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Parametric properties of semi-nonparametric distributions, with applications to option valuation*

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Abstract

We derive the statistical properties of the SNP densities of Gallant and Nychka (1987). We show that these densities, which are always positive, are more flexible than truncated Gram-Charlier expansions with positivity restrictions. We use the SNP densities for financial derivatives valuation. We relate real and risk-neutral measures, obtain closed-form prices for European options, and analyse the semi-parametric properties of our pricing model. In an empirical application to S&P500 index options, we compare our model to the standard and Practitioner's Black-Scholes formulas, truncated expansions, and the Generalised Beta and Variance Gamma models.

Keywords: Kurtosis, Density Expansions, Gram-Charlier, Skewness, S&P index options

JEL: G13, C16

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1 Introduction

In recent years, many studies have attempted to overcome the limitations of the popular normality assumption on the returns of stocks and other financial assets, which is often rejected in the empirical finance literature even after controlling for volatility clustering effects. Although this assumption may still be reasonable if the interest focuses on the first two conditional moments (see Bollerslev and Wooldridge, 1992), in many financial applications the features under study involve higher order moments such as skewness and kurtosis. An important example is option pricing theory. The Black and Scholes (1973) pricing formula, which relies on the normality of returns, remains the benchmark model due to its analytical tractability. Unfortunately, this framework is unable to capture some important puzzles, such as smiles and smirks.

However, any successful generalisation of the Gaussian assumption must satisfy two crucial requirements: modelling flexibility and analytical tractability. Both needs are satisfied by the Gram-Charlier expansions introduced in option pricing theory by Jarrow and Rudd (1982), and more recently used by Corrado and Su (1996, 1997), Capelle-Blanchard, Jurczenko, and Maillet (2001), and Jurczenko, Maillet, and Negrea (2002a). As is well known, many density functions can be expressed as a possibly infinite expansion of the Gaussian density. In practice, however, the expansion is usually truncated after the fourth power, even though such truncated expansions often imply negative densities over some interval of their domain of variation, as Jondeau and Rockinger (2001) emphasize. This feature is particularly worrying in option pricing applications because it allows for arbitrage opportunities. For instance, the price of a butterfly spread with positive payoff over an interval of negative density would necessarily be negative in those circumstances. As a solution to this problem, Jondeau and Rockinger (2001) propose to restrict the parameters of the expansion so that the density always remains positive. Unfortunately, their approach is difficult to implement even when the truncation order is low.

In this context, we propose the use of semi-nonparametric distributions (SNP) as an alternative expansion of the Gaussian density function that is always positive by construction. This distribution was introduced by Gallant and Nychka (1987) for non-parametric estimation purposes (see also Fenton and Gallant, 1996; Gallant and Tauchen, 1999). However, it has not been treated from a purely parametric point of view, that is, as if it reflected the actual data generating process instead of an approximating kernel. We assume that under the real measure asset returns follow a SNP distribution conditional on the information available at each point in time. We study first the statistical properties of this distribution, as well as its relationship to the Gram-Charlier densities. Then, we combine it with an exponentially affine assumption on the stochastic discount factor, which enable us to transform the real measure into the risk neutral measure required for the valuation of derivative assets, and obtain closed-form expressions for European option prices. We also compare the SNP with two other popular distributions in the option pricing literature: the Generalised Beta (GB) (see Bookstaber and McDonald, 1987; Liu et al., 2006, among others); and the Variance Gamma (VG) model of Madan and Milne (1991) and Madan, Carr, and Chang (1998). In addition, we use the Marron and Wand (1992) test suite to assess the semiparametric properties of our option pricing model when the true model is not SNP. We also assess the ability of our model to fit the low frequency smiles generated by a high frequency SNP process with stochastic volatility. Furthermore, we carry out an empirical application to the S&P 500 options data of Dumas, Fleming, and Whaley (1998), in which we estimate the implied volatilities and shape parameters of our model and evaluate the performance of the SNP pricing formulas. Finally, we provide a generalised version of the SNP distribution.

The paper is structured as follows. In the next section, we study the statistical properties of SNP densities, and compare them with those of Gram-Charlier expansions. In section 3, we first relate the real and risk neutral measures, and then focus on pricing European options. Section 4 studies the semiparametric properties of our methodology, while section 5 presents the empirical application. Finally, in section 6 we present our generalised SNP density, followed by our conclusions in section 7. Proofs and auxiliary results can be found in appendices.

2 Density definition

We want to analyse the statistical properties of the affine transformation $z = a + bx$, when the density of x belongs to the semi-nonparametric class introduced by Gallant and Nychka (1987). Specifically,

$$f(x;\boldsymbol{\nu}) = \frac{\phi(x)}{\boldsymbol{\nu}'\boldsymbol{\nu}} \left(\sum_{i=0}^{m} \nu_i H_i(x) \right)^2, \tag{1}$$

where $\boldsymbol{\nu} = (\nu_0, \nu_1, \cdots, \nu_m)' \in \mathbb{R}^{m+1}$, $\phi(\cdot)$ denotes the probability density function (pdf) of a standard normal random variable, and $H_i(x)$ is the normalised Hermite polynomial of order i. These polynomials can be defined recursively for $i \geq 2$ as

$$H_i(x) = \frac{xH_{i-1}(x) - \sqrt{i-1}H_{i-2}(x)}{\sqrt{i}}, \tag{2}$$

with initial conditions $H_0(x) = 1$ and $H_1(x) = x$. Importantly, $\{H_i(x)\}_{i\in\mathbb{N}}$ constitutes an orthonormal basis with respect to the weighting function $\phi(x)$, as illustrated by the following condition:

$$\int_{-\infty}^{+\infty} H_i(x) H_j(x) \phi(x) dx = \mathbf{1}(i = j)$$

where $\mathbf{1}(\cdot)$ is the usual indicator function. The change of variable formula implies that the density function of z will be

$$g(z; \boldsymbol{\nu}, a, b) = \frac{1}{b}\frac{1}{\boldsymbol{\nu}'\boldsymbol{\nu}}\phi\left(\frac{z-a}{b}\right)\left[\sum_{i=0}^{m}\nu_i H_i\left(\frac{z-a}{b}\right)\right]^2, \tag{3}$$

where we could interpret a as a location parameter and b as a scale parameter. Note that both (1) and (3) are homogeneous of degree zero in $\boldsymbol{\nu}$, which implies that there is a scale indeterminacy that we must solve by imposing a single normalising restriction on these parameters, such as $\nu_0 = 1$, or preferably $\boldsymbol{\nu}'\boldsymbol{\nu} = 1$, which we can ensure by working with hyperspherical coordinates (see e.g. Fang, Kotz, and Ng, 1990, Theorem 2.11).

If we expand the squared expression in (1), we can obtain the following result:

Proposition 1 *Let x be a SNP random variable with density $f(x; \boldsymbol{\nu})$ given by (1). Then:*

$$f(x; \boldsymbol{\nu}) = \phi(x) \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}) H_k(x), \tag{4}$$

where $\gamma_0(\boldsymbol{\nu}) = 1$,

$$\gamma_k(\boldsymbol{\nu}) = \frac{\boldsymbol{\nu}' A_k \boldsymbol{\nu}}{\boldsymbol{\nu}'\boldsymbol{\nu}}, \quad k \geq 1 \tag{5}$$

and A_k is a $(m+1) \times (m+1)$ symmetric matrix whose typical element is

$$a_{ij,k} = \frac{(i!j!k!)^{1/2}}{\left(\frac{i+j-k}{2}\right)!\left(\frac{i+k-j}{2}\right)!\left(\frac{k+j-i}{2}\right)!}$$

if $k \in \Gamma$ and zero otherwise, with

$$\Gamma = \left\{k \in \mathbb{N} : |i-j| \leq k \leq i+j; \quad \frac{i-j+k}{2} \in \mathbb{N}\right\}.$$

For instance, the values of $\gamma_k(\boldsymbol{\nu})$ when $m = 2$ are:

$$\gamma_1(\boldsymbol{\nu}) = 2\nu_1\left(\nu_0 + \sqrt{2}\nu_2\right)/\boldsymbol{\nu}'\boldsymbol{\nu}, \quad \gamma_2(\boldsymbol{\nu}) = \sqrt{2}\left(\nu_1^2 + 2\nu_2^2 + \sqrt{2}\nu_0\nu_2\right)/\boldsymbol{\nu}'\boldsymbol{\nu},$$
$$\gamma_3(\boldsymbol{\nu}) = 2\sqrt{3}\nu_1\nu_2/\boldsymbol{\nu}'\boldsymbol{\nu}, \quad \gamma_4(\boldsymbol{\nu}) = \sqrt{6}\nu_2^2/\boldsymbol{\nu}'\boldsymbol{\nu}.$$

2.1 Moments of x and z

The first four non-central moments of x, $\mu'_x(k)$, can be obtained by using the relationship between the powers of x and the Hermite polynomials:

$$\begin{array}{l} \mu'_x(1) \equiv E_f(x) = E_f[H_1(x)], \\ \mu'_x(2) \equiv E_f(x^2) = \sqrt{2}E_f[H_2(x)] + 1, \\ \mu'_x(3) \equiv E_f(x^3) = \sqrt{3!}E_f[H_3(x)] + 3E_f[H_1(x)], \\ \mu'_x(4) \equiv E_f(x^4) = \sqrt{4!}E_f[H_4(x)] + 6\sqrt{2}E_f[H_2(x)] + 3, \end{array} \tag{6}$$

where the operator $E_f[\cdot]$ takes the expectation of its argument with respect to the density function $f(x;\boldsymbol{\nu})$ in (1). Then, from the previous non-central moments, the corresponding central ones, $\mu_x(k)$, can be easily obtained (see e.g. Stuart and Ord, 1977). Finally, we can also compute the skewness and kurtosis coefficients, denoted by sk and ku, respectively. But since $\mu'_x(k)$ in (6) depends on $\{E_f[H_i(x)]\}_{i\in\mathbb{N}}$, we first need to find these moments:

Proposition 2 *Let x denote the SNP random variable x with density function (1). Then, the expected value of the $k-th$ order Hermite polynomial are given by:*

$$E_f[H_k(x)] = \gamma_k(\boldsymbol{\nu}), \tag{7}$$

if $k \leq 2m$, and zero otherwise, where $\gamma_k(\boldsymbol{\nu})$ is defined in (5).

On this basis, we can easily compute the first four non-centred moments of x for the important special case of $m = 2$:

Lemma 1 *If the density function of the random variable x is given by (1) with $m = 2$, then*

$$\begin{array}{ll} \mu'_x(1) = \frac{2\nu_1}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left(\nu_0 + \sqrt{2}\nu_2\right), & \mu'_x(2) = \frac{2}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left(\nu_1^2 + 2\nu_2^2 + \sqrt{2}\nu_2\nu_0\right) + 1, \\ \mu'_x(3) = \frac{6\nu_1}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left(\nu_0 + 2\sqrt{2}\nu_2\right), & \mu'_x(4) = \frac{12}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left(\nu_1^2 + 3\nu_2^2 + \sqrt{2}\nu_2\nu_0\right) + 3. \end{array}$$

More generally, we can show that:

Proposition 3 *The moment generating function of the SNP density (1) is $E_f[\exp(tx)] = \exp(t^2/2)\Lambda(\boldsymbol{\nu},t)$, while its characteristic function is $\psi_{SNP}(it) = \exp(-t^2/2)\Lambda(\boldsymbol{\nu},it)$, where*

$$\Lambda(\boldsymbol{\nu},t) = \sum_{k=0}^{2m}\gamma_k(\boldsymbol{\nu})\frac{t^k}{\sqrt{k!}}, \tag{8}$$

$\gamma_k(\boldsymbol{\nu})$ is defined in (5), and i is the usual imaginary unit.

Since z is an affine transformation of x, it is trivial to find the non-central moments of z, $\mu'_z(k)$, as a function of those of x. Specifically,

$$\mu'_z(n) \equiv E_f[(a+bx)^n] = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i\mu'_x(i).$$

In addition, we can always choose the location and dispersion coefficients a and b such that z has zero mean and unit variance. In particular, if we denote by z^* the standardised variable

$$z^* = \frac{x - \mu'_x(1)}{\sqrt{\mu_x(2)}}, \tag{9}$$

then its density function can be directly obtained from (3) with

$$a(\boldsymbol{\nu}) = -\mu'_x(1) / \sqrt{\mu_x(2)}, \quad b(\boldsymbol{\nu}) = 1/\sqrt{\mu_x(2)}. \tag{10}$$

We can also use Proposition 3 to derive the distribution of linear combinations of SNP variables. In particular, we can show that the distribution of the sum of n *iid* SNP variables of order m can be expressed as a Gram-Charlier expansion of order nm that is always positive by construction.

Proposition 4 *Define* $q = \sum_{k=1}^{n} p_k x_k$*, where* $\{x_k\}_{k=1,\cdots,n}$ *are iid random variables whose distribution is a SNP of order* m *with shape parameters* $\boldsymbol{\nu}$*. Then, the distribution of* q *is a Gram-Charlier expansion of order* $2mn$ *whose density function can be expressed as*

$$\varphi(q) = \frac{\phi\left(\frac{q}{\|\mathbf{p}\|}\right)}{\|\mathbf{p}\|} \sum_{j=0}^{2mn} d_j(\boldsymbol{\nu}, \mathbf{p}) H_j\left(\frac{q}{\|\mathbf{p}\|}\right), \tag{11}$$

where $\mathbf{p} = (p_1, \cdots, p_n)'$, $\|p\| = \sqrt{\sum_{k=1}^{n} p_k^2}$ *and*

$$d_j(\boldsymbol{\nu}, \mathbf{p}) = \sqrt{j}\frac{d^j}{dx^j}\left\{\prod_{i=1}^{k}\left[\sum_{k=0}^{2m}\frac{\gamma_k(\boldsymbol{\nu})}{\sqrt{k!}}(p_i x)^k\right]\right\}\Bigg|_{x=0} \tag{12}$$

We will exploit this property to analyse the effect of time aggregation on SNP returns.

2.2 Gram-Charlier expansion of the semi-nonparametric density

Under certain regularity conditions (see e.g. Stuart and Ord, 1977, p. 234), a density function $h(y)$ can be expressed as the product of a standard normal density times an infinite series of Hermite polynomials:

$$h(y) = \phi(y) \sum_{k=0}^{\infty} c_k H_k(y), \tag{13}$$

$$c_k = \int_{-\infty}^{\infty} H_k(y) h(y)\, dy = E_h(H_k(y)). \tag{14}$$

This is the so-called Gram-Charlier series of Type A.

With this in mind, we will first determine the Gram-Charlier expansion of the SNP density of z, and then we will particularise it for the standardised random variable z^* in (9). In the case of z, we will use the fact that, according to (3) and (4), its density can be written as

$$g\left(z;\boldsymbol{\nu},a,b\right)=\frac{1}{b}\phi\left(\frac{z-a}{b}\right)\sum_{i=0}^{2m}\gamma_{i}\left(\boldsymbol{\nu}\right)H_{i}\left(\frac{z-a}{b}\right), \tag{15}$$

where $\gamma_i(\boldsymbol{\nu})$ is defined in (5). Then, if we compare (14) and (15), we can write c_k for z as

$$c_k(\boldsymbol{\nu})=\frac{1}{b}\sum_{i=0}^{2m}\gamma_i(\boldsymbol{\nu})\int_{-\infty}^{\infty}\phi\left(\frac{z-a}{b}\right)H_i\left(\frac{z-a}{b}\right)H_k\left(z\right)dz, \qquad \forall\, k\geq 0,$$

which, with the simple change of variable $x=\left(z-a\right)/b$, becomes

$$c_k(\boldsymbol{\nu})=\sum_{i=0}^{2m}\gamma_i(\boldsymbol{\nu})E_{\phi}\left[H_i\left(x\right)H_k\left(a+bx\right)\right], \qquad \forall\, k\geq 0, \tag{16}$$

where $E_{\phi}\left[\cdot\right]$ is an expectation with respect to the standard normal density. The following proposition gives a general formula for these expectations:

Proposition 5 *Let $x\sim N(0,1)$ with density $\phi(x)$. Then:*

$$E_{\phi}\left[H_i\left(x\right)H_k\left(a+bx\right)\right]=\sqrt{\frac{k!}{i!}}\sum_{j=0}^{\left\lfloor\frac{k-i}{2}\right\rfloor}\frac{H_{k-i-2j}\left(a\right)}{j!\sqrt{(k-i-2j)!}2^j}b^{i+2j}$$

for $i\leq k$ and zero otherwise, where $H_i\left(\cdot\right)$ is the i-th order standardised Hermite polynomial in (2) and $\lfloor\cdot\rfloor$ rounds its argument to the nearest integer toward zero.

In consequence, the coefficients of z defined in (16) will be

$$c_k(\boldsymbol{\nu})=\sum_{i=0}^{\min(k,2m)}\sum_{j=0}^{\left\lfloor\frac{k-i}{2}\right\rfloor}\frac{\gamma_i\left(\boldsymbol{\nu}\right)}{j!2^j}\sqrt{\frac{k!}{\left(i!\right)\left(k-i-2j\right)!}}H_{k-i-2j}\left(a\right)b^{i+2j}. \tag{17}$$

Finally, we can easily find the coefficients of the Gram-Charlier expansion of z^* by substituting a and b by their respective values in (10). This expansion will generally be infinite except for one particular case. Specifically, if $\nu_1=\nu_2=0$ and $m>2$, then it can be shown that $c_k(\boldsymbol{\nu})=0$ for $k>2m$, since $a(\boldsymbol{\nu})=0$ and $b(\boldsymbol{\nu})=1$ in that case. Lim, Martin, and Martin (2005) have explored this restricted parametrisation with $m=4$ for option pricing purposes. In this paper, though, we will not impose any restrictions on the parameters of the SNP density.

