, r-1\}$ ($|\mu| \equiv \sum_{j=1}^q \mu_j$), $\int u_1^{\mu_1} \dots u_q^{\mu_q} K(u) du = 0$; 2) $\exists \mu \in \mathbb{N}^q : |\mu| = r$ and $\int u_1^{\mu_1} \dots u_q^{\mu_q} K(u) du \neq 0$; and 3) $\int \|u\|^r K(u) du < \infty$.

³Precisely, rescaled versions of (3) are classically used to implement tests of model misspecification (see, e.g., Pagan and Ullah (1999) for a comprehensive survey on those tests). Corradi and Swanson (2005) have recently developed new specification tests for diffusion processes based on *cumulative* probability functions.

To circumvent this problem, we consider a measure of distance alternative to (3). A simple possibility is an empirical distance in which the nonparametric estimate π_T is matched by the model’s density smoothed with the same kernel and conditional on a given bandwidth value:

$$L_T(\theta) = \int_{\mathbb{R}^q} [K * \pi(x; \theta) - \pi_T(x)]^2 w_T(x) dx. \quad (5)$$

Fan (1994) developed a class of bias-corrected goodness of fit tests based on the previous empirical distance and weighting function $w_T \equiv \pi_T$. And Härdle and Mammen (1993) devised a similar bias-correction procedure for testing the closeness of a parametric regression function to a nonparametric one.

A key idea in this paper is to combine the appealing idea underlying the estimator θ_T^I in (4) with the bias-corrected empirical measure in (5). To achieve this objective, we consider an estimator minimizing the distance in (5) rather than in (3), and consider a general empirical weighting function w_T . Specifically, define the following estimator:

$$\theta_T^L = \arg \min_{\theta \in \Theta} L_T(\theta), \quad (6)$$

where $w_T(x) \xrightarrow{p} w(x)$ uniformly, and w is another positive function. In (5), kernel smoothing operates in the same manner on model-implied density and on data-based density estimates. Therefore, bandwidth conditions affect the two estimators θ_T^I and θ_T^L in a quite different manner. Table 1 summarizes these bandwidth conditions. Under our regularity conditions, consistency of θ_T^L holds *independently of bandwidth behavior*. That is, up to identifiability (see assumption 3-(a) below), λ can be any strictly positive number. On the contrary, consistency of θ_T^I requires the additional conditions that $\lambda_T \rightarrow 0$ and $T\lambda_T^q \rightarrow \infty$.⁴

The “twin-smoothing” procedure underlying the estimator θ_T^L in (6) is intimately related to the general indirect inference strategy put forward in the seminal papers of Gouriéroux, Monfort and Renault (1993) and Smith (1993). In the language of indirect inference, we are matching a model-implied (infinite-dimensional) auxiliary parameter ($K * \pi(x; \theta)$) to the corresponding (infinite-dimensional) parameter computed on real data ($\pi_T(x)$). These auxiliary parameters can be estimated with an arbitrary bandwidth choice; yet, and up to identifiability, our estimator is still consistent in exactly the same spirit of the indirect inference principle.

Our basic idea is also related to the kernel-based indirect inference approach developed by Billio and Monfort (2003). Billio-Monfort estimator matches conditional expectations of arbitrary test-functions estimated through nonparametric methods - one conditional expectation computed

⁴Other estimators related to (4) suffer from exactly the same drawback. Two examples are 1) estimators based on nonparametric density estimates of the log-likelihood function obtained through simulations; and 2) estimators based on the so-called Kullback-Leibler distance (or relative entropy) $\int_{\mathbb{R}^q} \log[\pi(x, \theta) / \pi_0(x)] \pi(x, \theta) dx$. We are grateful to Oliver Linton for having suggested the latter example to us.

on true data and one conditional expectation computed on simulated data. This makes asymptotic bias issues irrelevant for their estimator. One important difference between our estimator θ_T^L in (6) and Billio-Monfort estimator is that our estimator is consistent at the usual parametric rate. The rate of convergence of Billio-Monfort estimator is contaminated by the rate of convergence of their bandwidth sequence to zero - although in practice the convergence of their bandwidth can be made very slow. Intuitively, Billio-Monfort estimator matches a finite number of test-functions. Instead, we match a continuum of moment conditions. But at the same time, this attractive feature of our estimator (matching a continuum of moments) brings an additional computational cost related to the evaluation of the Riemann integral in (5). Finally, our idea to directly focus on matching objects related to density functions resembles the “effective calibration” strategy of Gallant (2001). The main difference is that Gallant (2001) considers matching cumulative distribution functions. As we demonstrate in later sections, the advantage to focus on density functions is that it allows us to address efficiency issues.

Similarly as for consistency, θ_T^L and θ_T^I are asymptotically normally distributed under different bandwidth restrictions. Our estimator θ_T^L is asymptotically normal under the standard assumptions that $\lambda_T \rightarrow 0$ and $\sqrt{T}\lambda_T^q \rightarrow \infty$.⁵ Instead, θ_T^I is asymptotically normal under one additional condition on the order of the kernel (i.e. $\sqrt{T}\lambda_T^r \rightarrow 0$). Intuitively, this order condition guarantees that a density bias estimate vanishes at an appropriate rate without affecting the asymptotic behavior of θ_T^I . In contrast, density bias issues are totally absent if one implements estimator θ_T^L . As summarized in Table 1, bandwidth restrictions are only required to make our estimator θ_T^L asymptotically normal - not consistent. And as we demonstrate in the Monte Carlo experiments of Section 5, bandwidth restrictions in Table 1 are considerably less critical for asymptotic normality than for consistency.

Table 1 - Bandwidth assumptions and asymptotic behavior of θ_T^I and θ_T^L

	Consistency	Asymptotic normality
θ_T^I	$T\lambda_T^q \rightarrow \infty, \lambda_T \rightarrow 0$	$\sqrt{T}\lambda_T^q \rightarrow \infty, \lambda_T \rightarrow 0, \text{ and } \sqrt{T}\lambda_T^r \rightarrow 0$
θ_T^L	no asymptotic bandwidth restrictions	$\sqrt{T}\lambda_T^q \rightarrow \infty, \lambda_T \rightarrow 0$

3.2 Simulated Nonparametric Estimators

Our fundamental objective is to extend the previous ideas to general situations. Specifically, suppose that the analytical solution for density $\pi(x; \theta)$ in (5) is unknown, but that it is still possible to simulate from that density. Accordingly, the first step of our estimation strategy requires simulated paths of the observable variables in (1). To generate S simulated paths for a

⁵More sophisticated versions of our estimator are asymptotically normal under an additional assumption guaranteeing that certain derivatives of density estimates are well-behaved (i.e. $\sqrt{T}\lambda_T^{q+1} \rightarrow \infty$) (see theorem 1).

given parameter value θ , we draw $y_0(\theta)$ from its stationary distribution, and compute recursively

$$y_{t+1}(\theta) = f(y_t(\theta), \tilde{\epsilon}_{t+1}; \theta), \quad t = 0, 1, \dots, T,$$

where $\{\tilde{\epsilon}_t\}_{t=1}^{T+1}$ is a sequence of random numbers drawn from the distribution of ϵ . Let $x^i(\theta) = \{x_t^i(\theta)\}_{t=t_i}^T$ ($i = 1, \dots, S$), where $x_t^i(\theta)$ is the i -th simulation of the t -th observation when the parameter vector is θ , and define $y^i(\theta)$ in a similar way. Let $\pi_T^i(x; \theta) \equiv (T\lambda_T^q)^{-1} \sum_{t=t_i}^T K((x_t^i(\theta) - x)/\lambda_T)$, where K and λ are the same kernel and bandwidth functions used to compute the nonparametric density estimate $\pi_T(\cdot)$ on sample data.

We are now in a position to provide the definition of the first estimator considered in this paper:

Definition 1. (SNE) *For each fixed integer S , the Simulated Nonparametric Estimator (SNE) is the sequence $\{\theta_{T,S}\}_T$ given by:*

$$\theta_{T,S} = \arg \min_{\theta \in \Theta} \int_X [\tilde{\pi}_T(x; \theta) - \pi_T(x)]^2 w_T(x) dx, \quad (7)$$

where $\tilde{\pi}_T(\cdot; \cdot) \equiv S^{-1} \sum_{i=1}^S \pi_T^i(\cdot; \cdot)$ and $w_T(\cdot) > 0$ is a sequence of bounded and integrable functions satisfying $w_T(x) \xrightarrow{P} w(x)$, x -pointwise, for some function w .

The appealing feature of this estimator is that π_T^i and π_T are computed with the *same kernel and bandwidth*. Such a twin kernel smoothing procedure operates on sample and model generated data in exactly the same manner as in (5). Consequently, the asymptotic properties of θ_T in (7) and θ_T^L in (6) are quite comparable. Moreover, consistency of θ_T does not require an infinite number of simulations S . Even in correspondence of a finite number of simulations, the objective function in (7) is asymptotically equivalent to the objective function in (5). These two features of the SNE make our estimation strategy quite distinct from the estimation strategy introduced by Fermanian and Salanié (2004) - in which the likelihood function is directly approximated by kernel estimates of model-simulated data. But our approach also entails the additional computational cost related to the evaluation of the Riemann integral in (7).

We consider kernels satisfying the following regularity conditions:⁶

⁶Assumption K is needed to prove the lemmata in appendix A through Andrews (1995) strategy of proof. Andrews (1995) (assumption NP4(b), p. 566-567) required that $\int (1 + \|z\|^r) \sup_{b \geq 1} |\Phi(bz)| dz < \infty$. Assumption K imposes the weaker condition that $\int |\Phi(z)| dz < \infty$. We deleted the $(1 + \|z\|^r)$ multiplicand because y is strong mixing by assumption 2. The $\sup_{b \geq 1}$ requirement is not to be ignored in all applications with data-dependent bandwidths.

Assumption K. *Kernels K are bounded, continuously differentiable with bounded derivatives up to the fourth order, and absolutely integrable with an absolutely integrable Fourier transform $\Phi(z) \equiv (2\pi)^q \int \exp(\mathbf{i}z^\top u) K(u) du$.*

Let $L_T(\theta) \equiv \int [\tilde{\pi}_T(x; \theta) - \pi_T(x)]^2 w_T(x) dx$. Let the expectation of the kernel for a given bandwidth value λ be denoted as:

$$m(x; \theta) \equiv K * \pi(x; \theta) = \frac{1}{\lambda^q} \int K\left(\frac{x-u}{\lambda}\right) \pi(u; \theta) du.$$

Accordingly, set $L(\theta) \equiv \int [m(x; \theta) - m(x; \theta_0)]^2 w(x) dx$. Criteria are required to satisfy the following regularity *and* identifiability conditions:

Assumption 3 (a). *For all $\theta \in \Theta$, $L_T(\theta)$ is measurable and continuous on Θ a.s. Moreover, $L(\theta)$ is continuous on Θ , \exists unique $\theta_0 : L(\theta_0) = 0$, and $\liminf_{T \rightarrow \infty} \min_{\theta \in N_T^c} L_T(\theta) > 0$, where N_T^c is the complement in Θ of a neighborhood of $\theta_{T,S}$.*

The first part of assumption 3-(a) is needed to ensure existence of our SNE, and holds under mild conditions on the primitive model. For example, it holds under the previous kernel assumption K, and the assumption that function f in (1) is continuous on Θ . The second part of this assumption merits further discussion. We are designing our estimator in such a way that bandwidth choice is virtually irrelevant for consistency. But to accomplish this task, we need to make sure that the (infinite-dimensional) ‘‘auxiliary’’ parameter $K * \pi$ has information content on the ‘‘structural’’ parameter θ . The last part of assumption 3-(a) then makes our SNE *identifiably unique*.⁷ Consistency of the SNE requires the following additional assumption:

Assumption 3 (b). *There exists a $\alpha > 0$ and a sequence κ_T bounded in probability as T becomes large such that for all $(\varphi, \theta) \in \Theta \times \Theta$, $|L_T(\varphi) - L_T(\theta)| \leq \kappa_T \cdot \|\varphi - \theta\|_2^\alpha$.*

Assumption 3-(b) is a standard high level assumption. In Appendix F, we have developed specific examples of primitive conditions ensuring that assumption 3-(b) does hold. We now turn to formulate one assumption we use to prove asymptotic normality of the SNE. Let $\mathcal{K}_T^j(x; \theta) \equiv \left| K'((x_t^i(\theta) - x)/\lambda_T)(\partial y_{\ell,t}^i(\theta)/\partial \theta_j) \right|$ ($j = 1, \dots, p_\theta$ and $i = 1, \dots, S$), where $y_{\ell,t}^i(\theta)$ is the i -th simulation at t of the ℓ -th component of x_t in (2) ($\ell = 1, \dots, q^*$). We have:

⁷See, e.g., Gallant and White (1988, definition 3.2 p. 19). One referee suggested that identifiability may break down if the bandwidth λ is larger than the support of data. In appendix F.1, we formalize this referee’s suggestion and provide an example of kernels, bandwidth *levels* and data generating process (with *bounded* support) such that identifiability does break down. In appendix F, we also argue that if kernels satisfy assumption K, the identifiably uniqueness condition in assumption 3-(a) holds with *sufficiently small* bandwidth values (not necessarily shrinking to zero).

Assumption 4 (a). For all $j = 1, \dots, p_\theta$ and $(x, \theta) \in X \times \Theta$, $\mathcal{K}_T^j(x; \theta)$ is continuous, bounded and satisfies assumption 2; and $\partial \mathcal{K}_T^j(x; \theta) / \partial \theta_m$ is bounded for all $m = 1, \dots, p_\theta$; $\partial^{\rho+1} \pi(x; \theta) / \partial \theta \partial x^\rho$ is uniformly bounded for some $\rho \geq r$. **(b)** $\sup_{x \in X} |w_T(x) - w(x)| = O_p(T^{-\frac{1}{2}} \lambda_T^{-q}) + O_p(\lambda_T^r)$.

All in all, assumption 4 on \mathcal{K}_T^j is needed to make the first order conditions satisfied by the SNE analytically tractable. (Basically, it allows one to interchange the order of derivation and integration in $\nabla_\theta L_T(\theta)$.) The assumption on $\partial^{\rho+1} \pi(x; \theta) / \partial \theta \partial x^\rho$ ensures uniform convergence of score functions to their asymptotic counterparts (see lemmata 5 to 10 in appendix A). Finally, the assumption on the weighting function w_T is obviously under the investigator's control. As an example, one may take $w_T(x) \equiv \pi_T(x) \gamma(x)$, where γ is another function. By lemma 1 in appendix A, this choice satisfies assumption 4-(b).

The following result provides the asymptotic properties of the SNE:

Theorem 1. Let assumptions 1-(a), 2 and 3 hold; then, the SNE is (weakly) consistent. Furthermore, let $\Psi(x) \equiv [\int |\nabla_\theta \pi(u; \theta_0)|_2 w(u) du]^{-1} \nabla_\theta \pi(x; \theta_0) w(x)$. Then, under the additional assumption 1-(b) and 4, and the conditions that $\lambda \equiv \lambda_T \rightarrow 0$ and $T^{\frac{1}{2}} \lambda_T^{q+1} \rightarrow \infty$ as $T \rightarrow \infty$,

$$\sqrt{T}(\theta_{T,S} - \theta_0) \xrightarrow{d} N\left(0, \left(1 + \frac{1}{S}\right) V\right),$$

where $V \equiv \text{var}[\Psi(x_1)] + \sum_{k=1}^{\infty} \{cov[\Psi(x_1), \Psi(x_{1+k})] + cov[\Psi(x_{1+k}), \Psi(x_1)]\}$, provided it exists finitely.

Proof. In appendix B. ■

The asymptotic theory underlying the SNE displays four basic distinctive features. First, *and up to identifiability* (see assumption 3-(a)), consistency does *not* rely on any condition regarding the bandwidth parameter. The only bandwidth conditions we actually need only ensure that the SNE is asymptotically normal. In particular, the order of the kernel plays no role within our asymptotic theory.⁸ We shall see that this conclusion is only slightly modified even in more sophisticated versions of our basic estimator (see theorems 2 and 3 below).

Second, the (unscaled) variance V of theorem 1 collapses to the variance of the estimator in Aït-Sahalia (1996) in the scalar case and when $w_T = \pi_T$. However, we emphasize that the two estimators are radically different. Aït-Sahalia (1996) requires an analytical form of the model's density and, consequently, consistency of his estimator may only follow if both $\lambda_T \rightarrow 0$ and $T \lambda_T^q \rightarrow \infty$. The twin-smoothing procedure makes our SNE considerably less sensitive to

⁸The main technical reason explaining this result is that conditions such as $\sqrt{T} \cdot \lambda_T^r \rightarrow 0$ would be important if the theory required a functional limit theorem for $\sqrt{T}(\int \pi_T - \int \pi_0)$. We do not need such a demanding result. We only need a functional limit theorem for $\sqrt{T}(\int \pi_T - \int E(\pi_T))$.

bandwidth issues - a fact also documented in our Monte Carlo experiments.⁹ Furthermore, the SNE can address estimation of multivariate models driven by partially observed state variables with unknown distribution. Also, we explicitly consider matching *joint* densities of data, not marginal densities. Finally, the SNE minimizes a measure of closeness of two nonparametric density estimates - one on true data and the second on simulated data. Under correct model's specification, the resulting biases in the two kernel estimates cancel out each other, and asymptotic normality can then be obtained *without* relying on any bias-reducing devices. For all these reasons, the SNE is potentially apt to exhibit a finite sample behavior that is well approximated by the asymptotic theory. And such a finite sample behavior is indeed documented by our Monte Carlo experiments in Section 5.

Third, our SNE makes use of general weighting functions. If $w_T = \pi_T$, the corresponding SNE would overweight discrepancies occurring where observed data have more mass. More generally, Theorem 1 reveals that the asymptotic variance of the estimator depends indeed on the limiting weighting function w at hand. However, a weighting function minimizing such an asymptotic variance is unknown, even in the case of fully observable processes.¹⁰ In the next Section, we show that this problem can considerably be simplified through an appropriate change of the objective function in (7).¹¹

Fourth, the estimator's variance has to be rescaled by $(1 + S^{-1})$ - similarly as in the familiar asymptotics of Indirect Inference estimators (e.g., Gouriéroux, Monfort and Renault (1993)). This scaling term arises because the model's joint density is recovered by means of simulations.

As for other nonparametric density based-estimators, the SNE is subject to the curse of dimensionality. But as in related contexts, the SNE can be extended to mitigate this issue. As an example, we may let,

$$\theta_{T,S} = \arg \min_{\theta \in \Theta} \sum_{\ell=1}^l L_T^{(\ell)}(\theta), \quad L_T^{(\ell)}(\theta) \equiv \int_{X^\ell} \left[\tilde{\pi}_T(x^\ell; \theta) - \pi_T(x^\ell) \right]^2 w_T(x^\ell) dx^\ell, \quad X^\ell \subseteq \mathbb{R}^{2q^*},$$

where $x_t^\ell = (y_t^o, y_{t-\ell}^o)$ (see eq. (2)). In proposing the above estimator, we imitated Fermanian and Salanié (2004, Section 4), who also considered addressing dimensionality issues through the use of lagged observable variables. But even when the dimension of the model's observables q^* is small, in practice l should be a small number given the current state of computational power. In

⁹Accordingly, the Aït-Sahalia's estimator could also be modified through the bias correction procedure we suggested in (5).

¹⁰An exception arises exactly in the i.i.d. case. Under regularity conditions given in section 3.3, the optimal weighting function would be given by $w_T(x) = \mathbb{T}_T(x) \cdot \pi_T(x)^{-1}$, where $\mathbb{T}_T(\cdot)$ is a trimming function converging pointwise to 1 as $T \rightarrow \infty$.

¹¹Pastorello, Patilea and Renault (2003) have recently proposed a "latent backfitting" method to estimate partially observed systems through information provided by standard economic theory. In appendix G, we have extended the theory in the main text of this paper to the ideal situation in which partially observed systems are estimated in conjunction with asset pricing models holding without measurement error.

tests involving stochastic volatility models, the SNE computed with $l = 1$ (i.e. with a matching of the joint density of two adjacent observations) had a very encouraging behavior (see Section 5).

Dimensionality issues related to the spatial dimension q^* can be mitigated in the same vein. For example, one may consider all (low dimensional) combinations of elements of the vector of observables. An estimator mitigating dimensionality issues related to the spatial dimension q^* could then be,

$$\theta_{T,S} = \arg \min_{\theta \in \Theta} \sum_{\ell=-l}^l \sum_{k=1}^{C(q^*,2)} \int_{X^{k,\ell}} \left[\tilde{\pi}_T(x^{k,\ell}; \theta) - \pi_T(x^{k,\ell}) \right]^2 w_T(x^{k,\ell}) dx^{k,\ell}; \quad X^{k,\ell} \subseteq \mathbb{R}^2,$$

where $C(q^*, 2) \equiv \binom{q^*}{2}$ is the number of all combinations of *two* components out of the observables vector $y^o = (y_1^o, \dots, y_{q^*}^o)^\top$ and $x^{k,\ell}$ is one enumerated combination.¹² In words, the previous estimator reduces the problem of matching q^* -dimensional density estimates to a problem of matching all 2-dimensional joint density estimates among every single pair of observables. The previous device does address the dimensionality issue indeed; but again, a full use of it in large scale systems is hindered by the current state of computational power. For space reasons, we could not include this discussion in the paper.

3.3 Conditional Density SNE, and Efficiency

This section introduces a modification of the SNE, and addresses efficiency issues within the case of fully observable diffusions. We show that by casting the estimation problem as a matching of *conditional* densities (instead of *joint* ones), our resulting estimator is asymptotically (first-order) efficient whenever the state y in (1) is fully observable.

To prepare the analysis, consider again vector $x \in X \subseteq \mathbb{R}^q$ in (2). For each t , partition x_t as $x_t = (z_t, v_t)$, where $z_t \equiv y_t^o \in Z \subseteq \mathbb{R}^{q^*}$ is the vector of observable variables, and $v_t \in V \subseteq \mathbb{R}^{q-q^*}$, is the vector of predetermined variables:

$$v_t \equiv (y_{t-1}^o, \dots, y_{t-l}^o), \quad t = t_l \equiv 1 + l, \dots, T.$$

Consider the following conditional density matching estimator:

¹²To further illustrate this example, suppose that $q^* = 4$; then $C(q^*, 2) = 6$ and the enumerated combinations are, for fixed $\ell \in \{-l, \dots, 0, \dots, l\}$, $(y_{1,t}^o, y_{2,t-\ell}^o)$, $(y_{1,t}^o, y_{3,t-\ell}^o)$, $(y_{1,t}^o, y_{4,t-\ell}^o)$, $(y_{2,t}^o, y_{3,t-\ell}^o)$, $(y_{2,t}^o, y_{4,t-\ell}^o)$, $(y_{3,t}^o, y_{4,t-\ell}^o)$.

Definition 2. (CD-SNE) For each fixed integer S , the Conditional Density SNE (CD-SNE) is the sequence $\{\theta_{T,S}\}_T$ given by:

$$\theta_{T,S} = \arg \min_{\theta \in \Theta} \int_Z \int_V [\tilde{\pi}_T(z|v; \theta) - \pi_T(z|v)]^2 w_T(z, v) \mathbb{T}_{T,\delta}^2(v; \theta) dz dv, \quad (8)$$

where $\pi_T(z|v) \equiv \pi_T(z, v)/\pi_T(v)$, $\tilde{\pi}_T(z|v; \theta) \equiv S^{-1} \sum_{i=1}^S \pi_T^i(z, v; \theta)/\pi_T^i(v; \theta)$, $\{\mathbb{T}_{T,\delta}\}_T$ is a sequence of trimming functions satisfying assumption **T** below, and $w_T > 0$ is a sequence of weighting functions satisfying assumption 4-(b).

The CD-SNE relies on nonparametric conditional density estimates obtained as ratios between joints over marginals. Small values of the denominators in $\pi_T(z|v)$ may hinder numerical stability of the estimator, and the asymptotic theory. Therefore, we need to control the tail behavior of marginal density estimates. The role of trimming function $\mathbb{T}_{T,\delta}$ is to accommodate this task. Trimming functions are widely used in related contexts (see, e.g., Stone (1975), Bickel (1982), or more recently, Linton and Xiao (2000) and Fermanian and Salanié (2004)). In this paper, we consider trimming devices related to the original work of Andrews (1995).

Assumption T. Let g be a bounded, twice differentiable density function with support $[0, 1]$, $g(0) = g(1) = 0$, and let $g_\delta(u) \equiv \frac{1}{\delta} g(\frac{u}{\delta} - 1)$. We set, $\mathbb{T}_{T,\delta}(v; \theta) \equiv \prod_{i=0}^S \mathbb{T}(\pi_T^i(v; \theta))$ ($\pi_T^0(\cdot) \equiv \pi_T(\cdot)$), where $\mathbb{T}(\ell) \equiv \int_0^\ell g_{\delta_T}(u) du$, for some sequence $\delta_T \rightarrow 0$.

By construction, $\mathbb{T}_{T,\delta}$ is increasing, smooth and satisfies, $\mathbb{T}_{T,\delta}(v; \theta) = 0$ on $\{v : \pi_T^i(v; \theta) < \delta_T, i = 0, 1, \dots, S\}$; and $\mathbb{T}_{T,\delta}(v; \theta) = 1$ on $\{v : \pi_T^i(v; \theta) > 2\delta_T, i = 0, 1, \dots, S\}$. As $T \rightarrow \infty$ and $\delta_T \rightarrow 0$, and under additional regularity conditions, $\pi_T(z|v) \xrightarrow{P} \pi(z|v)$ - uniformly over expanding sets on which the trimming function $\mathbb{T}_{T,\delta}$ is nonzero (see lemma 3 in appendix A). In appendix C (see assumptions *T1-(a,b)*) we gather all regularity conditions on the asymptotic behavior of δ_T we need to demonstrate consistency and asymptotic normality of the CD-SNE.¹³

Let $\bar{L}_T(\theta) \equiv \iint [\tilde{\pi}_T(z|v; \theta) - \pi_T(z|v)]^2 w_T(x) dx$. We define the asymptotic counterpart of \bar{L}_T as $\bar{L}(\theta) \equiv \iint [n(z, v; \theta) - n(z, v; \theta_0)]^2 w(x) dx$, where $n(z, v; \theta) \equiv m(z, v; \theta)/m(v; \theta)$. To prove consistency of the CD-SNE, we need conditions paralleling the ones in assumption 3:

Assumption 5. \bar{L}_T and \bar{L} are as L_T and L in assumption 3-(a), and for all $(\varphi, \theta) \in \Theta \times \Theta$, $|\bar{L}_T(\varphi) - \bar{L}_T(\theta)| \leq \kappa_T \cdot \|\varphi - \theta\|_2^\alpha$, where α and κ_T are as in assumption 3-(b).

¹³Linton and Xiao (2000) suggested the following example of trimming functions with a closed-form solution. Let the Beta-type density $g(u) \propto z^k (1-z)^k$ (for some integer k); then $\mathbb{T}(\ell)$ is a $(2k+1)$ -polynomial in $(\ell - \delta_T)/\delta_T$.

The following result provides the asymptotic properties of the CD-SNE.

Theorem 2. *Let assumptions 1-(a), 2, 5 and assumption T1-(a) in appendix C hold; then the CD-SNE is (weakly) consistent. Under the additional assumptions 1-(b) and 4, and assumption T1-(b) in appendix C,*

$$\sqrt{T}(\theta_{T,S} - \theta_0) \xrightarrow{d} N(0, V),$$

where $V \equiv D_3^{-1} \cdot \text{var}[\frac{1}{S} \sum_{i=1}^S (D_1^i - D_2^i) - (D_1^0 - D_2^0)] \cdot D_3^{\top -1}$, provided it exists finitely; and the terms $\{D_1^i\}_{i=0}^S$, $\{D_2^i\}_{i=0}^S$ and D_3 are given in appendix C.2.

Proof. In appendix C. ■

The variance structure of the CD-SNE differs from the one in the asymptotic distribution of the SNE (see Section 3.2). In the CD-SNE case, one has to cope with additional terms arising because conditional densities are estimated as ratios of two densities (joints over marginals). These additional terms are $\{D_2^i\}_{i=0}^S$. As we show in appendix D.2, there exist weighting functions w_T making these terms identically zero. In those cases, the variance terms in theorem 2 have the same representation as the variance terms in Section 3.2. Proposition 2 in appendix D.2 summarizes our results on these issues.

We now argue that as soon as y in (1) is fully observable, there exists a weighting function w_T making the CD-SNE asymptotically attain the Cramer-Rao lower bound. Precisely, let,

$$w_T(z, v) = \frac{\pi_T(v)^2}{\pi_T(z, v)} \mathbb{T}_{T,\alpha}(z, v), \quad \mathbb{T}_{T,\alpha}(z, v) \equiv \mathbb{T}_\alpha(\pi_T(z, v; \theta)), \quad (9)$$

where $\mathbb{T}_\alpha(\ell) \equiv \int_0^\ell g_{\alpha T}(u) du$, and $g_{\alpha T}$ is as in assumption **T**. Similarly as for the CD-SNE in definition 2, $\mathbb{T}_{T,\alpha}(z, v)$ is a trimming function needed to control the tail behavior of the joint density estimate on sample data. If w_T is as in (9), the criterion in (8) reduces to:

$$\int_Z \int_V \left[\frac{\tilde{\pi}_T(z|v; \theta)}{\pi_T(z|v)} - 1 \right]^2 \pi_T(z, v) \mathbb{T}_{T,\delta}^2(v; \theta) \mathbb{T}_{T,\alpha}(z, v) dz dv,$$

which asymptotically becomes a Neyman's chi-squared measure of distance. A Taylor's expansion of the first order conditions satisfied by the CD-SNE around θ_0 yields that in large samples,

$$\begin{aligned} -J_T(\theta_0) \cdot \sqrt{T}(\theta_{T,S} - \theta_0) &\stackrel{d}{=} \iint_{\mathbb{T}} \left[\frac{\tilde{\pi}_T(z|v; \theta_0)}{\pi_T(z|v)} - 1 \right] \left[\frac{\nabla_\theta \tilde{\pi}_T(z|v; \theta_0)}{\pi_T(z|v)} \right] \pi_T(z, v) dz dv \\ &\stackrel{d}{=} \iint_{\mathbb{T}} [\tilde{\pi}_T(z, v; \theta_0) - \pi_T(z, v)] \cdot \nabla_\theta \ln \pi(z|v; \theta_0) \cdot dz dv \\ &= \frac{1}{S} \sum_{i=1}^S H_T^i(\theta_0) - H_T^0(\theta_0) \end{aligned} \quad (10)$$

where

$$\begin{aligned}
J_T(\theta_0) &= \iint_{\mathbb{T}} \left| \frac{\nabla_{\theta} \tilde{\pi}_T(z|v; \theta_0)}{\pi_T(z|v)} \right|_2 \pi_T(z, v) dz dv \\
H_T^i(\theta_0) &= \iint_{\mathbb{T}} \{ \pi_T^i(z, v; \theta_0) - E[\pi_T^i(z, v; \theta_0)] \} \cdot \nabla_{\theta} \ln \pi(z|v; \theta_0) \cdot dz dv, \quad i = 0, 1, \dots, S
\end{aligned}$$

(with $\pi_T^0 \equiv \pi_T$) and integrals with a subscript \mathbb{T} are integrals trimmed under the action of functions $\mathbb{T}_{T,\alpha}$ and $\mathbb{T}_{T,\delta}$. But $J_T(\theta_0)$ and $H_T^i(\theta_0)$ satisfy $J_T(\theta_0) \xrightarrow{P} E[|\nabla_{\theta} \ln \pi(z_1|v_1; \theta_0)|_2]$ and $H_T^i(\theta_0) \xrightarrow{d} N(0, \text{var}(\nabla_{\theta} \ln \pi(z|v; \theta_0)))$ ($i = 0, 1, \dots, S$) (see appendixes C.2 and D.2 for technical details on such a law of large numbers and central limit theorem¹⁴). Since the system is fully observable and Markov, $z_t = y_t$, and $\nabla_{\theta} \ln \pi(y_t|y_{t-1}; \theta_0)$ is a martingale difference with respect to the sigma-fields generated by y . Therefore, the variance of the CD-SNE (rescaled by $(1 + S^{-1})$) does attain the Cramer-Rao lower bound $E[|\nabla_{\theta} \ln \pi(y_2|y_1; \theta_0)|_2]^{-1}$.