2.3 Comparison with other distributions

Consider a truncated Gram-Charlier expansion of the form

$$h\left(z^{+}\right)=\phi\left(z^{+}\right)\left[1+\sum_{i=3}^{n}c_{i}H_{i}\left(z^{+}\right)\right]. \tag{18}$$

The moments of this distribution can be obtained by using the relationships given in (6) and exploiting the orthonormality of Hermite polynomials. In this sense, notice that (18) does not include the first and second Hermite polynomials (i.e. $c_1 = c_2 = 0$) to ensure that this density has zero mean and unit variance by construction.In addition, if $n = 2m$, (18) involves exactly the same number of parameters as our standardised SNP variable z^*. However, as Jondeau and Rockinger (2001) point out, it is necessary to impose further restrictions on the parameters c_i $(i = 3, 4, \cdots, n)$ to ensure that the pdf in (18) is non-negative for all values of $z^+ \in (-\infty, \infty)$. Unfortunately, they only determined those restrictions for $n = 4$, because it becomes exceedingly difficult to find them for higher n. In contrast, we can leave the vector of parameters $\boldsymbol{\nu}$ free, except for a scale restriction, because positivity is always satisfied by a SNP density regardless of the expansion order.

Given that both z^* and z^+ have zero mean and unit variance, one may ask which of them leads to more general higher order moments. We will initially answer this question in terms of the third and fourth moments that these distributions can generate by plotting in Figure 1 the envelope of all the combinations of skewness and kurtosis for $m = 2, 3$ and 4 and $n = 4$. We have used the procedure devised by Jondeau and Rockinger (2001) to obtain the frontier for a positive Gram-Charlier distribution with $n = 4$, while we rely on (6) to represent the frontier of SNP densities with $m = 2, 3$ and 4. To allow for $\nu_0 = 0$, we simulate 10 million parameters $\boldsymbol{\nu}$ in the unit sphere and compute the envelope of the values of skewness and kurtosis obtained from the simulated parameters. In addition, we have computed the regions of skewness and kurtosis generated by the VG distribution and the log of a GB variate. Finally, we also represent the skewness-kurtosis frontier that no density function can surpass (see e.g. Stuart and Ord, 1977). The advantage of the density in (18) is that the skewness and kurtosis coefficients can be directly obtained from c_3 and c_4. Nevertheless, the combinations of skewness and kurtosis that the variable z^+ can generate are well within the combinations spanned by the SNP standardised variable z^* with exactly the same number of free parameters, as we can see in Figure 1.

For instance, while z^+ could never be platykurtic, z^* can indeed have kurtosis coefficients lower than 3. More importantly, the differences in minimum and maximum skewness are also substantial. Of course, by using the SNP instead of the Gram-Charlier expansion, we lose the direct interpretation of the parameters as skewness and kurtosis. However, this is also the case with many other non-Gaussian distributions, such as symmetric and asymmetric Student t distributions, and even the GB or VG ones. Finally, it is worth recalling that the SNP distribution guarantees positive densities regardless of m. In this sense, Figure 1 shows that we could achieve much more flexibility with just one or two additional parameters. As for the other two models, we can observe that neither the GB nor the VG distributions can generate kurtosis below 3. It is also worth remarking that although the VG can generate infinite kurtosis, it cannot yield as high a skewness as the SNP for empirically relevant levels of kurtosis. In this sense, it can be shown that the frontier of the VG is obtained when this distribution converges to a Gamma. The GB also has limited flexibility, although it allows for higher skewness than Gram-Charlier expansions once positivity restrictions are imposed. In this case, it can be shown that the upper border of its frontier is obtained when the distribution of the log of a GB variate converges to an asymmetric double exponential, which becomes a single exponential at the two points of highest absolute skewness.

To get a clearer sense of the underlying differences between the distributions of z^+ and z^*, we can compare the Gram-Charlier expansion of z^* with (18). Since both variables are standardised, both have $c_0 = 1$ and $c_1 = c_2 = 0$. The third and fourth coefficients are functions of the skewness and kurtosis of the distributions, which we have already compared in the previous paragraph. Still, the main difference between z^* and z^+ is found in the higher order coefficients. In particular, whereas (18) imposes that $c_k = 0$ for all $k > n$, such a restriction no longer holds for z^*.

3 Option valuation

3.1 From the real to the risk neutral measure, and vice versa

Consider a frictionless market with a risk free asset and a risky asset with price S_t at time t. For any $T > t$, we can always express S_T in terms of S_t under the real measure $\mathbb{P}$ as:

$$S_T \equiv S_t \exp\left[\left(\mu_t - \sigma_t^2/2\right)\tau + \sigma_t\sqrt{\tau}z^*\right], \tag{19}$$

where $\tau = T - t$ and z^* is a random variable with zero mean and unit variance conditional on the information available at time t. In this context, μ_t and σ_t, which in general will be functions of the information known at t, represent the conditional mean and volatility per unit of time of $\log(S_T/S_t)$. In what follows, we will assume that $z^* = a(\boldsymbol{\nu}_t) + b(\boldsymbol{\nu}_t) x^{\mathbb{P}}$, where $a(\boldsymbol{\nu}_t)$ and $b(\boldsymbol{\nu}_t)$ are defined in (10), and $x^{\mathbb{P}}$ is a SNP variate with shape parameters $\boldsymbol{\nu}_t$. With this notation, we can write the log-return as $y_T = \log(S_T/S_t) = \delta_{\mathbb{P}t} + \lambda_{\mathbb{P}t} x^{\mathbb{P}}$, where $\delta_{\mathbb{P}t} = (\mu_t - \sigma_t^2/2)\,\tau + \sigma_t\sqrt{\tau} a(\boldsymbol{\nu}_t)$ and $\lambda_{\mathbb{P}t} = \sigma_t\sqrt{\tau} b(\boldsymbol{\nu}_t)$.

Our solution to the option pricing problem will be based on the use of a stochastic discount factor with an exponential affine form:

$$M_{t,T} = \exp(\alpha_t y_T + \beta_t \tau). \tag{20}$$

where again α_t and β_t can be functions of the information known at time t. Such a specification corresponds to the Esscher transform used in insurance (see Esscher, 1932). In option pricing applications, this approach was pioneered by Gerber and Shiu (1994), and has also been followed by Buhlman, Delbaen, Embrechts, and Shyraev (1996, 1998), Gourieroux and Monfort (2006a,b) and Bertholon, Monfort, and Pegoraro (2003) among others. The following result provides the conditions for absence of arbitrage.

Proposition 6 *Let r_t be the risk-free rate and I_t the information set at time t. If the conditional distribution of the log-return of the risky asset is a SNP of order m, then the stochastic discount factor (20) satisfies the arbitrage free conditions,*

$$E_{\mathbb{P}}\left[M_{t,T}\exp(r_t\tau)\right| I_t] = 1, \quad E_{\mathbb{P}}\left[M_{t,T}\exp(y_T)\right| I_t] = 1, \tag{21}$$

if and only if

$$\sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}_t) \frac{(\alpha_t \lambda_{\mathbb{P}t})^k}{\sqrt{k!}} = \exp\left[-\alpha_t\delta_{\mathbb{P}t} - \frac{1}{2}\alpha_t^2\lambda_{\mathbb{P}t}^2 - \beta_t\tau - r_t\tau\right], \tag{22}$$

$$\sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}_t) \frac{(1+\alpha_t)^k \lambda_{\mathbb{P}t}^k}{\sqrt{k!}} = \exp\left[-(1+\alpha_t)\,\delta_{\mathbb{P}t} - \frac{1}{2}(1+\alpha_t)^2\lambda_{\mathbb{P}t}^2 - \beta_t\tau\right]. \tag{23}$$

From these two constraints, we can easily express β_t as a function of α_t. Hence, α_t can be obtained by solving a single non-linear equation, which is an implicit function of the remaining parameters of the model.

In this context, if $\mathbb{Q}$ denotes the risk neutral measure whose numeraire is the risk free asset, the real and risk-neutral measures can be easily related by means of the Radon-Nykodym derivative, which in this case is proportional to the discount factor

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{M_{t,T}}{E_{\mathbb{P}}(M_{t,T})}.$$

Hence

$$E_{\mathbb{Q}}\left[F\left(S_T\right)|I_t\right] = E_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}F\left(S_T\right)|I_t\right], \tag{24}$$

where $F(\cdot)$ is an arbitrary function and $E_{\mathbb{P}}\left(M_{t,T}|I_t\right) = \exp(-r_t\tau)$, so that the discount factor correctly prices the risk-free asset. As a result, we can obtain the risk-neutral density from (24) as

$$f^{\mathbb{Q}}\left(y_T|I_t\right) = \exp(r_t\tau)M_{t,T}f^{\mathbb{P}}\left(y_T|I_t\right). \tag{25}$$

On this basis, we can fully characterise the risk-neutral measure as follows:

Proposition 7 *If the asset price S_T is given by (19) under the real measure $\mathbb{P}$, where the conditional distribution of its log return between t and T is a SNP of order m with shape parameters $\boldsymbol{\nu}_t$, then it can be written under the risk neutral measure $\mathbb{Q}$ as*

$$S_T = S_t \exp\left[\left(\mu_t^{\mathbb{Q}} - \frac{\left(\sigma_t^{\mathbb{Q}}\right)^2}{2}\right)\tau + \sigma_t^{\mathbb{Q}}\sqrt{\tau}\kappa^*\right], \tag{26}$$

where

$$\mu_t^{\mathbb{Q}} = \mu_t + \frac{\sigma_t^2}{2}\left[\left(\frac{b\left(\boldsymbol{\nu}_t\right)}{b\left(\boldsymbol{\theta}_t\right)}\right)^2 - 1\right] + \frac{\sigma_t}{\sqrt{\tau}}\left[a\left(\boldsymbol{\nu}_t\right) - a\left(\boldsymbol{\theta}_t\right)\frac{b\left(\boldsymbol{\nu}_t\right)}{b\left(\boldsymbol{\theta}_t\right)}\right] + \alpha_t\sigma_t^2 b^2\left(\boldsymbol{\nu}_t\right), \tag{27}$$

$$\sigma_t^{\mathbb{Q}} = \sigma_t b\left(\boldsymbol{\nu}_t\right)/b\left(\boldsymbol{\theta}_t\right), \tag{28}$$

and κ^* is a standardised SNP variable of order m with shape parameters $\boldsymbol{\theta}_t = (\theta_{0t}, \theta_{1t}, \cdots, \theta_{mt})'$, such that

$$\theta_{it} = \sum_{k=i}^{m} \frac{\nu_{kt}}{(k-i)!}\sqrt{\frac{k!}{i!}}\left(\alpha_t\lambda_{\mathbb{P}t}\right)^{k-i}. \tag{29}$$

Therefore, in a SNP context the change of measure affects not only the mean and the variance of the log price, but also the higher moments, as can be seen from the differences between $\boldsymbol{\theta}_t$ and $\boldsymbol{\nu}_t$. For the case of $m = 2$, for instance, we can show that the relation between $\boldsymbol{\theta}_t$ and $\boldsymbol{\nu}_t$ is $\theta_{0t} = \nu_{0t} + \nu_{1t}\alpha_t\lambda_{\mathbb{P}t} + \nu_{2t}\alpha_t^2\lambda_{\mathbb{P}t}^2/\sqrt{2}$, $\theta_{1t} = \nu_{1t} + \nu_{2t}\sqrt{2}\alpha_t\lambda_{\mathbb{P}t}$ and $\theta_{2t} = \nu_{2t}$. However, note that the SNP distribution is shared by the real and risk-neutral measures. Also, it is important to emphasise that this change of measure is always feasible because there are no restrictions on the shape parameters of the SNP distribution. Our results can be extended to more complicated specifications of the stochastic discount factor. For instance, an exponential quadratic form would also yield a SNP distribution of the same order under the risk-neutral distribution (the details are available upon request). In those cases, though, we would need to consider a larger number of assets in order to identify the parameters of the pricing kernel.

Obviously, our framework also allows us to value derivative assets by focusing on the risk-neutral measure directly without any reference to its relationship with the real measure, as in Jondeau and Rockinger (2001) or Jurczenko, Maillet, and Negrea (2002a,b). To follow this second approach, we just have to regard $\boldsymbol{\theta}_t$, $\mu_t^{\mathbb{Q}}$ and $\sigma_t^{\mathbb{Q}}$ as the structural parameters. The following proposition gives the expression that the risk-neutral drift must have to satisfy the martingale restriction (see Longstaff, 1995):

Proposition 8 *If asset price S_T is given by (26) under the risk-neutral measure $\mathbb{Q}$, where the conditional distribution of its log return between t and T is a SNP of order m with shape parameters $\boldsymbol{\theta}_t$, then the drift $\mu_t^{\mathbb{Q}}$ will satisfy the martingale restriction if and only if:*

$$\mu_t^{\mathbb{Q}} = r_t - (1/\tau)\left[\sigma_t^{\mathbb{Q}}\sqrt{\tau}a\left(\boldsymbol{\theta}_t\right) + (1/2)\left(\sigma_t^{\mathbb{Q}}\right)^2\tau\left(b^2\left(\boldsymbol{\theta}_t\right) - 1\right) + \log\Lambda(\boldsymbol{\theta}_t, \lambda_{\mathbb{Q}t})\right], \tag{30}$$

where $\lambda_{\mathbb{Q}t} = \sigma_t^{\mathbb{Q}}\sqrt{\tau}b\left(\boldsymbol{\theta}_t\right)$ and $\Lambda\left(\cdot,\cdot\right)$ is defined in (8).

Not surprisingly, we show in appendix C that (27) and (30) coincide, which confirms that both strategies are indeed equivalent. This equivalence result has important computational advantages in empirical applications such as ours that only use option price data, because it allows one to estimate the option values from the risk neutral parameters without having to solve the nonlinear equations (22) and (23) within the optimisation algorithm. At the same time, if we had data on the underlying we could obtain the implied real-measure parameters. In particular, for a given drift μ_t, risk-free rate r_t and risk neutral parameters $\sigma_t^{\mathbb{Q}}$ and $\boldsymbol{\theta}_t$, we can recover the parameters of the real measure σ_t and $\boldsymbol{\nu}_t$, together with the coefficient of relative risk aversion α_t, from the following system of equations

$$\begin{aligned}\left(\mu_t - \sigma_t^2/2\right)\tau + \sigma_t\sqrt{\tau}a\left(\boldsymbol{\nu}_t\right) &= \delta_{\mathbb{Q}t} - \alpha_t\lambda_{\mathbb{Q}t}^2,\\ \sigma_t\sqrt{\tau}b\left(\boldsymbol{\nu}_t\right) &= \lambda_{\mathbb{Q}t},\\ \nu_{it} &= \sum_{k=i}^{m}\frac{\theta_{it}}{(k-i)!}\sqrt{\frac{k!}{i!}}(-1)^{k-i}\left(\lambda_{\mathbb{Q}t}\alpha_t\right)^{k-i},\end{aligned} \tag{31}$$

where $\delta_{\mathbb{Q}t} = \left(\mu_t^{\mathbb{Q}} - \sigma_t^{\mathbb{Q}\,2}/2\right)\tau + \sigma_t^{\mathbb{Q}}\sqrt{\tau}a\left(\boldsymbol{\theta}_t\right)$. Finally, the discount factor β_t can be obtained from either (22) or (23).

3.2 Option pricing

Let C_t be the value at time t of a European call option with strike price K and expiration at time T, and let S_t denote the underlying asset value. We can express C_t as

$$C_t = \exp(-r_t\tau)E_{\mathbb{Q}}\left[\left(S_T - K\right)^+\middle|\, I_t\right], \tag{32}$$

where $(\cdot)^{+} = \max(\cdot, 0)$. It is important to emphasise again the conditional nature of (32), which implies that all the parameters of the model can potentially depend on the information available at time t. If define the region $A = \{S_T > K\}$ we can rewrite (32) as

$$C_t = \exp(-r_t\tau)E_{\mathbb{Q}}\left[S_T\mathbf{1}(A)|\, I_t\right] - K\exp(-r_t\tau)E_{\mathbb{Q}}\left[\mathbf{1}(A)|\, I_t\right]. \quad (33)$$

Following Geman, Karouri, and Rochet (1995), we can further simplify the calculations by changing the numeraire to the ratio of the risky asset prices S_T/S_t, which gives an alternative risk-neutral measure $\mathbb{Q}_1$. Then, if we use the Radon-Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{Q}_1} = \frac{B_T}{B_t}\frac{S_t}{S_T} = \exp(r_t\tau)\frac{S_t}{S_T}, \quad (34)$$

we can easily express any expectation under $\mathbb{Q}$ in terms of $\mathbb{Q}_1$. Specifically, we will have that

$$E_{\mathbb{Q}}\left[S_T\mathbf{1}(A)|\, I_t\right] = E_{\mathbb{Q}_1}\left[\left.\frac{d\mathbb{Q}}{d\mathbb{Q}_1}S_T\mathbf{1}(A)\right|\, I_t\right] = S_t\exp(r_t\tau)E_{\mathbb{Q}_1}\left[\mathbf{1}(A)|\, I_t\right],$$

which, once introduced in (33), gives us the general formula

$$\begin{aligned} C_t &= S_tE_{\mathbb{Q}_1}\left[\mathbf{1}(A)|\, I_t\right] - K\exp(-r_t\tau)E_{\mathbb{Q}}\left[\mathbf{1}(A)|\, I_t\right] \\ &= S_t\Pr_{\mathbb{Q}_1}\left[S_T > K|\, I_t\right] - K\exp(-r_t\tau)\Pr_{\mathbb{Q}}\left[S_T > K|\, I_t\right]. \quad (35) \end{aligned}$$

The analytical tractability of the SNP distribution allows us to obtain closed form expressions for the probabilities in (35):

Proposition 9 *The price at time t of a European call option with strike K written on the stock S_T defined by (26) under the risk-neutral measure can be expressed as:*

$$C_t^{SNP} = S_t\Pr_{\mathbb{Q}_1}\left[x > d_t|\, I_t\right] - K\exp(-r_t\tau)\Pr_{\mathbb{Q}}\left[x > d_t|\, I_t\right], \quad (36)$$

where

$$\begin{aligned} \Pr_{\mathbb{Q}}\left[x > d_t|\, I_t\right] &= \Phi(-d_t) + \phi(d_t)\sum_{k=1}^{2m}\frac{\gamma_k(\boldsymbol{\theta}_t)}{\sqrt{k}}H_{k-1}(d_t), \\ \Pr_{\mathbb{Q}_1}\left[x > d_t|\, I_t\right] &= \exp(-r_t\tau + \delta_{\mathbb{Q}t})\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta}_t)I_{k,t}^{*}, \end{aligned}$$

$$I_{k,t}^{*} = \frac{1}{\sqrt{k}}\exp(\lambda_{\mathbb{Q}t}d_t)H_{k-1}(d_t)\phi(d_t) + \frac{\lambda_{\mathbb{Q}t}}{\sqrt{k}}I_{k-1,t}^{*}; \quad I_{0,t}^{*} = \exp(\lambda_{\mathbb{Q}t}^2/2)\Phi(\lambda_{\mathbb{Q}t} - d_t), \quad (37)$$

$$\delta_{\mathbb{Q}t} = \left(\mu_t^{\mathbb{Q}} - \frac{\sigma_t^{\mathbb{Q}2}}{2}\right)\tau + a(\boldsymbol{\theta}_t)\sigma_t^{\mathbb{Q}}\tau,$$

$$d_t = \frac{\log(K/S_t) - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}}; \;\; \lambda_{\mathbb{Q}t} = b(\boldsymbol{\theta}_t)\sigma_t^{\mathbb{Q}}\sqrt{\tau}$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal density.