The previous arguments are obviously heuristic. For example, one critical issue is to ensure that as $\alpha_T \rightarrow 0$, the weighting function in (9) $w_T(z, v) \xrightarrow{P} w(z, v)$ - uniformly over expanding sets on which $\mathbb{T}_{T,\alpha}$ is nonzero (see lemma 2 in appendix A). In appendix D, we gather all joint asymptotic restrictions on α_T and δ_T leading to consistency and asymptotic normality of the CD-SNE with weighting function as in (9) (see assumption $T\text{-}\mathcal{L}(a, b)$). We have:

Theorem 3. (Cramer-Rao lower bound) *Suppose that the state is fully observable (i.e., $q^* = d$). Let the CD-SNE match one-step ahead conditional densities (i.e., $(z, v) \equiv (y_t, y_{t-1})$ in (8)) and let w_T be as in (9). Let assumptions 1-(a), 2, 5 and assumption T2-(a) in appendix D hold; then, the CD-SNE is (weakly) consistent. Under the additional assumptions 1-(b), 4-(a) and assumption T2-(b) in appendix D, the CD-SNE is as in theorem 2, and it attains the Cramer-Rao lower bound as $S \rightarrow \infty$.*

Proof. In appendix D. ■

The previous efficiency result follows because the weighting function in (9) makes the CD-SNE asymptotically equivalent to the score as soon as the system is *fully* observable (see eq. (10)). We emphasize that this property corresponds to the classical first-order efficiency criterion in Rao (1962). Furthermore, results by which estimators based on closeness-of-density retain efficiency properties are not a novelty in the statistical literature. In the context of independent observations with discrete distributions, Lindsay (1994) presented a class of estimators encompassing a number of minimum disparity estimators based on Hellinger's distance, Pearson's chi-square, Neyman's

¹⁴The central limit theorem can be understood heuristically as follows. Consider approximating $H_T(\theta_0)$ with $\tilde{H}_T \equiv \iint \omega(x) dA_T(x)$, where $x \equiv (z, v)$, $A_T(x) = \sqrt{T}[F_T(x) - E(F_T(x))]$, $\omega \equiv \nabla_{\theta} \ln \pi$, and $F_T(x) = \frac{1}{T} \sum_{t=t_1}^T \mathbb{1}_{x_t \leq x}$. We have, $\tilde{H}_T = T^{-\frac{1}{2}} \sum_{t=t_1}^T \iint \omega(x) (d\mathbb{1}_{x_t \leq x} - E(d\mathbb{1}_{x_t \leq x})) = T^{-\frac{1}{2}} \sum_{t=t_1}^T [\omega(x_t) - E(\omega(x_t))]$, where the last equality holds because $d\mathbb{1}_{x_t \leq x} = \delta(x - x_t) dx$, where $\delta(\cdot)$ is the Dirac's delta. Now apply the central limit theorem to conclude.

chi-square, Kullback-Leibler distance, and maximum likelihood. Lindsay showed that while all these estimators are first-order efficient, they may differ in terms of second-order efficiency, and robustness. Basu and Lindsay (1994) extended this theory to the case of continuous densities. Such an extension can be used to illustrate some fundamental properties of our estimator. In the i.i.d. case, our CD-SNE can be thought of as a member belonging to a general class of minimum disparity estimators θ_T defined by the following estimating equation:

$$0 = \int_{\mathbb{T}} \mathcal{A}(\phi(x)) [\nabla_{\theta} (K * \pi(x; \theta_T))] dx, \quad \phi(x) \equiv \frac{\pi_T(x) - K * \pi(x; \theta_T)}{K * \pi(x; \theta_T)},$$

where \mathcal{A} is an increasing continuous function in $(-1, \infty)$.¹⁵ Under regularity conditions, function \mathcal{A} determines how sensitive an estimator is to the presence of outliers. Indeed, function ϕ is high exactly when a point in the sample space has been accounted much more than predicted by the model. Accordingly, a robust estimator is one able to mitigate the effect of large values of ϕ . As a benchmark example, the likelihood disparity sets $\mathcal{A}(\phi) = \phi$. Estimators with the property that $\mathcal{A}(\phi) \ll \phi$ for large ϕ are more robust to the presence of outliers than maximum likelihood. For instance, the Hellinger's distance sets $\mathcal{A}(\phi) = 2[\sqrt{\phi + 1} - 1]$, and the Kullback-Leibler distance has $\mathcal{A}(\phi) = \ln(1 + \phi)$. It is easily seen that if $w_T = \pi_T(x)^{-1} \mathbb{T}_{T,\alpha}(x)$, our L_T is asymptotically a Neyman's chi-squared measure of distance, with $\mathcal{A}(\phi) = \phi / (1 + \phi)$. These simple facts suggest that the class of estimators that we consider displays interesting robustness properties.

Naturally, the aim of theorem 3 was to extend the above class of estimators to the case of dynamic models. However, we do not further investigate robustness properties of our estimators. Using robustness, and/or second-order efficiency criteria as discrimination devices of alternative parameter estimators of dynamic models is an interesting area that we leave for future research.

4 Applications to continuous-time financial models

All available simulation-based techniques (and the methods developed in this article) rest on the obvious assumption that the model of interest can be simulated in a simple manner. Unfortunately, continuous-time models can not even be simulated - except in the trivial case in which the transition density is known.¹⁶ The simple reason is that a continuous-time model can only be *imperfectly* simulated through some discretization device. In this section, we show that our theory still works if we allow the discretization to shrink to zero at an appropriate rate.

¹⁵As $\lambda \downarrow 0$, \mathcal{A} and ϕ collapse to Lindsay's (1994) adjustment function and Pearson's residual, respectively.

¹⁶To date, estimation methods specifically designed to deal with diffusion processes include moments generating techniques (e.g., Hansen and Scheinkman (1995), Singleton (2001)), approximations to maximum likelihood (e.g., Pedersen (1995) and Santa-Clara (1995), and Ait-Sahalia (2002, 2003)) and, on a radically different perspective, Markov Chain Monte Carlo approaches (e.g., Elerian, Chib and Shephard (2001)).

4.1 The model

Let $\Theta \subset \mathbb{R}^{p_\theta}$ be a compact parameter set, and for a given parameter vector $\theta_0 \in \Theta$, consider the following data generating process $y = \{y(\tau)\}_{\tau \geq 0}$:

$$dy(\tau) = b(y(\tau), \theta_0) d\tau + a(y(\tau), \theta_0) dW(\tau), \quad \tau \geq 0, \quad (11)$$

where W is a standard d -dimensional Brownian motion; b and a are vector and matrix valued functions in \mathbb{R}^d and $\mathbb{R}^{d \times d}$, respectively; a is full rank almost surely; and y takes values in $Y \subseteq \mathbb{R}^d$. As in Section 3, we partition y as $y = (y^o : y^u)$, where $y^o \in Y^o \subseteq \mathbb{R}^{q^*}$ is the subvector of observable variables. Data are assumed to be sampled at regular intervals, and we still let $q \equiv q^*(1+l)$ and $x_t \equiv (y_t^o, \dots, y_{t-l}^o)$ ($t = 1+l, \dots, T$), where $\{y_t^o\}_{t=1}^T$ is the observations sequence and T is the sample size. We consider the following regularity condition:

Maintained assumptions. *System (11) has a strong solution and it is strictly stationary. Furthermore, assumptions 1 and 2 (with mixing coefficients $\bar{\beta}_k$ and exponent $\bar{\mu} > 1$, say) hold in the context of model (11).*

Chen, Hansen and Carrasco (1999) provide primitive conditions guaranteeing that assumption 2 holds in the case of scalar diffusions. A scalar diffusion is β -mixing with exponential decay if their “pull measure”, defined as $\frac{b}{a} - \frac{1}{2} \frac{\partial a}{\partial y}$, is negative (positive) at the right (left) boundary (the authors also provide conditions ensuring β -mixing with polynomial decay in the case of zero pull measure at one of the boundaries (see their remark 5)). As regards multidimensional diffusions, β -mixing with exponential decay can be checked through results developed by Meyn and Tweedie (1993) for exponential ergodicity, as in Carrasco, Hansen and Chen (1999). Finally, Carrasco, Hansen and Chen (1999) provide more specific results pertaining to partially observed diffusions.

4.2 Estimation

To generate simulated paths of the observable variables in (11), various discretization schemes can be used (see, e.g., Kloeden and Platen (1999)). In this paper, we consider the simple Euler-Maruyama discrete time approximation to (11):

$${}_h y_{h(k+1)} - {}_h y_{hk} = b({}_h y_{hk}, \theta) \cdot h + a({}_h y_{hk}, \theta) \cdot \sqrt{h} \cdot \epsilon_{k+1}, \quad k = 0, 1, \dots, \quad (12)$$

where h is the discretization step and $\{\epsilon_k\}_{k=1, \dots}$ is a sequence of independent \mathbb{R}^d -valued i.i.d. random variables. Let $x_h^i(\theta) = \{x_{t,h}^i(\theta)\}_{t=t_l}^T$ denote the “pseudo”-skeleton of the i -th simulation path ($i = 1, \dots, S$) at the parameter value θ .¹⁷ That is, $x_{t,h}^i(\theta)$ is the i -th simulation of the t -th

¹⁷We used the wording “pseudo”-skeleton because h is nonzero.

observation when the parameter vector is θ . Finally, define $y_h^i(\theta)$ in the same way.

The behavior of the high frequency simulator is regulated by the following conditions:

Assumption D.1. For all $\theta \in \Theta$, **(a)** The high frequency simulator (12) converges weakly¹⁸ to the solution of (11) i.e., for each i , $y_h^i(\theta) \Rightarrow y(\theta)$ as $h \downarrow 0$. **(b)** The diffusion and drift functions a and b are Lipschitz continuous in y ; their components are four times continuously differentiable in y ; and a , b and their partial derivatives up to the fourth order have polynomial growth in y . **(c)** Finally, as $h \downarrow 0$ and $T \rightarrow \infty$: **(c.1)** $h \cdot \sqrt{T} \rightarrow 0$; or **(c.2)** $h \cdot T \rightarrow 0$.

The maintained assumption that (11) is stationary implies that the “observed skeleton” of the diffusion inherits the same features of the continuous-time process. Since the simulation step h can not be zero in practice, we extend assumption 2 to cover the “pseudo”-skeleton behavior:

Assumption D.2. For all $\theta \in \Theta$, $\exists h^0 > 0$ depending on θ : for all $h \in (0, h^0)$, $y_h^i(\theta)$ is β -mixing with mixing coefficients $\beta_k(h) : \lim_{k \rightarrow \infty} \max_{h \in (0, h^0)} k^{\mu_h} \beta_k(h) \rightarrow 0$ for some sequence $\{\mu_h\}_h > 1$; and $\lim_{h \downarrow 0} \mu_h = \bar{\mu}$, $\lim_{h \downarrow 0} \beta_k(h) = \bar{\beta}_k$, where $\bar{\mu}$ and $\bar{\beta}_k$ are as in the maintained assumptions.

Primitive conditions ensuring that assumption D.1-(a) holds are well-known and can be found, for instance, in Kloeden and Platen (1999). Primitive conditions guaranteeing that assumption D.2 holds are also well-known (see, e.g., Tjøstheim (1990) for conditions ensuring that (12) is exponentially ergodic for fixed h). Assumptions D.1-(b,c) make our estimators asymptotically free of biases arising from the *imperfect* simulation of model (11) (model (11) is imperfectly simulated so long as $h > 0$). Precisely, such biases arise through terms taking the form $\sqrt{T}[E(K(x_{t,h}^i(\theta_0))) - E(K(x_t))]$, where K is a symmetric bounded kernel. But by results summarized in Kloeden and Platen (1999, chapter 14), $\sqrt{T}[E(K(x_{t,h}^i(\theta_0))) - E(K(x_t))] = O(h \cdot \sqrt{T})$ whenever assumptions D.1-(a,b) hold and K is as differentiable as a and b are in assumption D.1-(b). The role of assumption D.1-(c) is then to asymptotically eliminate such bias terms. Naturally, more precise high frequency simulators would allow h to shrink to zero at an even lower rate. Finally, assumption D.1-(b) can considerably be weakened. For example, one may simply require that a, b be Hölder continuous, as in Kloeden and Platen (1999, theorem 14.1.5 p. 460). These extensions are not considered here to keep the presentation as simple as possible.

Let $L_{T,h}$ and $\bar{L}_{T,h}$ be the criterions of the SNE (definition 1) and the CD-SNE (definition 2), and consider a sequence $\{h_T\}_T$ of discretization stepsizes converging to zero. Let $\mathcal{K}_{T,h}^j(x; \theta)$ be defined similarly as $\mathcal{K}_T^j(x; \theta)$ in Section 3.2. We need the following regularity conditions:

¹⁸Let $(y_{hk})_{k=1}^\infty$ be a discrete time Markov process, and $(y(\tau))_{\tau \geq 0}$ be a diffusion process. When the probability laws generating the entire sample paths of $(y_{hk})_{k=1}^\infty$ converge to the probability laws generating $(y(\tau))_{\tau \geq 0}$ as $h \downarrow 0$, $(y_{hk})_{k=1}^\infty$ is said to converge weakly to $(y(\tau))_{\tau \geq 0}$.

Assumption D.3 (a) Either **(a.1)** $L_{T,h}$ satisfies assumption 3; or **(a.2)** $\bar{L}_{T,h}$ satisfies assumption 5. **(b)** With $\mathcal{K}_{T,h}^j$ replacing \mathcal{K}_T^j , **(b.1)** assumption 4 holds; or **(b.2)** assumption 4-(a) holds.

Assumptions D.1-D.3 are the additional assumptions we need to prove that our estimators work as in the previous Section 3. Precisely, the following theorem is proven in Appendix E.

Theorem D.1. *Let assumptions D.1-(a,b) and D.2 hold. Then, under the additional assumptions D.1-(c.1) and D.3-(a.1,b.1), the SNE is as in theorem 1; under the additional assumptions D.1-(c.2) and D.3-(a.2,b.1), the CD-SNE is as in theorem 2; and under the additional assumptions D.1-(c.2) and D.3-(a.2,b.2), the CD-SNE is as in theorem 3.*

5 Monte Carlo experiments

In this section we conduct Monte Carlo experiments to investigate finite sample properties of our estimators. We wish to address four points: First, we wish to ascertain whether the finite sample properties of our estimators are accurately approximated by the asymptotic theory. Second, we study how our SNE and CD-SNE compare with alternative estimators such as the Fermanian and Salanié (2004) NPSML estimator, and even the MLE. Third, we examine how the SNE and the CD-SNE compare with each other. And fourth, we investigate how bandwidth choice and the possible curse of dimensionality impart on our estimators' finite sample performance.

To address these points, we consider four distinct models: Two continuous-time models commonly utilized in finance (namely, the standard Vasicek model and one simple extension of the Vasicek model with stochastic volatility); and two discrete-time stochastic volatility models (one univariate and one bivariate). Our experiments on all these models share some common features. First, nonparametric density estimates are implemented through Gaussian kernels. Second, our bandwidth choice closely follows the suggestions made by Chen, Linton and Robinson (2001) in the context of conditional density estimation with dependent data; precisely, for each Monte Carlo replication, we select the bandwidth by searching over values minimizing the asymptotic mean integrated squared error of the conditional density estimated on *sample* data. Third, we trim 2% of the observations. Fourth, we set the number of path simulations equal to 5 in all experiments (i.e. $S = 5$). Fifth, in cases in which our estimators can not be efficient, asymptotic standard deviations are approximated through Newey-West windows of ∓ 12 . Sixth, we run 1000 Monte Carlo replications in each experiment. Finally, the experiments related to the continuous-time models are implemented with data sampled at weekly frequency; and models simulated through the Euler-Maruyama scheme (12) with stepsize $h = 1/(5 \times 52)$.¹⁹

¹⁹In the most demanding applications (diffusion processes and sample sizes of 1000 observations), computation time on a Pentium 4 with 1.7GHz is between 3 and 6 minutes. Computation time may vary according to the dimension of the parameter vector, the programming language, the optimization algorithm and sometimes, the

5.1 Continuous-time models

We start by considering the celebrated Vasicek model of the short-term interest rate,

$$dr(\tau) = (b_1 - b_2 r(\tau)) d\tau + a_1 \times dW(\tau), \quad (13)$$

where b_1 , b_2 and a_1 are parameters and W is a Brownian motion. This model is the continuous-time counterpart of a discrete-time AR(1) model. Given its simplicity, it is a natural starting point. Moreover, this model can also be easily estimated by maximum likelihood. Therefore, it is a useful benchmark. The parametrization we choose for this model is $b_1 = 3.00$, $b_2 = 0.50$ and $a_1 = 3.00$. These parameter values imply that the model-generated data have approximately the same mean, variance and autocorrelations as the US short-term interest rate.

We consider four estimators. The first estimator is the CD-SNE in (8) implemented with the weighting function in (9). As we explained in Section 3.3, this estimator matches the model conditional density to the conditional density $\pi_T(r_t|r_{t-1})$ estimated from sample data. As we also demonstrated in Section 3.3, the use of the weighting function $w_T(r_t, r_{t-1}) = \frac{\pi_T(r_{t-1})^2}{\pi_T(r_t, r_{t-1})}$ makes the resulting CD-SNE first order efficient in this case.

The second estimator is the SNE in (7) obtained by matching the joint density of any two adjacent observations $\pi_T(r_t, r_{t-1})$. We use $w_T(r_t, r_{t-1}) = \pi_T(r_t, r_{t-1})$ as a weighting function. According to our theory, the resulting estimator is not first-order efficient. This experiment will thus help us to understand the effects of suboptimal choice of the objective function on the finite sample properties of our estimators.

The third estimator, labelled Analytical-NE, is a modification of the SNE in which the simulated nonparametric estimate $S^{-1} \sum_{i=1}^S \pi_T^i(r_t, r_{t-1}; \theta)$ is replaced with its analytical counterpart $\pi^{\text{vas}}(r_t, r_{t-1}; \theta)$.²⁰ Precisely, the objective function of the Analytical-NE takes the form,

$$\int_{(r_t, r_{t-1}) \in \mathbb{R}^2} [\pi^{\text{vas}}(r_t, r_{t-1}; \theta) - \pi_T(r_t, r_{t-1})]^2 \pi_T(r_t, r_{t-1}) dr_t dr_{t-1}. \quad (14)$$

Naturally, the Analytical-NE is practically unfeasible in most models of interest. We consider this estimator because it provides us with useful information about the importance of the ‘‘twin-smoothing’’ procedure discussed in Section 3.1 - i.e. the importance to apply the same kernel smoothing procedure to sample data and model-related data.

The fourth and last estimator we consider is the MLE.

Table 2 provides results of our Monte Carlo experiments when model (13) is estimated through

spread of the uniform distribution we draw the initial guesses from (see the footnotes 20-22 below).

In the Monte Carlo experiments of this section, our estimators are implemented with Fortran-90. The objective functions are optimized through a Quasi-Newton algorithm, with a convergence criterion of the order of 10^{-5} .

²⁰As is well known, the transition density $\pi^{\text{vas}}(r_s|r_t; \theta)$ from date t to date s ($s > t$) is Gaussian with expectation equal to $b_1/b_2 + [r(t) - (b_1/b_2)] \exp(-b_2(s-t))$ and variance equal to $[a_1^2/(2b_2)] [1 - \exp(-2b_2(s-t))]$. The marginal density is obtained by letting $s \rightarrow \infty$.

the previous methods. We report mean, median, and sample standard deviation of the estimates over the Monte Carlo replications.²¹ As regards the CD-SNE and the SNE, Table 2 also reports: 1) asymptotic standard deviations (obtained through the relevant theory developed in Section 3); and 2) coverage rates for 90% confidence intervals computed through the usual asymptotic approximation to the distribution of the estimator - that is, the estimate plus or minus 1.645 times the asymptotic standard deviation.

When the size of the simulated samples is 1000, the performance of the CD-SNE and MLE are comparable in terms of variability of the estimates. Specifically, the CD-SNE has a lower standard deviation than the MLE as regards the estimation of the parameter b_2 tuning the persistence of r ; and the MLE is more precise than the CD-SNE as regards the estimation of the diffusion parameter a_1 . As it turns out, the sample standard deviation of the CD-SNE estimates of a_1 is larger than its asymptotic counterpart, and this is reflected in a coverage rate below the nominal one. As regards biases, the MLE tends to under-estimate the dependence of the data and largely over-estimate the constant b_1 in the drift term. Interestingly, this phenomenon does not emerge when the model is estimated with the CD-SNE.

As expected, the results in Table 2 clearly demonstrate that moving from CD-SNE to SNE causes an increase in the variability of the estimates; this result is pronounced for the diffusion parameter a_1 . Furthermore, the Analytical-NE produces a much larger variability of the estimates; even more interestingly, it estimates the parameters with large biases: in particular, minimizing (14) over-estimates the diffusion coefficient a_1 by 0.55 and the constant b_1 in the drift term by 0.47. These results are perfectly consistent with our theoretical explanation of a second order biases arising when the model density and the sample density are not smoothed with the same kernel. As is well-known, the practical performance of nonparametric methods hinges on the proper choice of the bandwidth parameter. Table 2 also shows the effects of bandwidth selection on the small samples performance on the CD-SNE. We have implemented two experiments: in the first one, estimation is performed with a bandwidth level which is double the size suggested by Chen, Linton and Robinson (2001) - which we utilized earlier; in the second experiment, the bandwidth size is half the one we utilized earlier. The results in Table 2 suggest that while these bandwidth choices produce some effects on the estimates, those effects are marginal. In particular, under-smoothing the data introduces some volatility in the density estimates - which is reflected in a higher standard deviation of the parameters estimates. And over-smoothing the data tends to increase the mean bias of the parameter estimates.

Finally, Table 2 also documents the performance of the CD-SNE, SNE and MLE in shorter samples of 500 observations. As expected, the variability of the estimates increases with all these methods. As regards the estimates of the b_1 and b_2 parameters, the mean bias of the MLE almost

²¹Initial values of the parameters are drawn from a uniform distribution on $[1.5, 4.5]$ (for b_1 and a_1); and on $[0.1, 0.9]$ (for b_2). The correlations (over the Monte Carlo replications) between initial values and final estimates are 0.07 (for the SNE) and 0.08 (for the CD-SNE) on average over the parameters.

doubles with respect to the longer sample; and the mean biases of the CD-SNE remain small relatively to the corresponding MLE mean biases.

A simple extension of model (13) is one in which the instantaneous volatility of the short-term rate r is proportional to an unobservable process $\{\sigma(\tau)\}_{t \geq 0}$ with constant elasticity of variance,

$$\begin{cases} dr(\tau) &= (b_1 - b_2 r(\tau)) d\tau + a_1 \times \sigma(\tau) dW_1(\tau) \\ d\sigma(\tau) &= b_3 \times (1 - \sigma(\tau)) d\tau + a_2 \times \sigma(\tau) dW_2(\tau) \end{cases} \quad (15)$$

where W_1 and W_2 are two uncorrelated Brownian motions, and b_3 and a_2 are parameters related to the volatility dynamics. Naturally, the presence of the unobservable volatility component in model (15) now makes MLE an unfeasible estimation alternative.

The parametrization of the stochastic volatility model (15) is $b_1 = 3.00$, $b_2 = 0.5$, $a_1 = 3.00$, $b_3 = 1.0$ and $a_2 = 0.3$. This parametrization implies that the unobservable volatility process is strongly dependent, but not as strongly as the observable process r itself. The parameters' values we are using are consistent with estimates of similar models on US short-term interest rates data.

We consider two estimators. The first estimator is the CD-SNE matching the model's conditional density to the conditional density $\pi_T(r_t | r_{t-1})$ of any two adjacent observations; we implement the CD-SNE with the weighting function in (9) of Section 3.3. The second estimator is the SNE implemented by matching the joint density $\pi_T(r_t, r_{t-1})$ of two adjacent observations; we use $\pi_T(r_t, r_{t-1})$ as a weighting function. The performance of both estimators is gauged in samples of 1000 and 500 observations, and the results are reported in Table 3.²²

As regards the larger simple size case and the CD-SNE, the standard deviation and the bias associated with the parameters b_1 and b_2 of the observable process are of the same order of magnitude as in Table 2; the presence of the unobservable volatility component makes the estimate of a_1 become more imprecise than the corresponding estimates in Table 2. As regards the bias terms, there is a tendency to over-estimate the parameter b_3 ; this phenomenon becomes more pronounced in the smaller sample size.

In contrast with our previous results obtained with the Vasicek model (13), we do not observe a clear ranking between the properties of the CD-SNE and the SNE. This phenomenon is particularly clear when the two estimators' properties are compared in terms of the variance of the estimates. Intuitively, the unobservable volatility process $\{\sigma(\tau)\}$ destroys the Markovianity property of the short-term interest rate $\{r(\tau)\}$. Precisely, the joint process $\{r(\tau), \sigma(\tau)\}$ in (15) is clearly Markov, but the "marginal" process $\{r(\tau)\}$ is not. Therefore, the conditions in Theorem 3 for asymptotic efficiency of the CD-SNE are not met. As a result, there is no reason for the CD-SNE to outperform the SNE. This makes the SNE an interesting alternative to look at in

²²Initial values of the parameters are drawn from a uniform distribution on $[1.5, 4.5]$ (for b_1 and a_1); on $[0.1, 0.9]$ (for b_2); on $[0.5, 1.5]$ (for b_3); and on $[0.1, 0.5]$ (for a_2). The correlation (over the Monte Carlo replications) between initial values and final estimates are 0.12 (for the SNE) and 0.11 (for the CD-SNE) on average over the parameters.

practical applications such as the ones considered in this section. The Monte Carlo experiments for discrete-time models reported below do reinforce this conclusion.

5.2 Discrete-time models

Discrete-time stochastic volatility models are also very often utilized in financial applications. The first model we consider in this section is the following one,

$$\begin{cases} y_t &= \sigma_b \times \exp(y_t^*/2) \times \epsilon_{1t} \\ y_t^* &= \phi \times y_{t-1}^* + \sigma_e \times \epsilon_{2t} \end{cases} \quad (16)$$

where $\{y_t\}_{t=1,2,\dots}$ is the observable variable; $\{y_t^*\}_{t=1,2,\dots}$ is the (latent) volatility process; ϵ_{1t} and ϵ_{2t} are two standard normal i.d. innovations; and ϕ , σ_b and σ_e are the parameters of interest. Our economic interpretation of the observable variable y_t is one of the unpredictable part of some long-lived asset return. One important reason leading us to focus on model (16) is that this model has become a workhorse in previous Monte Carlo studies - for example, Fermanian and Salanié (2004) tested their NPSML estimator on this model.

The parametrization of the discrete-time model (16) is $\phi = 0.95$, $\sigma_b = 0.025$ and $\sigma_e = 0.260$. We consider sample sizes of 500 observations. Table 4 reports the results of our Monte Carlo experiments when model (16) is estimated through the CD-SNE and the SNE. As in our previous Monte Carlo experiments on continuous-time models, we implement the CD-SNE by matching the model's conditional density to the conditional density $\pi_T(y_t|y_{t-1})$ of two adjacent observations, and utilize the weighting function (9) $\frac{\pi_T(y_{t-1})^2}{\pi_T(y_t, y_{t-1})}$ of Section 3.3. Similarly, we implement the SNE by matching the model's joint density to the joint density $\pi_T(y_t, y_{t-1})$ of two adjacent observations, and use $\pi_T(y_t, y_{t-1})$ as a weighting function.²³ Table 4 also reports the finite sample properties of three alternative estimation methods available in the literature, and summarized by Fermanian and Salanié (2004) (see *their* Table 4).

The results in Table 4 reveal that the finite sample properties of the CD-SNE and the SNE are very satisfactory, also in comparison with alternative estimation methods. In particular, the sample variability of the estimates of ϕ and σ_b obtained with our methods is in line with the asymptotic counterpart. As it turns out, it is relatively more difficult to estimate the volatility parameter σ_e of the latent process $\{y_t^*\}$; this results in a sample standard deviation larger than its asymptotic counterpart for both the CD-SNE and the SNE.

In our last Monte Carlo experiment, we explore how our methods are affected by the dimensionality of nonparametric density estimates. We consider a simple model in which *two* (unpredictable parts of) asset returns exhibit stochastic volatility. We make the simplifying assumption

²³Initial values of the the parameters are drawn from a uniform distribution on $[0.15, 0.35]$ (for σ_e); on $[0.9, 0.99]$ (for ϕ); and on $[0.015, 0.035]$ (for σ_b). The correlation (over the Monte Carlo replications) between initial values and final estimates are 0.09 (for the SNE) and 0.11 (for the CD-SNE) on average over the parameters.

that the two asset returns volatilities are driven by a common volatility factor,

$$\begin{cases} y_{1t} &= \sigma_{b1} \times \exp(y_t^*/2) \times \epsilon_{1t} \\ y_{2t} &= \sigma_{b2} \times \exp(y_t^*/2) \times \epsilon_{2t} \\ y_t^* &= \phi \times y_{t-1}^* + \sigma_e \times \epsilon_{3t} \end{cases} \quad (17)$$

where $\{y_{it}\}_{t=1,2,\dots}$ ($i = 1, 2$) are the observable variables; $\{y_t^*\}_{t=1,2,\dots}$ is the (latent) volatility process; ϵ_{1t} , ϵ_{2t} and ϵ_{3t} are three standard normal i.i.d. innovations; and σ_{bi} ($i = 1, 2$), ϕ and σ_e are the parameters of interest.