As expected, (36) reduces to the Black and Scholes (1973) formula when $\theta_{0t} = 1$ and $\theta_{kt} = 0 \ \forall k \geq 1$. Importantly, if we treat the coefficients γ_k of the Gram-Charlier expansion (4) as shape parameters themselves, instead of functions of either $\boldsymbol{\nu}_t$ or $\boldsymbol{\theta}_t$, we can show that (36) is also valid when the distribution of the underlying asset return is a finite Gram-Charlier expansion. As a consequence, we can use Proposition 9 to obtain closed form option prices when returns follow a high frequency process with *iid* SNP innovations, since Proposition 4 shows that their distribution at low frequencies is a Gram-Charlier expansion.

In Figure 2 we compare the range of call prices that the SNP density can produce with the corresponding ranges obtained for the Gram-Charlier expansion with positivity restrictions and the GB model. Not surprisingly, the higher flexibility of the SNP in modelling skewness and kurtosis that we saw in Figure 1 results in a wider range of call prices. The only exception is the VG model, which can reach the arbitrage bounds, but only under the limiting case in which the underlying distribution converges to a Bernoulli whose skewness tends to +/- infinity. Importantly, a larger value of m also leads to an SNP with even broader range. Nevertheless, there is a close relationship between the different pricing models: the Gram-Charlier call price formula can be obtained as a fourth-order Taylor expansion of (36), while Black-Scholes corresponds to a second-order one (see appendix A for further details).

4 Semiparametric properties of the SNP option pricing model

4.1 Estimation with a misspecified model

Fenton and Gallant (1996) and Gallant and Tauchen (1999) used the Marron and Wand (1992) test suite to analyse the semiparametric properties of SNP distributions in density estimation and in the implementation of the Efficient Method of Moments, respectively. However, their semiparametric properties in option pricing applications have not been studied. In this section, we will assess the performance of our option pricing model when the true distribution is not SNP.

Specifically, we will assume that the true distribution of the underlying asset return is one of the first nine non-Gaussian distributions proposed by Marron and Wand (1992). For each of them, we generate 1000 call option prices from the true model, with a range of moneyness uniformly distributed between ± 3 times the standard deviation of the underlying asset return. Finally, we estimate the parameters of the following misspecified models by minimising the root mean square pricing errors (RMSE's): Black-Scholes, Gram-Charlier with two shape parameters, SNP with $m = 2$, and SNP of order m^* such that the RMSE divided by the mean option price is less than 10 basis points.

Some selected results are displayed in Figure 3. The left panels show the shape of the true density, whereas the right panels display the true implied volatilities together with the ones estimated with the misspecified models. Since none of these models is Gaussian, Black-Scholes performs poorly in most cases. The models with two shape parameters perform reasonably well in some examples, such as the skewed and kurtotic unimodal cases. However, in some other examples, such as the strongly skewed, the Gram-Charlier parameter estimates cannot guarantee the positivity of the density. The consequence is that Gram-Charlier implied volatilities suddenly jump to zero for some ranges of the moneyness. In contrast, our pricing model does not suffer from this restriction. Of course, if we let $m \to \infty$ then we will be able to exactly reproduce all the volatility smiles. However, we are able to show that even for finite m, the SNP already performs very well. In this sense, we can check that we obtain substantial improvements in fit in all cases as we increase the order of the SNP.

4.2 Temporal aggregation

From Proposition 4, we know that the distribution of aggregated SNP returns is not a SNP of the same order, not even when they are *iid*. In this subsection, we assess the ability of our model applied to low frequency data to fit option prices that have been generated with a high frequency process with SNP log-returns. To do so, we model the weekly process of log-returns with a non-*iid* SNP distribution of order $m = 2$, whose volatility follows a persistent binary Markov chain. The skewness and kurtosis of this process conditional on the volatilities of the two states that we consider are characterised by the shape parameters of the SNP distribution, while the probabilities of remaining in each of these states determine the unconditional variance and the persistence of the stochastic volatility. All these parameters have been calibrated using S&P 500 weekly return data from 1950 to 2006. Although for the purposes of our exercise we could have considered a continuous distribution for volatility, we have chosen a Markov chain only because we can obtain closed form option prices. Specifically, we consider every possible volatility path that the Markov chain can generate. Along any of those paths, Proposition 4 implies that the distribution of the log return between the initial and final date is just a Gram-Charlier expansion, for which Proposition 9 applies if we impose (30) to ensure that the martingale restriction is satisfied. Finally, we can express the call price as a weighted sum of the option prices in each possible path, with weights that correspond to their unconditional probabilities of occurring. We have generated from this process 1000 option prices maturing in one and three months with the same range of moneyness as in the previous subsection. We fit SNP's of increasing order to these prices until the RMSE divided by the mean option price is less than 10 basis points.

As shown in Figure 4, a SNP with $m = 4$ is enough to yield a RMSE below our target for both maturities. This is somewhat surprising if we take into account that, for a given volatility path, the distribution of the one and three month log-returns are Gram-Charlier expansions of order 16 and 48, respectively (see Proposition 4). Thus, we believe that the time incoherence problem should not be an issue of major concern in our context. We can also notice in Figure 4 the flattening of the smile at the longer horizon, which is consistent with the empirical evidence (see e.g. Das and Sundaram, 1999).

5 Empirical performance of SNP option pricing

In this section, we apply the SNP option valuation formula (36) in Proposition 9 to S&P 500 index options using the same database as Dumas, Fleming, and Whaley (1998). Option prices were collected every Wednesday between 2:45 p.m. and 3:45 p.m. from June 1988 to December 1993, which makes a total number of 292 days. Options are European-style and expire on the third Friday of each contract month. We will focus on call options, and use the bid-ask mid price for estimation purposes. The riskless interest rate will be proxied by the T-bill rate implied by the average of the bid and ask discounts reported in the *Wall Street Journal*. To account for the presence of dividends, the implied forward price is computed as the current stock price S_t minus the present value of dividends $\bar{D}_t$ times the interest accrued until maturity, i.e. $F_{t,T} = (S_t - \bar{D}_t)\exp(r_t\tau)$ (see Dumas, Fleming, and Whaley, 1998, for further details).

We will compare the performance of the SNP option valuation framework with the following competing models: the standard Black and Scholes (1973) model, the Gram-Charlier expansion with positivity restrictions, the GB and VG models, and finally a variant of the Black-Scholes model where the volatility is assumed to be a quadratic function of moneyness. We will call this methodology Practitioners' Black-Scholes, a name inspired by its wide use in the financial industry. In order to guarantee positivity, we will consider the parametrisation

$$\sigma(x) = \rho_0 + \rho_1(x - \rho_2)^2 \tag{38}$$

where $\rho_0 > 0, \rho_1 \geq 0$ and $x = F_{t,T}/K$. Finally, note that since we are using implied forward prices, an adjustment in the spirit of Black (1976) is needed in all cases.

We consider separate estimations for short and long maturities. Specifically, we estimate the implied volatility and the remaining shape parameters of each model by minimising the sum of squared pricing errors between the observed option prices and the ones implied by the models. To select the short maturity group, we begin by considering call options that mature in 45 days for the first day in the sample. We track those options every week until two weeks before they expire. Then, we move to the next group of options that are 45 days away from expiration and start the tracking process again. At the end, we have data on 3,462 call option prices with median time to expiration of 24 days, and a number of options per day that ranges from 4 to 25, with a median of 11. In the long maturity group we follow an analogous selection process. In particular, we have selected 4,306 call option prices with a median time to maturity of 150 days. The number of prices per day also ranges between 4 and 25, but the median is now 15. Our empirical results are essentially unaffected by conditioning our estimation procedure on having at least 6 or 7 options per day. The main reason is that only 11 (10) out of the 292 days in our database have less than 6 options available across strikes for the short (long) maturity group.

Tables 1a to 1d report the RMSE's of the six competing models when we allow all the parameters of the conditional distribution of returns to vary each Wednesday, which is consistent with the conditional nature of our pricing framework. We also provide information on the degree of fit achieved for different degrees of moneyness using the six categories proposed by Bakshi, Cao, and Chen (1997), together with the number of options in each category. Tables 1a and 1c report in-sample RMSE's based on the first four years of data. In contrast, Tables 1b and 1d report out-of-sample results based on pricing errors for each Wednesday in the last year of the sample using the parameters estimated on the previous Wednesday. In the short maturity group, Practitioners's Black-Scholes and the SNP are the two best performing models in-sample, followed by the VG and GB models. However, if we look at the out-of-sample results, we can observe that Practitioner's Black-Scholes shows a strong parameter instability, whereas the other three models are much more stable. In the long maturity group, again the SNP, GB and VG models yield the lowest RMSE's, although VG yields a slightly better fit in this case. Nevertheless, the differences between these three models are very small, whereas the RMSE's of the Black-Scholes, Gram-Charlier with positivity restrictions and Practitioners' Black-Scholes models are clearly higher.

In Figures 5a and 5b we have plotted the skewness and kurtosis values implied by the SNP, Gram-Charlier with positivity restrictions, GB and VG models for each day in the in-sample period. Several important patterns arise from these figures. First, there is high dispersion in the estimated higher order moments, although skewness is usually negative and kurtosis is typically higher than 3. Second, skewness and kurtosis tend to be lower when the time to expiration is longer. Furthermore, skewness and kurtosis in Gram-Charlier densities with positivity restrictions are usually on the frontier of values compatible with these densities. This is also observed with the VG and specially with the GB model. In particular, market prices often suggest a more (negative) skewness than these models are able to account for. However, some SNP estimates are also located on the frontier, especially in the short maturity group. Although we could easily enlarge the SNP frontier by simply increasing the order m (see Figure 1), it is interesting to analyse in more detail the possible sources of the high sampling variability.

To do so, we have carried out the following bootstrap exercise. First, we group the SNP pricing errors obtained for short maturities in the six moneyness categories already considered. Then, we simulate prices for a specific but broadly representative day (November 13, 1991), by adding random pricing errors to the 19 prices of that day estimated with the SNP model. In this sense, we sample the errors that we add to each price from the same moneyness category to which that price belongs. In this way, we take into account possible distributional differences between pricing errors for, say, deep in the money and out of the money options. Finally, we re-estimate the SNP model on the simulated data. We plot the implied skewness and kurtosis for 1,000 such simulations in Figure 5c. As we can observe, the estimates are again highly disperse, and basically cover the whole region of negative skewness. Nevertheless, the true option prices have constant parameters by construction, which approximately correspond to skewness of -1.5 and kurtosis of 7.7 (see Figure 5c).

Therefore, it may well be the case that even if the true parameters are constant, the high variation in skewness and kurtosis that we observe in Figures 5a and 5b simply results from the relatively low number of prices with which we are estimating the daily models. For that reason, we also study the performance of all the different models under the assumption that the conditional distribution of standardised log-returns (or ρ_1 and ρ_2 in (38)) is time invariant, while volatility (or the intercept ρ_0 in Practitioner's Black-Scholes) is allowed to change over time as before. Again, we carry out an in-sample and an out-of-sample analysis, which show that the SNP, GB and VG models perform more or less on the same level, while the remaining models yield less satisfactory results (see Tables 2a to 2d). We can also note that, by increasing the order of the SNP we can improve its performance without deteriorating its out of sample stability.

If we compare the SNP pricing errors in Tables 2b and 2d with those of Tables 1b and 1d, we can observe that the assumption of constant shape parameters does indeed yield better out-of-sample results. Importantly, the SNP with fixed parameters generally performs better out-of-sample than the remaining models with time varying parameters. In terms of skewness and kurtosis, Figure 5d shows that SNP estimations are no longer at the frontier. In contrast, Gram-Charlier and GB estimates are very close or exactly on their respective frontiers.

As a sanity check, Figure 6 confirms that the main differences between Black-Scholes and the remaining non-Gaussian models lie in the tails of the distributions, and not so much in the temporal evolution of the option-implied volatilities.

Another interesting issue is whether the main reason for the rejection of the Black-Scholes model is skewness or excess kurtosis. To find out, we have re-estimated our SNP model for $m = 2$ with fixed parameters imposing zero skewness first, and then kurtosis equal to 3. Interestingly, it turns out that when we force the skewness to be zero we obtain the Black-Scholes special case. In contrast, if we fix the kurtosis to 3, we obtain substantial negative skewness for both the short and long maturity groups. Hence, it seems that negative skewness plays a more fundamental role in determining option prices than excess kurtosis. This result is likely to be related to the specific features of equity index options, which are typically characterised by significant smirks rather than purely symmetric smiles, especially after the 1987 stock market crash (see e.g. Brown and Jackwerth, 2004). However, it is beyond the scope of this paper to assess whether this result is specific to the equity-index market.

Finally, we compare the estimated conditional risk-neutral densities in Figures 7a to 7d for the same day as in the bootstrap exercise, having obtained the density implied by the Practitioner's Black-Scholes model from the second derivative of the call price with respect to the strike (see Breeden and Litzenberger, 1978). All the models except Black-Scholes imply negatively skewed and peaked densities, but they are reasonably similar at the centre, except for the much higher peaks in the VG densities. In fact, it turns out that this density has a pole near zero for the long maturity group. However, zooms of the left tails show that the Practitioner's Black-Scholes model attaches unreasonably high probabilities to extreme negative events. This result is consistent with the fact that the Practitioner's Black-Scholes method gives relatively good results in-sample but unrealistic implications for out-of-the-money calls. In Figure 8 we compare the smiles that each model can generate with the bid, ask and mid-price quotes for our chosen representative day. This figure shows a highly asymmetric smile, which Practitioner's Black-Scholes tries to fit with quadratic curve, at the cost of not providing very reliable results at the extremes (see in particular the out-of-the money area). This picture also shows that the rather limited amount of skewness allowed by "positive" Gram-Charlier densities prevents them from reproducing the empirical smile as we get deeper in the money. However, lack of liquidity is stronger in deep in-the-money options, so the real importance of this result must be taken with some caution.

6 Extensions

The SNP density of order m is constructed by multiplying the Gaussian density by a squared polynomial of order m. The fact that the polynomial of the expansion is a perfect square is a sufficient but not necessary condition for positivity of the final density. Hence, we can create a generalised SNP (GSNP) density by multiplying the Gaussian density with an otherwise unrestricted positive polynomial $P_{2m}(x)$ of order $2m$. This distribution will include as particular cases both the SNP and the Gram-Charlier density with positivity restrictions.

The positivity of $P_{2m}(x)$ is ensured if its roots are either real and double, or complex conjugates. In contrast, in the SNP case the complex roots must always be double. Meddahi (2001) shows that a necessary and sufficient condition for $P_{2m}(x)$ to be positive is that it can be written as the sum of two squared polynomials of order m.

Interestingly, we can interpret the GSNP density as a mixture of two SNP densities with the same location and scale:

Proposition 10 *The GSNP density can be written as*

$$f_{GSNP}(x;\boldsymbol{\nu}_1,\boldsymbol{\nu}_2) = p(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)f(x;\boldsymbol{\nu}_1) + [1 - p(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)]f(x;\boldsymbol{\nu}_2)$$

where $\boldsymbol{\nu}_1$ *and* $\boldsymbol{\nu}_2$ *are vectors of dimension* m *and* $m-1$*, respectively,* $f(x;\boldsymbol{\nu}_1)$ *is defined in (1), and*

$$p(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2) = \frac{\boldsymbol{\nu}_1'\boldsymbol{\nu}_1}{\boldsymbol{\nu}_1'\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2'\boldsymbol{\nu}_2}$$

This interpretation can be exploited to extend the results of the paper to this generalised class of distributions. Nevertheless, despite the increased generality of the GSNP, we have found that it does not seem to provide a higher flexibility in terms of skewness, kurtosis or range of option prices than a standard SNP density of the same order. A more thorough study of the characteristics of the GSNP density is left for future research.

7 Conclusions

The SNP distribution was introduced by Gallant and Nychka (1987) for nonparametric estimation purposes. In contrast, we propose to use it as a parametric model. This distribution shares the analytical tractability of truncated Gram-Charlier expansions, but its density is always positive.

We obtain its moments, show its flexibility in terms of skewness and kurtosis and derive the distribution of linear combinations. Next, we focus our attention on option pricing, and show that if the log of the underlying asset price has a conditional SNP distribution under the real measure, and the stochastic discount factor is exponentially affine, then the log of the underlying asset price will also have a conditional SNP distribution of the same order under the risk-neutral measure. On this basis, we obtain closed form expressions for European option prices. We also show that our SNP option pricing formula can approximate arbitrarily well the prices of options whose true densities are not SNP. Furthermore, we apply our pricing formulas to obtain exact option prices in a high frequency SNP model. In this sense, we show that a low order SNP can approximate very well the behaviour of low frequency option prices generated by a stochastic volatility high frequency process with SNP innovations.