The presence of a common source of stochastic volatility in asset returns can be rationalized by many recent theoretical models of long-lived asset price fluctuations. For example, models with external habit formation predict that a common volatility factor arises because all assets in the economy are consistently priced by a single pricing kernel. Therefore, time-varying volatility in the pricing kernel induced by habit formation propagates to all the asset returns (see, e.g., Menzly, Santos and Veronesi (2004)). Naturally, a sensible model for applied work is one in which returns volatilities also feature idiosyncratic components. But here we simply aim at isolating the effects of the curse of dimensionality on our estimators finite sample performance and, for obvious computational reasons, the Monte Carlo design has to be as simple as possible.

Similarly as for the previous experiments, we consider sample sizes of 500 observations, and parametrize model (17) as follows: $\phi = 0.95$, $\sigma_{b1} = \sigma_{b2} = 0.025$ and $\sigma_e = 0.260$. We examine finite sample properties of both the CD-SNE and the SNE. The CD-SNE is implemented by matching the conditional density $\pi_T(y_{1t}, y_{2t} | y_{1t-1}, y_{2t-1}) = \frac{\pi_T(y_{1t}, y_{2t}, y_{1t-1}, y_{2t-1})}{\pi_T(y_{1t-1}, y_{2t-1})}$ of two adjacent pairs of observations - with the weighting function (9). The SNE is implemented by matching the joint density $\pi_T(y_{1t}, y_{2t}, y_{1t-1}, y_{2t-1})$ of two adjacent pairs of observations - with weighting function $\pi_T(y_{1t}, y_{2t}, y_{1t-1}, y_{2t-1})$.²⁴ The results are displayed in Table 5.

The increase in dimensionality may produce two effects on the estimates. On the one hand, the observation of two asset returns may facilitate our understanding of the dynamic properties of the common unobserved volatility process. On the other hand, the larger dimension of the nonparametric density estimates may impinge upon the precision of the estimates. The results in Table 5 suggest that these effects do arise in our experiments. Overall, an increase in dimensionality does not seem to have jeopardized the performance of our estimators in this experiment.

6 Conclusions

This paper has introduced new methods to estimate the parameters of partially observed dynamic models. The building block of these methods is indeed very simple. It consists in simulating the

²⁴Initial values of the parameters are drawn as in the previous footnote. Correlations between initial guesses and final estimates are also of the same order of magnitude as in the previous footnote.

model of interest for the purpose of recovering the corresponding density function. Our estimators are the ones which make densities on simulated data as close as possible to their empirical counterparts. We made use of classical ideas in the statistical literature to build up convenient measures of closeness of densities. Our estimators are easy to implement, fast to compute and in the special case of fully observed Markov systems, they can attain the same asymptotic efficiency as the maximum likelihood estimator. Furthermore, Monte Carlo experiments revealed that their finite sample performance is very satisfactory, even in comparison to maximum likelihood.

Using simulations to recover model-implied density is not only convenient “just” because it allows one to recover estimates of densities unknown in closed-form. We demonstrated that our “twin-smoothing” procedure makes this feature of our methods stands as a great improvement upon alternative techniques matching “closed-form” model-implied densities to data-implied densities. Consistently with our asymptotic theory, finite sample results suggest that a careful choice of both the measures of closeness of density functions and the bandwidth functions does enhance the performance of our estimators, but mainly in terms of their precision. Furthermore, our trick to use simulations to recover model-implied densities makes our estimators attain a high degree of accuracy in terms of unbiasedness, even in cases of unsophisticated objective functions and/or bandwidth selection procedures.

In our numerical experiments, we emphasized applications related to some typical models arising in financial economics. But we also demonstrated that our approach is quite general, and can be used to address related estimation problems. As an example, the typical Markov models arising in applied macroeconomics may also be estimated with our methods. In these cases, too, the previous asymptotic efficiency and encouraging finite sample properties make our methods stand as a promising advance into the literature of simulation-based inference methods.

Appendix

A. Lemmata

Lemma 1. *Let assumptions 1-(a), 2 and K hold and for each t , let $x_t \equiv (z_t, v_t)$, as in the main text. We have,*

- (a) $\sup_{x \in \mathbb{R}^q} |\pi_T(x) - m_0(x)| = O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q}\right)$.
- (b) $\sup_{x \in \mathbb{R}^q} |\pi_T(x) - \pi_0(x)| = O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q}\right) + O_p(\lambda_T^r)$.

Proof. Part (a) and Part (b) of this lemma are special cases of lemma A-2 (p. 588) and lemma A-1 (p. 586) in Andrews (1995). ■

Lemma 2. *Let assumptions 1-(a), 2 and K hold, and set $A_T \equiv \{(z, v) \in Z \times V : \pi_T(z, v) > \alpha_T\}$, where $\lim_{T \rightarrow \infty} \alpha_T \rightarrow 0$, $\lim_{T \rightarrow \infty} T^{\frac{1}{2}} \lambda_T^q \alpha_T^3 \rightarrow \infty$ and $\lim_{T \rightarrow \infty} \lambda_T^{q*} \alpha_T \rightarrow 0$. We have:*

- (a) *Let $\lim_{T \rightarrow \infty} \lambda_T \geq 0$; then,*

$$\sup_{(z,v) \in A_T} \left[\frac{1}{m_0(z, v)} \left| \frac{\pi_T(v)}{\pi_T(z|v)} - \frac{m_0(v)}{n_0(z, v)} \right| \right] \xrightarrow{p} 0,$$

where $m_0(\cdot) \equiv m(\cdot; \theta_0)$ and $n_0(\cdot) \equiv n(\cdot; \theta_0)$.

- (b) *Let $\lim_{T \rightarrow \infty} \lambda_T = 0$, and $\lim_{T \rightarrow \infty} \alpha_T^3 \lambda_T^{-r} = \infty$; then,*

$$\sup_{(z,v) \in A_T} \left[\frac{1}{\pi_0(z, v)} \left| \frac{\pi_T(v)}{\pi_T(z|v)} - \frac{\pi_0(v)}{\pi_0(z|v)} \right| \right] \xrightarrow{p} 0.$$

Proof. (Part (a)) We shall make a repeated use of the identity: $\frac{a}{b} - \frac{\bar{a}}{\bar{b}} = \frac{1}{\bar{b}}(a - \bar{a}) - \frac{a}{\bar{b}\bar{b}}(b - \bar{b})$, where a, b, \bar{a} and \bar{b} are any four strictly positive numbers. Let $A_{1T}(\varepsilon) \equiv \{(z, v) \in Z \times V : m_0(z, v) \geq \varepsilon \alpha_T\}$, $A_{2T}(\varepsilon) \equiv \{(z, v) \in Z \times V : \pi_T(z, v) \geq \varepsilon \alpha_T\}$ and $\hat{A}_T \equiv \hat{A}_T(\varepsilon) \equiv A_{1T}(\varepsilon) \cap A_{2T}(\varepsilon)$ for some $\varepsilon > 0$. We have:

$$\begin{aligned} \sup_{(z,v) \in \hat{A}_T} \left| \frac{\pi_T(v)}{\pi_T(z|v)} - \frac{m_0(v)}{n_0(z, v)} \right| &\leq \sup_{(z,v) \in \hat{A}_T} \left[\frac{1}{n_0(z, v)} |\pi_T(v) - m_0(v)| \right] \\ &+ \sup_{(z,v) \in \hat{A}_T} \left[\frac{\pi_T(v)}{\pi_T(z|v) n_0(z, v)} |\pi_T(z|v) - n_0(z, v)| \right]; \end{aligned}$$

and for all $(z, v) \in \hat{A}_T$,

$$\frac{\pi_T(v)}{n_0(z, v)} |\pi_T(z|v) - n_0(z, v)| \leq \frac{\pi_T(v)}{m_0(z, v)} |\pi_T(z, v) - m_0(z, v)| + \frac{\pi_T(z, v)}{m_0(z, v)} |\pi_T(v) - m_0(v)|.$$

Hence,

$$\begin{aligned} & \sup_{(z, v) \in \hat{A}_T} \left[\frac{1}{m_0(z, v)} \left| \frac{\pi_T(v)}{\pi_T(z|v)} - \frac{m_0(v)}{n_0(z, v)} \right| \right] \\ & \leq \sup_{(z, v) \in \hat{A}_T} \left[\frac{m_0(v) + \pi_T(v)}{m_0(z, v)^2} |\pi_T(v) - m_0(v)| \right] + \sup_{(z, v) \in \hat{A}_T} \left[\frac{\pi_T(v)^2}{\pi_T(z, v) m_0(z, v)^2} |\pi_T(z, v) - m_0(z, v)| \right] \\ & \leq c_1 \alpha_T^{-2} \sup_{(z, v) \in \hat{A}_T} |\pi_T(v) - m_0(v)| + c_2 \alpha_T^{-3} \sup_{(z, v) \in \hat{A}_T} |\pi_T(z, v) - m_0(z, v)| \\ & = O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \alpha_T^{-2} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \alpha_T^{-3} \right), \end{aligned} \quad (\text{A1})$$

where $c_1 \equiv \varepsilon^{-1} \sup_{v \in V} (m_0(v) + \pi_T(v))$, $c_2 \equiv \varepsilon^{-1} \sup_{v \in V} \pi_T(v)^2$, and the last equality follows by lemma 1-(a). Now suppose that $\lim_{T \rightarrow \infty} T^{\frac{1}{2}} \lambda_T^q \alpha_T^3 = \lim_{T \rightarrow \infty} (T^{\frac{1}{2}} \lambda_T^{q-q^*} \alpha_T^2) (\lambda_T^{q^*} \alpha_T) = \infty$. Since $\lim_{T \rightarrow \infty} \lambda_T^{q^*} \alpha_T = 0$, then $\lim_{T \rightarrow \infty} T^{\frac{1}{2}} \lambda_T^{q-q^*} \alpha_T^2 = \infty$. We are left to show that eq. (A1) holds when substituting \hat{A}_T with the feasible trimming set $A_T = A_{2T}(1)$. The argument is nearly identical to Andrews (1995, proof of thm. 1, p. 588). We have $\hat{A}_T \supseteq \hat{A}_T^* \equiv A_{1T}(\varepsilon) \cap A_{2T}(2\varepsilon)$ and so eq. (A1) holds with \hat{A}_T^* replacing \hat{A}_T . Moreover, by lemma 1-(a), and one argument similar to Andrews (1995, p. 588), $A_{2T}(2\varepsilon) \subseteq A_{1T}(\varepsilon)$ with probability (wp) 1 as $T \rightarrow \infty$. Therefore, eq. (A1) holds with \hat{A}_T replaced by $A_{2T}(2\varepsilon)$ wp 1 as $T \rightarrow \infty$, and the result follows by setting $\varepsilon = \frac{1}{2}$.

The proof of Part (b) is nearly identical. Define trimming sets A_T , \hat{A}_T , A_{1T} and A_{2T} as before, with the exception that function $m_0(z, v)$ in A_{1T} is replaced with function $\pi_0(z, v)$. By lemma 1-(b) and the same arguments leading to (A1),

$$\begin{aligned} \sup_{(z, v) \in \hat{A}_T} \left| \frac{\pi_T(v)}{\pi_T(z|v)} - \frac{\pi_0(v)}{\pi_0(z|v)} \right| &= O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \alpha_T^{-1} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \alpha_T^{-2} \right) \\ &+ O_p \left(\alpha_T^{-1} \lambda_T^r \right) + O_p \left(\alpha_T^{-2} \lambda_T^r \right). \end{aligned}$$

Part (b) then follows by replacing \hat{A}_T with the feasible trimming set A_T , exactly as we did in Part (a). ■

Lemma 3. *Let assumptions 1-(a), 2 and K hold, and set $B_T \equiv \{v \in V : \pi_T^i(v; \theta) > \delta_T, i = 0, 1, \dots, S, \text{ all } \theta \in \Theta\}$ ($\pi_T^0 \equiv \pi_T$), where $\lim_{T \rightarrow \infty} \delta_T \rightarrow 0$, $\lim_{T \rightarrow \infty} T^{\frac{1}{2}} \lambda_T^{q-q^*} \delta_T^2 \rightarrow \infty$ and $\lim_{T \rightarrow \infty} T^{\frac{1}{2}} \lambda_T^q \delta_T \rightarrow \infty$. We have:*

(a) Let $\lim_{T \rightarrow \infty} \lambda_T \geq 0$; then,

$$\sup_{(z,v) \in Z \times B_T} \left| \frac{\pi_T(z,v)}{\pi_T(v)} - n_0(z,v) \right| \xrightarrow{p} 0.$$

(b) Let $\lim_{T \rightarrow \infty} \lambda_T = 0$, and $\lim_{T \rightarrow \infty} \delta_T^2 \lambda_T^{-r} = \infty$; then,

$$\sup_{(z,v) \in B_T} \left| \frac{\pi_T(z,v)}{\pi_T(v)} - \pi(z|v) \right| \xrightarrow{p} 0.$$

Proof. (Part (a)) The argument is nearly identical to the one utilized to show lemma 2-(a), and so the proof is sketchy. Let $B_{1T}(\varepsilon) \equiv \{v \in V : m_0(v) \geq \varepsilon \delta_T\}$, $B_{2T}(\varepsilon) \equiv \{v \in V : \pi_T^i(v; \theta) \geq \varepsilon \delta_T, i = 0, 1, \dots, S, \text{ all } \theta \in \Theta\}$, and $\hat{B}_T \equiv B_{1T}(\varepsilon) \cap B_{2T}(\varepsilon)$ for some $\varepsilon > 0$. We have,

$$\begin{aligned} & \sup_{(z,v) \in Z \times \hat{B}_T} |\pi_T(z|v) - n_0(z,v)| \\ & \leq \sup_{(z,v) \in Z \times \hat{B}_T} \left[\frac{1}{m_0(v)} |\pi_T(z,v) - m_0(z,v)| \right] + \sup_{(z,v) \in Z \times \hat{B}_T} \left[\frac{\pi_T(z,v)}{m_0(v) \pi_T(v)} |\pi_T(v) - m_0(v)| \right] \\ & = O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \delta_T^{-1} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-2} \right), \end{aligned} \quad (\text{A2})$$

where the last equality follows by lemma 1-(a). By the same arguments in the proof of lemma 2-(a), eq. (A2) also holds wp 1 as $T \rightarrow \infty$ when \hat{B}_T is replaced with B_T . The proof of Part (b) is obtained through lemma 1-(b), and is otherwise nearly identical to the proof of lemma 2-(b). ■

Lemma 4. Let assumptions 1-(a), 2 and K hold, and let $\lim_{T \rightarrow \infty} \alpha_T \rightarrow 0$, $\lim_{T \rightarrow \infty} \delta_T \rightarrow 0$, $\lim_{T \rightarrow \infty} T^{\frac{1}{2}} \lambda_T^q \alpha_T^2 \delta_T \rightarrow \infty$ and $\lim_{T \rightarrow \infty} T^{\frac{1}{2}} \lambda_T^{q-q^*} \alpha_T^2 \delta_T^2 \rightarrow \infty$. We have:

(a) Let $\lim_{T \rightarrow \infty} \lambda_T \geq 0$; then, for each $i = 1, \dots, S$, and $\theta \in \Theta$,

$$\sup_{(z,v) \in A_T \cap B_T} \left[\frac{1}{m_0(z,v) n_0(z,v)} \left| \frac{\pi_T^i(z,v;\theta)}{\pi_T^i(v;\theta)} - n(z,v;\theta) \right| \right] \xrightarrow{p} 0.$$

(b) Let $\lim_{T \rightarrow \infty} \lambda_T = 0$, $\lim_{T \rightarrow \infty} \alpha_T^2 \delta_T \lambda_T^{-r} = \infty$ and $\lim_{T \rightarrow \infty} \alpha_T^2 \delta_T^2 \lambda_T^{-r} = \infty$; then, for each $i = 1, \dots, S$, and $\theta \in \Theta$,

$$\sup_{(z,v) \in A_T \cap B_T} \left[\frac{1}{m_0(z,v) n_0(z,v)} \left| \frac{\pi_T^i(z,v;\theta)}{\pi_T^i(v;\theta)} - \pi(z|v;\theta) \right| \right] \xrightarrow{p} 0.$$

Proof. (Part (a)) As in the proof of lemma 3, the proof is sketchy as it is nearly identical to the one in lemma 2-(a). For each $i = 1, \dots, S$, and $\theta \in \Theta$,

$$\begin{aligned}
& \sup_{(z,v) \in \hat{A}_T \cap \hat{B}_T} \left[\frac{m_0(v)}{m_0(z,v)^2} \left| \frac{\pi_T^i(z,v;\theta)}{\pi_T^i(v;\theta)} - n(z,v;\theta) \right| \right] \\
& \leq \sup_{(z,v) \in \hat{A}_T \cap \hat{B}_T} \left[\frac{m_0(v)}{m_0(z,v)^2 m(v;\theta)} \left| \pi_T^i(z,v;\theta) - m(z,v;\theta) \right| \right] \\
& + \sup_{(z,v) \in \hat{A}_T \cap \hat{B}_T} \left[\frac{\pi_T^i(z,v;\theta) m_0(v)}{m_0(z,v)^2 m(v;\theta) \pi_T^i(v;\theta)} \left| \pi_T^i(v;\theta) - m(v;\theta) \right| \right] \\
& = O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \alpha_T^{-2} \delta_T^{-1} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \alpha_T^{-2} \delta_T^{-2} \right),
\end{aligned}$$

where the last equality holds by lemma 1-(a). Conclude as in the previous lemmas 2 and 3. Part (b) is nearly identical given lemma 1-(b). ■

Lemma 5. *Let assumptions 1, 2, K hold. For each t , let $x_t \equiv (z_t, v_t)$, as in the main text, and let $\mathcal{K}_T^j(z, v; \theta)$ satisfy the mixing condition in assumption 2 ($j = 1, \dots, p_\theta$). Finally let $\partial^{\rho+1} \pi(x; \theta) / \partial \theta \partial x^\rho$ be uniformly bounded for some $\rho \geq r$. Then, for all $\theta \in \Theta$ and $i = 1, \dots, S$,*

$$\sup_{x \in \mathbb{R}^q} \left| \nabla_{\theta_j} \tilde{\pi}_T(x; \theta) - \nabla_{\theta_j} \pi(x; \theta) \right| = O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \right) + O_p(\lambda_T^r), \quad j = 1, \dots, p_\theta.$$

Proof. We have,

$$\begin{aligned}
& \sup_{x \in \mathbb{R}^q} \left| \nabla_{\theta_j} \tilde{\pi}_T(x; \theta) - \nabla_{\theta_j} \pi(x; \theta) \right| \\
& \leq \frac{1}{S} \sum_{i=1}^S \sup_{x \in \mathbb{R}^q} \left| \nabla_{\theta_j} \pi_T^i(x; \theta) - E[\nabla_{\theta_j} \pi_T^i(x; \theta)] \right| + \frac{1}{S} \sum_{i=1}^S \sup_{x \in \mathbb{R}^q} \left| \nabla_{\theta_j} \pi(x; \theta) - E[\nabla_{\theta_j} \pi_T^i(x; \theta)] \right|.
\end{aligned}$$

For each $i = 1, \dots, S$, and $\theta \in \Theta$,

$$\begin{aligned}
& \sup_{x \in \mathbb{R}^q} \left| \nabla_{\theta_j} \pi_T^i(x; \theta) - E[\nabla_{\theta_j} \pi_T^i(x; \theta)] \right| \\
& = \sup_{x \in \mathbb{R}^q} \left| \frac{1}{T \lambda_T^{q+1}} \sum_{t=t_i}^T \frac{\partial x_t^i(\theta)}{\partial \theta} \cdot K' \left(\frac{x_t^i(\theta) - x}{\lambda_T} \right) - E \left[\frac{\partial x_t^i(\theta)}{\partial \theta} \cdot K' \left(\frac{x_1^i(\theta) - x}{\lambda_T} \right) \right] \right| \\
& = O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \right),
\end{aligned}$$

where the second equality follows by lemma A-2 (p. 588) in Andrews (1995), and the mixing condition on \mathcal{K}_T . As for the bias term,

$$\begin{aligned} E[\nabla_{\theta_j} \pi_T^i(x; \theta)] &= \nabla_{\theta_j} E[\pi_T^i(x; \theta)] \\ &= \nabla_{\theta_j} \pi(x; \theta) + \frac{\lambda_T^r}{r!} \int \frac{\partial}{\partial \theta_j} \sum_{i_1, \dots, i_r=1}^q \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_r}} \pi(x + \lambda_T^* z; \theta) z_{i_1} \cdots z_{i_r} K(z) dz, \end{aligned}$$

where the first equality follows by dominated convergence, and $\lambda_T^* \in (0, \lambda_T)$. The result follows by uniform boundedness of $\partial^{r+1} \pi(x; \theta) / \partial \theta \partial x^r$. ■

In lemmas 6 through 10 below, α_T and δ_T denote the same sequences introduced in the previous lemmas 2 and 3.

Lemma 6. *Let the assumptions in lemma 5 hold. Then, for all $\theta \in \Theta$ and $j = 1, \dots, p\theta$*

$$\begin{aligned} &\sup_{(z,v) \in Z \times B_T} |\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta) - \nabla_{\theta_j} \pi(z|v; \theta)| \\ &= O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \delta_T^{-2}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \delta_T^{-2}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-3}\right) + O_p\left(\lambda_T^r \delta_T^{-3}\right). \end{aligned}$$

Proof. We show that this lemma holds with the unfeasible trimming set \hat{B}_T . The extension to the feasible set B_T follows by the same arguments in lemma 1. We have,

$$\sup_{(z,v) \in Z \times \hat{B}_T} |\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta) - \nabla_{\theta_j} \pi(z|v; \theta)| \leq \frac{1}{S} \sum_{i=1}^S \sup_{(z,v) \in Z \times B_T} |\nabla_{\theta_j} \pi_T^i(z|v; \theta) - \nabla_{\theta_j} \pi(z|v; \theta)|,$$

and

$$\begin{aligned} &\sup_{(z,v) \in Z \times \hat{B}_T} |\nabla_{\theta_j} \pi_T^i(z|v; \theta) - \nabla_{\theta_j} \pi(z|v; \theta)| \\ &\leq \sup_{(z,v) \in Z \times \hat{B}_T} \left[\frac{1}{\pi_T^i(v; \theta)} |\nabla_{\theta_j} \pi_T^i(z, v; \theta) - \nabla_{\theta_j} \pi(z, v; \theta)| \right] \\ &+ \sup_{(z,v) \in Z \times \hat{B}_T} \left[\frac{\pi_T^i(z, v; \theta)}{\pi_T^i(v; \theta)^2} |\nabla_{\theta_j} \pi_T^i(v; \theta) - \nabla_{\theta_j} \pi(v; \theta)| \right] \\ &+ \sup_{(z,v) \in Z \times \hat{B}_T} \left[|\nabla_{\theta_j} \pi(z, v; \theta)| \left| \frac{1}{\pi_T^i(v; \theta)} - \frac{1}{\pi(v; \theta)} \right| + |\nabla_{\theta_j} \pi(v; \theta)| \left| \frac{\pi_T^i(z, v; \theta)}{\pi_T^i(v; \theta)^2} - \frac{\pi(z, v; \theta)}{\pi(v; \theta)^2} \right| \right] \\ &\equiv T_1 + T_2 + T_3. \end{aligned}$$

By lemma 5, and boundedness of $\nabla_{\theta_j} \pi(z, v; \theta)$, $T_1 = O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \delta_T^{-1}\right) + O_p\left(\lambda_T^r \delta_T^{-1}\right)$ and $T_2 = O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \delta_T^{-2}\right) + O_p\left(\lambda_T^r \delta_T^{-2}\right)$. As regards the T_3 term we have, by lemma 1,

$$\begin{aligned} & \sup_{(z,v) \in Z \times \hat{B}_T} \left| \frac{1}{\pi_T^i(v; \theta)} - \frac{1}{\pi(v; \theta)} \right| \\ & \leq \sup_{(z,v) \in Z \times \hat{B}_T} \left[\frac{1}{\pi_T^i(v; \theta) \pi(v; \theta)} |\pi_T^i(v; \theta) - \pi(v; \theta)| \right] = O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-2}\right) + O_p\left(\lambda_T^r \delta_T^{-2}\right), \end{aligned}$$

and,

$$\begin{aligned} & \sup_{(z,v) \in Z \times \hat{B}_T} \left| \frac{\pi_T^i(z, v; \theta)}{\pi_T^i(v; \theta)^2} - \frac{\pi(z, v; \theta)}{\pi(v; \theta)^2} \right| \\ & \leq \sup_{(z,v) \in Z \times \hat{B}_T} \left[\frac{1}{\pi_T^i(v; \theta)^2} |\pi_T^i(z, v; \theta) - \pi(z, v; \theta)| \right] + \sup_{(z,v) \in Z \times \hat{B}_T} \left| \frac{1}{\pi_T^i(v; \theta)^2} - \frac{1}{\pi(v; \theta)^2} \right| \pi(z, v; \theta) \\ & \leq \sup_{(z,v) \in Z \times \hat{B}_T} \left[\frac{1}{\pi_T^i(v; \theta)^2} |\pi_T^i(z, v; \theta) - \pi(z, v; \theta)| \right] + \delta_T^{-3} \sup_{(z,v) \in Z \times \hat{B}_T} |\pi_T^i(v; \theta) - \pi(v; \theta)| \pi(z, v; \theta) \\ & = O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q} \delta_T^{-2}\right) + O_p\left(\lambda_T^r \delta_T^{-2}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-3}\right) + O_p\left(\lambda_T^r \delta_T^{-3}\right), \end{aligned}$$

where the last equality follows by lemma 1, and the second inequality holds because

$$\begin{aligned} & \sup_{v \in \hat{B}_T} \left| \frac{1}{\pi_T^i(v; \theta)^2} - \frac{1}{\pi(v; \theta)^2} \right| \\ & = \sup_{v \in \hat{B}_T} \left[\frac{1}{\pi_T^i(v; \theta) \pi(v; \theta)} \left| \frac{\pi(v; \theta)}{\pi_T^i(v; \theta)} - \frac{\pi_T^i(v; \theta)}{\pi(v; \theta)} \right| \right] \\ & \leq \delta_T^{-2} \cdot \sup_{v \in \hat{B}_T} \left[\frac{1}{\pi(v; \theta)} + \frac{1}{\pi_T^i(v; \theta)} \right] |\pi_T^i(v; \theta) - \pi(v; \theta)| \end{aligned}$$

Hence by boundedness of $\nabla_{\theta_j} \pi(z, v; \theta)$ and $\nabla_{\theta_j} \pi(z; \theta)$, $T_3 = O_p\left(\lambda_T^r \delta_T^{-3}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q} \delta_T^{-2}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-3}\right)$. ■

Lemma 7. *Let the assumptions in lemma 5, and assumption 4-(b) hold. Then, for all $\theta \in \Theta$ and $j = 1, \dots, p_\theta$*

$$\begin{aligned} & \sup_{(z,v) \in Z \times B_T} \left| \frac{\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) w_T(z, v)}{\pi_T(v)} - \frac{\nabla_{\theta_j} \pi(z|v; \theta_0) w(z, v)}{\pi(v; \theta_0)} \right| \\ & = O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \delta_T^{-3}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \delta_T^{-3}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-4}\right) + O_p\left(\lambda_T^r \delta_T^{-4}\right). \end{aligned}$$

Proof. We proceed as in the proof of lemma 6, and demonstrate the result with the unfeasible trimming set \hat{B}_T . This is without loss of generality. We have,

$$\begin{aligned}
& \sup_{(z,v) \in Z \times \hat{B}_T} \left| \frac{\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) w_T(z, v)}{\pi_T(v)} - \frac{\nabla_{\theta_j} \pi(z|v; \theta_0) w(z, v)}{\pi(v; \theta_0)} \right| \\
& \leq \sup_{(z,v) \in Z \times \hat{B}_T} \left[\frac{w_T(z, v)}{\pi_T(v)} |\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) - \nabla_{\theta_j} \pi(z|v; \theta_0)| \right] \\
& + \sup_{(z,v) \in Z \times \hat{B}_T} \frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)|}{\pi(v; \theta_0)} |w_T(z, v) - w(z, v)| \\
& + \sup_{(z,v) \in Z \times \hat{B}_T} \frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)|}{\pi_T(v) \pi(v; \theta_0)} |\pi_T(v) - \pi(v; \theta_0)| \\
& = O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \delta_T^{-3} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \delta_T^{-3} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-4} \right) + O_p \left(\lambda_T^r \delta_T^{-4} \right) \\
& + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \delta_T^{-1} \right) + O_p \left(\lambda_T^r \delta_T^{-1} \right) \\
& + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \delta_T^{-2} \right) + O_p \left(\lambda_T^r \delta_T^{-2} \right),
\end{aligned}$$

by lemma 1, lemma 6, assumption 4(b), and boundedness of $|\nabla_{\theta} \pi(z|v; \theta_0)|$. ■

Lemma 8. *Let the assumptions in lemma 7 hold. Then, for all $\theta \in \Theta$ and $j = 1, \dots, p_\theta$*

$$\begin{aligned}
& \sup_{(z,v) \in Z \times B_T} \left| \frac{\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) E[\pi_T(z, v)] w_T(z, v)}{\pi_T^1(v; \theta_0) \cdot \pi_T(v)} - \frac{\nabla_{\theta_j} \pi(z|v; \theta_0) \pi_0(z, v) w(z, v)}{\pi(v; \theta_0)^2} \right| \\
& = O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \delta_T^{-4} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \delta_T^{-4} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-5} \right) + O_p \left(\lambda_T^r \delta_T^{-5} \right).
\end{aligned}$$

Proof. As in the previous two lemmas, we demonstrate the result with the unfeasible trimming set \hat{B}_T (w.l.o.g.). We have,

$$\begin{aligned}
& \sup_{(z,v) \in Z \times \hat{B}_T} \left| \frac{\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) E[\pi_T(z, v)] w_T(z, v)}{\pi_T^1(v; \theta_0) \cdot \pi_T(v)} - \frac{\nabla_{\theta_j} \pi(z|v; \theta_0) \pi_0(z, v) w(z, v)}{\pi(v; \theta_0)^2} \right| \\
\leq & \sup_{(z,v) \in Z \times \hat{B}_T} \frac{E[\pi_T(z, v)] w_T(z, v)}{\pi_T^1(v; \theta_0) \pi(v; \theta_0)} \cdot |\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) - \nabla_{\theta_j} \pi(z|v; \theta_0)| \\
& + \sup_{(z,v) \in Z \times \hat{B}_T} \frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)| w_T(z, v)}{\pi_T^1(v; \theta_0) \pi(v; \theta_0)} \cdot |E[\pi_T(z, v)] - \pi_0(z, v)| \\
& + \sup_{(z,v) \in Z \times \hat{B}_T} \frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)| \pi_0(z, v)}{\pi_T^1(v; \theta_0) \pi(v; \theta_0)} \cdot |w_T(z, v) - w(z, v)| \\
& + \sup_{(z,v) \in Z \times \hat{B}_T} \frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)| \pi_0(z, v) w(z, v)}{\pi_T^1(v; \theta_0) \pi(v; \theta_0)^2} \cdot |\pi_T^1(v; \theta_0) - \pi(v; \theta_0)| \\
& + \sup_{(z,v) \in Z \times \hat{B}_T} \left[\frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)| \pi_0(z, v) w(z, v)}{\pi_T(v) \pi(v; \theta_0) \pi_T^1(v; \theta_0)} \cdot |\pi_T(v) - \pi(v; \theta_0)| \right] \\
= & O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \delta_T^{-4} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \delta_T^{-4} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-5} \right) + O_p \left(\lambda_T^r \delta_T^{-5} \right) \\
& + O_p \left(\lambda_T^r \delta_T^{-2} \right) \\
& + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \delta_T^{-2} \right) + O_p \left(\lambda_T^r \delta_T^{-2} \right) \\
& + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-3} \right) + O_p \left(\lambda_T^r \delta_T^{-3} \right) \\
& + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-3} \right) + O_p \left(\lambda_T^r \delta_T^{-3} \right),
\end{aligned}$$

by lemma 1, lemma 6, assumption 4-(b), and boundedness of $|\nabla_{\theta_j} \pi(z|v; \theta_0)|$. ■