Finally, we carry out an empirical application to the S&P 500 options data of Dumas, Fleming, and Whaley (1998). We compare the performance of our pricing formulas with the Black and Scholes (1973) model, the Practitioner's Black-Scholes procedure, the Gram-Charlier density with positivity restrictions, as well as the GB and VG models. We estimate the shape parameters and the implied volatility that minimise the sum of squared pricing errors of these models. We find that the SNP, together with the GB and the VG, are the best performing models, both in and out of sample. We also find a high dispersion in the daily estimates of skewness and kurtosis, which is probably due to sampling variability. In this sense, we find that the pricing performance of our model improves out-of-sample if we keep the shape parameters constant over time. It is also worth mentioning that skewness seems to be relatively more important than excess kurtosis for the empirical rejection of the Black-Scholes model. This result is probably due to the asymmetric smiles that are typically observed for equity index options.

We propose a generalised version of the SNP distribution that nests all positive Gram-Charlier expansions. We show that it can be generated as a mixture of two SNP variables with the same location and scale, which allows us to extend our previous results to this density.

A fruitful avenue for future research would be to exploit the relationship between real and risk-neutral measures in the estimation of our option pricing model by combining data on the underlying asset price, which is informative about the real measure, with option price data, which contains information about the risk-neutral measure (see Jackwerth, 2000). It would also be interesting to explore possible time varying specifications for the parameters of the model, such as GARCH parametrisations for the volatility (see Heston and Nandi, 2000), and analogous extensions for the remaining shape parameters, as in Hansen (1994) or Jondeau and Rockinger (2005). Similarly, it would also be worth exploring the flexibility of the SNP and generalised SNP distributions for risk-management purposes.

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A Relationship with Gram-Charlier option pricing models

We can express (36) in terms of the infinite Gram-Charlier expansion the SNP distribution as follows:

Proposition 11 *The call price* C_t^{SNP} *in (36) can be rewritten in terms of an infinite expansion* $C_t^{SNP} = \xi_{0t} + \xi_{3t} sk_t + \xi_{4t}(ku_t - 3) + \zeta_t$, *where*

$$\begin{aligned}
\zeta_t &= e^{-r_t\tau}\sum_{k=5}^{\infty} c_k(\boldsymbol{\theta}_t)\int_{\omega_t}^{\infty}\left(S_T\left(\kappa^*\right)-K\right)H_k\left(\kappa^*\right)\phi\left(\kappa^*\right)d\kappa^*,\\
\xi_{0t} &= S_t e^{\left(\mu_t^{\mathbb{Q}}-r_t\right)\tau}\Phi\left(d_{1t}^*\right) - Ke^{-r_t\tau}\Phi\left(d_{1t}^* - \sigma_{t,\tau}^{\mathbb{Q}}\right),\\
\xi_{3t} &= (1/3!)\sigma_{t,\tau}^{\mathbb{Q}} S_t e^{\left(\mu_t^{\mathbb{Q}}-r_t\right)\tau}\left[\sigma_{t,\tau}^{\mathbb{Q}\,2}\Phi\left(d_{1t}^*\right) + \left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}^*\right)\phi\left(d_{1t}^*\right)\right],\\
\xi_{4t} &= (1/4!)\sigma_{t,\tau}^{\mathbb{Q}} S_t e^{\left(\mu_t^{\mathbb{Q}}-r_t\right)\tau}\left[\sigma_{t,\tau}^{\mathbb{Q}\,3}\Phi\left(d_{1t}^*\right) + \left(3\sigma_{t,\tau}^{\mathbb{Q}\,2} - 3d_{1t}^*\sigma_{t,\tau}^{\mathbb{Q}} + d_{1t}^{*2} - 1\right)\phi\left(d_{1t}^*\right)\right],
\end{aligned}$$

$\omega_t = \sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}^*, \sigma_{t,\tau}^{\mathbb{Q}} = \sigma_t^{\mathbb{Q}}\sqrt{\tau}, d_{1t}^* = [\log\left(S_t/K\right) + \left(\mu_t^{\mathbb{Q}} + \sigma_t^{\mathbb{Q}\,2}/2\right)\tau]/\sigma_{t,\tau}^{\mathbb{Q}}$ *and* $S_T\left(\kappa^*\right)$, *defined in (26), is regarded as a function of the standardised random variable* κ^*, *while the coefficients* $c_k(\boldsymbol{\theta}_t)$ *are given in (17).*

We can use Proposition 11 to relate our pricing model to the model of Corrado and Su (1996, 1997), who consider a fourth order Gram-Charlier density ($c_k = 0$, for $k \geq 5$ in (13)), without imposing positivity restrictions. In this respect, it is important to mention that the original Corrado-Su formula, apart from containing a mistake in the definition of the Hermite polynomials, does not satisfy the martingale restriction (32). Both problems are dealt by Jurczenko, Maillet, and Negrea (2002b). The following result shows that the martingale restriction in Jurczenko, Maillet, and Negrea (2002b) can be regarded as a truncated version of our drift (30):

Lemma 2 *The drift of the risk neutral price model can be written as*
$\mu_t^{\mathbb{Q}} = r_t - (1/\tau)\log\left[1 + (sk_t/3!)\sigma_{t,\tau}^{\mathbb{Q}\,3} + ((ku_t-3)/4!)\sigma_{t,\tau}^{\mathbb{Q}\,4} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right)\right]$.

On this basis, it is easy to show that the modified Corrado-Su formula is an approximated version of our call formula in which we only retain the first four elements of a Taylor expansion in $\sigma_{t,\tau}^{\mathbb{Q}}$ of the SNP call pricing formula:

Proposition 12 *Consider the call price* C_t^{SNP} *in Proposition 11. Then, if we neglect the term* ζ_t, C_t^{SNP} *can be written as* $C_t^{SNP} = C_t^{*CS} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right)$, *where* C_t^{*CS} *is the modified Corrado-Su formula (see Jurczenko et al., 2002b)*

$$C_t^{*CS} = C_t^{*BS} + sk_t\, Q_{3t}^* + (ku_t - 3)\, Q_{4t}^*, \tag{A1}$$

$$C_t^{*BS} = S_t\Phi(d_t^*) - Ke^{-r_t\tau}\Phi\left(d_t^* - \sigma_{t,\tau}^{\mathbb{Q}}\right),$$

$$d_t^* = \frac{1}{\sigma_{t,\tau}^{\mathbb{Q}}}\left[\log\left(\frac{S_t}{K}\right) + \left(r_t + \frac{\sigma_t^{\mathbb{Q}\,2}}{2}\right)\tau\right] - \frac{1}{\sigma_{t,\tau}^{\mathbb{Q}}}\log\left(1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}\right),$$

$$Q_{3t}^* = (1/3!)\sigma_{t,\tau}^{\mathbb{Q}} S_t \left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_t^*\right)\phi\left(d_t^*\right)\left(1 + (1/3!)\, sk_t\, \sigma_{t,\tau}^{\mathbb{Q}\,3} + (1/4!)(ku_t-3)\sigma_{t,\tau}^{\mathbb{Q}\,4}\right)^{-1},$$

and

$$\begin{aligned} Q_{4t}^* &= (1/4!)\sigma_{t,\tau}^{\mathbb{Q}} S_t \left(3\sigma_{t,\tau}^{\mathbb{Q}\,2} - 3d_t^*\sigma_{t,\tau}^{\mathbb{Q}} + d_t^{*2} - 1\right)\phi\left(d_t^*\right) \\ &\quad \times \left(1 + (1/3!)\, sk_t\, \sigma_{t,\tau}^{\mathbb{Q}\,3} + (1/4!)(ku_t-3)\sigma_{t,\tau}^{\mathbb{Q}\,4}\right)^{-1}. \end{aligned}$$

The main difference between the SNP model and the modified Corrado-Su formula results from the fact that Corrado and Su do not impose positivity restrictions on the density. In fact, a statistically correct version of the Corrado-Su model should impose the positivity restrictions of Jondeau and Rockinger (2001). Hence, our SNP assumption, which implicitly guarantees a non-negative density, leads to a slightly more complex formula for the same number of parameters (i.e., for $m = 2$). However, as Proposition 12 shows, if we eliminate the higher order terms in the infinite expansion of Proposition 11, the same fundamental effects of skewness and kurtosis emerge. Furthermore, if we neglect the terms $\sigma_{t,\tau}^{\mathbb{Q}\,k}$ for $k \geq 3$ in a Taylor expansion of (A1) we can relate the SNP and the Black-Scholes model with the following result:

Proposition 13 *We can write C_t^{SNP} as*

$$C_t^{SNP} = C_t^{BS} + \beta_{3t} sk_t + \beta_{4t}(ku_t - 3) + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right), \tag{A2}$$

where C_t^{BS} is the Black-Scholes formula, $d_{1t} = [\log(S_t/K) + (r_t + (1/2)\sigma_t^{\mathbb{Q}\,2})\tau]/(\sigma_t^{\mathbb{Q}}\sqrt{\tau})$ and

$$\beta_{3t} = (1/3!)S_t\sigma_{t,\tau}^{\mathbb{Q}}\left(\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}\right)\phi\left(d_{1t}\right) + (1/3!)K\exp(-r_t\tau)\phi\left(d_{1t}\right)\sigma_{t,\tau}^{\mathbb{Q}\,2},$$

and $\beta_{4t} = (1/4!)S_t\sigma_{t,\tau}^{\mathbb{Q}}\left(d_{1t}^2 - 3d_1\sigma_{t,\tau}^{\mathbb{Q}} - 1\right)\phi\left(d_{1t}\right).$

An analogous result is provided in Jurczenko, Maillet, and Negrea (2002b) for the modified Corrado-Su formula, under the name of "*Simplified Corrado-Su formula*". However, we will not obtain exactly the formula since Jurczenko, Maillet, and Negrea (2002b) approximate d_t^* by d_{1t}, which implies that they are effectively discarding some terms in $\sigma_{t,\tau}^{\mathbb{Q}\,2}$. We can also provide an approximate expression for the implied volatility in the SNP model:

Proposition 14 *Let C_t^{SNP} denote the market price on a European call option. Then the implied volatility Ψ_t for a given moneyness and time to maturity can be written as*

$$\Psi_t \simeq \sigma_t^{\mathbb{Q}}\sqrt{\tau} + \widetilde{\beta}_{3t} sk_t + \widetilde{\beta}_{4t}(ku_t - 3), \tag{A3}$$

where $\widetilde{\beta}_{3t} = (1/3!)\sigma_{t,\tau}^{\mathbb{Q}}\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}\right) + (1/3!)(K/S_t)\exp(-r_t\tau)\sigma_{t,\tau}^{\mathbb{Q}\,2},$
and $\tilde{\beta}_{4t} = (1/4!)\sigma_{t,\tau}^{\mathbb{Q}}\left(d_{1t}^2 - 3d_{1t}\sigma_{t,\tau}^{\mathbb{Q}} - 1\right).$

Table 1

(a) In-sample RMSE for the short maturity group with time-varying parameters.

Moneyness	BS	Pr. BS	GC^+	SNP(m=2)	G. Beta	V. Gamma	N
< 0.94	0.488	0.127	0.213	0.104	0.086	0.113	65
0.94-0.97	0.542	0.137	0.201	0.110	0.106	0.141	287
0.97-1.00	0.489	0.143	0.175	0.124	0.125	0.179	450
1.00-1.03	0.291	0.176	0.144	0.140	0.213	0.173	439
1.03-1.06	0.662	0.160	0.167	0.129	0.172	0.189	434
>1.06	0.732	0.284	0.435	0.334	0.436	0.330	1,176
Total	0.611	0.218	0.309	0.236	0.306	0.249	2,851

(b) Out-of-sample RMSE for the short maturity group with time-varying parameters.

Moneyness	BS	Pr. BS	GC^+	SNP(m=2)	G. Beta	V. Gamma	N
< 0.94	0.637	0.079	0.310	0.148	0.117	0.545	2
0.94-0.97	0.855	0.238	0.683	0.334	0.329	0.643	40
0.97-1.00	1.044	0.531	0.783	0.581	0.534	0.998	91
1.00-1.03	0.836	0.721	0.751	0.752	0.848	0.815	107
1.03-1.06	1.035	1.131	0.670	0.732	0.687	0.893	108
>1.06	1.064	5.911	0.882	0.823	0.847	0.860	263
Total	1.005	3.925	0.797	0.737	0.753	0.867	611

(c) In-sample RMSE for the long maturity group with time-varying parameters.

Moneyness	BS	Pr. BS	GC^+	SNP(m=2)	G. Beta	V. Gamma	N
< 0.94	1.878	0.330	0.848	0.251	0.169	0.141	360
0.94-0.97	1.634	0.298	0.625	0.191	0.149	0.145	365
0.97-1.00	1.196	0.251	0.366	0.175	0.195	0.161	457
1.00-1.03	0.630	0.209	0.323	0.202	0.188	0.144	474
1.03-1.06	0.968	0.244	0.454	0.166	0.174	0.152	440
>1.06	1.662	0.393	0.447	0.277	0.324	0.254	1,599
Total	1.464	0.327	0.500	0.235	0.251	0.201	3,695

(d) Out-of-sample RMSE for the long maturity group with time-varying parameters.

Moneyness	BS	Pr. BS	GC^+	SNP(m=2)	G. Beta	V. Gamma	N
< 0.94	2.045	0.401	0.832	0.596	0.324	0.605	36
0.94-0.97	2.153	0.929	0.935	0.900	0.692	1.126	59
0.97-1.00	1.654	0.977	0.987	0.894	0.876	0.813	66
1.00-1.03	1.102	0.840	1.077	1.079	1.020	0.797	94
1.03-1.06	1.358	0.764	0.969	0.953	0.943	0.792	97
>1.06	1.838	1.985	0.906	1.155	0.893	0.916	259
Total	1.703	1.438	0.952	1.037	0.880	0.876	611

Notes: In-sample analysis uses different parameters for each Wednesday from 1988 to 1992, while Out-of-sample tables use the parameters estimated on the previous Wednesday during 1993. Moneyness is defined as the ratio of the implicit forward price of the underlying asset to the strike price. BS, Pr. BS, GC^+ , G. Beta and V. Gamma denote, respectively, Black-Scholes, Practitioners' Black-Scholes, Gram-Charlier with positivity restrictions, Generalised Beta and Variance Gamma models. N denotes the number of option prices per moneyness category.

Table 2

(a) In-sample RMSE for the short maturity group with fixed shape parameters.

Moneyness	BS	Prac. BS	GC^+	SNP m=2	SNP m=3	SNP m=4	Gen. Beta	Var. Gamma	N
< 0.94	0.488	0.211	0.228	0.230	0.206	0.193	0.215	0.285	65
0.94-0.97	0.542	0.296	0.235	0.256	0.242	0.236	0.221	0.237	287
0.97-1.00	0.489	0.285	0.250	0.246	0.243	0.241	0.260	0.277	450
1.00-1.03	0.291	0.213	0.202	0.206	0.196	0.191	0.222	0.221	439
1.03-1.06	0.662	0.295	0.283	0.292	0.282	0.278	0.262	0.270	434
>1.06	0.732	0.503	0.473	0.443	0.422	0.408	0.464	0.451	1,176
Total	0.611	0.384	0.357	0.343	0.328	0.319	0.351	0.349	2,851

(b) Out-of-sample RMSE for the short maturity group with fixed shape parameters.

Moneyness	BS	Prac. BS	GC^+	SNP m=2	SNP m=3	SNP m=4	Gen. Beta	Var. Gamma	N
< 0.94	0.637	0.086	0.232	0.132	0.110	0.097	0.241	0.316	2
0.94-0.97	0.855	0.286	0.410	0.427	0.391	0.367	0.387	0.451	40
0.97-1.00	1.044	0.715	0.668	0.695	0.678	0.660	0.642	0.620	91
1.00-1.03	0.836	0.739	0.723	0.719	0.723	0.720	0.762	0.757	107
1.03-1.06	1.035	0.694	0.637	0.632	0.630	0.627	0.652	0.644	108
>1.06	1.064	0.859	0.882	0.815	0.775	0.740	0.862	0.846	263
Total	1.005	0.762	0.759	0.729	0.706	0.685	0.754	0.743	611

(c) In-sample RMSE for the long maturity group with fixed shape parameters.