Lemma 9. *Let the assumptions in lemma 5 hold. Let $v \mapsto \xi_{1T}(v)$ ($v \in V \subseteq \mathbb{R}^{q-q^*}$) be a sequence of real, bounded functions satisfying $\sup_{v \in V} |\xi_{1T}(v) - \xi_1(v)| = O_p(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)}) + O_p(\lambda_T^r)$, for some function ξ_1 . Then, for all $\theta \in \Theta$ and $j = 1, \dots, p_\theta$,*

$$\begin{aligned}
& \sup_{(z,v) \in A_T \times B_T} \left| \frac{\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) \pi_T(v) \xi_{1T}(v)}{\pi_T(z, v)} - \frac{\nabla_{\theta_j} \pi(z|v; \theta_0) \pi_0(v) \xi_1(v)}{\pi_0(z, v)} \right| \\
= & O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \alpha_T^{-1} \delta_T^{-2} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \alpha_T^{-1} \delta_T^{-2} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \alpha_T^{-1} \delta_T^{-3} \right) \\
& + O_p \left(\lambda_T^r \alpha_T^{-1} \delta_T^{-3} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \alpha_T^{-1} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \alpha_T^{-2} \right) + O_p \left(\lambda_T^r \alpha_T^{-2} \right).
\end{aligned}$$

Proof. Similarly as in the previous lemmas, we demonstrate the result with the unfeasible trimming set $\hat{A}_T \times \hat{B}_T$ (w.l.o.g.). We have,

$$\begin{aligned}
& \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \left| \frac{\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) \pi_T(v) \xi_{1T}(v)}{\pi_T(z,v)} - \frac{\nabla_{\theta_j} \pi(z|v; \theta_0) \pi_0(v) \xi_1(v)}{\pi_0(z,v)} \right| \\
\leq & \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \frac{\pi_T(v) \xi_{1T}(v)}{\pi_T(z,v)} \left| \nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) - \nabla_{\theta_j} \pi(z|v; \theta_0) \right| \\
& + \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \left| \nabla_{\theta_j} \pi(z|v; \theta_0) \right| \left| \frac{\pi_T(v) \xi_{1T}(v)}{\pi_T(z,v)} - \frac{\pi_0(v) \xi_1(v)}{\pi_0(z,v)} \right| \\
\equiv & S_{1T} + S_{2T}.
\end{aligned}$$

By lemma 6,

$$\begin{aligned}
S_{1T} = & O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \delta_T^{-2} \alpha_T^{-1} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \delta_T^{-2} \alpha_T^{-1} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-3} \alpha_T^{-1} \right) \\
& + O_p \left(\lambda_T^r \delta_T^{-3} \alpha_T^{-1} \right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
S_{2T} \leq & \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)| \xi_{1T}(v)}{\pi_T(z,v)} |\pi_T(v) - \pi_0(v)| \\
& + \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)| \pi_0(v)}{\pi_T(z,v)} |\xi_{1T}(v) - \xi_1(v)| \\
& + \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)| \pi_0(v) \xi_1(v)}{\pi_T(z,v) \pi_0(z,v)} |\pi_T(z,v) - \pi_0(z,v)| \\
\equiv & S_{21T} + S_{22T}.
\end{aligned}$$

By lemma 1, and the assumption on function ξ_{1T} , $S_{21T} = O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \alpha_T^{-1} \right) + O_p \left(\lambda_T^r \alpha_T^{-1} \right)$. Again by lemma 1, $S_{22T} = O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \alpha_T^{-2} \right) + O_p \left(\lambda_T^r \alpha_T^{-2} \right)$. The result follows by boundedness of $|\nabla_{\theta_j} \pi(z|v; \theta_0)|$. ■

Lemma 10. *Let the assumptions in lemma 5 hold, and let $\xi_{1T}(v)$ be the sequence of functions in lemma 9. Then, for all $\theta \in \Theta$ and $j = 1, \dots, p\theta$,*

$$\begin{aligned} & \sup_{(z,v) \in A_T \times B_T} \left| \frac{\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) E[\pi_T(z, v)] \xi_{1T}(v) \pi_T(v)}{\pi_T^1(v; \theta_0) \pi_T(z, v)} - \nabla_{\theta_j} \pi(z|v; \theta_0) \xi_1(v) \right| \\ &= O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \alpha_T^{-1} \delta_T^{-3}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q} \alpha_T^{-2} \delta_T^{-1}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \alpha_T^{-1} \delta_T^{-4}\right) \\ &+ O_p\left(\lambda_T^r \alpha_T^{-1} \delta_T^{-4}\right) + O_p\left(\lambda_T^r \alpha_T^{-2} \delta_T^{-1}\right) + O_p\left(\lambda_T^r \delta_T^{-1} \alpha_T^{-1}\right). \end{aligned}$$

Proof. Similarly as in the previous lemmas, we demonstrate the result with the unfeasible trimming set $\hat{A}_T \times \hat{B}_T$ (w.l.o.g.). We have,

$$\begin{aligned} & \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \left| \frac{\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) E[\pi_T(z, v)] \xi_{1T}(v) \pi_T(v)}{\pi_T^1(v; \theta_0) \pi_T(z, v)} - \nabla_{\theta_j} \pi(z|v; \theta_0) \xi_1(v) \right| \\ \leq & \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \frac{E[\pi_T(z, v)]}{\pi_T^1(v; \theta_0)} \cdot \left| \frac{\nabla_{\theta_j} \tilde{\pi}_T(z|v; \theta_0) \pi_T(v) \xi_{1T}(v)}{\pi_T(z, v)} - \frac{\nabla_{\theta_j} \pi(z|v; \theta_0) \pi_0(v) \xi_1(v)}{\pi_0(z, v)} \right| \\ & + \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \left| \nabla_{\theta_j} \pi(z|v; \theta_0) \xi_1(v) \right| \left| \frac{\pi_0(v) E[\pi_T(z, v)] - \pi_0(z, v) \pi_T^1(v; \theta_0)}{\pi_0(z, v) \pi_T^1(v; \theta_0)} \right| \\ \equiv & Q_{1T} + Q_{2T}. \end{aligned}$$

By lemma 9,

$$\begin{aligned} Q_{1T} &= O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q-1} \alpha_T^{-1} \delta_T^{-3}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \alpha_T^{-1} \delta_T^{-3}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \alpha_T^{-1} \delta_T^{-4}\right) \\ &+ O_p\left(\lambda_T^r \alpha_T^{-1} \delta_T^{-4}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \alpha_T^{-1} \delta_T^{-1}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q} \alpha_T^{-2} \delta_T^{-1}\right) + O_p\left(\lambda_T^r \alpha_T^{-2} \delta_T^{-1}\right). \end{aligned}$$

Moreover,

$$\begin{aligned} Q_{2T} &\leq \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)| \xi_1(v) \pi_0(v)}{\pi_0(z, v) \pi_T^1(v; \theta_0)} |E[\pi_T(z, v)] - \pi_0(z, v)| \\ &+ \sup_{(z,v) \in \hat{A}_T \times \hat{B}_T} \frac{|\nabla_{\theta_j} \pi(z|v; \theta_0)| \xi_1(v)}{\pi_T^1(v; \theta_0)} |\pi_T^1(v; \theta_0) - \pi_0(v)| \\ &= O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-1}\right) + O_p\left(\lambda_T^r \delta_T^{-1} \alpha_T^{-1}\right), \end{aligned}$$

by lemma 1, and boundedness of $|\nabla_{\theta_j} \pi(z|v; \theta_0)|$. ■

B. Proof of theorem 1

B.1 Consistency

Proposition 1. *Let assumptions 1, 2 and 3-(a) hold. Then $\forall \theta \in \Theta$, $L_T(\theta) \xrightarrow{P} L(\theta)$ as $T \rightarrow \infty$.*

According to a well-known result (see Newey (1991, thm. 2.1 p. 1162)), the following conditions are equivalent:

$$\text{C1: } \lim_{T \rightarrow \infty} P(\sup_{\theta \in \Theta} |L_T(\theta) - L(\theta)| > \epsilon) = 0.$$

$$\text{C2: } \forall \theta \in \Theta, L_T(\theta) \xrightarrow{P} L(\theta), \text{ and } L_T(\theta) \text{ is stochastically equicontinuous.}$$

By Newey and McFadden (1994, lemma 2.9 p. 2138), assumption 3-(b) guarantees that $L_T(\theta)$ is stochastically equicontinuous, and so weak consistency follows from the equivalence of C1 and C2 above, assumption 3-(a,b), compactness of Θ , and a classical argument (e.g., White (1994, theorem 3.4)). So we are only left to prove proposition 1.

Proof of proposition 1. We have:

$$|L_T(\theta) - L(\theta)| \leq \int |g_T(x; \theta)| dx,$$

where

$$\begin{aligned} g_T(x; \theta) &\equiv \sigma_{1T}(x; \theta) + [|\tilde{\pi}_T(x; \theta) - \pi_T(x)| - |m(x; \theta) - m(x; \theta_0)|] \cdot [\rho_T(x; \theta) + \rho(x; \theta)] \\ &\leq \sigma_{1T}(x; \theta) + [|\tilde{\pi}_T(x; \theta) - m(x; \theta)| - |\pi_T(x) - m(x; \theta_0)|] \cdot [\rho_T(x; \theta) + \rho(x; \theta)] \\ &\equiv \sigma_{1T}(x; \theta) + \sigma_{2T}(x; \theta) \\ \sigma_{1T}(x; \theta) &\equiv |\tilde{\pi}_T(x; \theta) - \pi_T(x)| \cdot |m(x; \theta) - m(x; \theta_0)| \cdot |w_T(x) - w(x)| \\ \rho_T(x; \theta) &\equiv |\tilde{\pi}_T(x; \theta) - \pi_T(x)| \cdot w_T(x) \\ \rho(x; \theta) &\equiv |m(x; \theta) - m(x; \theta_0)| \cdot w(x) \end{aligned}$$

We claim that for all $\theta \in \Theta$, $\int \sigma_{1T} \xrightarrow{P} 0$. Indeed, for fixed θ , σ_{1T} is clearly bounded by integrable functions independent of T . As $T \rightarrow \infty$, $\sigma_{1T}(x; \theta) \xrightarrow{P} 0$, x -pointwise. By dominated convergence, $\lim_{T \rightarrow \infty} E[\sigma_{1T}(x; \theta)] = E[\lim_{T \rightarrow \infty} \sigma_{1T}(x; \theta)] = 0$ all $(x, \theta) \in X \times \Theta$. By Fubini, $E[\int \sigma_{1T}(x; \theta) dx] = \int E[\sigma_{1T}(x; \theta)] dx$ all $\theta \in \Theta$. Again by dominated convergence,

$$\lim_{T \rightarrow \infty} E \left[\int \sigma_{1T}(x; \theta) dx \right] = \lim_{T \rightarrow \infty} \int E[\sigma_{1T}(x; \theta)] dx = \int \lim_{T \rightarrow \infty} E[\sigma_{1T}(x; \theta)] dx = 0, \forall \theta \in \Theta.$$

By Markov's inequality:

$$\forall \epsilon > 0, P \left\{ \int \sigma_{1T}(x; \theta) dx > \epsilon \right\} \leq \frac{E \left[\int \sigma_{1T}(x; \theta) dx \right]}{\epsilon}, \quad \forall \theta \in \Theta.$$

Hence, for all $\theta \in \Theta$, $\int \sigma_{1T} \xrightarrow{p} 0$. The proof for the σ_{2T} term is similar. The additional argument is the observation that for all $x, \theta \in X \times \Theta$, $\max[m(x; \theta) - \tilde{\pi}_T(x; \theta), 0] \leq m(x; \theta)$, which is clearly integrable, and so by $\tilde{\pi}_T(x; \theta) \xrightarrow{p} m(x; \theta)$, x -pointwise, and dominated convergence,

$$\text{for all } \theta \in \Theta, \int |m(x; \theta) - \tilde{\pi}_T(x; \theta)| dx = 2 \int \max[m(x; \theta) - \tilde{\pi}_T(x; \theta), 0] dx \xrightarrow{p} 0.$$

Hence by arguments nearly identical to the ones leading to $\int \sigma_{1T} \xrightarrow{p} 0$, we also have that for all $\theta \in \Theta$, $\int \sigma_{2T} \xrightarrow{p} 0$, and the proof is complete. The case $\lambda \equiv \lambda_T \downarrow 0$ is identical. ■

B.2 Asymptotic normality

Let $\mathbf{0}_n$ denote a column vector of n zeros. By assumption 4-(a), the order of derivation and integration in $\nabla_{\theta} L_T(\theta)$ may be interchanged (see Newey and McFadden (1994, lemma 3.6 p. 2152-2153)), and the first order conditions satisfied by the SNE are,

$$\mathbf{0}_{p_{\theta}} = \int [\tilde{\pi}_T(x; \theta_{T,S}) - \pi_T(x)] \nabla_{\theta} \tilde{\pi}_T(x; \theta_{T,S}) w_T(x) dx.$$

Let $\theta(c) \equiv c \circ (\theta_0 - \theta_{T,S}) + \theta_{T,S}$, where, for any $c \in (0, 1)^{p_{\theta}}$ and $\theta \in \Theta$, $c \circ \theta$ denotes the vector in Θ whose i -th element is $c^{(i)} \theta^{(i)}$. By assumption 4-(a), there exists a c^* in $(0, 1)^{p_{\theta}}$ such that:

$$\begin{aligned} \mathbf{0}_{p_{\theta}} &= \sqrt{T} \int [\tilde{\pi}_T(x; \theta_0) - \pi_T(x)] \nabla_{\theta} \tilde{\pi}_T(x; \theta_0) w_T(x) dx \\ &+ \left[\int |\nabla_{\theta} \tilde{\pi}_T(x; \bar{\theta})|_2 w_T(x) dx + (\bar{\theta} - \theta_0) \cdot k_{1T}(\bar{\theta}) + k_{2T}(\bar{\theta}) \right] \cdot \sqrt{T}(\theta_{T,S} - \theta_0), \end{aligned} \quad (\text{B1})$$

where $\bar{\theta} \equiv \theta(c^*)$, $|b|_2$ denotes the outer product $b \cdot b^{\top}$ of a column vector b , and for some θ^* ,

$$\begin{aligned} |k_{1T}(\bar{\theta})| &\leq \int |\nabla_{\theta} \tilde{\pi}_T(x; \theta^*)| |\nabla_{\theta\theta} \tilde{\pi}_T(x; \bar{\theta})| w_T(x) dx \\ |k_{2T}(\bar{\theta})| &\leq \int |\tilde{\pi}_T(x; \theta_0) - \pi_T(x)| |\nabla_{\theta\theta} \tilde{\pi}_T(x; \bar{\theta})| w_T(x) dx \end{aligned}$$

By assumption 4-(a), the term $\nabla_{\theta\theta}\tilde{\pi}_T(x;\bar{\theta})$ is bounded in probability as T becomes large. Hence a) so is $|k_{1T}(\bar{\theta})|$; and b) by lemma 1, $|k_{2T}(\bar{\theta})| \xrightarrow{p} \mathbf{0}_{p_\theta \times p_\theta}$. Moreover,

$$\int |\nabla_{\theta}\tilde{\pi}_T(x;\bar{\theta})|_2 w_T(x)dx = \int |\nabla_{\theta}\tilde{\pi}_T(x;\theta_0)|_2 w_T(x)dx + R_T(\bar{\theta}),$$

where

$$|R_T(\bar{\theta})|_{i,j} \leq \int ||\nabla_{\theta}\tilde{\pi}_T(x;\bar{\theta})|_2 - |\nabla_{\theta}\tilde{\pi}_T(x;\theta_0)|_2|_{i,j} w_T(x)dx.$$

Since $\int |w_T - w| \xrightarrow{p} 0$ and $\bar{\theta} \xrightarrow{p} \theta_0$, then by lemma 5, $|R_T(\bar{\theta})|_{i,j} \xrightarrow{p} 0$ for all i, j . Hence,

$$\int |\nabla_{\theta}\tilde{\pi}_T(x;\bar{\theta})|_2 w_T(x)dx + (\bar{\theta} - \theta_0) k_{1T}(\bar{\theta}) + k_{2T}(\bar{\theta}) \xrightarrow{p} \int |\nabla_{\theta}\pi(x;\theta_0)|_2 w(x)dx. \quad (\text{B2})$$

Next, consider the first term in (B1). For all $x \in X$ and fixed T , $E[\pi_T^i(x;\theta_0)] = E[\pi_T(x)]$ ($i = 1, \dots, S$). Hence,

$$\begin{aligned} & \sqrt{T} \int [\tilde{\pi}_T(x;\theta_0) - \pi_T(x)] \nabla_{\theta}\tilde{\pi}_T(x;\theta_0)w_T(x)dx \\ &= \int \sqrt{T} [\tilde{\pi}_T(x;\theta_0) - E(\tilde{\pi}_T(x;\theta_0))] \nabla_{\theta}\tilde{\pi}_T(x;\theta_0)w_T(x)dx \\ & - \int \sqrt{T} [\pi_T(x) - E(\pi_T(x))] \nabla_{\theta}\tilde{\pi}_T(x;\theta_0)w_T(x)dx. \end{aligned} \quad (\text{B3})$$

Let \mathbb{G} be a measurable V-C subgraph class of uniformly bounded functions (see, e.g., Arcones and Yu (1994, definition 2.2 p. 51)). By Arcones and Yu (1994, corollary 2.1 p. 59-60), for each $G \in \mathbb{G}$, $T^{-1/2} \sum_{t=t_l}^T [G(x_t) - EG]$ converges in law to a Gaussian process under assumption 2. Now $\lambda_T^{-q} K((x_t - x)/\lambda_T) \in \mathbb{G}$. Let $F(x;\theta) = \int_0^x \pi(v;\theta)dv$, $F_T(x) = \int_0^x \pi_T(v)dv$ and $F(x) = \int_0^x \pi_0(v)dv$. Under the theorem's assumptions,

$$A_T \equiv \sqrt{T}(F_T(x) - E(F_T(x))) \Rightarrow \omega^0(F(x)),$$

where $\omega^0(F)$ is a Generalized Brownian Bridge with covariance kernel,

$$\min(F(x), F(y)) [1 - F(y)] + \sum_{k=1}^{\infty} \left[F^k(x, y) + F^k(y, x) - 2F(x)F(y) \right],$$

and $F^k(x, y) \equiv P(x_0 \leq x, x_k \leq y)$. We have,

$$\begin{aligned}
J_T &\equiv \sqrt{T} \int [\pi_T(x) - E(\pi_T(x))] \nabla_{\theta} \tilde{\pi}_T(x; \theta_0) w_T(x) dx \\
&= \int [w_T(x) - w(x)] [\nabla_{\theta} \tilde{\pi}_T(x; \theta_0) - \nabla_{\theta} \pi(x; \theta_0)] dA_T(x) \\
&+ \int [\nabla_{\theta} \tilde{\pi}_T(x; \theta_0) - \nabla_{\theta} \pi(x; \theta_0)] w(x) dA_T(x) \\
&+ \int [w_T(x) - w(x)] \nabla_{\theta} \pi(x; \theta_0) dA_T(x) + \int \nabla_{\theta} \pi(x; \theta_0) w(x) dA_T(x) \\
&\equiv J_{1T} + J_{2T} + J_{3T} + J_{4T}.
\end{aligned}$$

By the continuous mapping theorem,

$$J_{4T} \xrightarrow{d} J_4 \equiv \int \nabla_{\theta} \pi(x; \theta_0) w(x) d\omega^0(F(x)).$$

By w_T and w bounded, and lemma 5, $J_{iT} = [O_p(T^{-\frac{1}{2}} \lambda_T^{-q-1}) + O_p(\lambda_T^r)] \mathbf{1}_{p_{\theta}}$, $i = 1, 2$. By assumption 4-(b), $J_{3T} = [O_p(T^{-\frac{1}{2}} \lambda_T^{-q}) + O_p(\lambda_T^r)] \mathbf{1}_{p_{\theta}}$. By the theorem's conditions, therefore, $J_T \xrightarrow{d} N(0, V_J)$, $V_J \equiv \text{var}(J_4)$. By the same computations in Aït-Sahalia (1994) (proof of thm. 1 p. 21-22) and Aït-Sahalia (1996) (proof of eq. (12), p. 420-421),

$$\begin{aligned}
V_J &= \text{var} [\nabla_{\theta} \pi(x_1; \theta_0) w(x_1)] + \sum_{k=1}^{\infty} \{ \text{cov} [\nabla_{\theta} \pi(x_1; \theta_0) w(x_1), \nabla_{\theta} \pi(x_{1+k}; \theta_0) w(x_{1+k})] \\
&+ \text{cov} [\nabla_{\theta} \pi(x_{1+k}; \theta_0) w(x_{1+k}), \nabla_{\theta} \pi(x_1; \theta_0) w(x_1)] \}. \tag{B4}
\end{aligned}$$

Finally, let $F_T^i(x; \theta) \equiv \int_0^x \pi_T^i(v; \theta) dv$, $i = 1, \dots, S$. As for A_T , $A_T^i(x; \theta_0) \equiv \sqrt{T} [F_T^i(x; \theta_0) - E(F_T^i(x; \theta_0))] \Rightarrow \omega_i^0(F(x))$, where $\omega_i^0(F)$ are independent Generalized Brownian Bridges. Hence,

$$\sqrt{T} \sum_{i=1}^S [F_T^i(x; \theta_0) - E(F_T^i(x; \theta_0))] \Rightarrow \sum_{i=1}^S \omega_i^0(F(x)).$$

Since $E(F_T^i(x; \theta_0)) = E(F_T^j(x; \theta_0))$ for all $i, j = 1, \dots, S$, we have, similarly as for J_T term,

$$\int [\sqrt{T} (\tilde{\pi}_T(x; \theta_0) - E(\tilde{\pi}_T(x; \theta_0)))] \nabla_{\theta} \tilde{\pi}_T(x; \theta_0) w_T(x) dx \xrightarrow{d} N\left(0, \frac{1}{S} V_J\right),$$

where V_J is as in (B4). Finally, A and A_T^i , $i = 1, \dots, S$, are all independent. Therefore, by (B3),

$$\sqrt{T} \int [\tilde{\pi}_T(x; \theta_0) - \pi_T(x)] \nabla_{\theta} \tilde{\pi}_T(x; \theta_0) w_T(x) dx \xrightarrow{d} N\left(0, \left(1 + \frac{1}{S}\right) V_J\right). \tag{B5}$$

Hence by (B1), (B2), (B5) and Slutsky's theorem, $\sqrt{T}(\theta_{T,S} - \theta_0) \xrightarrow{d} N(0, (1 + \frac{1}{S})V)$, where

$$V \equiv \left[\int |\nabla_{\theta} \pi(x; \theta_0)|_2 w(x) dx \right]^{-1} \cdot V_J \cdot \left[\int |\nabla_{\theta} \pi(x; \theta_0)|_2 w(x) dx \right]^{\top - 1}.$$

Remark 0. A crucial step of the previous proof is the weak convergence $\sqrt{T}[F_T(x) - E(F_T(x))] \Rightarrow \mathcal{G}(F)$. Because $\sqrt{T}(F_T - F) = \sqrt{T}[F_T - E(F_T)] + \sqrt{T}[E(F_T) - F]$, we see that $\sqrt{T}[F_T(x) - F(x)] \Rightarrow \mathcal{G}(F)$ under the more stringent condition that $\lim_{T \rightarrow \infty} \sqrt{T}\lambda_T^r \rightarrow 0$. This condition is needed to asymptotically zero the bias term $\sqrt{T}[E(F_T) - F]$, and is exactly assumption A4(r,0) in Aït-Sahalia (1994, lemma 1 p. 20). As we noted in the main text, we do not need such a more severe condition because bias effects cancel out each other through the decomposition in eq. (B3).

C. Proof of theorem 2

The following assumption contains one set of regularity conditions mentioned in the statement of theorem 2.

Assumption T1. *We have,*

(a) $\delta_T \rightarrow 0$ and $T^{\frac{1}{2}}\delta_T^2 \rightarrow \infty$.

(b) *In addition to assumption T1-(a), $\lambda \equiv \lambda_T \rightarrow 0$, $T^{\frac{1}{2}}\lambda_T^{q-q^*}\delta_T^5 \rightarrow \infty$, $T^{\frac{1}{2}}\lambda_T^{q+1}\delta_T^4 \rightarrow \infty$, and $\delta_T^5\lambda_T^{-r} \rightarrow \infty$.*

C.1 Consistency

Similarly as for the SNE, the objective function of the CD-SNE \bar{L}_T satisfies $|\bar{L}_T(\theta) - \bar{L}(\theta)| \leq \iint (s_{1T}(z, v; \theta) + s_{2T}(z, v; \theta)) dzdv$, where

$$\begin{aligned} s_{1T}(z, v; \theta) &\equiv |\tilde{\pi}_T(z|v; \theta) - \pi_T(z|v)| \mathbb{T}_{T,\delta}(v; \theta) \cdot |n(z, v; \theta) - n(z, v; \theta_0)| \cdot |w_T(z, v) - w(z, v)|; \\ s_{2T}(z, v; \theta) &\equiv |[\tilde{\pi}_T(z|v; \theta)\mathbb{T}_{T,\delta}(v; \theta) - n(z, v; \theta)] - [\pi_T(z|v)\mathbb{T}_{T,\delta}(v; \theta) - n(z, v; \theta_0)]| \\ &\quad \times [r_T(z, v; \theta) + r(z, v; \theta)]; \\ r_T(z, v; \theta) &\equiv |\tilde{\pi}_T(z|v; \theta) - \pi_T(z|v)| \mathbb{T}_{T,\delta}(v; \theta) \cdot w_T(z, v); \\ r(z, v; \theta) &\equiv |n(z, v; \theta) - n(z, v; \theta_0)| \cdot w(z, v). \end{aligned}$$

We now show that $\iint (s_{1T} + s_{2T}) \xrightarrow{P} 0$ for all $\theta \in \Theta$. We study the two integrals separately.

- For all $(z, v, \theta) \in Z \times V \times \Theta$, $s_{1T}(z, v; \theta) \leq \ell_T(z, v; \theta) \cdot r_{2T}(z, v; \theta)$, where

$$\begin{aligned} \ell_T(z, v; \theta) &\equiv |n(z, v; \theta) - n(z, v; \theta_0)| \cdot |w_T(z, v) - w(z, v)| \\ r_{2T}(z, v; \theta) &\equiv \frac{1}{S} \sum_{i=1}^S |\pi_T^i(z|v; \theta) - n(z, v; \theta)| \mathbb{T}_{T,\delta}(v; \theta) + |\pi_T(z|v) - n(z, v; \theta_0)| \mathbb{T}_{T,\delta}(v; \theta) \\ &\quad + |n(z, v; \theta) - n(z, v; \theta_0)| \mathbb{T}_{T,\delta}(v; \theta). \end{aligned} \tag{C1}$$

For each $\theta \in \Theta$, function ℓ_T is bounded by integrable functions independent of T , and $\ell_T \xrightarrow{P} 0$ (z, v) -pointwise. Moreover,

$$\sup_{(z,v) \in Z \times V} |\pi_T^i(z|v; \theta) - n(z, v; \theta)| \mathbb{T}_{T,\delta}(v; \theta) \xrightarrow{P} 0, \quad i = 1, \dots, S,$$

as a consequence of lemma 3-(a), and the conditions in the theorem. This result clearly holds for the second term in (C1) as well. Finally, $|n(\cdot, \cdot; \theta) - n(\cdot, \cdot; \theta_0)|$ is bounded. Therefore, $\iint s_{1T}(z, v; \theta) \xrightarrow{P} 0$ for all $\theta \in \Theta$.

- For all $(z, v, \theta) \in Z \times V \times \Theta$,

$$\begin{aligned}
& s_{2T}(z, v; \theta) \\
& \leq \left[\frac{1}{S} \sum_{i=1}^S |\pi_T^i(z|v; \theta) \mathbb{T}_{T, \delta}(v; \theta) - n(z, v; \theta)| + |\pi_T(z|v) \mathbb{T}_{T, \delta}(v; \theta) - n(z, v; \theta_0)| \right] r_{3T}(z, v; \theta),
\end{aligned} \tag{C2}$$

where $r_{3T}(z, v; \theta) \equiv r(z, v; \theta) + r_T(z, v; \theta) \leq r(z, v; \theta) + r_{2T}(z, v; \theta) w_T(z, v)$. For each $i = 1, \dots, S$, and $(z, v, \theta) \in Z \times V \times \Theta$,

$$\begin{aligned}
& |\pi_T^i(z|v; \theta) \mathbb{T}_{T, \delta}(v; \theta) - n(z, v; \theta)| r_{3T}(z, v; \theta) \\
& \leq n(z, v; \theta) [1 - \mathbb{T}_{T, \delta}(v; \theta)] \cdot [r(z, v; \theta) + r_{2T}(z, v; \theta) w_T(z, v)] \\
& \quad + |\pi_T^i(z|v; \theta) - n(z, v; \theta)| \mathbb{T}_{T, \delta}(v; \theta) \cdot [r(z, v; \theta) + r_{2T}(z, v; \theta) w_T(z, v)] \\
& \equiv s_{21T}(z, v; \theta) + s_{22T}(z, v; \theta),
\end{aligned}$$

where the inequality holds by the triangle inequality. Since w_T , r and n are bounded, and w_T and r are also integrable, $\iint s_{22T}(z, v; \theta) \xrightarrow{P} 0$ for all $\theta \in \Theta$ by lemma 3-(a). As for the s_{21T} term, clearly $|1 - \mathbb{T}_{T, \delta}(v; \theta)| \leq 1$. Moreover, $1 - \mathbb{T}_{T, \delta}(v; \theta) \xrightarrow{P} P_\pi \{ \pi_0(v_1) < \lim_{T \rightarrow \infty} \delta_T \} - \int_{v: \pi_0(v) \in (\lim_{T \rightarrow \infty} \delta_T, 2 \lim_{T \rightarrow \infty} \delta_T)} \mathbb{T}_{T, \delta}(v; \theta) P_\pi(dv)$, where P_π is the stationary measure of v . Hence, by the conditions in the theorem and again lemma 3-(a), $\iint s_{21T}(z, v; \theta) \xrightarrow{P} 0$ for all $\theta \in \Theta$. By reiterating the previous arguments, one shows that the same result holds for the second term in (C2) and therefore, $\iint s_{2T}(z, v; \theta) \xrightarrow{P} 0$ for all $\theta \in \Theta$.

The case $\lambda \equiv \lambda_T \downarrow 0$ is dealt with similarly through lemma 3-(b) instead of lemma 3-(a), and the proof of consistency is complete by the same arguments in appendix B.1.

C.2 Asymptotic normality

The following remarks are useful.

Remark 1. We have,

$$\begin{aligned}
\nabla_{\theta} \mathbb{T}_{T,\delta}(v; \theta_0) &= \nabla_{\theta} \prod_{i=0}^S \mathbb{T}(\pi_T^i(v; \theta_0)) \\
&= \mathbb{T}(\pi_T(v)) \nabla_{\theta} \prod_{i=1}^S \mathbb{T}(\pi_T^i(v; \theta_0)) \\
&= \mathbb{T}(\pi_T(v)) \sum_{i=1}^S [\nabla_{\theta} \mathbb{T}(\pi_T^i(v; \theta_0))] \prod_{j \neq i} \mathbb{T}(\pi_T^j(v; \theta_0)) \\
&= \mathbb{T}(\pi_T(v)) \sum_{i=1}^S g_{\delta_T}(\pi_T^i(v; \theta_0)) [\nabla_{\theta} \pi_T^i(v; \theta_0)] \prod_{j \neq i} \mathbb{T}(\pi_T^j(v; \theta_0)) \\
&= \sum_{i=1}^S g_{\delta_T}(\pi_T^i(v; \theta_0)) [\nabla_{\theta} \pi_T^i(v; \theta_0)] \mathbb{T}_{T,\delta}^{(-i)}(v; \theta_0)
\end{aligned}$$

where

$$\mathbb{T}_{T,\delta}^{(-i)}(v; \theta_0) \equiv \prod_{j=0; j \neq i}^S \mathbb{T}(\pi_T^j(v; \theta_0)),$$

and g_{δ} is the function introduced in assumption **T** of the main text.

Remark 2. For all $\ell = 1, \dots, S$, we have,

$$\begin{aligned}
\nabla_{\theta_i} \pi_T^{\ell}(z|v; \theta) &= \frac{\nabla_{\theta_i} \pi_T^{\ell}(z, v; \theta)}{\pi_T^{\ell}(v; \theta)} - \frac{\pi_T^{\ell}(z, v; \theta)}{\pi_T^{\ell}(v; \theta)^2} \nabla_{\theta_i} \pi_T^{\ell}(v; \theta), \\
\nabla_{\theta_i \theta_j} \pi_T^{\ell}(z|v; \bar{\theta}) &= \frac{\nabla_{\theta_j} \nabla_{\theta_i} \pi_T^{\ell}(z, v; \theta)}{\pi_T^{\ell}(v; \theta)} \\
&\quad - \frac{\nabla_{\theta_i} \pi_T^{\ell}(z, v; \theta) \nabla_{\theta_j} \pi_T^{\ell}(v; \theta) + \nabla_{\theta_j} \pi_T^{\ell}(z, v; \theta) \nabla_{\theta_i} \pi_T^{\ell}(v; \theta) + \pi_T^{\ell}(z, v; \theta) \nabla_{\theta_j \theta_i} \pi_T^{\ell}(v; \theta)}{\pi_T^{\ell}(v; \theta)^2} \\
&\quad + 2 \frac{\pi_T^{\ell}(z, v; \theta) \nabla_{\theta_i} \pi_T^{\ell}(v; \theta) \nabla_{\theta_j} \pi_T^{\ell}(v; \theta)}{\pi_T^{\ell}(v; \theta)^3},
\end{aligned}$$

at all points of continuity.

We now demonstrate our asymptotic normality claims. By remark 2 and assumption 4-(a), we may interchange the order of derivation and integration in $\nabla_{\theta} \bar{L}(\theta)$ (similarly as for the SNE

in appendix B.2). The CD-SNE thus satisfies the following first order conditions,

$$\begin{aligned} \mathbf{0}_{p_\theta} &= \frac{1}{S} \sum_{i=1}^S \iint \left[\frac{\pi_T^i(z, v; \theta_{T,S})}{\pi_T^i(v; \theta_{T,S})} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] \nabla_\theta \tilde{\pi}_T(z|v; \theta_{T,S}) w_T(z, v) \mathbb{T}_{T,\delta}^2(v; \theta_{T,S}) dz dv \\ &+ \iint [\tilde{\pi}_T(z|v; \theta_{T,S}) - \pi_T(z|v)]^2 w_T(z, v) \mathbb{T}_{T,\delta}(v; \theta_{T,S}) \nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_{T,S}) dz dv. \end{aligned}$$

For some convex combination $\bar{\theta}$ of θ_0 and $\theta_{T,S}$,

$$\begin{aligned} \mathbf{0}_{p_\theta} &= \frac{1}{S} \sum_{i=1}^S \sqrt{T} \iint \left[\frac{\pi_T^i(z, v; \theta_0)}{\pi_T^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] \nabla_\theta \tilde{\pi}_T(z|v; \theta_0) w_T(z, v) \mathbb{T}_{T,\delta}^2(v; \theta_0) dz dv \\ &+ B_T + C_T \cdot \sqrt{T} (\theta_{T,S} - \theta_0), \end{aligned} \quad (\text{C3})$$

where

$$B_T \equiv \sqrt{T} \iint [\tilde{\pi}_T(z|v; \theta_0) - \pi_T(z|v)]^2 w_T(z, v) \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_0) dz dv.$$

and,

$$\begin{aligned} C_T &\equiv \iint \nabla_\theta \left\{ [\tilde{\pi}_T(z|v; \bar{\theta}) - \pi_T(z|v)] \nabla_\theta \tilde{\pi}_T(z|v; \bar{\theta}) \mathbb{T}_{T,\delta}^2(v; \bar{\theta}) \right\} w_T(z, v) dz dv \\ &+ \iint \nabla_\theta \left\{ [\tilde{\pi}_T(z|v; \bar{\theta}) - \pi_T(z|v)]^2 \mathbb{T}_{T,\delta}(v; \bar{\theta}) \nabla_\theta \mathbb{T}_{T,\delta}(v; \bar{\theta}) \right\} w_T(z, v) dz dv. \end{aligned}$$

We now study these two terms (B_T & C_T), and then elaborate on the first order conditions in (C3).

The B_T term

We have,

$$\begin{aligned} B_T &= \sqrt{T} \iint \frac{1}{S} \sum_{i=1}^S \left[\frac{\pi_T^i(z, v; \theta_0)}{\pi_T^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] [\tilde{\pi}_T(z|v; \theta_0) - \pi_T(z|v)] \\ &\quad \times [w_T(z, v) - w(z, v)] \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_0) dz dv \\ &+ \sqrt{T} \iint \frac{1}{S} \sum_{i=1}^S \left[\frac{\pi_T^i(z, v; \theta_0)}{\pi_T^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] [\tilde{\pi}_T(z|v; \theta_0) - \pi_T(z|v)] \\ &\quad \times w(z, v) \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_0) dz dv \\ &\equiv B_{1T} + B_{2T}. \end{aligned} \quad (\text{C4})$$

We show first that $B_{1T} \xrightarrow{P} \mathbf{0}_{p_\theta}$. Clearly,

$$\begin{aligned}
B_{1T} &= \sqrt{T} \iint \frac{1}{S} \sum_{i=1}^S \left[\frac{\pi_T^i(z, v; \theta_0)}{\pi_T^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] [\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)] \\
&\quad \times [w_T(z, v) - w(z, v)] \mathbb{T}_{T, \delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T, \delta}(v; \theta_0) dz dv \\
&\quad - \sqrt{T} \iint \frac{1}{S} \sum_{i=1}^S \left[\frac{\pi_T^i(z, v; \theta_0)}{\pi_T^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] [\pi_T(z|v) - \pi_0(z|v)] \\
&\quad \times [w_T(z, v) - w(z, v)] \mathbb{T}_{T, \delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T, \delta}(v; \theta_0) dz dv
\end{aligned} \tag{C5}$$

Moreover, $E(\pi_T(z, v)) = E(\pi_T^i(z, v; \theta_0))$. Therefore, at all points of continuity,

$$\begin{aligned}
&\frac{\pi_T^i(z, v; \theta_0)}{\pi_T^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \\
&= \frac{\pi_T^i(z, v; \theta_0) - E(\pi_T^i(z, v; \theta_0))}{\pi_T^i(v; \theta_0)} - \frac{\pi_T(z, v) - E(\pi_T(z, v))}{\pi_T(v)} + \frac{E(\pi_T(z, v)) [\pi_T(v) - \pi_T^i(v; \theta_0)]}{\pi_T^i(v; \theta_0) \cdot \pi_T(v)}.
\end{aligned} \tag{C6}$$

By replacing (C6) into (C5) leaves,

$$\begin{aligned}
B_{1T} &= \frac{1}{S} \sum_{i=1}^S \iint \eta_T^i(z, v) dA_T^i(z, v; \theta_0) - \iint \eta_T^0(z, v) dA_T(z, v) \\
&\quad + \frac{1}{S} \sum_{i=1}^S \iint \eta_{iT}^{00}(z, v) [dA_T(v) - dA_T^i(v; \theta_0)] \\
&\quad \equiv \check{B}_{1T}^{(1)} + \check{B}_{1T}^{(2)} + \check{B}_{1T}^{(3)}
\end{aligned} \tag{C7}$$

where, for $i = 0, 1, \dots, S$,

$$\begin{aligned}
\eta_T^i(z, v) &= \frac{1}{\pi_T^i(v; \theta_0)} \{ [\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)] - [\pi_T(z|v) - \pi_0(z|v)] \} \\
&\quad \times \mathbb{T}_{T, \delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T, \delta}(v; \theta_0) \cdot [w_T(z, v) - w(z, v)]
\end{aligned}$$

$(\pi_T^0(v; \theta_0) = \pi_T(v))$ and, for $i = 1, \dots, S$,

$$\begin{aligned}
\eta_{iT}^{00}(z, v) &= \frac{E(\pi_T(z, v))}{\pi_T^i(v; \theta_0) \cdot \pi_T(v)} \{ [\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)] - [\pi_T(z|v) - \pi_0(z|v)] \} \\
&\quad \times \mathbb{T}_{T, \delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T, \delta}(v; \theta_0) \cdot [w_T(z, v) - w(z, v)]
\end{aligned}$$

and, finally, $A_T^i(z, v; \theta_0)$, $A_T(z, v)$, $A_T(v)$ and $A_T^i(v; \theta_0)$ are defined similarly as in appendix

B.2. In particular, these terms converge weakly to Gaussian processes. For example, $dA_T(z, v) = \{\pi_T(z, v) - E[\pi_T(z, v)]\} dzdv$, and A_T converges weakly to a Generalized Brownian Bridge.

By Remark 1,

$$\begin{aligned}
b_{1T} &\equiv \frac{1}{\pi_T^i(v; \theta_0)} \{|\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)| + |\pi_T(z|v) - \pi_0(z|v)|\} \mathbb{T}_{T,\delta}(v; \theta_0) |\nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_0)| \\
&\leq \frac{1}{\pi_T^i(v; \theta_0)} \{|\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)| + |\pi_T(z|v) - \pi_0(z|v)|\} \mathbb{T}_{T,\delta}(v; \theta_0) \\
&\quad \times \sum_{j=1}^S \left| \nabla_\theta \pi_T^j(v; \theta_0) - \nabla_\theta \pi_0(v) \right| g_{\delta_T} \left(\pi_T^j(v; \theta_0) \right) \mathbb{T}_{T,\delta}^{(-j)}(v; \theta_0) \\
&\quad + \frac{1}{\pi_T^i(v; \theta_0)} \{|\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)| + |\pi_T(z|v) - \pi_0(z|v)|\} \mathbb{T}_{T,\delta}(v; \theta_0) \\
&\quad \times \sum_{j=1}^S |\nabla_\theta \pi_0(v)| g_{\delta_T} \left(\pi_T^j(v; \theta_0) \right) \mathbb{T}_{T,\delta}^{(-j)}(v; \theta_0) \\
&\equiv b_{11T} + b_{12T}.
\end{aligned}$$

By assumption \mathbf{T} , $g_{\delta_T} = \delta_T^{-1} \times \check{k}_T$, where \check{k}_T is bounded in probability. Therefore, by lemmas 3 and 5,

$$\begin{aligned}
&\sup_{(z,v) \in Z \times V} b_{11T} \\
&= \delta_T^{-1} \times \left[O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \delta_T^{-1} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-2} \right) + O_p \left(\lambda_T^r \delta_T^{-2} \right) \right] \times \left[O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \right) + O_p \left(\lambda_T^r \right) \right] \\
&\xrightarrow{p} \mathbf{0}_{p_\theta},
\end{aligned}$$

where the convergence follows by the conditions in the theorem. By similar arguments, and boundedness of $\nabla_\theta \pi_0(v)$, $\sup_{(z,v) \in Z \times V} b_{12T} \xrightarrow{p} \mathbf{0}_{p_\theta}$. Therefore by the previous results on b_{1T} and assumption 4-(b),

$$\sup_{(z,v) \in Z \times V} |\eta_T^i(z, v)| \leq \left(\sup_{(z,v) \in Z \times V} b_{1T} \right) \cdot \left(\sup_{(z,v) \in Z \times V} |w_T(z, v) - w(z, v)| \right) \xrightarrow{p} \mathbf{0}_{p_\theta}, \quad i = 0, 1, \dots, S.$$

Hence $(\check{B}_{1T}^{(1)} + \check{B}_{1T}^{(2)}) \xrightarrow{p} \mathbf{0}_{p_\theta}$ in (C7). Next, we show that in (C7), $\check{B}_{1T}^{(3)} \xrightarrow{p} \mathbf{0}_{p_\theta}$ as well. We have,

$$\begin{aligned}
&|\eta_{iT}^{00}(z, v)| \\
&\leq \frac{E(\pi_T(z, v))}{\pi_T^i(v; \theta_0) \cdot \pi_T(v)} \{|\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)| + |\pi_T(z|v) - \pi_0(z|v)|\} \\
&\quad \times \mathbb{T}_{T,\delta}(v; \theta_0) |\nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_0)| \cdot |w_T(z, v) - w(z, v)|.
\end{aligned}$$

By Remark 1,

$$\begin{aligned}
& \frac{1}{\pi_T^i(v; \theta_0) \cdot \pi_T(v)} \{ |\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)| + |\pi_T(z|v) - \pi_0(z|v)| \} \\
& \quad \times \mathbb{T}_{T,\delta}(v; \theta_0) |\nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_0)| \\
& \leq \frac{1}{\pi_T^i(v; \theta_0) \cdot \pi_T(v)} \{ |\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)| + |\pi_T(z|v) - \pi_0(z|v)| \} \\
& \quad \times \mathbb{T}_{T,\delta}(v; \theta_0) \sum_{j=1}^S \left| \nabla_\theta \pi_T^j(v; \theta_0) - \nabla_\theta \pi_0(v) \right| g_{\delta_T} \left(\pi_T^j(v; \theta_0) \right) \mathbb{T}_{T,\delta}^{(-j)}(v; \theta_0) \\
& + \frac{1}{\pi_T^i(v; \theta_0) \cdot \pi_T(v)} \{ |\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)| + |\pi_T(z|v) - \pi_0(z|v)| \} \\
& \quad \times \mathbb{T}_{T,\delta}(v; \theta_0) \sum_{j=1}^S |\nabla_\theta \pi_0(v)| g_{\delta_T} \left(\pi_T^j(v; \theta_0) \right) \mathbb{T}_{T,\delta}^{(-j)}(v; \theta_0) \\
& \equiv b_{1T}^* + b_{2T}^*.
\end{aligned}$$

By arguments nearly identical to the ones used for the η_T^i terms,

$$\begin{aligned}
& \sup_{(z,v) \in Z \times V} b_{1T}^* \\
& = \delta_T^{-2} \times \left[O_p \left(T^{-\frac{1}{2}} \lambda_T^{-q} \delta_T^{-1} \right) + O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-2} \right) + O_p \left(\lambda_T^r \delta_T^{-2} \right) \right] \times \left[O_p \left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)-1} \right) + O_p \left(\lambda_T^r \right) \right] \\
& \xrightarrow{p} \mathbf{0}_{p_\theta}.
\end{aligned}$$

Once again, the convergence follows by the conditions in the theorem; and by similar arguments, and boundedness of $\nabla_\theta \pi_0(v)$, $\sup_{(z,v) \in Z \times V} b_{2T}^* \xrightarrow{p} \mathbf{0}_{p_\theta}$ as well. By assumption 4(b), $\sup_{(z,v) \in Z \times V} |\eta_{iT}^{00}(z, v)| \xrightarrow{p} \mathbf{0}_{p_\theta}$ and hence $\tilde{B}_{1T}^{(3)} \xrightarrow{p} \mathbf{0}_{p_\theta}$ in (C7). We have thus established that $B_{1T} \xrightarrow{p} \mathbf{0}_{p_\theta}$. By eq. (C4), we then have that $B_T \xrightarrow{p} \mathbf{0}_{p_\theta}$ whenever $B_{2T} \xrightarrow{p} \mathbf{0}_{p_\theta}$. We now show that this is the case. Indeed, the B_{2T} term has the same representation as in eq. (C7), but with functions η_S given by:

$$\begin{aligned}
\eta_T^i(z, v) &= \frac{1}{\pi_T^i(v; \theta_0)} \{ [\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)] - [\pi_T(z|v) - \pi_0(z|v)] \} \\
& \quad \times \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_0) w(z, v) \quad (i = 0, 1, \dots, S) \\
\eta_{iT}^{00}(z, v) &= \frac{E(\pi_T(z, v))}{\pi_T^i(v; \theta_0) \cdot \pi_T(v)} \{ [\tilde{\pi}_T(z|v; \theta_0) - \pi_0(z|v)] - [\pi_T(z|v) - \pi_0(z|v)] \} w(z, v) \\
& \quad \times \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_0) w(z, v) \quad (i = 1, \dots, S)
\end{aligned}$$

Therefore $B_{2T} \xrightarrow{p} \mathbf{0}_{p_\theta}$ by boundedness of w , and the same arguments used to show that $B_{1T} \xrightarrow{p} \mathbf{0}_{p_\theta}$.

Hence,

$$B_T \xrightarrow{p} \mathbf{0}_{p_\theta}. \quad (\text{C8})$$

The C_T term

We have,

$$C_T = C_{1T} + C_{2T},$$

where

$$\begin{aligned} C_{1T} &\equiv \iint \nabla_\theta \{ [\tilde{\pi}_T(z|v; \bar{\theta}) - \pi_T(z|v)] \nabla_\theta \tilde{\pi}_T(z|v; \bar{\theta}) \mathbb{T}_{T,\delta}^2(v; \bar{\theta}) \} w_T(z, v) dz dv; \\ C_{2T} &\equiv \iint \nabla_\theta \{ [\tilde{\pi}_T(z|v; \bar{\theta}) - \pi_T(z|v)]^2 \mathbb{T}_{T,\delta}(v; \bar{\theta}) \nabla_\theta \mathbb{T}_{T,\delta}(v; \bar{\theta}) \} w_T(z, v) dz dv. \end{aligned}$$

We study these two integrals separately.

- By performing the inner differentiation,

$$\begin{aligned} C_{1T} &= \iint \{ |\nabla_\theta \tilde{\pi}_T(z|v; \bar{\theta})|_2 + [\tilde{\pi}_T(z|v; \bar{\theta}) - \pi_T(z|v)] \nabla_{\theta\theta} \tilde{\pi}_T(z|v; \bar{\theta}) \} \mathbb{T}_{T,\delta}^2(v; \bar{\theta}) w_T(z, v) dz dv \\ &+ 2 \iint [\tilde{\pi}_T(z|v; \bar{\theta}) - \pi_T(z|v)] \nabla_\theta \tilde{\pi}_T(z|v; \bar{\theta}) \mathbb{T}_{T,\delta}(v; \bar{\theta}) \nabla_\theta \mathbb{T}_{T,\delta}(v; \bar{\theta}) w_T(z, v) dz dv \\ &\equiv C_{11T} + C_{12T}. \end{aligned}$$

We have,

$$C_{11T} = \iint |\nabla_\theta \tilde{\pi}_T(z|v; \bar{\theta}) \mathbb{T}_{T,\delta}(v; \bar{\theta})|_2 w_T(z, v) dz dv + (\bar{\theta} - \theta_0) \cdot K_{1T}(\bar{\theta}) + K_{2T}(\bar{\theta}),$$

where, for some θ^* ,

$$\begin{aligned} |K_{1T}(\bar{\theta})| &\leq \frac{1}{S} \sum_{i=1}^S \iint |\nabla_\theta \pi_T^i(z|v; \theta^*)| |\nabla_\theta \tilde{\pi}_T(z|v; \bar{\theta})| w_T(z, v) \mathbb{T}_{T,\delta}^2(v; \bar{\theta}) dz dv \\ |K_{2T}(\bar{\theta})| &\leq \frac{1}{S} \sum_{i=1}^S \iint \left| \frac{\pi_T^i(z, v; \theta_0)}{\pi_T^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right| |\nabla_\theta \tilde{\pi}_T(z|v; \bar{\theta})| w_T(z, v) \mathbb{T}_{T,\delta}^2(v; \bar{\theta}) dz dv \end{aligned}$$

By remark 2 and assumption 4-(a), $|\nabla_\theta \tilde{\pi}_T(z|v; \bar{\theta})| \mathbb{T}_{T,\delta}^2(v; \bar{\theta})$ is bounded in probability as T becomes large. Hence by lemma 3 and integrability of w_T , $|K_{2T}(\bar{\theta})|_{i,j} \xrightarrow{p} \mathbf{0}_{p_\theta \times p_\theta}$ all

i, j . Moreover,

$$\begin{aligned} & \iint |\nabla_{\theta} \tilde{\pi}_T(z|v; \bar{\theta}) \mathbb{T}_{T,\delta}(v; \bar{\theta})|_2 w_T(z, v) dz dv \\ &= \iint |\nabla_{\theta} \tilde{\pi}_T(z|v; \theta_0) \mathbb{T}_{T,\delta}(v; \theta_0)|_2 w_T(z, v) dz dv + \mathcal{R}_T(\bar{\theta}) + o_p(1) \mathbf{1}_{p_{\theta} \times p_{\theta}}, \end{aligned}$$

where

$$|\mathcal{R}_T(\bar{\theta})|_{i,j} \leq \iint \left| |\nabla_{\theta} \tilde{\pi}_T(z|v; \bar{\theta})|_2 - |\nabla_{\theta} \pi(z|v; \theta_0)|_2 \right|_{i,j} \mathbb{T}_{T,\delta}^2(v; \bar{\theta}) w_T(z, v) dz dv.$$

By $\iint |w_T - w| \xrightarrow{p} 0$, lemma 6 and $\bar{\theta} \xrightarrow{p} \theta$, $\mathcal{R}_T(\bar{\theta})_{i,j} \xrightarrow{p} 0$ all i, j . Finally, $K_{1T}(\bar{\theta})$ is bounded in probability as T becomes large (component-wise). Hence,

$$C_{11T} = \iint |\nabla_{\theta} \tilde{\pi}_T(z|v; \theta_0) \mathbb{T}_{T,\delta}(v; \theta_0)|_2 w_T(z, v) dz dv + o_p(1) \mathbf{1}_{p_{\theta} \times p_{\theta}}.$$

As regards the C_{12T} term, we have that $C_{12T} \xrightarrow{p} \mathbf{0}_{p_{\theta} \times p_{\theta}}$ by exactly the same arguments we made to show that $B_{1T} \xrightarrow{p} \mathbf{0}_{p_{\theta}}$ in eq. (C7) above. Hence, $C_{1T} \xrightarrow{p} \mathbf{0}_{p_{\theta} \times p_{\theta}}$.

- By differentiating,

$$\begin{aligned} & C_{2T} \\ &= 2 \iint [\tilde{\pi}_T(z|v; \bar{\theta}) - \pi_T(z|v)] \nabla_{\theta} \tilde{\pi}_T(z|v; \bar{\theta}) \mathbb{T}_{T,\delta}(v; \bar{\theta}) \nabla_{\theta} \mathbb{T}_{T,\delta}(v; \bar{\theta}) w_T(z, v) dz dv \\ &+ \iint [\tilde{\pi}_T(z|v; \bar{\theta}) - \pi_T(z|v)]^2 [|\nabla_{\theta} \mathbb{T}_{T,\delta}(v; \bar{\theta})|_2 + \mathbb{T}_{T,\delta}(v; \bar{\theta}) \nabla_{\theta\theta} \mathbb{T}_{T,\delta}(v; \bar{\theta})] w_T(z, v) dz dv. \end{aligned}$$

We have, $C_{2T} \xrightarrow{p} \mathbf{0}_{p_{\theta} \times p_{\theta}}$ by the same arguments produced to deal with C_{1T} term, by Remark 1, by $\pi^i \leq \delta_T \Rightarrow g = 0$ (assumption **T**), by noticing again that by Remark 1, $\nabla_{\theta\theta} \mathbb{T}_{T,\delta}(v; \bar{\theta}) = \delta_T^{-2} \hat{k}_T$ (where \hat{k}_T is a term bounded in probability), and finally by the conditions T1 in the theorem.

Therefore,

$$C_T = \iint |\nabla_{\theta} \tilde{\pi}_T(z|v; \theta_0) \mathbb{T}_{T,\delta}(v; \theta_0)|_2 w_T(z, v) dz dv + o_p(1) \mathbf{1}_{p_{\theta} \times p_{\theta}}. \quad (\text{C9})$$

First order conditions

By the previous results on B_T and C_T (in (C8) and (C9)), the first order conditions in eq. (C3) are,

$$\begin{aligned} \mathbf{0}_{p_\theta} &= \frac{1}{S} \sum_{i=1}^S \sqrt{T} \iint \left[\frac{\pi_T^i(z, v; \theta_0)}{\pi_T^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] \nabla_\theta \tilde{\pi}_T(z|v; \theta_0) w_T(z, v) \mathbb{T}_{T, \delta}^2(v; \theta_0) dz dv + o_p(1) \mathbf{1}_{p_\theta} \\ &+ \left[\iint |\nabla_\theta \tilde{\pi}_T(z|v; \theta_0) \mathbb{T}_{T, \delta}(v; \theta_0)|_2 w_T(z, v) dz dv + o_p(1) \mathbf{1}_{p_\theta \times p_\theta} \right] \cdot \sqrt{T} (\theta_{T, S} - \theta_0). \end{aligned} \quad (\text{C10})$$

We now elaborate on eq. (C10). By replacing (C6) into eq. (C10) we obtain,

$$\begin{aligned} & o_p(1) \mathbf{1}_{p_\theta} \\ &= \frac{1}{S} \sum_{i=1}^S \sqrt{T} \iint \left[\frac{\pi_T^i(z, v; \theta_0) - E(\pi_T^i(z, v; \theta_0))}{\pi_T^i(v; \theta_0)} \right] \nabla_\theta \tilde{\pi}_T(z|v; \theta_0) w_T(z, v) \mathbb{T}_{T, \delta}^2(v; \theta_0) dz dv \\ &- \frac{1}{S} \sum_{i=1}^S \sqrt{T} \iint \left[\frac{\pi_T(z, v) - E(\pi_T(z, v))}{\pi_T(v)} \right] \nabla_\theta \tilde{\pi}_T(z|v; \theta_0) w_T(z, v) \mathbb{T}_{T, \delta}^2(v; \theta_0) dz dv \\ &+ \frac{1}{S} \sum_{i=1}^S \sqrt{T} \iint \left[\frac{E(\pi_T(z, v)) [\pi_T(v) - \pi_T^i(v; \theta_0)]}{\pi_T^i(v; \theta_0) \cdot \pi_T(v)} \right] \nabla_\theta \tilde{\pi}_T(z|v; \theta_0) w_T(z, v) \mathbb{T}_{T, \delta}^2(v; \theta_0) dz dv \\ &+ \left[\iint |\nabla_\theta \tilde{\pi}_T(z|v; \theta_0) \mathbb{T}_{T, \delta}(v; \theta_0)|_2 w_T(z, v) dz dv + o_p(1) \mathbf{1}_{p_\theta \times p_\theta} \right] \cdot \sqrt{T} (\theta_{T, S} - \theta_0) \\ &\equiv \frac{1}{S} \sum_{i=1}^S (D_{1T}^i + D_{2T}^i) - D_{1T}^0 + [D_{3T} + o_p(1) \mathbf{1}_{p_\theta \times p_\theta}] \cdot \sqrt{T} (\theta_{T, S} - \theta_0), \end{aligned}$$

where

$$\begin{aligned} D_{1T}^i &\equiv \iint \frac{\nabla_\theta \tilde{\pi}_T(z|v; \theta_0) w_T(z, v)}{\pi_T^i(v; \theta_0)} \mathbb{T}_{T, \delta}^2(v; \theta_0) dA_T^i(z, v; \theta_0); \\ D_{1T}^0 &\equiv \iint \frac{\nabla_\theta \tilde{\pi}_T(z|v; \theta_0) w_T(z, v)}{\pi_T(v)} \mathbb{T}_{T, \delta}^2(v; \theta_0) dA_T(z, v); \\ D_{2T}^i &\equiv \iint \frac{\nabla_\theta \tilde{\pi}_T(z|v; \theta_0) E[\pi_T(z, v)] w_T(z, v)}{\pi_T^i(v; \theta_0) \cdot \pi_T(v)} \mathbb{T}_{T, \delta}^2(v; \theta_0) dz [dA_T(v) - dA_T^i(v; \theta_0)]; \\ D_{3T} &\equiv \iint |\nabla_\theta \tilde{\pi}_T(z|v; \theta_0) \mathbb{T}_{T, \delta}(v; \theta_0)|_2 w_T(z, v) dz dv; \end{aligned}$$

and $A_T^i(z, v; \theta_0)$, $A_T(z, v)$, $A_T(v)$ and $A_T^i(v; \theta_0)$ are as in the definition in eq. (C7) above. Please

also note that,

$$D_{2T}^i = \int_V \gamma_T(v) \mathbb{T}_{T,\delta}^2(v; \theta_0) dA_T(v) - \int_V \gamma_T(v) \mathbb{T}_{T,\delta}^2(v; \theta_0) dA_T^i(v; \theta_0)$$

where

$$\gamma_T(v) \equiv \int_Z \frac{\nabla_{\theta} \tilde{\pi}_T(z|v; \theta_0) E[\pi_T(z, v)] w_T(z, v)}{\pi_T^i(v; \theta_0) \cdot \pi_T(v)} dz.$$

Next let $\omega_i^0 \circ F(z, v)$, $i = 0, 1, \dots, S$, denote independent Generalized Brownian Bridges. Let, also, $\omega^0 \circ F(v)$ and $\omega_i^0 \circ F(v)$, $i = 1, \dots, S$, denote independent Generalized Brownian Bridges. Finally, let

$$\gamma(v) \equiv \int_Z \frac{\nabla_{\theta} \pi(z|v; \theta_0) \pi_0(z, v) w(z, v)}{\pi(v; \theta_0)^2} dz. \quad (\text{C11})$$

We now demonstrate that for $i = 0, 1, \dots, S$,

$$D_{1T}^i \xrightarrow{d} D_1^i \equiv \iint \frac{\nabla_{\theta} \pi(z|v; \theta_0) w(z, v)}{\pi(v; \theta_0)} d\omega_i^0(F(z, v)); \quad (\text{C12-a})$$

that for $i = 1, \dots, S$,

$$D_{2T}^i \xrightarrow{d} D_2^0 - D_2^i \equiv \int_V \gamma(v) d\omega^0(F(v)) - \int_V \gamma(v) d\omega_i^0(F(v)); \quad (\text{C12-b})$$

and finally that,

$$D_{3T} \xrightarrow{d} D_3 \equiv \iint |\nabla_{\theta} \pi(z|v; \theta_0)|_2 w(z, v) dz dv. \quad (\text{C12-c})$$

We show eq. (C12-a) for the D_{1T}^0 term only (the proof for the other D_{1T}^i terms ($i = 1, \dots, S$) being identical). We have,

$$D_{1T}^0 = \iint \frac{\nabla_{\theta} \pi(z|v; \theta_0) w(z, v)}{\pi(v; \theta_0)} dA_T^i(z, v; \theta_0) + \hat{D}_{1T}^0 \quad (\text{C13})$$

where,

$$\begin{aligned} \hat{D}_{1T}^0 &\equiv \iint \left[\frac{\nabla_{\theta} \tilde{\pi}_T(z|v; \theta_0) w_T(z, v)}{\pi_T(v)} - \frac{\nabla_{\theta} \pi(z|v; \theta_0) w(z, v)}{\pi(v; \theta_0)} \right] \mathbb{T}_{T,\delta}^2(v; \theta_0) dA_T^0(z, v; \theta_0) \\ &\quad - \iint \frac{\nabla_{\theta} \pi(z|v; \theta_0) w(z, v)}{\pi(v; \theta_0)} [1 - \mathbb{T}_{T,\delta}^2(v; \theta_0)] dA_T^0(z, v; \theta_0). \end{aligned}$$

By $|1 - \mathbb{T}_{T,\delta}^2(v; \theta_0)| < 1$, $|1 - \mathbb{T}_{T,\delta}^2(v; \theta_0)| \xrightarrow{p} 0$, and lemma 7, we have that $\hat{D}_{1T}^0 \xrightarrow{p} 0$. Eq. (C12-a) then follows by eq. (C13) and the continuous mapping theorem.