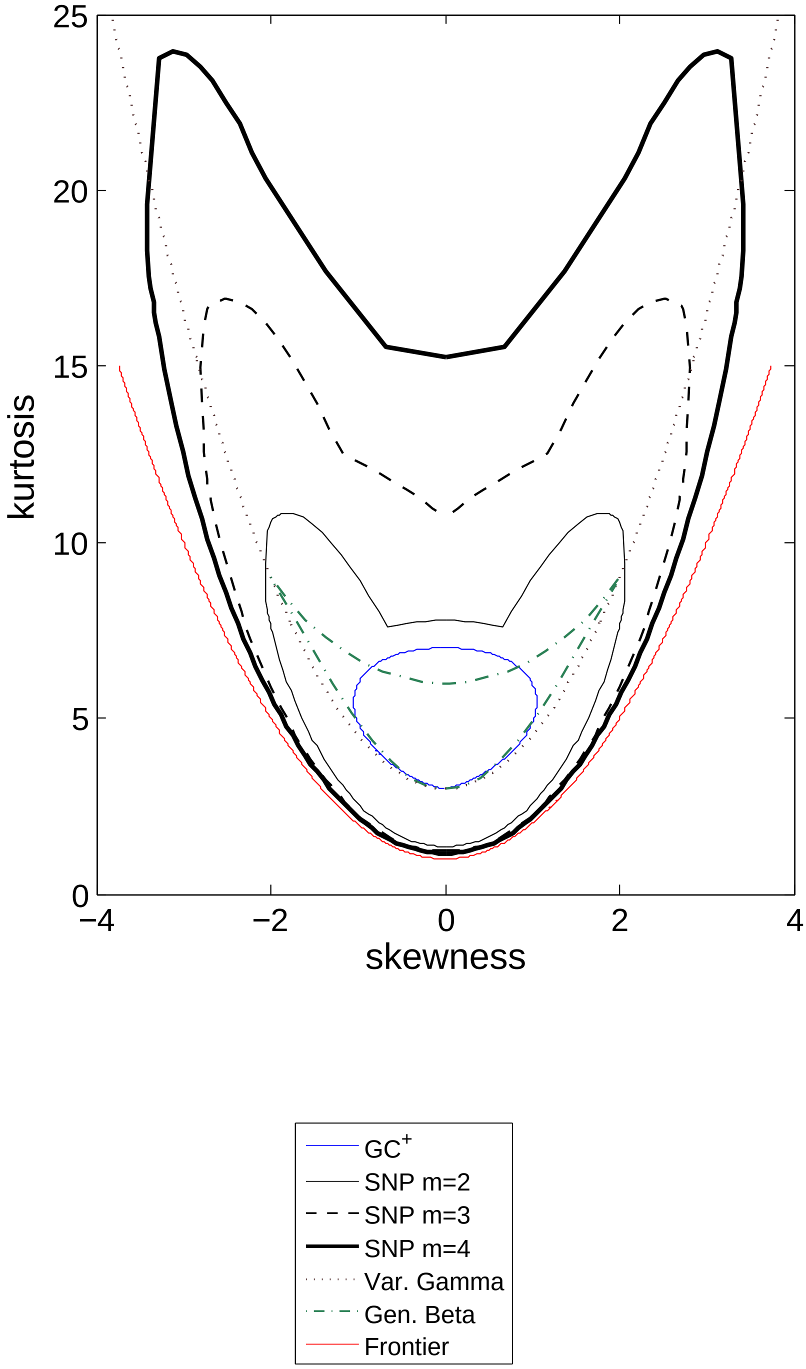
Moneyness	BS	Prac. BS	GC^+	SNP m=2	SNP m=3	SNP m=4	Gen. Beta	Var. Gamma	N
< 0.94	1.878	0.582	0.851	0.554	0.551	0.508	0.496	0.497	360
0.94-0.97	1.634	0.521	0.637	0.450	0.438	0.438	0.444	0.450	365
0.97-1.00	1.196	0.399	0.406	0.349	0.338	0.337	0.338	0.339	457
1.00-1.03	0.630	0.256	0.342	0.252	0.218	0.217	0.218	0.223	474
1.03-1.06	0.968	0.302	0.448	0.250	0.230	0.230	0.224	0.225	440
>1.06	1.662	0.583	0.530	0.512	0.461	0.455	0.453	0.454	1,599
Total	1.464	0.496	0.540	0.441	0.404	0.400	0.398	0.400	3,695

(d) Out-of-sample RMSE for the long maturity group with fixed shape parameters.

Moneyness	BS	Prac. BS	GC^+	SNP m=2	SNP m=3	SNP m=4	Gen. Beta	Var. Gamma	N
< 0.94	2.045	0.585	0.765	0.420	0.328	0.319	0.366	0.404	36
0.94-0.97	2.153	1.039	0.899	0.720	0.707	0.711	0.706	0.701	59
0.97-1.00	1.654	1.081	1.012	0.988	0.996	1.001	0.995	0.992	66
1.00-1.03	1.102	0.765	1.046	0.989	0.980	0.982	0.976	0.972	94
1.03-1.06	1.358	0.678	0.923	0.882	0.883	0.886	0.886	0.884	97
>1.06	1.838	0.613	0.849	0.903	0.841	0.832	0.841	0.847	259
Total	1.703	0.757	0.912	0.886	0.856	0.854	0.857	0.858	611

Notes: In-sample analysis (1988 to 1992) allows volatility to be time varying, but the other shape parameters are kept fixed. Out-of-sample estimates (1993) use for each week the volatility from the previous week and the fixed shape parameters estimated from the first five years. Moneyness is the ratio of the implicit forward price to the strike price. BS, Pr. BS, GC^+ , G. Beta and V. Gamma denote, respectively, Black-Scholes, Practitioners' Black-Scholes, Gram-Charlier with positivity restrictions, Generalised Beta and Variance Gamma models. N denotes the number of option prices per moneyness category.

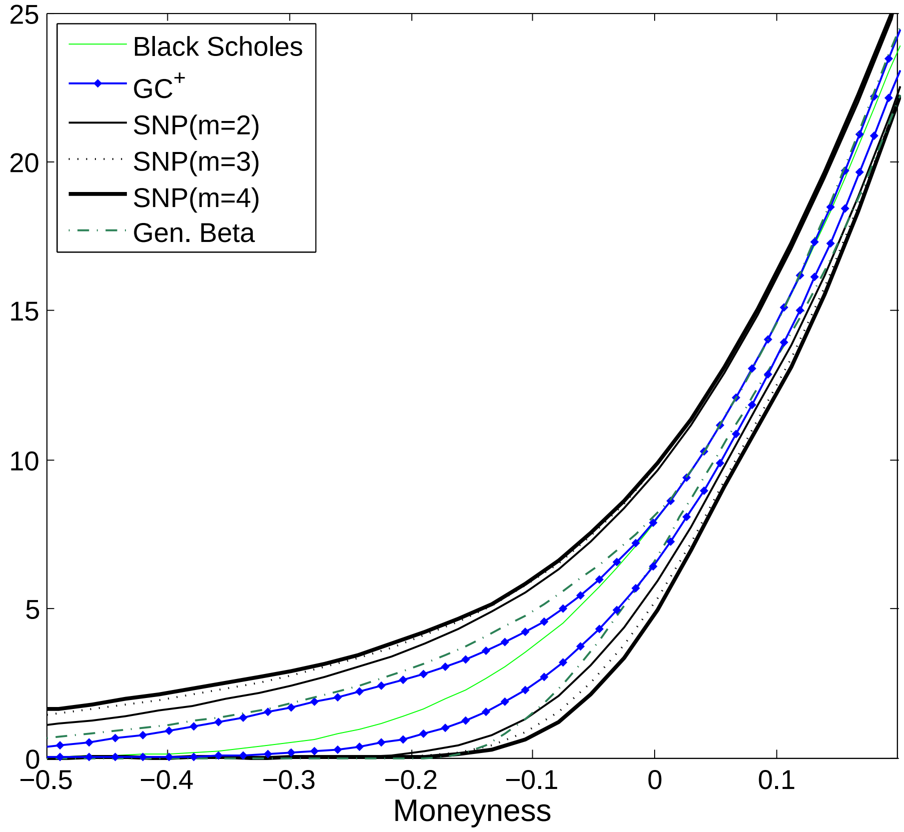
Figure 1
Regions of skewness and kurtosis



Note: GC$^+$ denotes a Gram-Charlier expansion of order $n = 4$ with positivity restrictions, while Gen. Beta denotes the distribution of the log of a Generalised Beta.

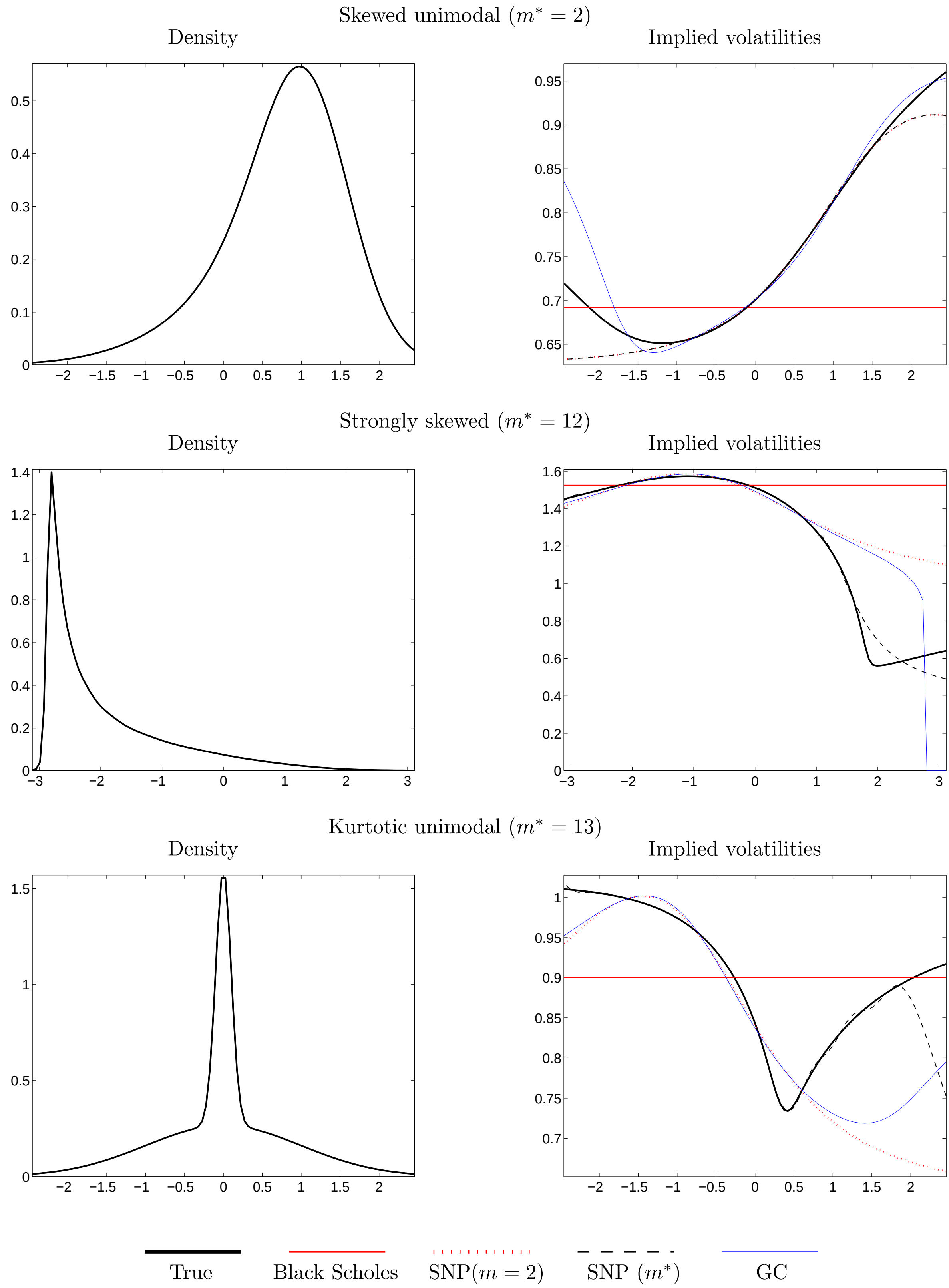
Figure 2

Flexibility to model departures from Black-Scholes



Note: This figure shows the minimum and maximum European call prices that each distribution can yield for a strike price of 100, a maturity of 3 months and a risk free interest rate of 3%. GC^+ denotes a Gram-Charlier expansion of order $n = 4$ with positivity restrictions.

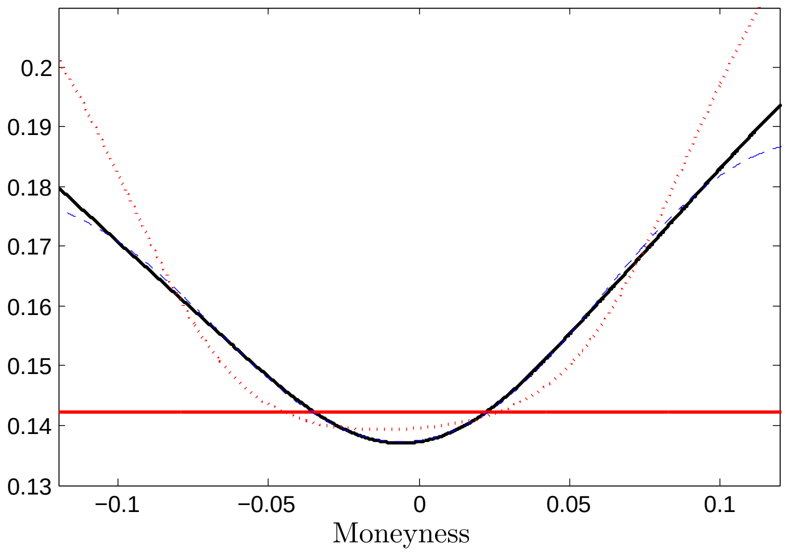
Figure 3: Estimation of options from Marron-Wand test suite



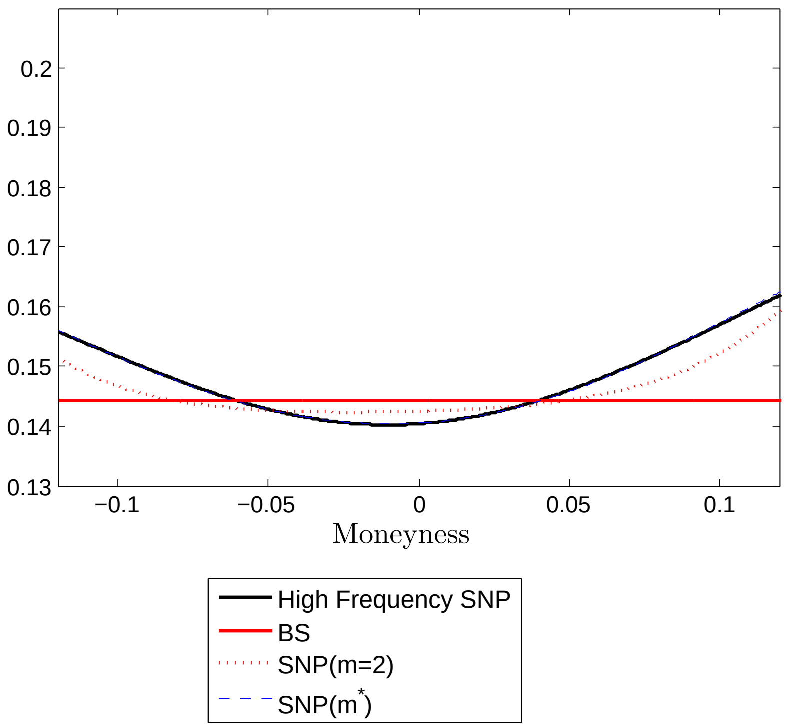
Notes: Marron-Wand densities are represented in the left panels. The corresponding true implied volatilities are plotted on the right panels, together with the ones obtained by estimating the SNP and Gram-Charlier option pricing models. SNP (m^*) denotes the SNP model of lowest order that makes the root mean square pricing error divided by the mean call price smaller than 10 basis point. The remaining non-Gaussian models only use two shape parameters.

Figure 4: Fit of the implied volatility of a multiperiod SNP process

(a) One-month options ($m^* = 4$)



(b) Three-month options ($m^* = 4$)



Notes: SNP (m^*) denotes the SNP model of lowest order that makes the root mean square pricing error divided by the mean call price smaller than 10 basis point. The option prices of the high frequency SNP model have been generated by assuming that the weekly log-returns under the risk-neutral measure are SNP of order 2 whose skewness and kurtosis is -0.4 and 6.5, respectively. Finally, the volatility follows a Markov chain with two states: $\sigma_1 = 0.1960$ and $\sigma_2 = 0.1023$. The probabilities of remaining in states 1 and 2 are $p = 0.9787$ and $q = 0.9847$, respectively. The risk-free rate is set at $r = 3\%$.