We now turn to demonstrate the convergence in eq. (C12-b). We have,

$$D_{2T}^1 = \iint \frac{\nabla_{\theta} \pi(z|v; \theta_0) \pi_0(z, v) w(z, v)}{\pi(v; \theta_0)^2} dz [dA_T(v) - dA_T^1(v; \theta_0)] + \hat{D}_{2T}^1,$$

where

$$\begin{aligned} \hat{D}_{2T}^1 &\equiv \iint \left[\frac{\nabla_{\theta} \tilde{\pi}_T(z|v; \theta_0) E[\pi_T(z, v)] w_T(z, v)}{\pi_T^1(v; \theta_0) \cdot \pi_T(v)} - \frac{\nabla_{\theta} \pi(z|v; \theta_0) \pi_0(z, v) w(z, v)}{\pi(v; \theta_0)^2} \right] \mathbb{T}_{T, \delta}^2(v; \theta_0) dz \\ &\times [dA_T(v) - dA_T^1(v; \theta_0)] \\ &- \iint \frac{\nabla_{\theta} \pi(z|v; \theta_0) \pi_0(z, v) w(z, v)}{\pi(v; \theta_0)^2} [1 - \mathbb{T}_{T, \delta}^2(v; \theta_0)] dz [dA_T(v) - dA_T^1(v; \theta_0)]. \end{aligned}$$

The result follows by lemma 8 and the same arguments used to show eq. (C12-a).

Finally, eq. (C12-c) follows by $\iint |w_T - w| \xrightarrow{p} 0$ and lemma 6 and arguments nearly identical to the those we used to demonstrate (C12-a) and (C12-b).

The normality claim in theorem 2 now immediately follows. As in appendix B.2, the terms D_1^i , $i = 0, 1, \dots, S$, are all independent and asymptotically centered Gaussian. Therefore, $\sqrt{T}(\theta_{T,S} - \theta_0)$ is asymptotically centered normal with variance

$$V \equiv D_3^{-1} \cdot \text{var} \left[\frac{1}{S} \sum_{i=1}^S (D_1^i - D_2^i) - (D_1^0 - D_2^0) \right] \cdot D_3^{\top -1}.$$

D. Proof of theorem 3

The following assumption contains one set of regularity conditions mentioned in the statement of theorem 3.

Assumption T2. *We have,*

- (a) $\alpha_T \rightarrow 0$, $\delta_T \rightarrow 0$, $T^{\frac{1}{2}}\alpha_T^3 \rightarrow \infty$, $T^{\frac{1}{2}}\delta_T^2 \rightarrow \infty$, and $T^{\frac{1}{2}}\alpha_T^2\delta_T^2 \rightarrow \infty$.
- (b) *In addition to assumption T2-(a),* $\lambda \equiv \lambda_T \rightarrow 0$, $T^{\frac{1}{2}}\lambda_T^q\alpha_T^3 \rightarrow \infty$, $T^{\frac{1}{2}}\lambda_T^q\alpha_T^2\delta_T \rightarrow \infty$, $T^{\frac{1}{2}}\lambda_T^{q+1}\alpha_T\delta_T^3 \rightarrow \infty$, $T^{\frac{1}{2}}\lambda_T^{q-q^*}\alpha_T^2\delta_T^2 \rightarrow \infty$, $T^{\frac{1}{2}}\lambda_T^{q-q^*}\alpha_T\delta_T^4 \rightarrow \infty$, $\alpha_T^3\lambda_T^{-r} \rightarrow \infty$, $\delta_T^2\lambda_T^{-r} \rightarrow \infty$, $\alpha_T^2\delta_T^2\lambda_T^{-r} \rightarrow \infty$, and $\alpha_T\delta_T^4\lambda_T^{-r} \rightarrow \infty$.

D.1 Consistency

By appendixes B.1 and C.1, we only have to show that for all $\theta \in \Theta$, $\iint s_{iT}(z, v; \theta) dz dv \xrightarrow{P} 0$, $i = 1, 2$, where s_{iT} are defined in appendix C.1, with $w_T(z, v) = [\pi_T(v)/\pi_T(z|v)]\mathbb{T}_{T,\alpha}(z, v)$, $w(z, v) = m_0(v)/n_0(z, v)$. We proceed as in appendix C.1, and study these two integrals separately.

- For all $(z, v, \theta) \in Z \times V \times \Theta$,

$$\begin{aligned}
s_{1T}(z, v; \theta) &\leq |\tilde{\pi}_T(z|v; \theta) - \pi_T(z|v)| \mathbb{T}_{T,\delta}(v; \theta) m_0(z, v) |n(z, v; \theta) - n(z, v; \theta_0)| \\
&\quad \times \frac{1}{m_0(z, v)} \left| \frac{\pi_T(v)}{\pi_T(z|v)} - \frac{m_0(v)}{n_0(z, v)} \right| \mathbb{T}_{T,\alpha}(z, v) \\
&\quad + |\tilde{\pi}_T(z|v; \theta) - \pi_T(z|v)| \mathbb{T}_{T,\delta}(v; \theta) |n(z, v; \theta) - n(z, v; \theta_0)| \frac{m_0(v)}{n_0(z, v)} \\
&\quad \times [1 - \mathbb{T}_{T,\alpha}(z, v)] \\
&\leq \ell_{1T}(z, v; \theta) \cdot \ell_{2T}(z, v; \theta) \cdot m_0(z, v) |n(z, v; \theta) - n(z, v; \theta_0)| \\
&\quad + \ell_{2T}(z, v; \theta) \cdot [n(z, v; \theta) - n(z, v; \theta_0)]^2 m_0(z, v) \mathbb{T}_{T,\delta}(v; \theta) \\
&\quad + \ell_{3T}(z, v; \theta) |n(z, v; \theta) - n(z, v; \theta_0)| m_0(z, v) m_0(v) [1 - \mathbb{T}_{T,\alpha}(z, v)] \\
&\quad + \frac{m_0(v)}{n_0(z, v)} [n(z, v; \theta) - n(z, v; \theta_0)]^2 \mathbb{T}_{T,\delta}(v; \theta) [1 - \mathbb{T}_{T,\alpha}(z, v)] \\
&\equiv s_{11T}(z, v; \theta) + s_{12T}(z, v; \theta) + s_{13T}(z, v; \theta) + s_{14T}(z, v; \theta),
\end{aligned}$$

where

$$\begin{aligned}\ell_{1T}(z, v; \theta) &\equiv \left[\frac{1}{S} \sum_{i=1}^S |\pi_T^i(z|v; \theta) - n(z, v; \theta)| + |\pi_T(z|v) - n(z, v; \theta_0)| \right] \mathbb{T}_{T, \delta}(v; \theta) \\ \ell_{2T}(z, v; \theta) &\equiv \frac{1}{m_0(z, v)} \left| \frac{\pi_T(v)}{\pi_T(z|v)} - \frac{m_0(v)}{n_0(z, v)} \right| \mathbb{T}_{T, \alpha}(z, v) \\ \ell_{3T}(z, v; \theta) &\equiv \frac{\ell_{1T}(z, v; \theta)}{m_0(z, v) n_0(z, v)}\end{aligned}$$

By lemmas 2-(a), 3-(a) and 4-(a), $\iint s_{1jT} \xrightarrow{P} 0$ for all $\theta \in \Theta$ and $j = 1, 2, 3$. As regards the s_{14T} term, notice that function $n(z, v; \theta_0)^{-1} [n(z, v; \theta) - n(z, v; \theta_0)]^2 m_0(v)$ is the integrand of the asymptotic objective function, which is bounded and integrable by assumption. Moreover, $|\mathbb{T}_{T, \delta}(v; \theta) [1 - \mathbb{T}_{T, \alpha}(z, v)]| \leq 1$, and $[1 - \mathbb{T}_{T, \alpha}(z, v)] \xrightarrow{P} P_\pi \{ \pi_0(z_1, v_1) < \lim_{T \rightarrow \infty} \alpha_T \} - \iint_{(z, v): \pi_0(z, v) \in (\lim_{T \rightarrow \infty} \alpha_T, 2 \lim_{T \rightarrow \infty} \alpha_T)} \mathbb{T}_{T, \alpha}(z, v) P_\pi(dz, dv)$, where P_π is now the stationary measure of (z, v) . Hence $\iint s_{14T} \xrightarrow{P} 0$ for all $\theta \in \Theta$, and so $\iint s_{1T} \xrightarrow{P} 0$ for all $\theta \in \Theta$ as well.

- For all $(z, v, \theta) \in Z \times V \times \Theta$,

$$\begin{aligned}s_{2T}(z, v; \theta) &\leq [r_T(z, v; \theta) + r(z, v; \theta)] \\ &\quad \times [|\tilde{\pi}_T(z|v; \theta) - n(z, v; \theta)| + |\pi_T(z|v) - n(z, v; \theta_0)|] \mathbb{T}_{T, \delta}(v; \theta) \\ &\quad + [r_T(z, v; \theta) + r(z, v; \theta)] [n(z, v; \theta) - n(z, v; \theta_0)] [1 - \mathbb{T}_{T, \delta}(v; \theta)] \\ &\equiv s_{21T}(z, v; \theta) + s_{22T}(z, v; \theta).\end{aligned}$$

For all $(z, v, \theta) \in Z \times V \times \Theta$,

$$\begin{aligned}
& s_{21T}(z, v; \theta) \\
& \leq |\tilde{\pi}_T(z|v; \theta) - n(z, v; \theta)| \mathbb{T}_{T, \delta}(v; \theta) \frac{1}{m_0(z, v)} \left| \frac{\pi_T(v)}{\pi_T(z|v)} - \frac{m_0(v)}{n_0(z, v)} \right| \mathbb{T}_{T, \alpha}(z, v) m_0(z, v) \\
& \times [|\tilde{\pi}_T(z|v; \theta) - n(z, v; \theta)| + |\pi_T(z|v) - n(z, v; \theta_0)|] \mathbb{T}_{T, \delta}(v; \theta) \\
& + |\pi_T(z|v) - n_0(z, v)| \mathbb{T}_{T, \delta}(v; \theta) \frac{1}{m_0(z, v)} \left| \frac{\pi_T(v)}{\pi_T(z|v)} - \frac{m_0(v)}{n_0(z, v)} \right| \mathbb{T}_{T, \alpha}(z, v) m_0(z, v) \\
& \times [|\tilde{\pi}_T(z|v; \theta) - n(z, v; \theta)| + |\pi_T(z|v) - n(z, v; \theta_0)|] \mathbb{T}_{T, \delta}(v; \theta) \\
& + |n(z, v; \theta) - n_0(z, v)| \mathbb{T}_{T, \delta}(v; \theta) \frac{1}{m_0(z, v)} \left| \frac{\pi_T(v)}{\pi_T(z|v)} - \frac{m_0(v)}{n_0(z, v)} \right| \mathbb{T}_{T, \alpha}(z, v) m_0(z, v) \\
& \times [|\tilde{\pi}_T(z|v; \theta) - n(z, v; \theta)| + |\pi_T(z|v) - n(z, v; \theta_0)|] \mathbb{T}_{T, \delta}(v; \theta) \\
& + |n(z, v; \theta) - n(z, v; \theta_0)| m_0(v) m_0(z, v) \\
& \times \frac{1}{n_0(z, v) m_0(z, v)} [|\tilde{\pi}_T(z|v; \theta) - n(z, v; \theta)| + |\pi_T(z|v) - n(z, v; \theta_0)|] \mathbb{T}_{T, \delta}(v; \theta) \\
& + |\tilde{\pi}_T(z|v; \theta) - \pi_T(z|v)| \mathbb{T}_{T, \delta}(v; \theta) \frac{m_0(v)}{n_0(z, v)} \mathbb{T}_{T, \alpha}(z, v) \\
& \times [|\tilde{\pi}_T(z|v; \theta) - n(z, v; \theta)| + |\pi_T(z|v) - n(z, v; \theta_0)|] \mathbb{T}_{T, \delta}(v; \theta) \\
& \equiv s_{211T}(z, v; \theta) + s_{212T}(z, v; \theta) + s_{213T}(z, v; \theta) + s_{214T}(z, v; \theta) + s_{215T}(z, v; \theta)
\end{aligned}$$

By lemmas 2-(a), 3-(a) and 4-(a), $\iint s_{21jT} \xrightarrow{P} 0$ (all $\theta \in \Theta$ and $j = 1, 2, 3, 4$). Similar results for the s_{215T} term lead to $\iint s_{215T} \xrightarrow{P} 0$ (all $\theta \in \Theta$), and so $\iint s_{21T} \xrightarrow{P} 0$ for all $\theta \in \Theta$.

Next, for all $(z, v, \theta) \in Z \times V \times \Theta$,

$$s_{22T}(z, v; \theta) \leq \left\{ s_{22T}^*(z, v; \theta) m_0(z, v) + [n(z, v; \theta) - n_0(z, v)]^2 \frac{m_0(v)}{n_0(z, v)} \right\} [1 - \mathbb{T}_{T, \delta}(v; \theta)],$$

where

$$\begin{aligned}
& s_{22T}^*(z, v; \theta) \\
& \equiv [|\tilde{\pi}_T(z|v; \theta) - n(z, v; \theta)| + |\pi_T(z|v) - n_0(z, v)| + |n(z, v; \theta) - n_0(z, v)|] \mathbb{T}_{T, \delta}(v; \theta) \\
& \times \frac{1}{m_0(z, v)} \left| \frac{\pi_T(v; \theta)}{\pi_T(z|v)} - \frac{m_0(v)}{n_0(z, v)} \right| \mathbb{T}_{T, \alpha}(z, v) \cdot |n(z, v; \theta) - n_0(z, v)| \\
& + \frac{1}{m_0(z, v) n_0(z, v)} |\tilde{\pi}_T(z|v; \theta) - \pi_T(z|v)| \mathbb{T}_{T, \delta}(v; \theta) m_0(v) \cdot |n(z, v; \theta) - n_0(z, v)| \mathbb{T}_{T, \alpha}(z, v).
\end{aligned}$$

Similarly as in the previous appendixes, $1 - \mathbb{T}_{T, \delta}(v; \theta) \xrightarrow{P} 0$ and since $1 - \mathbb{T}_{T, \delta}(v; \theta) \leq 1$, and both $m_0(z, v)$ and $n(z, v; \theta_0)^{-1} [n(z, v; \theta) - n(z, v; \theta_0)]^2 m_0(v)$ are bounded and integrable,

$\iint s_{22T} \xrightarrow{p} 0$ for all $\theta \in \Theta$. Hence, $\iint s_{2T} \xrightarrow{p} 0$ for all $\theta \in \Theta$.

The case $\lambda \equiv \lambda_T \downarrow 0$ is dealt with similarly through lemmas 2-(b), 3-(b) and 4-(b).

D.2 Asymptotic normality

We are going to provide a result slightly more general than needed - namely for more general weighting functions. Let $\xi(z, v) \equiv \pi_0(z, v) w(z, v) / \pi_0(v)^2$, and consider the definition of γ in appendix C.2 (see (C11)). In terms of this new function ξ , γ is

$$\gamma(v) \equiv \int_Z \nabla_{\theta} \pi(z|v; \theta_0) \xi(z, v) dz. \quad (\text{D1})$$

Next, let

$$W_T^{\xi} \equiv \left\{ w_T(z, v) : w_T(z, v) = \xi_{1T}(v) \cdot \frac{\pi_T(v)^2}{\pi_T(z, v)} \mathbb{T}_{T, \alpha}(z, v) \right\},$$

where function ξ_{1T} satisfies the conditions in lemma 9.

We study the asymptotic behavior of the CD-SNE for weighting functions $w_T \in W_T^{\xi}$. First, by remark 2 and assumption 4-(a), we may still interchange the order of differentiation and integration in the criterion. The first order conditions of the CD-SNE are still as in eq. (C3) for $w_T \in W_T^{\xi}$. So we only need to check that the B_T and C_T in eq. (C3) are asymptotically well-behaved. As regards B_T , the terms η_T^i ($i = 0, 1, \dots, S$) and η_T^{00} ($i = 1, \dots, S$) in eq. (C7) (appendix C.2) converge uniformly to zero by the same arguments in appendix C.2, and additionally by lemma 2. Hence $B_{1T} \xrightarrow{p} 0$ for $w_T \in W_T^{\xi}$ as well. Finally $B_{2T} \xrightarrow{p} \mathbf{0}_{p\theta}$ by the same arguments in appendix C.2, and additionally by lemma 4. As regards C_T , $K_{2T} \xrightarrow{p} 0$ and $C_{2T} \xrightarrow{p} 0$ by the same arguments in appendix C.2, and additionally by lemma 4. Hence in eq. (C9), $C_T = D_{3T} + o_p(1) \mathbf{1}_{p\theta \times p\theta}$, where

$$D_{3T} = \iint |\nabla_{\theta} \tilde{\pi}_T(z|v; \theta_0) \mathbb{T}_{T, \delta}(v; \theta_0)|_2 \frac{\pi_T(v)}{\pi_T(z|v)} \xi_{1T}(v) \mathbb{T}_{T, \alpha}(z, v) dz dv. \quad (\text{D2})$$

Next, consider the terms D_{jT}^i in appendix C.2, and let $w_T \in W_T^{\xi}$. By lemma 9 (applied to the D_{1T}^i term, $i = 0, 1, \dots, S$), lemma 10 (applied to the D_{2T}^i term, $i = 1, \dots, S$), assumption T2, and additional arguments nearly identical to the ones in appendix C.2, we have that for $i = 0, 1, \dots, S$,

$$D_{1T}^i \xrightarrow{d} D_1^i \equiv \iint \frac{\nabla_{\theta} \pi(z|v; \theta_0)}{\pi(z|v; \theta_0)} \xi(z, v) d\omega_i^0(F(z, v)),$$

and that for $i = 1, \dots, S$,

$$D_{2T}^i \xrightarrow{d} D_2^i \equiv \int_V \gamma(v) d\hat{\omega}_0^0(F(v)) - \int_V \gamma(v) d\hat{\omega}_i^0(F(v)). \quad (\text{D3})$$

Finally the D_{3T} term in eq. (D2) is,

$$\begin{aligned} D_{3T} &= \iint |\nabla_{\theta} \tilde{\pi}_T(z|v; \theta_0) \mathbb{T}_{T,\delta}(v; \theta_0)|_2 \frac{\pi_T(v)}{\pi_T(z|v)} \xi_{1T}(v) \mathbb{T}_{T,\alpha}(z, v) dz dv \\ &= \iint \left| \frac{\nabla_{\theta} \tilde{\pi}_T(z|v; \theta_0)}{\pi_T(z|v)} \mathbb{T}_{T,\delta}(v; \theta_0) \sqrt{\mathbb{T}_{T,\alpha}(z, v)} \right|_2 \xi_{1T}(v) \pi_T(z, v) dz dv, \end{aligned}$$

and by lemma 9,

$$D_{3T} \xrightarrow{P} D_3 \equiv \iint \left| \frac{\nabla_{\theta} \pi(z|v; \theta_0)}{\pi(z|v; \theta_0)} \right|_2 \xi_1(v) \pi_0(z, v) dz dv.$$

Moreover, for any $w_T \in W_T^{\xi}$, the limiting function in (D1) $\xi(z, v) = \xi_1(v)$. But for all $v \in V$, $\int_Z \nabla_{\theta} \pi(z|v; \theta_0) dz = 0$. Hence $\gamma(v) = 0$ for all $v \in V$, and then $D_2^i \equiv 0$ in eq. (D3). So we have shown the following result:

Proposition 2. *Under the assumptions of theorem 2 and assumption T2, CD-SNEs with weighting functions $w_T \in W_T^{\xi}$ are consistent and asymptotically normal with variance/covariance matrix*

$$V \equiv \left(1 + \frac{1}{S}\right) \cdot \left\{ \text{var}(\Psi_1) + \sum_{k=1}^{\infty} [\text{cov}(\Psi_1, \Psi_{1+k}) + \text{cov}(\Psi_{1+k}, \Psi_1)] \right\}$$

(provided it exists finitely), where $\Psi_i \equiv \Psi(z_i, v_i)$ and,

$$\Psi(z, v) \equiv \left[\iint \left| \frac{\nabla_{\theta} \pi(u_1|u_2; \theta_0)}{\pi(u_1|u_2; \theta_0)} \right|_2 \xi_1(u_2) \pi_0(u_1, u_2) dz dv \right]^{-1} \frac{\nabla_{\theta} \pi(z|v; \theta_0)}{\pi(z|v; \theta_0)} \xi_1(v). \quad (\text{D4})$$

Theorem 3 is a special case of proposition 2. Precisely, set $\xi_1(\cdot) = \xi_{1T}(\cdot) \equiv 1$ and $(z, v) = (y_2, y_1)$. Function Ψ in (D4) is then,

$$\Psi(y_2, y_1) = \left[\iint \left| \frac{\nabla_{\theta} \pi(z|v; \theta_0)}{\pi(z|v; \theta_0)} \right|_2 \pi(z, v; \theta_0) dz dv \right]^{-1} \frac{\nabla_{\theta} \pi(y_2|y_1; \theta_0)}{\pi(y_2|y_1; \theta_0)}.$$

The efficiency claim now immediately follows by the standard score martingale difference argument.

E. Diffusion models

This appendix contains the proof of theorem D.1. We employ the following pieces of notation. We let, $\pi_{T,h}^i(x; \theta) \equiv (T\lambda_T^q)^{-1} \sum_{t=t_i}^T K((x_{t,h}^i(\theta) - x)/\lambda_T)$, where $x_{t,h}^i(\theta)$ is as in the main text. Accordingly, we set $\tilde{\pi}_{T,h}(x; \theta) \equiv S^{-1} \sum_{i=1}^S \pi_{T,h}^i(x; \theta)$. Finally, we let $x(\theta)$ denote the observable skeleton when the parameter is θ .

E.1 Consistency

We only provide the proof of consistency for the SNE. The proofs of consistency for the CD-SNE follow by a mere change in notation.

Similarly as in appendix B.1, we only need to show that for all $\theta \in \Theta$, $L_{T,h}(\theta) \xrightarrow{P} L(\theta)$ as $T \rightarrow \infty$ and $h \downarrow 0$. Now by arguments similar to ones used in the proof of proposition 1,

$$|L_{T,h}(\theta) - L(\theta)| \leq \sum_{j=1}^2 \int \sigma_{jT,h}(x; \theta) dx$$

where

$$\begin{aligned} \sigma_{1T,h}(x; \theta) &\equiv |\tilde{\pi}_{T,h}(x; \theta) - \pi_T(x)| \cdot |m(x; \theta) - m(x; \theta_0)| \cdot |w_T(x) - w(x)| \\ \sigma_{2T,h}(x; \theta) &\equiv |[\tilde{\pi}_{T,h}(x; \theta) - m(x; \theta)] - [\pi_T(x) - m(x; \theta_0)]| \cdot [\rho_{T,h}(x; \theta) + \rho(x; \theta)] \\ \rho_T(x; \theta) &\equiv |\tilde{\pi}_{T,h}(x; \theta) - \pi_{T,h}(x)| \cdot w_T(x) \quad ; \quad \rho(x; \theta) \equiv |m(x; \theta) - m(x; \theta_0)| \cdot w(x) \end{aligned}$$

By assumption D1-(a), $x_h^i(\theta) \Rightarrow x(\theta)$ as $h \downarrow 0$ (all $\theta \in \Theta$), $i = 1, \dots, S$. By continuity of $\pi_{T,h}^i(x; \theta)$ with respect to the simulated points $\{x_{t,h}^i(\theta)\}_{t=t_i}^T$ and independence of the simulations, $\pi_{T,h}^i(x; \theta) \Rightarrow \pi_T^i(x; \theta) \equiv \sum_{t=t_i}^T K((x_t(\theta) - x)/\lambda_T) / (T\lambda_T^q)$ as $h \downarrow 0$ (all $\theta \in \Theta$), for all $i = 1, \dots, S$. Therefore, for all $x \in X$, $\sigma_{jT,h}(x; \theta) \Rightarrow \sigma_{jT}(x; \theta)$ as $h \downarrow 0$ (all $\theta \in \Theta$), $j = 1, 2$. Consistency then follows by the results proven in appendix B.1 that for all $\theta \in \Theta$, $\int \sigma_{jT}(x; \theta) dx \xrightarrow{P} 0$, $j = 1, 2$.

E.2 Asymptotic normality

We now produce proofs of asymptotic normality for the SNE, and the CD-SNE. (The proof for the CD-SNE with weighting function $w_T(z, v) = [\pi_T(v)/\pi_T(z|v)] \mathbb{T}_{T,\alpha}(z, v)$ follows by a mere change in notation, and the Markov property of a diffusion (see, e.g., Arnold (1992), thm. 9.2.3 p. 146)).

SNE

By the same arguments in appendix B.2, the first order conditions lead to the following counterpart to eq. (B3):

$$\begin{aligned}
& \sqrt{T} \int [\tilde{\pi}_{T,h}(x; \theta_0) - \pi_T(x)] \nabla_{\theta} \tilde{\pi}_{T,h}(x; \theta_0) w_T(x) dx \\
&= \int \sqrt{T} [\tilde{\pi}_{T,h}(x; \theta_0) - E(\tilde{\pi}_{T,h}(x; \theta_0))] \nabla_{\theta} \tilde{\pi}_{T,h}(x; \theta_0) w_T(x) dx \\
&- \int \sqrt{T} [\pi_T(x) - E(\pi_T(x))] \nabla_{\theta} \tilde{\pi}_{T,h}(x; \theta_0) w_T(x) dx \\
&+ \int \sqrt{T} [E(\tilde{\pi}_{T,h}(x; \theta_0)) - E(\pi_T(x))] \nabla_{\theta} \tilde{\pi}_{T,h}(x; \theta_0) w_T(x) dx \\
&\equiv A_{1T} + A_{2T} + A_{3T}.
\end{aligned}$$

(The presence of the additional term A_{3T} arises by imperfectness of simulations.) Under assumption D1-(b) and assumption K (the kernel is four times continuously differentiable), $A_{3T} = O_p(\sqrt{T}h)$ by Kloeden and Platen (1999, thm. 14.5.1 p. 473). By assumption D1-(c), $A_{3T} \xrightarrow{p} \mathbf{0}_{p_{\theta}}$. The terms A_{1T} and A_{2T} behave exactly as the two terms in the r.h.s. of eq. (B3) in appendix B.2.