Figure 5a

Skewness and kurtosis for the short maturity group with time-varying parameters

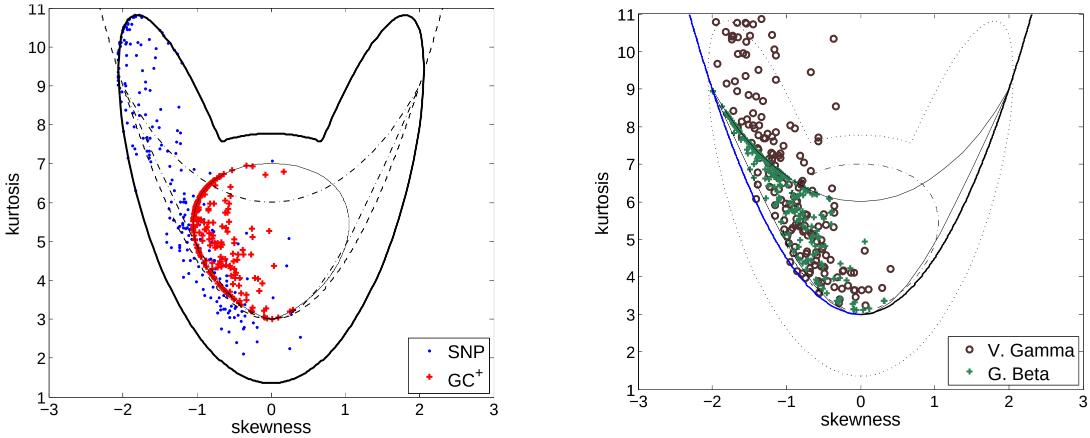


Figure 5b

Skewness and kurtosis for the long maturity group with time-varying parameters

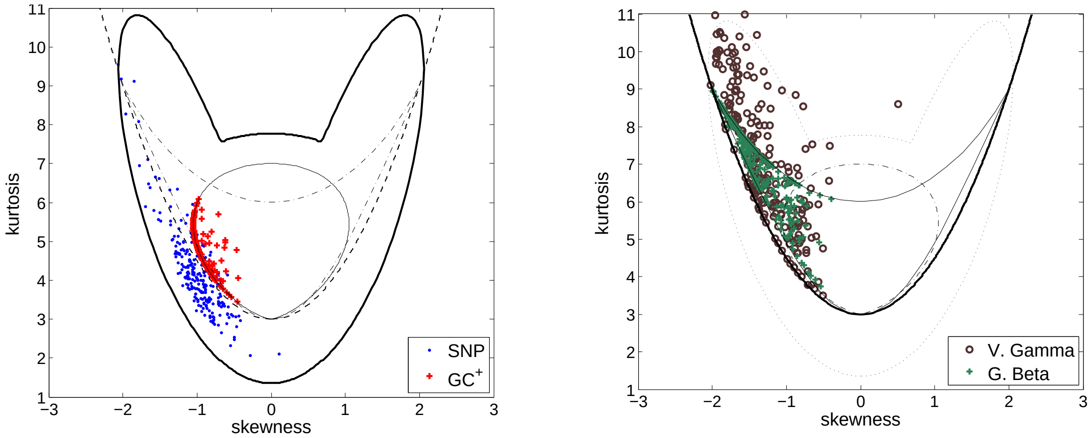


Figure 5c

Skewness and kurtosis of the bootstrapped call prices

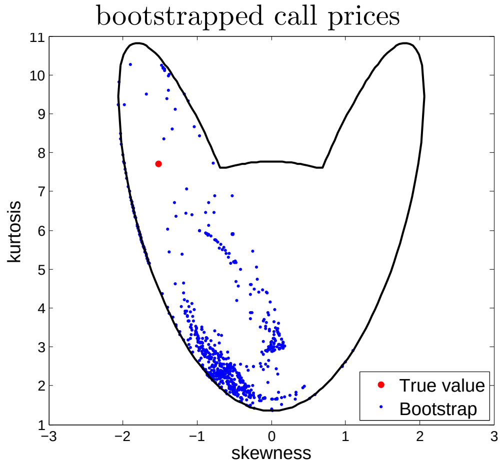
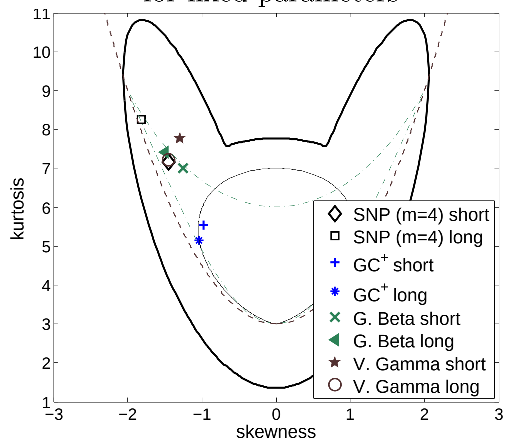


Figure 5d

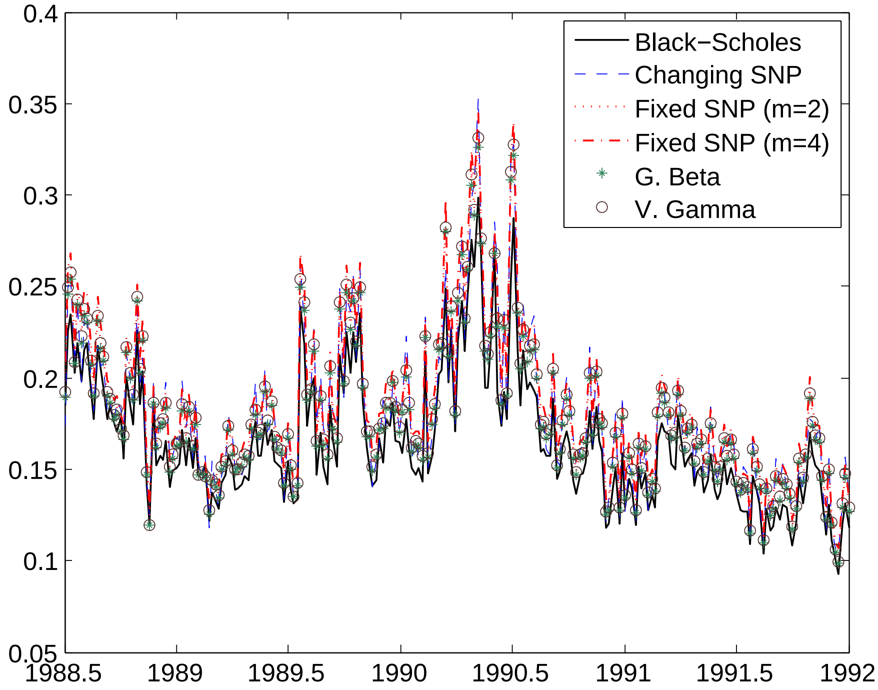
Skewness and kurtosis for fixed parameters



Notes: The results in Figures 5a and 5b correspond to separate estimations for each Wednesday in-sample, while to obtain Figure 5d all parameters except volatility are assumed to be constant over the whole sample. In Figures 5a and 5b SNP refers to a semi-nonparametric distribution of order 2. GC^{+}, G. Beta and V. Gamma denote, respectively, the Gram-Charlier expansion ($n = 4$) with positivity restrictions, the Generalised Beta and Variance Gamma models, while "Short" and "Long" denote the short and long maturity groups.

Figure 6

Option-implied volatilities for the short maturities



Note: "Fixed SNP" assumes that the shape parameters of the SNP are constant over time, while "Changing SNP" allows them to be time varying. Gen. Beta and V. Gamma denote, respectively, the Generalised Beta and Variance Gamma models.

Figure 7a
Risk-neutral density of $\log(S_T/S_t)$ for the short maturity group

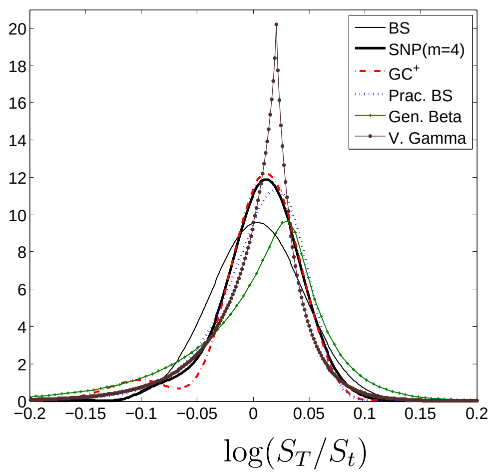


Figure 7b
Left tail of the risk-neutral density of $\log(S_T/S_t)$ for the short maturity group

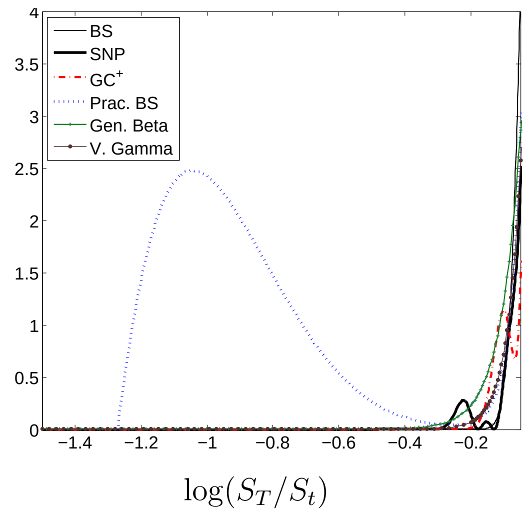


Figure 7c
Risk-neutral density of $\log(S_T/S_t)$ for the long maturity group

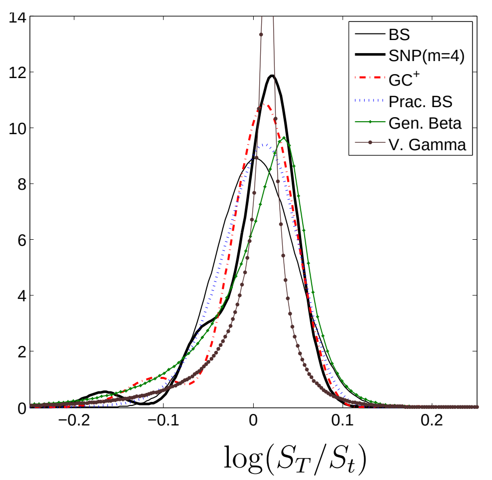
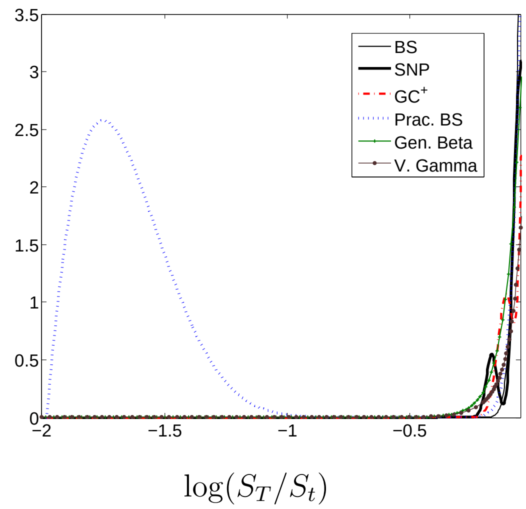


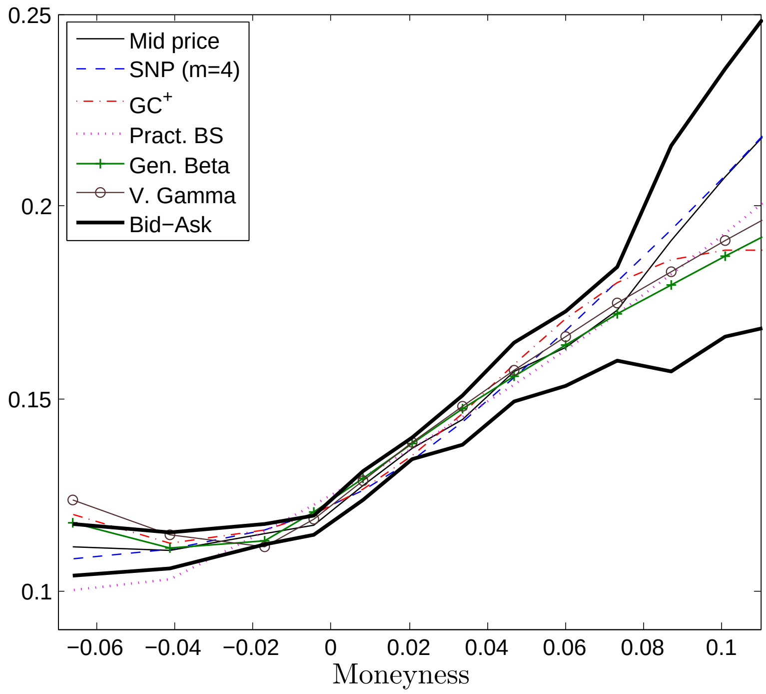
Figure 7d
Left tail of the risk-neutral density of $\log(S_T/S_t)$ for the long maturity group



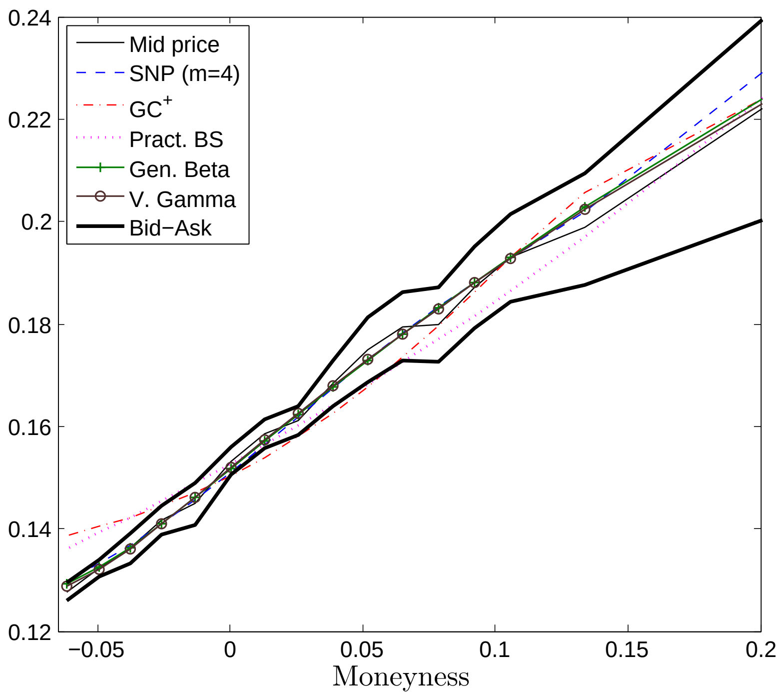
Notes: These results are based on the volatility estimated on November 13, 1991, but the shape parameters are estimated using data between 1988 and 1992. Pract. BS denotes a model in which volatility is a quadratic function of moneyness. SNP refers to a seminonparametric distribution of order 4, while GC^+, Gen. Beta and V. Gamma denote, respectively, the Gram-Charlier distribution ($n = 4$) with positivity restrictions, the Generalised Beta and Variance Gamma models.

Figure 8:
Implied volatility on November 13, 1991

(a) Short maturity group



(b) Long maturity group



Note: All the models use time varying volatilities but constant shape parameters. Moneyness defined as $\log(S_t/K) + r(T-t)$. Pract. BS denotes a model in which volatility is a quadratic function of moneyness. SNP (m=4) refers to a seminonparametric distribution of order 4, while GC^{+}, Gen. Beta and V. Gamma denote, respectively, the Gram-Charlier expansion ($n = 4$) with positivity restrictions, the Generalised Beta and Variance Gamma models.

Supplemental appendices for

Parametric properties of semi-nonparametric distributions, with applications to option valuation

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B Properties of Hermite polynomials

The j^{th} derivative of a Hermite polynomial of order k (see Stuart and Ord, 1977), is

$$\frac{d^j}{dx^j}H_k(x) = \sqrt{\frac{k!}{(k-j)!}}H_{k-j}(x)$$

if $j \leq k$, and zero otherwise. Using this result, $H_k(a+b)$ can be expressed as the following finite order Taylor expansion around a

$$\begin{aligned} H_k(a+b) &= \sum_{j=0}^{k} \frac{1}{j!} \left. \frac{d^j}{dx^j}H_k(x)\right|_{x=a} b^j \\ &= \sum_{j=0}^{k} \frac{1}{j!}\sqrt{\frac{k!}{(k-j)!}}H_{k-j}(a)b^j \end{aligned} \tag{B4}$$

C Proofs

Proposition 1

We know that

$$\frac{1}{\boldsymbol{\nu}'\boldsymbol{\nu}}\left[\sum_{i=0}^{m}\nu_i H_i(x)\right]^2 = \sum_{i=0}^{m}\sum_{j=0}^{m}\frac{\nu_i\nu_j}{\boldsymbol{\nu}'\boldsymbol{\nu}}H_i(x)\,H_j(x) = \sum_{k=0}^{2m}\gamma_k(\boldsymbol{\nu})\,H_k(x)\,, \tag{C5}$$

where it is verified that $\forall i,j$

$$H_i(x)\,H_j(x) = \sum_{q\in\Gamma}\frac{1}{\sqrt{q!}}\begin{pmatrix} q \\ \frac{i-j+q}{2}\end{pmatrix}\left(\prod_{s=0}^{(i-j+q)/2-1}(i-s)\prod_{s=0}^{(j-i+q)/2-1}(j-s)\right)^{1/2}H_q(x)\,, \tag{C6}$$

with

$$\Gamma = \left\{q\in\mathbb{N} : |i-j|\leq q\leq i+j;\quad \frac{i-j+q}{2}\in\mathbb{N}\right\}.$$

We can rewrite (C6) as

$$\begin{aligned} H_i(x)\,H_j(x) &= \sum_{q\in\Gamma}\frac{(i!j!q!)^{1/2}}{\left(\frac{i+j-q}{2}\right)!\left(\frac{i+q-j}{2}\right)!\left(\frac{q+j-i}{2}\right)!}H_q(x) \\ &= \sum_{q\in\Gamma}a_{ij,q}H_q(x) \end{aligned}$$

after verifying that $a_{ij,q} = a_{iq,j} = a_{ji,q} = a_{jq,i} = a_{qi,j} = a_{qj,i}$ by using some properties of the binomial coefficients. Hence, we will have that

$$\sum_{i=0}^{m}\sum_{j=0}^{m}\frac{\nu_i\nu_j}{\boldsymbol{\nu}'\boldsymbol{\nu}}H_i(x)\,H_j(x) = \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k\in\Gamma}\frac{\nu_i\nu_j}{\boldsymbol{\nu}'\boldsymbol{\nu}}a_{i,j,k}H_k(x)\,. \tag{C7}$$

Finally, if we equate (C5) and (C7), we obtain the desired result.

Proposition 2

Consider the expanded SNP density function (4). Then

$$\begin{aligned} E_f\left[H_k(x)\right] &= \int_{-\infty}^{\infty} \phi(x) H_k(x) \left(\sum_{i=0}^{2m} \gamma_k(\boldsymbol{\nu}) H_i(x)\right) dx \\ &= \sum_{i=0}^{2m} \gamma_k(\boldsymbol{\nu}) E_\phi\left[H_i(x) H_k(x)\right] \end{aligned}$$

We can easily obtain (7) by using the property that $E_\phi\left[H_i(x) H_k(x)\right] = 1$ if $i = k$ and zero otherwise.

Lemma 1

By using Proposition 2 we can directly obtain the matrices:

$$A_k = \begin{pmatrix} a_{00,k} & & \\ a_{10,k} & a_{11,k} & \\ a_{20,k} & a_{21,k} & a_{22,k} \end{pmatrix}$$

for $k = 1, \ldots, 4$ and $m = 2$. Specifically,

$$A_1 = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & \sqrt{2} & 0 \end{pmatrix}; \qquad A_2 = \begin{pmatrix} 0 & & \\ 0 & \sqrt{2} & \\ 1 & 0 & 2\sqrt{2} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & \sqrt{3} & 0 \end{pmatrix}; \qquad A_4 = \begin{pmatrix} 0 & & \\ 0 & 0 & \\ 0 & 0 & \sqrt{6} \end{pmatrix}.$$

On this basis, we can directly compute $E_f\left[H_k(x)\right]$ in (7). Finally, we can apply the equations in (6) to obtain the values of $\mu_x'(k)$.

Proposition 3

Note that

$$\begin{aligned} E_f\left(e^{tx}\right) &= \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}) \int_{-\infty}^{+\infty} e^{tx} H_k(x) \phi(x)\, dx \\ &= \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}) E_\phi\left[e^{tx} H_k(x)\right], \end{aligned} \tag{C8}$$

and that

$$\int H_k(x) \phi(x)\, dx = \frac{-1}{\sqrt{k}} H_{k-1}(x) \phi(x). \tag{C9}$$

If we consider (C9), and integrate by parts (C8), we obtain:

$$\begin{aligned} E_\phi\left[e^{tx} H_k(x)\right] &= \left[e^{tx} H_{k-1}(x) \phi(x) / \sqrt{k}\right]_{-\infty}^{+\infty} + \frac{t}{\sqrt{k}} E_\phi\left[e^{tx} H_{k-1}(x)\right] \\ &= \frac{t}{\sqrt{k}} E_\phi\left[e^{tx} H_{k-1}(x)\right]. \end{aligned}$$

where the subindex ϕ denotes integration with respect to the standard normal density. By l'Hospital rule, we can then verify that $e^{tx} H_{k-1}(x)\,\phi(x) \to 0 \;\; \forall k \geq 1$ when $x \to \pm\infty$. Hence,

$$E_\phi \left[e^{tx} H_k(x) \right] = \frac{t^k}{\sqrt{k!}} e^{t^2/2}. \tag{C10}$$

In addition, given (C8) and (C10), we will have that:

$$\begin{aligned} E\left(e^{\lambda x}\right) &= e^{t^2/2} \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\nu}) \frac{t^k}{\sqrt{k!}} \\ &= e^{\lambda^2/2} \Lambda(\boldsymbol{\theta}, t). \end{aligned}$$

On the other hand, the characteristic function can be written as

$$\begin{aligned} \psi_{snp}(t) &= \int_{-\infty}^{+\infty} \exp(itx)\,\phi(x) \sum_{j=0}^{2m} \gamma_j(\boldsymbol{\nu}) H_j(x)\,dx \\ &= \sum_{j=0}^{2m} \gamma_j(\boldsymbol{\nu}) \int_{-\infty}^{+\infty} \exp(itx)\,\phi(x)\,H_j(x)\,dx, \end{aligned}$$

where

$$\int_{-\infty}^{+\infty} \exp(itx)\,\phi(x)\,H_0(x)\,dx = \exp\left(\frac{-t^2}{2}\right)$$

coincides with the characteristic function of a standard normal variable. Then, using integration by parts we will have that

$$\begin{aligned} \int_{-\infty}^{+\infty} \exp(itx)\,\phi(x)\,H_1(x)\,dx &= -\exp(itx)\,\phi(x)\big]_{-\infty}^{+\infty} + it \int_{-\infty}^{+\infty} \exp(itx)\,\phi(x)\,dx \\ &= it \exp\left(\frac{-t^2}{2}\right). \end{aligned}$$

Finally, we can combine the relationships in (2) with

$$H_k'(x) = \sqrt{k} H_{k-1}(x),$$

to show by induction that

$$\int_{-\infty}^{+\infty} \exp(itx)\,\phi(x)\,H_k(x)\,dx = \frac{(it)^k}{\sqrt{k!}} \exp\left(\frac{-t^2}{2}\right).$$

Proposition 4

Since x_k are iid, we can use Proposition 3 to show that the characteristic function of q can be expressed as

$$\psi_q(t) = \prod_{k=1}^{n} \left[\exp\left(\frac{-p_k^2 t^2}{2}\right) \sum_{j=0}^{2m} \frac{(ip_k t)^j}{\sqrt{j!}} \gamma_j(\boldsymbol{\nu}) \right]. \tag{C11}$$

If we expand (C11), we will obtain:

$$\psi_q(t) = \exp\left(\frac{-\|p\|^2 t^2}{2}\right) \sum_{j=0}^{2mn} \frac{(it)^j}{\sqrt{j!}} \|p\|^j d_j(\boldsymbol{\nu}, \mathbf{p}), \tag{C12}$$

where the coefficients $d_j(\boldsymbol{\nu}, \mathbf{p})$ are such that

$$\prod_{k=1}^{n} \left[\sum_{j=0}^{2m} \frac{\gamma_j(\boldsymbol{\nu})}{\sqrt{j!}} (p_k z)^j\right] = \sum_{j=0}^{2mn} \frac{d_j(\boldsymbol{\nu}, \mathbf{p})}{\sqrt{j!}} z^j \tag{C13}$$

for all z. Hence, from (C13), it is straightforward to obtain (12). Finally, we can use Proposition 3 to show that the characteristic function of (11) is (C12), which proves that the density function of q is indeed (11).

Proposition 5

Consider the generating function of Hermite polynomials (see Bontemps and Meddahi, 2005):

$$\exp\left(zt - \frac{t^2}{2}\right) = \sum_{k=0}^{\infty} \frac{H_k(z)}{\sqrt{k!}} t^k. \tag{C14}$$

Notice that, using both the relation $z = a + bx$ and (C14), we can write the generating function as

$$\begin{aligned} \exp\left(zt - \frac{t^2}{2}\right) &= \exp\left(\frac{b^2t^2}{2}\right) \exp\left(btx - \frac{b^2t^2}{2}\right) \exp\left(at - \frac{t^2}{2}\right) \\ &= \exp\left(\frac{b^2t^2}{2}\right) \left\{\sum_{s=0}^{\infty} \frac{H_s(x)}{\sqrt{s!}} (bt)^s\right\} \left\{\sum_{m=0}^{\infty} \frac{H_m(a)}{\sqrt{m!}} t^m\right\}. \end{aligned} \tag{C15}$$

If we compute the expected value of the product of the generating function in (C14) times the Hermite polynomial of order i, both with argument x, where x is a standard normal variable, we get:

$$E_\phi\left[\exp\left((a+bx)t - \frac{t^2}{2}\right) H_i(x)\right] = \sum_{k=0}^{\infty} \frac{E_\phi\left[H_k(a+bx)H_i(x)\right]}{\sqrt{k!}} t^k. \tag{C16}$$

Analogously, we can obtain from (C15) that

$$\begin{aligned} E_\phi\left[\exp\left((a+bx)t - \frac{t^2}{2}\right) H_i(x)\right] &= \exp\left(\frac{b^2t^2}{2}\right) \left\{\sum_{m=0}^{\infty} \frac{H_m(a)}{\sqrt{m!}} t^m\right\} \\ &\quad \times \left\{\sum_{s=0}^{\infty} \frac{E_\phi\left[H_s(x)H_i(x)\right]}{\sqrt{s!}} (bt)^s\right\}. \end{aligned}$$

If we then combine the orthogonality property of the Hermite polynomials with the Taylor expansion for the above exponential function, we obtain

$$\begin{aligned} E_\phi\left[\exp\left((a+bx)t - \frac{t^2}{2}\right) H_i(x)\right] &= \frac{(bt)^i}{\sqrt{i!}} \exp\left(\frac{b^2t^2}{2}\right) \sum_{m=0}^{\infty} \frac{H_m(a)}{\sqrt{m!}} t^m \\ &= \frac{b^i}{\sqrt{i!}} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{H_m(a)}{j! 2^j \sqrt{m!}} b^{2j} t^{2j+i+m}. \end{aligned}$$

Finally, if we define $l = 2j + i + m$, we can write the above equation as

$$E_\phi \left[\exp\left((a+bx)t - \frac{t^2}{2} \right) H_i(x) \right] = \frac{b^i}{\sqrt{i!}} \sum_{j=0}^{\infty} \sum_{l=i+2j}^{\infty} \frac{H_{l-i-2j}(a)}{j! 2^j \sqrt{(l-i-2j)!}} b^{2j} t^l. \tag{C17}$$

Next, we can find the coefficients that multiply t^k for $k = 0, 1, 2, \cdots$, by comparing (C16) and (C17):

- When $i > k$:
$$E_\phi \left[H_k(a+bx) H_i(x) \right] = 0.$$
- When $i = k$:
$$E_\phi \left[H_i(a+bx) H_i(x) \right] = b^i.$$
- When $k > i$ and $k - i$ is an even number:
$$E_\phi \left[H_k(a+bx) H_i(x) \right] = b^i \sqrt{\frac{k!}{i!}} \sum_{j=0}^{\frac{k-i}{2}} \frac{H_{k-i-2j}(a)}{j! \sqrt{(k-i-2j)!} 2^j} b^{2j}.$$
- When $k > i$ and $k - i$ is an odd number:
$$E_\phi \left[H_k(a+bx) H_i(x) \right] = b^i \sqrt{\frac{k!}{i!}} \sum_{j=0}^{\frac{k-i-1}{2}} \frac{H_{k-i-2j}(a)}{j! \sqrt{(k-i-2j)!} 2^j} b^{2j}.$$

Proposition 6

Since we can write y_T as $y_T = \delta_{\mathbb{P}t} + \lambda_{\mathbb{P}t} x^{\mathbb{P}}$, the arbitrage free conditions become

$$\begin{aligned} E_{\mathbb{P}} \left[\exp\left(\alpha_t \lambda_{\mathbb{P}t} x^{\mathbb{P}} \right) \middle| I_t \right] &= \exp\left[-\alpha_t \delta_{\mathbb{P}t} - \beta_t \tau - r_t \tau \right], \\ E_{\mathbb{P}} \left[\exp\left((1+\alpha_t) \lambda_{\mathbb{P}t} x^{\mathbb{P}} \right) \middle| I_t \right] &= \exp\left[-(1+\alpha_t) \delta_{\mathbb{P}t} - \beta_t \tau \right]. \end{aligned}$$

Then, using Proposition 3, we can easily obtain (22) and (23) from the previous two equations.

Proposition 7

Using (3) and (25) we can write

$$\begin{aligned} f^{\mathbb{Q}}(y_T | I_t) &= \exp(r_t \tau) \exp(\alpha_t y_T + \beta_t \tau) \\ &\quad \times \frac{\phi\left(\frac{y_T - \delta_{\mathbb{P}t}}{\lambda_{\mathbb{P}t}} \right)}{\boldsymbol{\nu}_t' \boldsymbol{\nu}_t \lambda_{\mathbb{P}t}} \left[\sum_{i=0}^{m} \nu_{it} H_i \left(\frac{y_T - \delta_{\mathbb{P}t}}{\lambda_{\mathbb{P}t}} \right) \right]^2. \end{aligned} \tag{C18}$$

We can rearrange the elements in (C18) as

$$\begin{aligned} f^{\mathbb{Q}}(y_T | I_t) &= \exp(r_t \tau + \beta_t \tau) \exp\left(\alpha_t \delta_{\mathbb{P}t} + \frac{\alpha_t^2 \lambda_{\mathbb{P}t}^2}{2} \right) \\ &\quad \times \frac{\phi\left(\frac{y_T - \left(\delta_{\mathbb{P}t} + \alpha_t \lambda_{\mathbb{P}t}^2 \right)}{\lambda_{\mathbb{P}t}} \right)}{\boldsymbol{\nu}_t' \boldsymbol{\nu}_t \lambda_{\mathbb{P}t}} \left[\sum_{i=0}^{m} \nu_{it} H_i \left(\frac{y_T - \delta_{\mathbb{P}t}}{\lambda_{\mathbb{P}t}} \right) \right]^2 \qquad \text{(C19)} \\ &= \frac{\phi\left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} \right)}{\boldsymbol{\theta}_t' \boldsymbol{\theta}_t \lambda_{\mathbb{Q}t}} \left[\sum_{i=0}^{m} \theta_{it} H_i \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} \right) \right]^2, \qquad \text{(C20)} \end{aligned}$$

where $\delta_{\mathbb{Q}t} = \delta_{\mathbb{P}t} + \alpha_t \lambda_{\mathbb{P}t}^2$, $\lambda_{\mathbb{Q}t} = \lambda_{\mathbb{P}t}$. The parameters in the vector $\boldsymbol{\theta}_t = (\theta_{0t}, \theta_{1t}, \cdots, \theta_{mt})$ can be easily obtained by noting that we can always rewrite (C19) in terms of a squared sum of Hermite polynomials in $(y_T - \delta_{\mathbb{Q}t})/\lambda_{\mathbb{Q}t}$. That is, we can always find the value of $\boldsymbol{\theta}_t$ such that

$$\sum_{i=0}^{m} \theta_{it} H_i \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} \right) = \sum_{i=0}^{m} \nu_{it} H_i \left(\frac{y_T - \delta_{\mathbb{P}t}}{\lambda_{\mathbb{P}t}} \right). \tag{C21}$$

Starting from the right-hand side, we can write

$$\sum_{i=0}^{m} \nu_{it} H_i \left(\frac{y_T - \delta_{\mathbb{P}t}}{\lambda_{\mathbb{P}t}} \right) = \sum_{i=0}^{m} \nu_{it} H_i \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} + \alpha_t \lambda_{\mathbb{P}t} \right). \tag{C22}$$

Then, using (B4), we can show that (C22) equals

$$\sum_{k=0}^{m} \sum_{j=0}^{k} \nu_{kt} \frac{1}{j!} \sqrt{\frac{k!}{(k-j)!}} H_{k-j} \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} \right) (\alpha_t \lambda_{\mathbb{P}t})^j ,$$

which, through the change of indices $i = k - j$ becomes

$$\sum_{k=0}^{m} \sum_{i=0}^{k} \nu_{kt} \frac{1}{(k-i)!} \sqrt{\frac{k!}{i!}} H_i \left(\frac{y_T - \delta_{\mathbb{Q}t}}{\lambda_{\mathbb{Q}t}} \right) (\alpha_t \lambda_{\mathbb{P}t})^{k-i} . \tag{C23}$$

Now, if we compare (C23) with (C21), it is straightforward to find (29). Finally, we only need to check that the integrating constants are equal, i.e.

$$\boldsymbol{\theta}_t' \boldsymbol{\theta}_t = \boldsymbol{\nu}_t' \boldsymbol{\nu}_t \exp \left(-r_t \tau - \beta_t \tau - \alpha_t \delta_{\mathbb{P}t} - \frac{\alpha_t^2 \lambda_{\mathbb{P}t}^2}{2} \right). \tag{C24}$$

We have already shown that both (C19) and (C20) are proportional. Since both expressions are well defined densities in the sense that both integrate to one, (C24) must necessarily be satisfied. In consequence, y_T can be written under the risk neutral measure as

$$y_T = \delta_{\mathbb{Q}t} + \lambda_{\mathbb{Q}t} x^{\mathbb{Q}}, \tag{C25}$$

where $x^{\mathbb{Q}}$ is a non-standardised SNP variable with parameters $\boldsymbol{\theta}_t$. Hence, both the real and the risk-neutral measures have a SNP distribution of the same order. In particular, if we express the asset price S_T under the risk-neutral measure as in (26), where $\kappa^* = a(\boldsymbol{\theta}_t) + b(\boldsymbol{\theta}_t) x^{\mathbb{Q}}$, then we can easily relate the risk-neutral drift and volatility by the following relations

$$\left(\mu_t^{\mathbb{Q}} - \frac{\left(\sigma_t^{\mathbb{Q}}\right)^2}{2} \right) \tau + \sigma_t^{\mathbb{Q}} \sqrt{\tau} a(\boldsymbol{\theta}_t) = \delta_{\mathbb{Q}t}, \tag{C26}$$

$$\sigma_t^{\mathbb{Q}} \sqrt{\tau} b(\boldsymbol{\theta}_t) = \lambda_{\mathbb{Q}t}. \tag{C27}$$

From (C27), it is straightforward to obtain (28), while the relationship for the drift can easily be found by replacing (28) in (C26).

Proposition 8

Let us start with (27). As we know, (21) implies

$$\begin{aligned} 1 &= E_{\mathbb{P}}\left[M_{t,T}\exp(y_T)\right| I_t] \\ &= \exp(-r_t\tau)E_{\mathbb{Q}}\left[\exp(y_T)\right| I_t]. \end{aligned}$$

Hence, since y_T can be written as (C25) in the risk neutral measure, we can use (C10) to show that

$$\exp\left(r_t\tau - \delta_{\mathbb{P}t} - \alpha_t\lambda_{\mathbb{P}t}^2 - \frac{1}{2}\lambda_{\mathbb{P}t}^2\right) = \Lambda\left(\boldsymbol{\theta}_t, \lambda_{\mathbb{Q}t}\right), \tag{C28}$$

where $\Lambda\left(\boldsymbol{\theta}_t, \lambda_{\mathbb{Q}t}\right)$ is given in (8). From (C28), we can write

$$\alpha_t\sigma_t^2 b^2\left(\boldsymbol{\nu}_t\right) = r_t - \mu_t - \frac{\sigma_t^2}{2}\left(b^2\left(\boldsymbol{\nu}_t\right) - 1\right) - \frac{\sigma_t}{\sqrt{\tau}}a\left(\boldsymbol{\nu}_t\right) - \log\Lambda\left(\boldsymbol{\theta}_t, \lambda_{\mathbb{Q}t}\right),$$

which, once substituted in (27), yields (30).