CD-SNE

The first order conditions are still formally as in appendix C.2, and lead to the following expansion,

$$\begin{aligned}
\mathbf{0}_{p_{\theta}} &= \frac{1}{S} \sum_{i=1}^S \sqrt{T} \iint \left[\frac{\pi_{T,h}^i(z, v; \theta_0)}{\pi_{T,h}^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] \nabla_{\theta} \tilde{\pi}_{T,h}(z|v; \theta_0) w_T(z, v) \mathbb{T}_{T,\delta}^2(v; \theta_0) dz dv \\
&+ B_{T,h} + C_{T,h} \cdot \sqrt{T} (\theta_{T,S,h} - \theta_0),
\end{aligned}$$

for some convex combination $\bar{\theta}$ of θ_0 and $\theta_{T,S,h}$. Here, $\theta_{T,S,h}$ is the CD-SNE, and,

$$B_{T,h} \equiv \sqrt{T} \iint [\tilde{\pi}_{T,h}(z|v; \theta_0) - \pi_T(z|v)]^2 w_T(z, v) \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_{\theta} \mathbb{T}_{T,\delta}(v; \theta_0) dz dv \quad (\text{E1})$$

$$\begin{aligned}
C_{T,h} &\equiv \iint \nabla_{\theta} \left\{ [\tilde{\pi}_{T,h}(z|v; \bar{\theta}) - \pi_T(z|v)] \nabla_{\theta} \tilde{\pi}_{T,h}(z|v; \bar{\theta}) \mathbb{T}_{T,\delta}^2(v; \bar{\theta}) \right\} w_T(z, v) dz dv \quad (\text{E2}) \\
&+ \iint \nabla_{\theta} \left\{ [\tilde{\pi}_{T,h}(z|v; \bar{\theta}) - \pi_T(z|v)]^2 \mathbb{T}_{T,\delta}(v; \bar{\theta}) \nabla_{\theta} \mathbb{T}_{T,\delta}(v; \bar{\theta}) \right\} w_T(z, v) dz dv
\end{aligned}$$

As regards the $B_{T,h}$ term in (E1) we have,

$$\begin{aligned}
B_{T,h} &= \sqrt{T} \iint \frac{1}{S} \sum_{i=1}^S \left[\frac{\pi_{T,h}^i(z, v; \theta_0)}{\pi_{T,h}^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] [\tilde{\pi}_{T,h}(z|v; \theta_0) - \pi_T(z|v)] \\
&\quad \times [w_T(z, v) - w(z, v)] \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_{\theta} \mathbb{T}_{T,\delta}(v; \theta_0) dzdv \\
&+ \sqrt{T} \iint \frac{1}{S} \sum_{i=1}^S \left[\frac{\pi_{T,h}^i(z, v; \theta_0)}{\pi_{T,h}^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] [\tilde{\pi}_{T,h}(z|v; \theta_0) - \pi_T(z|v)] \\
&\quad \times w(z, v) \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_{\theta} \mathbb{T}_{T,\delta}(v; \theta_0) dzdv.
\end{aligned} \tag{E3}$$

Moreover, at all points of continuity,

$$\begin{aligned}
&\frac{\pi_{T,h}^i(z, v; \theta_0)}{\pi_{T,h}^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \\
&= \frac{\pi_{T,h}^i(z, v; \theta_0) - E(\pi_{T,h}^i(z, v; \theta_0))}{\pi_{T,h}^i(v; \theta_0)} - \frac{\pi_T(z, v) - E(\pi_T(z, v))}{\pi_T(v)} + \frac{E(\pi_T(z, v)) [\pi_T(v) - \pi_{T,h}^i(v; \theta_0)]}{\pi_{T,h}^i(v; \theta_0) \cdot \pi_T(v)} \\
&+ \frac{E(\pi_{T,h}^i(z, v; \theta_0)) - E(\pi_T(z, v))}{\pi_{T,h}^i(v; \theta_0)} \\
&\equiv \check{A}_{1T} + \check{A}_{2T} + \check{A}_{3T} + \check{A}_{4T}.
\end{aligned} \tag{E4}$$

(This expression is the counterpart to eq. (C6), and differs formally from (C6) because of the additional term \check{A}_{4T} arising by imperfectness of simulations.) By replacing this expression into (E3),

$$B_{T,h} = B_{T,h}^* + B_{T,h}^{**},$$

where

$$\begin{aligned}
B_{T,h}^* &= \sqrt{T} \iint \frac{1}{S} \sum_{i=1}^S \left[\frac{E(\pi_{T,h}^i(z, v; \theta_0)) - E(\pi_T(z, v))}{\pi_{T,h}^i(v; \theta_0)} \right] [\tilde{\pi}_{T,h}(z|v; \theta_0) - \pi_T(z|v)] \\
&\quad \times [w_T(z, v) - w(z, v)] \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_{\theta} \mathbb{T}_{T,\delta}(v; \theta_0) dzdv \\
&+ \sqrt{T} \iint \frac{1}{S} \sum_{i=1}^S \left[\frac{E(\pi_{T,h}^i(z, v; \theta_0)) - E(\pi_T(z, v))}{\pi_{T,h}^i(v; \theta_0)} \right] [\tilde{\pi}_{T,h}(z|v; \theta_0) - \pi_T(z|v)] \\
&\quad \times w(z, v) \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_{\theta} \mathbb{T}_{T,\delta}(v; \theta_0) dzdv \\
&\equiv B_{1T,h}^* + B_{2T,h}^*,
\end{aligned}$$

and $B_{T,h}^{**}$ is a term such that $B_{T,h}^{**} \xrightarrow{P} \mathbf{0}_{p_{\theta}}$ (by precisely the same arguments utilized to show that

$B_{jT} \xrightarrow{p} \mathbf{0}_{p_\theta}$ ($j = 1, 2$) in appendix C.2.) We claim that $B_{T,h}^* \xrightarrow{p} \mathbf{0}_{p_\theta}$ as well. Indeed,

$$\begin{aligned} |B_{1T,h}^*| &\leq \sqrt{T} \iint \frac{1}{S} \sum_{i=1}^S \frac{|E(\pi_{T,h}^i(z, v; \theta_0)) - E(\pi_T(z, v))| |\tilde{\pi}_{T,h}(z|v; \theta_0) - \pi_T(z|v)|}{\pi_{T,h}^i(v; \theta_0) \pi_0(z, v)} \\ &\quad \times [w_T(z, v) - w(z, v)] \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_0) \pi_0(z, v) dz dv \\ &= O_p(T \cdot h) \times \left[O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q} \delta_T^{-2} \alpha_T^{-1}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-3} \alpha_T^{-1}\right) + O_p\left(\alpha_T^{-1} \delta_T^{-3} \lambda_T^r\right) \right], \end{aligned}$$

where the $O_p(T \cdot h)$ arises by $\sqrt{T} |E(\pi_{T,h}^i(z, v; \theta_0)) - E(\pi_T(z, v))| = O_p(T \cdot h)$ (similarly as in the SNE case). Similarly,

$$\begin{aligned} |B_{2T,h}^*| &\leq \sqrt{T} \iint \frac{1}{S} \sum_{i=1}^S \frac{|E(\pi_{T,h}^i(z, v; \theta_0)) - E(\pi_T(z, v))| |\tilde{\pi}_{T,h}(z|v; \theta_0) - \pi_T(z|v)|}{\pi_{T,h}^i(v; \theta_0)} \\ &\quad \times w(z, v) \mathbb{T}_{T,\delta}(v; \theta_0) \nabla_\theta \mathbb{T}_{T,\delta}(v; \theta_0) dz dv \\ &= O_p(T \cdot h) \times \left[O_p\left(T^{-\frac{1}{2}} \lambda_T^{-q} \delta_T^{-2}\right) + O_p\left(T^{-\frac{1}{2}} \lambda_T^{-(q-q^*)} \delta_T^{-3}\right) + O_p\left(\delta_T^{-3} \lambda_T^r\right) \right] \end{aligned}$$

By the assumption in the theorem, $B_{jT,h}^* \xrightarrow{p} \mathbf{0}_{p_\theta}$ ($j = 1, 2$). Hence $B_{T,h} \xrightarrow{p} \mathbf{0}_{p_\theta}$ in (E1). Finally, the $C_{T,h}$ term in (E2) behaves precisely as the C_T term in appendix C.2 (see eq. (C9)). Therefore, the first order conditions of the CD-SNE are as in eq. (C10):

$$\begin{aligned} \mathbf{0}_{p_\theta} &= \frac{1}{S} \sum_{i=1}^S \sqrt{T} \iint \left[\frac{\pi_{T,h}^i(z, v; \theta_0)}{\pi_{T,h}^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] \nabla_\theta \tilde{\pi}_{T,h}(z|v; \theta_0) w_T(z, v) \mathbb{T}_{T,\delta}^2(v; \theta_0) dz dv + o_p(1) \mathbf{1}_{p_\theta} \\ &\quad + \left[\iint |\nabla_\theta \tilde{\pi}_{T,h}(z|v; \theta_0) \mathbb{T}_{T,\delta}(v; \theta_0)|_2 w_T(z, v) dz dv + o_p(1) \mathbf{1}_{p_\theta \times p_\theta} \right] \cdot \sqrt{T} (\theta_{T,S,h} - \theta_0). \quad (\text{E5}) \end{aligned}$$

To complete the proof we only need to deal with the first term in (E5). (The third in (E5) behaves precisely as the third term in (C10).) By plugging (E4) into the first term of (E5), we obtain,

$$\begin{aligned} &\frac{1}{S} \sum_{i=1}^S \sqrt{T} \iint \left[\frac{\pi_{T,h}^i(z, v; \theta_0)}{\pi_{T,h}^i(v; \theta_0)} - \frac{\pi_T(z, v)}{\pi_T(v)} \right] \nabla_\theta \tilde{\pi}_{T,h}(z|v; \theta_0) w_T(z, v) \mathbb{T}_{T,\delta}^2(v; \theta_0) dz dv \\ &= \frac{1}{S} \sum_{i=1}^S (D_{1T,h}^i + D_{2T,h}^i) - D_{1T,h}^0 + D_{T,h}^* \end{aligned}$$

where the terms $D_{1T,h}^i$ ($i = 0, 1, \dots, S$) and $D_{2T,h}^i$ ($i = 1, \dots, S$) behave precisely as the corre-

sponding terms D_{1T}^i and D_{2T}^i in appendix C.2, and,

$$D_{T,h}^* \equiv \frac{1}{S} \sum_{i=1}^S \sqrt{T} \iint \left[\frac{E(\pi_{T,h}^i(z, v; \theta_0)) - E(\pi_T(z, v))}{\pi_{T,h}^i(v; \theta_0)} \right] \nabla_{\theta} \tilde{\pi}_{T,h}(z|v; \theta_0) w_T(z, v) \mathbb{T}_{T,\delta}^2(v; \theta_0) dz dv.$$

So we are left to show that $D_{T,h}^* \xrightarrow{p} \mathbf{0}_{p_{\theta}}$. But this easily follows by the same arguments and conditions we previously utilized to demonstrate that $B_{T,h} \xrightarrow{p} \mathbf{0}_{p_{\theta}}$ in (E1).

F. Identifiability and bandwidth choice, modulus of continuity issues and Neyman Chi Square measures of distance

F.1 Identifiability and bandwidth choice

Our consistency proofs rely on the *identifiably uniqueness* conditions introduced in assumption 3-(a). This property may break down if the bandwidth λ is larger than the support of data.²⁵ Consider for example the uniform kernel,

$$K(u) = \frac{1}{2} \mathbb{I}_{|u| \leq 1}.$$

We have

$$\begin{aligned} \Delta m(x; \theta) &\equiv m(x; \theta) - m(x; \theta_0) \\ &= \frac{1}{\lambda} \int_{u \in X} K\left(\frac{x-u}{\lambda}\right) [\pi(u; \theta) - \pi(u; \theta_0)] du \\ &= \frac{1}{\lambda} \frac{1}{2} \int_{u \in X} \mathbb{I}_{|x-u| \leq \lambda} [\pi(u; \theta) - \pi(u; \theta_0)] du. \end{aligned}$$

With λ large enough (i.e. $\lambda > \max_{(x_1, x_2) \in X \times X} |x_1 - x_2|$), $\mathbb{I}_{|x-u| \leq \lambda} = 1$ a.e., and

$$\Delta m(x; \theta) = \frac{1}{\lambda} \frac{1}{2} \int_{u \in X} [\pi(u; \theta) - \pi(u; \theta_0)] du = 0.$$

That is, $\Delta m(x; \theta) = 0$ for all $\theta \in \Theta$. Clearly, this situation does not arise (i.e. the $\Delta m(x; \theta)$ surface is *not* flat) if, 1) for all $x \in X$, $|\pi(x; \theta) - \pi(x; \theta_0)| \Rightarrow \theta = \theta_0$; and 2) data have *unbounded* support. In particular, unbounded support would ensure that $\mathbb{I}_{|x_1 - x_2| \leq \lambda} = 0$ for some sets with strictly positive Lebesgue measure.

We now develop one basic example and illustrate how to cope with data having bounded support - while still allowing for nonzero bandwidth. To keep notation simple, set $q = 1$. Assume that data are i.i.d., and are generated by a Beta distribution with parameters α, β , where β_0 is known and equal to 2. That is,

$$\pi(x; \theta) = \frac{\Gamma(\theta + 2)}{\Gamma(\theta)} x^{\theta-1} (1-x), \quad \theta \equiv \alpha.$$

²⁵We are very grateful to an anonymous referee for bringing this point to our attention.

Let $\theta_0 = 2$. For all finite $\theta > 1$,

$$\Delta m(x; \theta) = \frac{\Gamma(\theta + 2)}{\Gamma(\theta)} \int_{\frac{x}{\lambda}}^{\frac{x-1}{\lambda}} K(\ell) (x - \ell\lambda)^{\theta-1} (1 - x + \ell\lambda) du - 6 \int_{\frac{x}{\lambda}}^{\frac{x-1}{\lambda}} K(\ell) (x - \ell\lambda) (1 - x + \ell\lambda) d\ell.$$

We may consider two extreme kernel smoothing procedures at this juncture.

- $K(u) = \frac{1}{2}\mathbb{I}_{|u| \leq 1}$. In this case,

$$m(x; \theta) = \frac{1}{2} \frac{\Gamma(\theta + 2)}{\Gamma(\theta)} \int_{\frac{x}{\lambda}}^{\frac{x-1}{\lambda}} \mathbb{I}_{|\ell| \leq 1} (x - \ell\lambda)^{\theta-1} (1 - x + \ell\lambda) d\ell.$$

Because $\ell \in (\frac{x}{\lambda}, \frac{x-1}{\lambda})$, $|\ell| \leq \frac{1}{\lambda}$. Then let λ be greater than the support of data, viz $\lambda > 1$. It follows that $|\ell| < 1$, and hence $\Delta m(x; \theta) = 0$ for all $\theta \geq 1$. But it is easily seen that if $\lambda < 1$, $\Delta m(x; \theta) = 0 \Rightarrow \theta = \theta_0$.

- $K(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}$. In this case,

$$\begin{aligned} \Delta m(x; \theta) &= \frac{\Gamma(\theta + 2)}{\sqrt{2\pi} \cdot \Gamma(\theta)} \int_{\frac{x}{\lambda}}^{\frac{x-1}{\lambda}} e^{-\frac{1}{2}\ell^2} (x - \ell\lambda)^{\theta-1} (1 - x + \ell\lambda) du \\ &\quad - \frac{6}{\sqrt{2\pi}} \int_{\frac{x}{\lambda}}^{\frac{x-1}{\lambda}} e^{-\frac{1}{2}\ell^2} (x - \ell\lambda) (1 - x + \ell\lambda) d\ell. \end{aligned}$$

For example, let $\lambda = 1$. Figure 1 plots $\Delta m(x; \theta)$ in this specific case. Note that for all x , $\Delta m(x; \theta) = 0$ only with $\theta = \theta_0 = 2$. Identifiability always occurs.²⁶

As these two simple points reveal, identifiability in data with *bounded* support can be dealt with through small bandwidth values (but not necessarily zero bandwidth values) and/or kernels with unbounded support.

²⁶Please also note that $M \equiv \int_0^1 m(x; \theta) dx = \int_0^1 \pi(u; \theta) g(u) du$, where $g(u) \equiv \int_0^1 K(x - u) dx$. Therefore, M doesn't need to equal one for $\lambda = 1$.

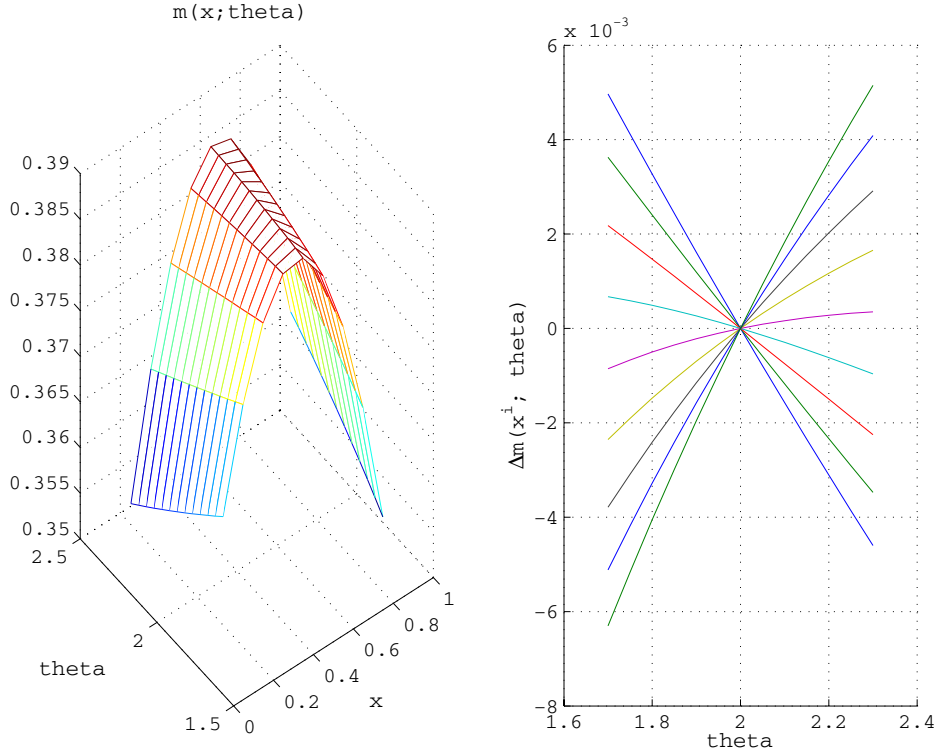


Figure 1 - Identifiability with Beta distributions. The left hand side panel depicts function $m(x; \theta) = \lambda^{-1} \int K\left(\frac{x-u}{\lambda}\right) \pi(u; \theta) du$ evaluated at points $x^i = 0.10, 0.20, \dots, 0.90$ and $\theta^i = 1.70, 1.75, \dots, 2.30$. The right hand side panel depicts the lines $\theta \mapsto \Delta m(x; \theta) = m(x, \theta) - m(x, \theta_0)$ evaluated at the same points x^i, θ^i . In all cases, $\pi(x; \theta) = \frac{\Gamma(\theta+2)}{\Gamma(\theta)} x^{\theta-1} (1-x)$ and $\theta_0 = 2$. In all cases, K is the Gaussian kernel and $\lambda = 1$.

F.2 Modulus of continuity issues

On Assumption 3-(b). An example of conditions under which assumption 3-(b) holds is a global modulus of continuity condition on $\tilde{\pi}_T(x; \cdot)$ similar to the standard one used by Duffie and Singleton (1993, p. 938) in a related problem:

$$\forall x \in X, \quad \forall (\varphi, \theta) \in \Theta \times \Theta, \quad |\tilde{\pi}_T(x; \varphi) - \tilde{\pi}_T(x; \theta)| \leq k_T(x) \cdot \|\varphi - \theta\|_2^\alpha, \quad \alpha > 0, \quad (\text{F1})$$

where $k_T(x)$ is a sequence of functions such that

$$\beta_{pT} \equiv \int k_T(x)^p w_T(x) dx < \infty, \quad \text{all } T \text{ and } p = 1, 2.$$

In turn, condition (F1) holds for $\alpha = 1$ whenever $\nabla_{\theta} \tilde{\pi}_T(x; \theta)$ is continuous and bounded (see, also, related results by Andrews (1992, p. 248-249)). Indeed, suppose that $\tilde{\pi}_T$ has bounded derivative w.r.t θ . By the mean value theorem,

$$\forall x \in X, \quad \forall (\varphi, \theta) \in \Theta \times \Theta, \quad \tilde{\pi}_T(x; \varphi) = \tilde{\pi}_T(x; \theta) + \nabla_{\theta} \tilde{\pi}_T(x; \bar{\theta}) \cdot (\varphi - \theta),$$

for some convex combination $\bar{\theta}$ of φ and θ . Hence,

$$\forall x \in X, \quad \forall (\varphi, \theta) \in \Theta \times \Theta, \quad |\tilde{\pi}_T(x; \varphi) - \tilde{\pi}_T(x; \theta)| \leq M \sum_{i=1}^{p_{\theta}} |\varphi - \theta|_i \leq M \sqrt{p_{\theta}} \|\varphi - \theta\|_2,$$

where $M \equiv \sup_{\theta \in \Theta} \{|\nabla_{\theta} \tilde{\pi}_T(x; \theta)|_i, i = 1, \dots, p_{\theta}, x \in X\}$, the second inequality follows by Cauchy-Schwartz inequality, and $M < \infty$ because Θ is compact. We now prove the claim that (F1) implies assumption 3-(b).

Modulus of continuity claim. (Ineq. (F1) implies assumption 3-(b)) For all $(\varphi, \theta) \in \Theta \times \Theta$,

$$\begin{aligned} L_T(\varphi) - L_T(\theta) &= \int [\tilde{\pi}_T(x; \varphi) - \tilde{\pi}_T(x; \theta)]^2 w_T(x) dx \\ &\quad + 2 \int [\tilde{\pi}_T(x; \varphi) - \tilde{\pi}_T(x; \theta)] [\tilde{\pi}_T(x; \theta) - \pi_T(x)] w_T(x) dx. \end{aligned}$$

Let $B \equiv \max_{(\varphi, \theta) \in \Theta \times \Theta} \|\varphi - \theta\|_2^{\alpha}$. By Θ compact, $B < \infty$. By condition (F1),

$$\begin{aligned} |L_T(\varphi) - L_T(\theta)| &\leq \|\varphi - \theta\|_2^{2\alpha} \cdot \beta_{2T} + 2 \|\varphi - \theta\|_2^{\alpha} \cdot \int k_T(x) |\tilde{\pi}_T(x; \theta) - \pi_T(x)| w_T(x) dx \\ &\leq \|\varphi - \theta\|_2^{2\alpha} \cdot \beta_{2T} + \|\varphi - \theta\|_2^{\alpha} \cdot \xi_T \cdot \beta_{1T} \\ &\leq \|\varphi - \theta\|_2^{\alpha} \cdot (B \cdot \beta_{2T} + \xi_T \cdot \beta_{1T}), \end{aligned}$$

where $\xi_T \equiv 2 \cdot \sup_{x \in X, \theta \in \Theta} |\tilde{\pi}_T(x; \theta) - \pi_T(x)| < \infty$. Since β_{1T} , β_{2T} and ξ_T are bounded in probability as T becomes large, so is $B \cdot \beta_{2T} + \xi_T \cdot \beta_{1T}$. Set then $\kappa_T \equiv B \cdot \beta_{2T} + \xi_T \cdot \beta_{1T}$ to conclude. ■

Modulus of continuity issues. We present one primitive condition ensuring that the modulus of continuity condition in assumption 5 holds true in the context of theorems 2 and 3, namely:

$$\forall (z, v) \in A_T, \quad \forall (\varphi, \theta) \in \Theta \times \Theta, \quad |\tilde{\pi}_T(z|v; \varphi) - \tilde{\pi}_T(z|v; \theta)| w_T(z, v) \leq k_T(z, v) \cdot \|\varphi - \theta\|_2^{\alpha},$$

where k_T is a sequence of functions satisfying:

$$\begin{aligned} M_{1T} &\equiv \iint k_T(z, v) |\tilde{\pi}_T(z|v; \varphi) - \tilde{\pi}_T(z|v; \theta)| \mathbb{I}_{A_T}(z, v) dzdv < \infty \\ M_{2T} &\equiv \iint k_T(z, v) |\tilde{\pi}_T(z|v; \theta) - \pi_T(z|v)| \mathbb{I}_{A_T}(z, v) dzdv < \infty \end{aligned}$$

Indeed, for all $(\varphi, \theta) \in \Theta \times \Theta$,

$$\begin{aligned} L_T(\varphi) - L_T(\theta) &= \iint [\tilde{\pi}_T(z|v; \varphi) - \tilde{\pi}_T(z|v; \theta)]^2 \mathbb{I}_{A_T}(z, v) w_T(z, v) dzdv \\ &\quad + 2 \iint [\tilde{\pi}_T(z|v; \varphi) - \tilde{\pi}_T(z|v; \theta)] [\tilde{\pi}_T(z|v; \theta) - \pi_T(z|v)] \mathbb{I}_{A_T}(z, v) w_T(z, v) dzdv \\ &\leq (M_{1T} + 2M_{2T}) \cdot \|\varphi - \theta\|_2^\alpha. \end{aligned}$$

Similarly as for theorem 1, the previous modulus of continuity condition holds whenever $\tilde{\pi}_T(z|v; \cdot)$ has bounded derivative on Θ for all $(z, v) \in A_T$.

F.3 On Neyman Chi Square measures of distance

We provide a simple example of parameter restrictions ensuring the existence of Neyman Chi-Square distances in a dynamic context. We consider a stationary Gaussian AR(1) model,

$$y_t = b_0 y_{t-1} + \epsilon_t, \quad \epsilon_t \sim \text{NID}(0, \sigma_0^2), \quad b_0 \in (-1, 1), \quad \sigma_0 > 0, \quad t = -\infty, \dots \quad (\text{F2})$$

Our Neyman Chi-Square measure of distance is,

$$\text{NCS}(\theta) = \iint \left[\frac{\pi(z|v; \theta) - \pi(z|v; \theta_0)}{\pi(z|v; \theta_0)} \right]^2 \pi(z, v; \theta_0) dzdv, \quad z \equiv y_t, v \equiv y_{t-1}, \theta = (b, \sigma).$$

We wish to find parameter restrictions such that function $f(z, v; \theta)$ defined as,

$$f(z, v; \theta) \equiv \frac{\pi(z|v; \theta) - \pi(z|v; \theta_0)}{\pi(z|v; \theta_0)} \sqrt{\pi(z, v; \theta_0)},$$

is bounded and integrable at all boundaries.

In the context of model (F2),

$$\begin{aligned} \pi(z|v; \theta) &= \frac{1}{\sqrt{2\pi\sigma}} \exp \left[-\frac{1}{2} \frac{(z - bv)^2}{\sigma^2} \right] \\ \pi(z, v; \theta) &= \frac{\sqrt{1 - b^2}}{2\pi\sigma^2} \exp \left(-\frac{z^2 - 2bvz + v^2}{2\sigma^2} \right) \end{aligned}$$

and,

$$f(z, v; \theta) \leq \frac{(1 - b_0^2)^{\frac{1}{4}}}{\sqrt{2\pi}\sigma_0} [f_1(z, v; \theta) + f_2(z, v; \theta)],$$

where

$$\begin{aligned} f_1(z, v; \theta) &\equiv \frac{\sigma_0}{\sigma} \exp\left(-\frac{c_1 z^2 + c_2 v z + c_3 v^2}{4\sigma^2 \sigma_0^2}\right) \\ c_1 &\equiv 2\sigma_0^2 - \sigma^2 \\ c_2 &\equiv 2\sigma^2 b_0 - 4\sigma_0^2 b \\ c_3 &\equiv 2\sigma_0^2 b^2 + (1 - 2b_0^2) \sigma^2 \\ f_2(z, v; \theta) &\equiv \exp\left(-\frac{z^2 - 2b_0 v z + v^2}{4\sigma_0^2}\right). \end{aligned}$$

Clearly, f_2 is bounded and integrable, and so is f_1 whenever $c_1 > 0$ and $c_2 > 0$, i.e. whenever,

$$\begin{cases} \sigma < \sqrt{2}\sigma_0 \\ 0 < 2\sigma_0^2 b^2 + (1 - 2b_0^2) \sigma^2 \end{cases} \quad (\text{F3})$$

If $b_0^2 \leq \frac{1}{2}$, the system of inequalities (F3) holds whenever $\sigma < \sqrt{2}\sigma_0$ and $b \in (-1, 1)$. If $b_0^2 \in (\frac{1}{2}, 1)$ and $b_0 > 0$, the corresponding admissible region is the shaded area in Figure 2 (the case $b_0 < 0$ is similar). Please notice that in this case, θ_0 can only lie below the straight line,

$$\sigma = \frac{\sqrt{2}\sigma_0}{\sqrt{-(1 - 2b_0^2)}} b.$$

For suppose not. Then $\sigma_0 > [-(1 - 2b_0^2)]^{-\frac{1}{2}} \sqrt{2}\sigma_0 b_0 \Leftrightarrow -1 + 2b_0^2 > 2b_0^2$, which is an absurdity.

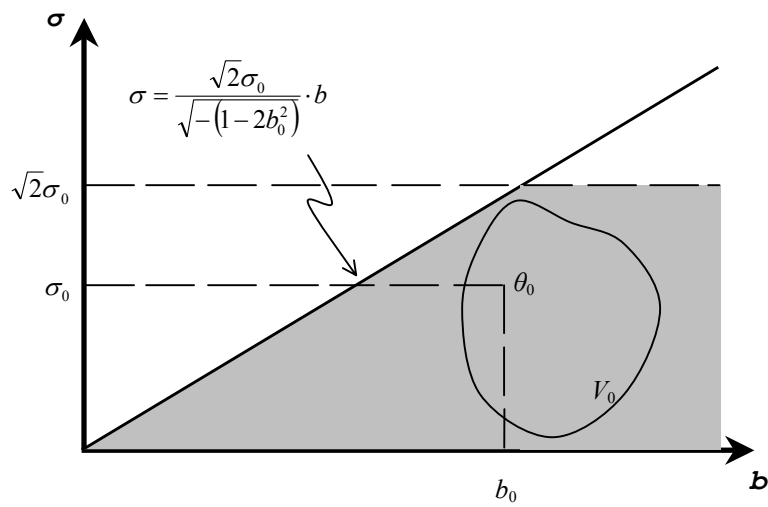


Figure 2 - Parameter restrictions ensuring existence of Neyman distance in the Gaussian AR(1) model (F2). In the $b_0 > 0$ case, Neyman distance $NCS(\theta)$ exists for all θ in the shaded area. The case $b_0 < 0$ is symmetric.

G. Asset pricing, prediction functions and statistical efficiency

In this appendix, we demonstrate that our CD-SNE is still first-order efficient as soon as an unobservable multidimensional process is estimated in conjunction with predictions functions suggested by standard asset pricing theories. As in the main text, we assume that the data generating process is a multidimensional partially observed diffusion process solution to,

$$dy(\tau) = b(y(\tau); \theta) d\tau + a(y(\tau); \theta) dW(\tau) \quad (\text{G1})$$

This appendix analyzes situations in which the original partially observed system (G1) can be estimated by augmenting it with a number of observable deterministic functions of the state. In many situations of interest, such deterministic functions are suggested by asset pricing theories in a natural way. Typical examples include derivative asset price functions or any deterministic function(s) of asset prices (e.g., asset returns, bond yields, implied volatility, etc.). The idea to use predictions of asset pricing theories to improve the fit of models with unobservable factors is not new (see, e.g., Christensen (1992), Pastorello, Renault and Touzi (2000), Chernov and Ghysels (2000), Singleton (2001, sections 3.2 and 3.3)), and Pastorello, Patilea and Renault (2003). In this appendix, we provide a theoretical description of the mechanism leading to efficiency within the class of our estimators.

We consider a standard Markov pricing setting. For fixed $t \geq 0$, we let M be the expiration date of a contingent claim with rational price process $c = \{c(y(\tau), M - \tau)\}_{\tau \in [t, M]}$, and let $\{\zeta(y(\tau))\}_{\tau \in [t, M]}$ and $\Pi(y)$ be the associated intermediate payoff process and final payoff function, respectively. Let $\partial/\partial\tau + L$ be the usual infinitesimal generator of (G1) taken under the risk-neutral measure. In a frictionless economy without arbitrage opportunities, c is the solution to the following partial differential equation:

$$\begin{cases} 0 = \left(\frac{\partial}{\partial\tau} + L - R \right) c(y, M - \tau) + \zeta(y), \quad \forall (y, \tau) \in Y \times [t, M] \\ c(y, 0) = \Pi(y), \quad \forall y \in Y \end{cases} \quad (\text{G2})$$

where $R \equiv R(y)$ is the short-term rate. We call *prediction function* any continuous and twice differentiable function $c(y; M - \tau)$ solution to the partial differential equation (G2).

We now augment system (G1) with $d - q^*$ prediction functions. Precisely, we let:

$$C(\tau) \equiv (c(y(\tau), M_1 - \tau), \dots, c(y(\tau), M_{d-q^*} - \tau)), \quad \tau \in [t, M_1]$$

where $\{M_i\}_{i=1}^{d-q^*}$ is an increasing sequence of fixed maturity dates. Furthermore, we define the

measurable vector valued function:

$$\phi(y(\tau); \theta, \gamma) \equiv (y^o(\tau), C(y(\tau))), \quad \tau \in [t, M_1], \quad (\theta, \gamma) \in \Theta \times \Gamma,$$

where $\Gamma \subset \mathbb{R}^{p_\gamma}$ is a compact parameter set containing additional parameters. These new parameters arise from the change of measure leading to the pricing model (G2), and are now part of our estimation problem.

We assume that the pricing model (G2) is correctly specified. That is, all contingent claim prices in the economy are taken to be generated by the prediction function $c(y, M - \tau)$ for some $(\theta_0, \gamma_0) \in \Theta \times \Gamma$. For simplicity, we also consider a stylized situation in which all contingent claims have the same contractual characteristics specified by $\mathcal{C} \equiv (\zeta, \Pi)$. More generally, one may define a series of classes of contingent claims $\{\mathcal{C}_j\}_{j=1}^J$, where class of contingent claims j has contractual characteristics specified by $\mathcal{C}_j \equiv (\zeta_j, \Pi_j)$.²⁷ The number of prediction functions that we would introduce in this case would be equal to $d - q^* = \sum_{j=1}^J M^j$, where M^j is the number of prediction functions within class of assets j . To keep the presentation simple, we do not consider such a more general situation here.

Our objective is to provide estimators of the parameter vector (θ_0, γ_0) under which observations were generated. In exactly the same spirit as for the estimators considered in the main text, we want our CD-SN estimator of (θ_0, γ_0) to make the finite dimensional distributions of ϕ implied by model (G1) and (G2) as close as possible to their sample counterparts. Let $\Phi \subseteq \mathbb{R}^d$ be the domain on which ϕ takes values. As illustrated in Figure 3, our program is to move from the “unfeasible” domain Y of the original state variables in y (observables and not) to the domain Φ on which all observable variables take value. Ideally, we would like to implement such a change in domain in order to recover as much information as possible on the original unobserved process in (G1). Clearly, ϕ is fully revealing whenever it is globally invertible. However, we will show that our methods can be implemented even when ϕ is only locally one-to-one. Further intuition on this distinction will be provided after the statement of theorem G.1 below.

An important feature of the theory in this appendix is that it does not hinge upon the availability of contingent prices data covering the same sample period covered by the observables in (G1). First, the price of a given contingent claim is typically not available for a long sample period. As an example, available option data often include option prices with a life span smaller than the usual sample span of the underlying asset prices; in contrast, it is common to observe long time series of option prices having the same maturity. Second, the price of a single contingent claim depends on time-to-maturity of the claim; therefore, it does not satisfy the stationarity assumptions maintained in this paper. To address these issues, we deal with data on assets having the same characteristics at each point in time. Precisely, consider the data generated by

²⁷As an example, assets belonging to class \mathcal{C}_1 can be European options; assets belonging to class \mathcal{C}_1 can be bonds; and so on.

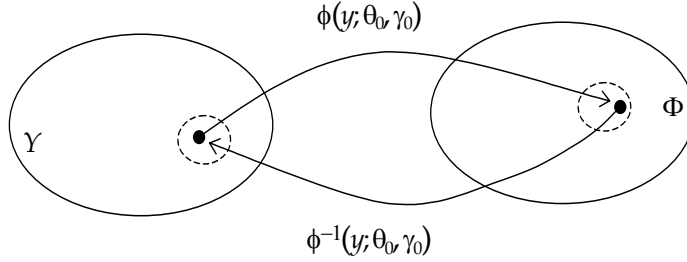


Figure 3 - Asset pricing, the Markov property, and statistical efficiency. Y is the domain on which the partially observed primitive state process $y \equiv (y^o \ y^u)^\top$ takes values, Φ is the domain on which the observed system $\phi \equiv (y^o \ C(y))^\top$ takes values in Markovian economies, and $C(y)$ is a contingent claim price process in \mathbb{R}^{d-q^*} . Let $\phi^c = (y^o, c(y, \ell_1), \dots, c(y, \ell_{d-q^*}))$, where $\{c(y, \ell_j)\}_{j=1}^{d-q^*}$ forms an intertemporal cohort of contingent claim prices, as in definition G.1. If *local restrictions* of ϕ are one-to-one and onto, the CD-SNE applied to ϕ^c is feasible. If ϕ is also *globally* invertible, the CD-SNE applied to ϕ^c achieves first-order asymptotic efficiency.

the following random processes:

Definition G.1. (Intertemporal (ℓ, N) -cohort of contingent claim prices) *Given a prediction function $c(y; M - \tau)$ and a N -dimensional vector $\ell \equiv (\ell_1, \dots, \ell_N)$ of fixed maturities, an intertemporal (ℓ, N) -cohort of contingent claim prices is any collection of contingent claim price processes $c(\tau, \ell) \equiv (c(y(\tau), \ell_1), \dots, c(y(\tau), \ell_N))$ ($\tau \geq 0$) generated by the pricing model (G2).*

Consider for example a sample realization of three-months at-the-money option prices, or a sample realization of six-months zero-coupon bond prices. Long sequences such as the ones in these examples are common to observe. If these sequences were generated by (G2), as in definition G.1, they would be deterministic functions of y , and hence stationary. We now develop conditions ensuring both feasibility and first-order efficiency of the CD-SNE procedure as applied to this kind of data. Let \bar{a} denote the matrix having the first q^* rows of a , where a is the diffusion matrix in (G1). Let ∇C denote the Jacobian of C with respect to y . We have:

Theorem G.1. (Asset pricing and Cramer-Rao lower bound) *Suppose to observe an intertemporal $(\ell, d - q^*)$ -cohort of contingent claim prices $c(\tau, \ell)$, and that there exist prediction functions C in \mathbb{R}^{d-q^*} with the property that for $\theta = \theta_0$ and $\gamma = \gamma_0$,*

$$\begin{pmatrix} \bar{a}(\tau) \cdot a(\tau)^{-1} \\ \nabla C(\tau) \end{pmatrix} \neq 0, \quad P \otimes d\tau\text{-a.s.} \quad \text{all } \tau \in [t, t+1], \quad (\text{G3})$$

where C satisfies the initial condition $C(t) = c(t, \ell) \equiv (c(y(t), \ell_1), \dots, c(y(t), \ell_{d-q^*}))$. Let $(z, v) \equiv (\phi_t^c, \phi_{t-1}^c)$, where $\phi_t^c = (y^o(t), c(y(t), \ell_1), \dots, c(y(t), \ell_{d-q^*}))$. Then, under the assumptions in theorem 3, the CD-SNE has the same properties as in theorem 2, with the variance terms being taken with respect to the fields generated by ϕ_t^c . Finally, suppose that ϕ_t^c is Markov, and set $w_T(z, v) = \pi_T(z)^2 / \pi_T(z, v) \mathbb{T}_{T, \alpha}(z, v)$. Then, the CD-SNE attains the Cramer-Rao lower bound (with respect to the fields generated by ϕ_t^c) as $S \rightarrow \infty$.

Proof. Let $\pi_t \equiv \pi_t(\phi(y(t+1), \mathbf{M} - (t+1)\mathbf{1}_{d-q^*}) | \phi(y(t), \mathbf{M} - t\mathbf{1}_{d-q^*}))$ denote the transition density of

$$\phi(y(t), \mathbf{M} - t\mathbf{1}_{d-q^*}) \equiv \phi(y(t)) \equiv (y^o(t), c(y(t), M_1 - t), \dots, c(y(t), M_{d-q^*} - t)),$$

where we have emphasized the dependence of ϕ on the time-to-expiration vector:

$$\mathbf{M} - t\mathbf{1}_{d-q^*} \equiv (M_1 - t, \dots, M_{d-q^*} - t).$$

By $a(\tau)$ full rank $P \otimes d\tau$ -a.s., and Itô's lemma, ϕ satisfies, for $\tau \in [t, t+1]$,

$$\begin{cases} dy^o(\tau) &= b^o(\tau)d\tau + F(\tau)a(\tau)dW(\tau) \\ dc(\tau) &= b^c(\tau)d\tau + \nabla c(\tau)a(\tau)dW(\tau) \end{cases}$$

where b^o and b^c are, respectively, q^* -dimensional and $(d - q^*)$ -dimensional measurable functions, and $F(\tau) \equiv \bar{a}(\tau) \cdot a(\tau)^{-1}$ $P \otimes d\tau$ -a.s. Under condition (G3), π_t is not degenerate. Furthermore, $C(y(t); \ell) \equiv C(t)$ is deterministic in $\ell \equiv (\ell_1, \dots, \ell_{d-q^*})$. That is, for all $(\bar{c}, \bar{c}^+) \in \mathbb{R}^d \times \mathbb{R}^d$, there exists a function μ such that for any neighbourhood $N(\bar{c}^+)$ of \bar{c}^+ , there exists another

neighborhood $N(\mu(\bar{c}^+))$ of $\mu(\bar{c}^+)$ such that,

$$\begin{aligned}
& \{\omega \in \Omega : \phi(y(t+1), \mathbf{M} - (t+1)\mathbf{1}_{d-q^*}) \in N(\bar{c}^+) \mid \phi(y(t), \mathbf{M} - t\mathbf{1}_{d-q^*}) = \bar{c}\} \\
&= \{\omega \in \Omega : (y^o(t+1), c(y(t+1), M_1 - t), \dots, c(y(t+1), M_{d-q^*} - t)) \in N(\mu(\bar{c}^+)) \\
&\quad \mid \phi(y(t), \mathbf{M} - t\mathbf{1}_{d-q^*}) = \bar{c}\} \\
&= \{\omega \in \Omega : (y^o(t+1), c(y(t+1), M_1 - t), \dots, c(y(t+1), M_{d-q^*} - t)) \in N(\mu(\bar{c}^+)) \\
&\quad \mid (y^o(t), c(y(t), M_1 - t), \dots, c(y(t), M_{d-q^*} - t)) = \bar{c}\}
\end{aligned}$$

where the last equality follows by the definition of ϕ . In particular, the transition laws of ϕ_t^c given ϕ_{t-1}^c are not degenerate; and ϕ_t^c is stationary. The feasibility of the CD-SNE is proved. The efficiency claim follows by the Markov property of ϕ , and the usual score martingale difference argument. ■

According to theorem G.1, our CD-SNE is feasible whenever ϕ is locally invertible for a time span equal to the sampling interval. As Figure 3 illustrates, condition (G3) is satisfied whenever ϕ is locally one-to-one and onto.²⁸ If ϕ is also globally invertible for the same time span, ϕ^c is Markov. The last part of this theorem then says that in this case, the CD-SNE is asymptotically efficient. We emphasize that such an efficiency result is simply about first-order efficiency in the *joint* estimation of θ and γ given the observations on ϕ^c . We are not claiming that our estimator is first-order efficient in the estimation of θ in the case in which y is fully observable.

Naturally, condition (G3) does not ensure that ϕ is globally one-to-one and onto. In other terms, ϕ might have many locally invertible restrictions.²⁹ In practice, ϕ might fail to be globally invertible because monotonicity properties of ϕ may break down in multidimensional diffusion models. In models with stochastic volatility, for example, option prices can be decreasing in the underlying asset price (see Bergman, Grundy and Wiener (1996)); and in the corresponding stochastic volatility yield curve models, medium-long term bond prices can be increasing in the short-term rate (see Mele (2003)). Intuitively, these pathologies may arise because there is no guarantee that the solution to a stochastic differential system is nondecreasing in the initial condition of one if its components - as it is instead the case in the scalar case.

When all components of vector y^o represent the prices of assets actively traded in frictionless markets, (G3) corresponds to a condition ensuring market completeness in the sense of Harrison

²⁸Local invertibility of ϕ means that for every $y \in Y$, there exists an open set Y_* containing y such that the restriction of ϕ to Y_* is invertible. And ϕ is locally invertible on Y_* if $\det J\phi \neq 0$ (where $J\phi$ is the Jacobian of ϕ), which is condition (G3).

²⁹As an example, consider the mapping $\mathbb{R}^2 \mapsto \mathbb{R}^2$ defined as $\phi(y_1, y_2) = (e^{y_1} \cos y_2, e^{y_1} \sin y_2)$. The Jacobian satisfies $\det J\phi(y_1, y_2) = e^{2y_1}$, yet ϕ is 2π -periodic with respect to y_2 . For example, $\phi(0, 2\pi) = \phi(0, 0)$.

and Pliska (1983). As an example, condition (G3) for Heston's (1993) model is $\partial c / \partial \sigma \neq 0$ $P \otimes dt$ -a.s., where σ denotes instantaneous volatility of the price process. This condition is satisfied by the Heston's model. In fact, Romano and Touzi (1997) showed that within a fairly general class of stochastic volatility models, option prices are *always* strictly increasing in σ whenever they are convex in Q . Theorem G.1 can be used to implement efficient estimators in other complex multidimensional models. Consider for example a three-factor model of the yield curve. Consider a state-vector (r, σ, ℓ) , where r is the short-term rate and σ, ℓ are additional factors (such as, say, instantaneous short-term rate volatility and a central tendency factor). Let $u^{(i)} = u(r(\tau), \sigma(\tau), \ell(\tau); M_i - \tau)$ be the time τ rational price of a pure discount bond expiring at $M_i \geq \tau$, $i = 1, 2$, and take $M_1 < M_2$. Let $\phi \equiv (r, u^{(1)}, u^{(2)})$. Condition (G3) for this model is then,

$$u_{\sigma}^{(1)} u_{\ell}^{(2)} - u_{\ell}^{(1)} u_{\sigma}^{(2)} \neq 0, \quad P \otimes dt\text{-a.s.} \quad \tau \in [t, t + 1], \quad (\text{G4})$$

where subscripts denote partial derivatives. It is easily checked that this same condition must be satisfied by models with correlated Brownian motions and by yet more general models. Classes of models of the short-term rate for which condition (G4) holds are more intricate to identify than in the European option pricing literature mentioned above (see Mele (2003)).

References

- Aït-Sahalia, Y., 1994, "The Delta Method for Nonparametric Kernel Functionals," working paper, Princeton University.
- Aït-Sahalia, Y., 1996, "Testing Continuous-Time Models of the Spot Interest Rate," *Review of Financial Studies*, 9, 385-426.
- Aït-Sahalia, Y., 2002, "Maximum Likelihood Estimation of Discretely Sampled Diffusions: a Closed-Form Approximation Approach," *Econometrica*, 70, 223-262.
- Aït-Sahalia, Y., 2003, "Closed-Form Likelihood Expansions for Multivariate Diffusions," working paper, Princeton University.
- Altissimo, F. and A. Mele, 2005, "Simulated Nonparametric Estimation of Dynamic Models with Applications to Finance (Unabridged version)," forthcoming as Financial Markets Group working paper, London School of Economics.
- Andrews, D.W.K., 1992, "Generic Uniform Convergence," *Econometric Theory*, 8, 241-257.
- Andrews, D.W.K., 1995, "Nonparametric Kernel Estimation for Semiparametric Models," *Econometric Theory*, 11, 560-596.
- Arcones, M.A. and B. Yu, 1994, "Central Limit Theorems for Empirical and U-Processes of Stationary Mixing Sequences," *Journal of Theoretical Probability*, 7, 47-71.
- Arnold, L., 1992, *Stochastic Differential Equations: Theory and Applications*. Malabar, Florida: Krieger Publishing Company.
- Bergman, Y. Z., B. D. Grundy, and Z. Wiener, 1996, "General Properties of Option Prices," *Journal of Finance*, 51, 1573-1610.
- Billio, M. and A. Monfort, 2003, "Kernel-Based Indirect Inference," *Journal of Financial Econometrics*, 1, 297-326.
- Basu, A. and B. G. Lindsay, 1994, "Minimum Disparity Estimation for Continuous Models: Efficiency, Distributions and Robustness," *Annals of the Institute of Statistical Mathematics*, 46, 683-705.
- Bickel, P.J., 1982, "On Adaptive Estimation," *Annals of Statistics*, 70, 647-671.
- Bickel, P.J. and M. Rosenblatt, 1973, "On Some Global Measures of the Deviations of Density Function Estimates," *Annals of Statistics*, 1, 1071-1095.
- Carrasco, M., L. P. Hansen and X. Chen, 1999, "Time Deformation and Dependence," working paper, University of Rochester.
- Carrasco, M., M. Chernov, J.-P. Florens and E. Ghysels, 2004, "Efficient Estimation of Jump-Diffusions and General Dynamic Models with a Continuum of Moment Conditions," working paper, University of Rochester.
- Chen, X., L. P. Hansen and M. Carrasco, 1999, "Nonlinearity and Temporal Dependence," working paper, University of Rochester.

- Chen, X, O. Linton, and P. M. Robinson, 2001, "The Estimation of Conditional Densities," *Journal of Statistical Planning and Inference*, 71-84.
- Chernov, M. and E. Ghysels, 2000, "A Study towards a Unified Approach to the Joint Estimation of Objective and Risk-Neutral Measures for the Purpose of Options Valuation," *Journal of Financial Economics*, 56, 407-458.
- Christensen, B. J., 1992, "Asset Prices and the Empirical Martingale Model," working paper, New York University.
- Corradi, V. and N. R. Swanson, 2005, "Bootstrap Specification Tests for Diffusion Processes," *Journal of Econometrics*, 124, 117-148.
- Duffie, D. and K.J. Singleton, 1993, "Simulated Moments Estimation of Markov Models of Asset Prices," *Econometrica*, 61, 929-952.
- Elerian, O., S. Chib and N. Shephard, 2001, "Likelihood Inference for Discretely Observed Nonlinear Diffusions," *Econometrica*, 69, 959-993.
- Fan, Y., 1994, "Testing the Goodness-of-Fit of a Parametric Density Function by Kernel Method," *Econometric Theory*, 10, 316-356.
- Fermanian, J.-D. and B. Salanié, 2004, "A Nonparametric Simulated Maximum Likelihood Estimation Method," *Econometric Theory*, 20, 701-734.
- Gallant, A. R., 2001, "Effective Calibration," working paper, University of North Carolina.
- Gallant, A. R. and H. White, 1988, *A Unified Theory of Estimation and Inference for Nonlinear Dynamic Models*, Oxford, Basil Blackwell.
- Gallant, A. R. and G. Tauchen, 1996, "Which Moments to Match ?," *Econometric Theory*, 12, 657-681.
- Gouriéroux, C., A. Monfort and E. Renault, 1993, "Indirect Inference," *Journal of Applied Econometrics*, 8, S85-S118.
- Hajivassiliou, V. and D. McFadden, 1998, "The Method of Simulated Scores for the Estimation of Limited-Dependent Variable Models," *Econometrica*, 66, 863-896.
- Hansen, L. and J. A. Scheinkman, 1995, "Back to the Future: Generating Moment Implications for Continuous-Time Markov Processes," *Econometrica*, 63, 767-804.
- Harrison, J. M. and S. R. Pliska, 1983, "A Stochastic Calculus Model of Continuous Trading: Complete Markets," *Stochastic Processes and their Applications* 15, 313-316.
- Härdle, W. and E. Mammen, 1993, "Comparing Nonparametric versus Parametric Regression Fits," *Annals of Statistics*, 21, 1926-1947.
- Heston, S., 1993, "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies*, 6, 327-343.
- Karatzas, I. and S. Shreve, 1991, *Brownian Motion and Stochastic Calculus*, Berlin: Springer Verlag.

- Kloeden, P.E. and E. Platen, 1999, *Numerical Solutions of Stochastic Differential Equations*, Berlin: Springer Verlag.
- Laroque, G. and B. Salanié, 1989, "Estimation of Multimarket Fix-Price Models: An Application of Pseudo-Maximum Likelihood Methods," *Econometrica*, 57, 831-860.
- Laroque, G. and B. Salanié, 1993, "Simulation-Based Estimation of Models with Lagged Latent Variables," *Journal of Applied Econometrics*, 8, S119-S133.
- Laroque, G. and B. Salanié, 1994, "Estimating the Canonical Disequilibrium Model: Asymptotic Theory and Finite Sample Properties," *Journal of Econometrics*, 62, 165-210.
- Lee, B-S. and B. F. Ingram, 1991, "Simulation Estimation of Time-Series Models," *Journal of Econometrics*, 47, 197-207.
- Lee, L. F., 1995, "Asymptotic Bias in Simulated Maximum Likelihood Estimation of Discrete Choice Models," *Econometric Theory*, 11, 437-483.
- Lindsay, B. G., 1994, "Efficiency versus Robustness: The Case for Minimum Hellinger Distance and Related Methods," *Annals of Statistics*, 22, 1081-1114.
- Linton, O. and Z. Xiao, 2000, "Second Order Approximation for Adaptive Regression Estimators," working paper, London School of Economics.
- McFadden, D., 1989, "A Method of Simulated Moments for Estimation of Discrete Response Models without Numerical Integration," *Econometrica*, 57, 995-1026.
- Mele, A., 2003, "Fundamental Properties of Bond Prices in Models of the Short-Term Rate," *Review of Financial Studies*, 16, 679-716.
- Menzly L., T. Santos and P. Veronesi, 2004, "Understanding Predictability," *Journal of Political Economy*, 112, 1-47.
- Meyn, S.P. and R. L. Tweedie, 1993, "Stability of Markovian Processes III: Foster-Lyapunov Criteria for Continuous-Time Processes," *Advances in Applied Probability*, 25, 518-548.
- Newey, W. K., 1991, "Uniform Convergence in Probability and Stochastic Equicontinuity," *Econometrica*, 59, 1161-1167.
- Newey, W. K. and D. L. McFadden, 1994, "Large Sample Estimation and Hypothesis Testing," in Engle, R.F. and D. L. McFadden (eds.), *Handbook of Econometrics*, vol. 4, chapter 36, 2111-2245. Amsterdam: Elsevier.
- Pagan, A. and A. Ullah, 1999, *Nonparametric Econometrics*, Cambridge: Cambridge University Press.
- Pakes, A. and D. Pollard, 1989, "Simulation and the Asymptotics of Optimization Estimators," *Econometrica*, 57, 1027-1057.
- Pastorello, S., E. Renault and N. Touzi, 2000, "Statistical Inference for Random-Variance Option Pricing," *Journal of Business and Economic Statistics* 18, 358-367.
- Pastorello, S., V. Patilea, and E. Renault, 2003, "Iterative and Recursive Estimation in Structural Non Adaptive Models," *Journal of Business and Economic Statistics*, 21, 449-509.

- Pedersen, A.R., 1995, "A New Approach to Maximum Likelihood Estimation for Stochastic Differential Equations Based on Discrete Observations," *Scandinavian Journal of Statistics*, 22, 55-71.
- Rao, C.R., 1962, "Efficient Estimates and Optimum Inference Procedures in Large Samples," *Journal of The Royal Statistical Society, Series B*, 24, 46-63.
- Romano, M. and N. Touzi, 1997, "Contingent Claims and Market Completeness in a Stochastic Volatility Model," *Mathematical Finance*, 7, 399-412.
- Santa-Clara, P. 1995, "Simulated Likelihood Estimation of Diffusions With an Application to the Short Term Interest Rate," Ph.D. dissertation, INSEAD.
- Singleton, K.J., 2001, "Estimation of Affine Asset Pricing Models Using the Empirical Characteristic Function," *Journal of Econometrics*, 102, 111-141.
- Smith, A., 1993, "Estimating Nonlinear Time Series Models Using Simulated Vector Autoregressions," *Journal of Applied Econometrics*, 8, S63-S84.
- Stone, C., 1975, "Adaptive Maximum Likelihood Estimation of a Location Parameter," *Annals of Statistics*, 3, 267-284.
- Tjøstheim D., 1990, "Non-Linear Time Series and Markov Chains," *Advances in Applied Probability*, 22, 587-611.
- White, H., 1994, *Estimation, Inference and Specification Analysis*, Cambridge: Cambridge University Press.

Tables 2 through 5

Table 2 - Monte Carlo experiments. (Vasicek model (13).) True parameter values are: $b_1 = 3.00$, $b_2 = 0.50$ and $a_1 = 3.00$.

Sample	Estimators		b_1	b_2	a_1
T=1000	CD-SNE	Mean	2.87	0.49	3.08
		Median	2.89	0.47	3.10
		Sample std. dev.	0.97	0.17	0.29
		Asymptotic std. dev.	1.10	0.19	0.23
		Coverage rate 90% conf. interval	0.95	0.92	0.82
	CD-SNE - Double bandwidth	Mean	2.65	0.43	3.23
		Median	2.56	0.44	3.16
		Sample std. dev.	0.84	0.17	0.28
	CD-SNE - Half bandwidth	Mean	2.98	0.54	2.97
		Median	2.93	0.56	3.04
		Sample std. dev.	1.06	0.23	0.40
	SNE	Mean	3.20	0.55	2.89
		Median	3.07	0.52	2.76
		Sample std. dev.	1.11	0.25	0.41
		Asymptotic std. dev.	1.24	0.22	0.31
		Coverage rate 90% conf. interval	0.95	0.85	0.81
	Analytical-NE	Mean	3.47	0.57	3.55
		Median	3.20	0.47	3.46
Sample std. dev.		2.09	0.64	0.62	
MLE	Mean	3.74	0.62	3.01	
	Median	3.93	0.63	2.99	
	Sample std. dev.	1.21	0.20	0.07	
T=500	CD-SNE	Mean	2.95	0.48	3.14
		Median	2.95	0.48	3.12
		Sample std. dev.	1.03	0.24	0.42
		Asymptotic std. dev.	1.36	0.26	0.32
		Coverage rate 90% conf. interval	0.94	0.94	0.83
	SNE	Mean	3.06	0.58	2.58
		Median	3.03	0.51	2.51
		Sample std. dev.	1.41	0.35	0.71
		Asymptotic std. dev.	1.65	0.31	0.57
		Coverage rate 90% conf. interval	0.97	0.84	0.76
	MLE	Mean	3.99	0.70	2.99
		Median	4.01	0.69	3.00
		Sample std. dev.	1.36	0.27	0.10

Table 3 - Monte Carlo experiments. (Continuous-time stochastic volatility model (15).) True parameter values are: $b_1 = 3.00$, $b_2 = 0.50$, $a_1 = 3.00$, $b_3 = 1.00$ and $a_2 = 0.30$.

Sample	Estimator		b_1	b_2	a_1	b_3	a_2
T=1000	CD-SNE	Mean	3.03	0.48	3.05	1.11	0.34
		Median	3.07	0.49	3.04	0.98	0.32
		Sample std. dev.	0.93	0.22	0.40	0.59	0.20
		Asymptotic std. dev.	1.17	0.22	0.32	0.45	0.16
		Coverage rate 90% conf. interval	0.95	0.88	0.83	0.82	0.83
	SNE	Mean	2.91	0.48	2.97	1.10	0.38
		Median	2.95	0.49	2.91	1.05	0.33
		Sample std. dev.	1.15	0.22	0.50	0.52	0.20
		Asymptotic std. dev.	1.20	0.23	0.31	0.50	0.18
		Coverage rate 90% conf. interval	0.91	0.91	0.78	0.84	0.88
T=500	CD-SNE	Mean	2.94	0.49	3.12	1.30	0.34
		Median	2.99	0.49	3.07	1.11	0.31
		Sample std. dev.	1.41	0.30	0.62	0.77	0.27
		Asymptotic std. dev.	1.69	0.31	0.44	0.63	0.22
		Coverage rate 90% conf. interval	0.95	0.89	0.80	0.83	0.85
	SNE	Mean	2.96	0.46	2.92	1.29	0.33
		Median	3.01	0.47	2.87	1.12	0.29
		Sample std. dev.	1.52	0.29	0.61	0.75	0.25
		Asymptotic std. dev.	1.75	0.32	0.43	0.70	0.25
		Coverage rate 90% conf. interval	0.94	0.92	0.81	0.87	0.89

Table 4 - Monte Carlo experiments. (Univariate discrete-time stochastic volatility model (16).) True parameter values are: $\phi = 0.95$, $\sigma_b = 0.025$ and $\sigma_e = 0.260$. Sample size: $T = 500$.

Estimator		ϕ	σ_b	σ_e
CD-SNE	Mean	0.909	0.024	0.229
	Median	0.939	0.023	0.210
	Sample std. dev.	0.102	0.003	0.131
	Asymptotic std. dev.	0.115	0.004	0.089
	Coverage rate 90% conf. interval	0.92	0.93	0.74
SNE	Mean	0.942	0.027	0.297
	Median	0.960	0.026	0.274
	Sample std. dev.	0.095	0.005	0.144
	Asymptotic std. dev.	0.121	0.005	0.093
	Coverage rate 90% conf. interval	0.94	0.89	0.72
QML*	Mean	0.906	...	0.302
	Sample std. dev.	0.18	...	0.17
MCL*	Mean	0.930	...	0.233
	Sample std. dev.	0.10	...	0.07
NPSML*	Mean	0.913	0.022	0.318
	Sample std. dev.	0.10	0.003	0.17

* QML stands for Quasi Maximum Likelihood; MCL for Monte Carlo Maximum Likelihood; and NPSML for Nonparametric Simulated Maximum Likelihood.

Table 5 - Monte Carlo experiments. (Bivariate discrete-time stochastic volatility model (17).) True parameter values are: $\phi = 0.95$, $\sigma_{b1} = 0.025$, $\sigma_{b2} = 0.025$ and $\sigma_e = 0.260$. Sample size: $T = 500$.

Estimator		ϕ	σ_{b1}	σ_{b2}	σ_e
CD-SNE	Mean	0.916	0.025	0.026	0.289
	Median	0.919	0.026	0.027	0.287
	Sample std. dev.	0.072	0.004	0.004	0.101
	Asymptotic std. dev.	0.080	0.004	0.004	0.113
	Coverage rate 90% conf. interval	0.92	0.83	0.88	0.91
SNE	Mean	0.913	0.027	0.027	0.365
	Median	0.938	0.026	0.027	0.331
	Sample std. dev.	0.084	0.004	0.004	0.164
	Asymptotic std. dev.	0.085	0.005	0.005	0.154
	Coverage rate 90% conf. interval	0.88	0.92	0.93	0.88