Proposition 9

Consider the general option formula (35) and equation (19), and express the set corresponding to $\{S_T > K\}$, denoted as A for brevity, as $\{x > d_t\}$, where d_t is given in Proposition 9. Then, (35) can be rewritten as

$$C_t^{SNP} = S_t \Pr_{\mathbb{Q}_1}\left[x > d_t\right| I_t] - Ke^{-r_t\tau}\Pr_{\mathbb{Q}}\left[x > d_t\right| I_t].$$

If we apply the limits of integration $+\infty$ and d_t to the indefinite integral (C9), taking into account that $H_k\left(x\right)\phi\left(x\right) \to 0$ when $x \to +\infty$ (use L'Hospital rule), then

$$\int_{d_t}^{\infty} H_k\left(x\right)\phi\left(x\right)dx = \frac{1}{\sqrt{k}}H_{k-1}\left(d_t\right)\phi\left(d_t\right), \qquad k \geq 1. \tag{C29}$$

Given (4), (C29) and the fact that $\gamma_0(\boldsymbol{\theta}_t) = 1$, we can easily compute:

$$\begin{aligned} \Pr_{\mathbb{Q}}\left[x > d_t\right| I_t] &= \sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta}_t)\int_{d_t}^{+\infty} H_k\left(x\right)\phi\left(x\right)dx \\ &= \Phi\left(-d_t\right) + \sum_{k=1}^{2m}\frac{\gamma_k(\boldsymbol{\theta}_t)}{\sqrt{k}}H_{k-1}\left(d_t\right)\phi\left(d_t\right). \end{aligned}$$

Next, we will solve $\Pr_{\mathbb{Q}_1}\left[x > d_t\right| I_t]$ by working under the $\mathbb{Q}$-measure, for which we must apply the Radon-Nikodym derivative, which in this case is just the inverse of (34), i.e.

$$\frac{d\mathbb{Q}_1}{d\mathbb{Q}} = e^{-r_t\tau}\frac{S_T}{S_t} = e^{-r_t\tau + \delta_{\mathbb{Q}t} + \lambda_{\mathbb{Q}t}x}.$$

Then,

$$\begin{aligned} E_{\mathbb{Q}_1}\left[\mathbf{1}(A)\right| I_t] &= E_{\mathbb{Q}}\left(\frac{d\mathbb{Q}_1}{d\mathbb{Q}}\mathbf{1}(A)\middle| I_t\right) \\ &= e^{-r_t\tau + \delta_{\mathbb{Q}t}}\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta}_t)\int_{d_t}^{\infty} e^{\lambda x}H_k\left(x\right)\phi\left(x\right)dx \\ &= e^{-r_t\tau + \delta_{\mathbb{Q}t}}\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta}_t)E_{\phi}\left[e^{\lambda_{\mathbb{Q}t}x}H_k\left(x\right)\mathbf{1}(A)\right]. \end{aligned} \tag{C30}$$

For the sake of brevity, define $I^*_{k,t}$ as $E_\phi\left[e^{\lambda_{\mathbb{Q}t}x}H_k\left(x\right)\mathbf{1}(A)\right]$. The next step consists in computing $I^*_{k,t}$ for each k. When $k=0$, the integral is easy to obtain, namely, $I^*_{0,t}=e^{\lambda^2_{\mathbb{Q}t}/2}\Phi\left(\lambda-d_t\right)$. But since $\gamma_0(\boldsymbol{\theta}_t)=1$, we can rewrite (C30) as

$$\Pr_{\mathbb{Q}_1}\left[x>d_t|\,I_t\right]=e^{-r_t\tau+\delta_{\mathbb{Q}t}}\left[e^{\lambda^2_{\mathbb{Q}t}/2}\Phi\left(\lambda_{\mathbb{Q}t}-d_t\right)+\sum_{k=1}^{2m}\gamma_k(\boldsymbol{\theta}_t)I^*_{k,t}\right].$$

Now, we will obtain the value of $I^*_{k,t}$ when $k\geq 1$. To do so, we will integrate by parts taking (C9) into account, which results in

$$\begin{aligned} I^*_{k,t} &= \int_{d_t}^{\infty} e^{\lambda_{\mathbb{Q}t}x}H_k\left(x\right)\phi\left(x\right)dx \qquad \text{(C31)}\\ &= -\left[e^{\lambda_{\mathbb{Q}t}x}H_{k-1}\left(x\right)\phi\left(x\right)/\sqrt{k}\right]_{d_t}^{\infty}+\frac{\lambda_{\mathbb{Q}t}}{\sqrt{k}}\int_{d_t}^{\infty}e^{\lambda_{\mathbb{Q}t}x}H_{k-1}\left(x\right)\phi\left(x\right)dx\\ &= -\left[e^{\lambda_{\mathbb{Q}t}x}H_{k-1}\left(x\right)\phi\left(x\right)/\sqrt{k}\right]_{d_t}^{\infty}+\frac{\lambda_{\mathbb{Q}t}}{\sqrt{k}}I^*_{k-1,t}. \end{aligned}$$

Since it is verified by applying L'Hospital rule that $e^{\lambda x}H_{k-1}\left(x\right)\phi\left(x\right)\rightarrow 0\ \ \forall k\geq 1$ when $x\rightarrow\infty$, then

$$I^*_{k,t}=\frac{1}{\sqrt{k}}e^{\lambda_{\mathbb{Q}t}d_t}H_{k-1}\left(d_t\right)\phi\left(d_t\right)+\frac{\lambda_{\mathbb{Q}t}}{\sqrt{k}}I^*_{k-1,t}.$$

Finally, we can recursively obtain the formula for $I^*_{k,t}$ given in (37).

Proposition 10

Since the roots of $P_{2m}(x)$ are real and double or complex conjugates, we can express this polynomial as

$$\begin{aligned} P_{2m}(x) &= \prod_{j=1}^{j=m}[(x-a_j)^2+b_j^2]\\ &= \prod_{j=1}^{j=m}[(x-a_j-ib_j)(x-a_j+ib_j)] \end{aligned}$$

Alternatively, we can write $P_{2m}(x)$ as a sum of two squared polynomials of order m:

$$\begin{aligned} P_{2m}(x) &= \underbrace{\prod_{j=1}^{j=m}(x-a_j-ib_j)}_{Q(x)}\underbrace{\prod_{j=1}^{j=m}(x-a_j+ib_j)}_{\overline{Q}(x)}\\ &= Re^2[Q(x)]+Im^2[Q(x)] \end{aligned}$$

where $\overline{Q}(x)$ is the complex conjugate of $Q(x)$. Furthermore, it can be shown that the order of $Re[Q(x)]=P_{1,m}(x)$ is m, while the order of $Im[Q(x)]=P_{2,m-1}(x)$ is $m-1$ at most. Hence, we can express the GSNP as:

$$f_{GSNP}(x;\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)=\phi(x)\left[P^2_{1,m}(x)+P^2_{2,m-1}(x)\right]$$

where $P_{i,m_i}(x)=k_i\left[\nu_{i0}+\nu_{i1}H_1(x)+\cdots\nu_{im_i}H_{m_i}(x)\right]$, for $i=1,2$, $m_1=m$ and $m_2=m-1$. Since this density is homogeneous of degree zero, we can chose $k_1=p(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)/(\boldsymbol{\nu}_1'\boldsymbol{\nu}_1)$, and $k_2=[1-p(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)]/(\boldsymbol{\nu}_1'\boldsymbol{\nu}_1)$ without lost of generality.

Proposition 11

Given (26) for S_T where κ^* has a pdf defined in (15), and considering (17), we have that

$$\begin{aligned} g\left(\kappa^*|I_t\right) &= \phi\left(\kappa^*\right)\sum_{k=0}^{\infty} c_k(\boldsymbol{\theta}_t)H_k\left(\kappa^*\right) \\ &= \phi\left(\kappa^*\right)\left[1+\frac{sk_t}{\sqrt{3!}}H_3\left(\kappa^*\right)+\frac{ku_t-3}{\sqrt{4!}}+\sum_{k=5}^{\infty} c_k(\boldsymbol{\theta}_t)H_k\left(\kappa^*\right)\right]. \end{aligned}$$

Therefore, the call price C_t^{SNP} can be rewritten as:

$$\begin{aligned} C_t^{SNP} &= \xi_{0t}+\xi_{3t}sk_t+\xi_{4t}(ku_t-3)+\zeta_t \\ &= e^{-r_t\tau}\int_{\omega_t}^{\infty}\left(S_T\left(\kappa^*\right)-K\right)\phi\left(\kappa^*\right)d\kappa^* \\ &\quad +\frac{sk_t}{\sqrt{3!}}e^{-r_t\tau}\int_{\omega_t}^{\infty}\left(S_T\left(\kappa^*\right)-K\right)H_3\left(\kappa^*\right)\phi\left(\kappa^*\right)d\kappa^* \\ &\quad +\frac{ku_t-3}{\sqrt{4!}}e^{-r_t\tau}\int_{\omega_t}^{\infty}\left(S_T\left(\kappa^*\right)-K\right)H_4\left(\kappa^*\right)\phi\left(\kappa^*\right)d\kappa^* \\ &\quad +e^{-r_t\tau}\sum_{k=5}^{\infty} c_k(\boldsymbol{\theta}_t)\int_{\omega_t}^{\infty}\left(S_T\left(\kappa^*\right)-K\right)H_k\left(\kappa^*\right)\phi\left(\kappa^*\right)d\kappa^*, \end{aligned}$$

where ω_t is such that $S_T(\omega_t)=K$. Next, we will compute the values of ξ_{0t},ξ_{3t} and ξ_{4t}.

- For ξ_{0t}:

$$\begin{aligned} \xi_{0t} &= e^{-r_t\tau}\int_{\omega_t}^{\infty}\left(S_T\left(\kappa^*\right)-K\right)\phi\left(\kappa^*\right)d\kappa^* \\ &= S_te^{-r_t\tau+\mu_{t,\tau}^{\mathbb{Q}}}\int_{\omega_t}^{\infty} e^{\sigma_{t,\tau}^{\mathbb{Q}}\kappa^*}\phi\left(\kappa^*\right)d\kappa^* - Ke^{-r_t\tau}\Phi\left(-\omega_t\right) \\ &= S_te^{\left(\mu_t^{\mathbb{Q}}-r_t\right)\tau}\Phi\left(d_{1t}^*\right)-Ke^{-r_t\tau}\Phi\left(d_{1t}^*+\sigma_{t,\tau}^{\mathbb{Q}}\right), \end{aligned}$$

  where $\mu_{t,\tau}^{\mathbb{Q}}=\left(\mu_t^{\mathbb{Q}}-\sigma_t^{\mathbb{Q}\,2}/2\right)\tau$ and $d_{1t}^*=\sigma_{t,\tau}^{\mathbb{Q}}-\omega_t$.

To obtain ξ_{3t} and ξ_{4t}, we will use (37) and (C29). Specifically:

- For ξ_{3t}:

$$\xi_{3t}=\frac{1}{\sqrt{3!}}e^{-r_t\tau}\int_{\omega_t}^{\infty}\left(S_T\left(\kappa^*\right)-K\right)H_3\left(\kappa^*\right)\phi\left(\kappa^*\right)d\kappa^*$$

$$=\frac{1}{\sqrt{3!}}\left\{S_te^{-r_t\tau+\mu_{t,\tau}^{\mathbb{Q}}}\int_{\omega_t}^{\infty} e^{\sigma_{t,\tau}^{\mathbb{Q}}\kappa^*}H_3\left(\kappa^*\right)\phi\left(\kappa^*\right)d\kappa^*-Ke^{-r_t\tau}\int_{\omega_t}^{\infty}H_3\left(\kappa^*\right)\phi\left(\kappa^*\right)d\kappa^*\right\}$$

$$=\frac{1}{\sqrt{3!}}\left\{S_te^{-r_t\tau+\mu_{t,\tau}^{\mathbb{Q}}}I_{3,t}^*-\frac{1}{\sqrt{3}}Ke^{-r_t\tau}H_2\left(\omega_t\right)\phi\left(\omega_t\right)\right\}, \qquad \text{(C32)}$$

  Since

$$e^{\sigma_{t,\tau}^{\mathbb{Q}}\omega_t}=\frac{Ke^{-\mu_{t,\tau}^{\mathbb{Q}}}}{S_t},$$

then

$$I_{3,t}^{*} = \frac{e^{\sigma_{t,\tau}^{\mathbb{Q}\,2}/2}}{\sqrt{3!}} \left[\sigma_{t,\tau}^{\mathbb{Q}\,3} \Phi\left(\sigma_{t,\tau}^{\mathbb{Q}} - \omega_t\right) + \frac{K e^{-\mu_t^{\mathbb{Q}}\tau}}{S_t} \phi\left(\omega_t\right) \sum_{j=0}^{2} \sqrt{j!}\, \sigma_{t,\tau}^{\mathbb{Q}\,2-j} H_j\left(\omega_t\right) \right].$$

Plugging $I_{3,t}^{*}$ into equation (C32), we finally obtain

$$\begin{aligned} \xi_{3t} &= \frac{e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau}}{3!} \left[S_t \sigma_{t,\tau}^{\mathbb{Q}\,3} \Phi\left(\sigma_{t,\tau}^{\mathbb{Q}} - \omega_t\right) + K e^{-\mu_t^{\mathbb{Q}}\tau} \phi\left(\omega_t\right) \sum_{j=0}^{2} \sqrt{j!}\, \sigma_{t,\tau}^{\mathbb{Q}\,2-j} H_j\left(\omega_t\right) \right] \\ &\quad - \frac{1}{\sqrt{3!}} \frac{1}{\sqrt{3}} K e^{-r_t\tau} H_2\left(\omega_t\right) \phi\left(\omega_t\right) \\ &= \frac{e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau}}{3!} S_t \sigma_{t,\tau}^{\mathbb{Q}\,3} \Phi\left(\sigma_{t,\tau}^{\mathbb{Q}} - \omega_t\right) + \frac{K}{3!} e^{-r_t\tau} \phi\left(\omega_t\right) \left[\sigma_{t,\tau}^{\mathbb{Q}\,2} + \sigma_{t,\tau}^{\mathbb{Q}} \omega_t \right]. \end{aligned} \tag{C33}$$

Following the same idea as Jurczenko, Maillet, and Negrea (2002a), we can write:

$$\left(\sigma_{t,\tau}^{\mathbb{Q}} - \omega_t\right)^2 = \omega_t^2 + 2\log\left(S_t e^{\mu_t^{\mathbb{Q}}\tau}/K\right),$$

so that

$$\phi\left(\sigma_{t,\tau}^{\mathbb{Q}} - \omega_t\right) = \left(K/S_t e^{\mu_t^{\mathbb{Q}}\tau}\right) \phi\left(\omega_t\right),$$

which implies that

$$K\phi\left(\omega_t\right) = S_t e^{\mu_t^{\mathbb{Q}}\tau} \phi\left(\sigma_{t,\tau}^{\mathbb{Q}} - \omega_t\right).$$

If we substitute the above equation into (C33), we obtain:

$$\begin{aligned} \xi_{3t} &= \frac{\sigma_{t,\tau}^{\mathbb{Q}}}{3!} S_t e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau} \left[\sigma_{t,\tau}^{\mathbb{Q}\,2} \Phi\left(\sigma_{t,\tau}^{\mathbb{Q}} - \omega_t\right) + \left(\sigma_{t,\tau}^{\mathbb{Q}} + \omega_t\right) \phi\left(\sigma_{t,\tau}^{\mathbb{Q}} - \omega_t\right) \right] \\ &= \frac{\sigma_{t,\tau}^{\mathbb{Q}}}{3!} S_t e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau} \left[\sigma_{t,\tau}^{\mathbb{Q}\,2} \Phi\left(d_{1t}^{*}\right) + \left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}^{*}\right) \phi\left(d_{1t}^{*}\right) \right]. \end{aligned}$$

- For ξ_{4t}:

$$\begin{aligned} \xi_{4t} &= \frac{1}{\sqrt{4!}} e^{-r_t\tau} \int_{\omega}^{\infty} \left(S_T\left(\kappa^{*}\right) - K\right) H_4\left(\kappa^{*}\right) \phi\left(\kappa^{*}\right) d\kappa^{*} \\ &= \frac{1}{\sqrt{4!}} \left\{ S_t e^{-r_t\tau + \mu_{t,\tau}^{\mathbb{Q}}} I_{4,t}^{*} - \frac{1}{\sqrt{4}} K e^{-r_t\tau} H_3\left(\omega_t\right) \phi\left(\omega_t\right) \right\}. \end{aligned}$$

Following the same procedure as with ξ_{3t}, we can show that:

$$\xi_{4t} = \frac{\sigma_{t,\tau}^{\mathbb{Q}}}{4!} S_t e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau} \left[\sigma_{t,\tau}^{\mathbb{Q}\,3} \Phi\left(d_{1t}^{*}\right) + \left(3\sigma_{t,\tau}^{\mathbb{Q}\,2} - 3d_{1t}^{*}\sigma_{t,\tau}^{\mathbb{Q}} + d_{1t}^{*2} - 1\right) \phi\left(d_{1t}^{*}\right) \right].$$

Lemma 2

From (30), we have

$$\mu_t^{\mathbb{Q}} = r_t - \frac{1}{\tau} \log\left[\exp\left(\sigma_{t,\tau}^{\mathbb{Q}} a(\boldsymbol{\theta}_t) + \frac{1}{2} \sigma_{t,\tau}^{\mathbb{Q}\,2} \left(b^2(\boldsymbol{\theta}_t) - 1\right) \right) \sum_{k=0}^{2m} \gamma_k(\boldsymbol{\theta}_t) \frac{(\sigma_{t,\tau}^{\mathbb{Q}} b(\boldsymbol{\theta}_t))^k}{\sqrt{k!}} \right], \tag{C34}$$

where

$$\exp\left(\sigma_{t,\tau}^{\mathbb{Q}} a(\boldsymbol{\theta}_t) + \frac{1}{2}\sigma_{t,\tau}^{\mathbb{Q}\,2}\left(b^2(\boldsymbol{\theta}_t) - 1\right)\right) = 1 + a(\boldsymbol{\theta}_t)\sigma_{t,\tau}^{\mathbb{Q}} + \frac{a^2(\boldsymbol{\theta}_t) + b^2(\boldsymbol{\theta}_t) - 1}{2}\sigma_{t,\tau}^{\mathbb{Q}\,2}$$
$$+\frac{a^3(\boldsymbol{\theta}_t) + 3a(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 3a(\boldsymbol{\theta}_t)}{6}\sigma_{t,\tau}^{\mathbb{Q}\,3}$$
$$+\frac{3b^4(\boldsymbol{\theta}_t) - 6b^2(\boldsymbol{\theta}_t) + 3 + 6a^2(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 6a^2(\boldsymbol{\theta}_t) + a^4(\boldsymbol{\theta}_t)}{24}\sigma_{t,\tau}^{\mathbb{Q}\,4}$$
$$+o(\sigma_{t,\tau}^{\mathbb{Q}\,4}).$$

Then, from Proposition 1 we obtain that

$$\begin{aligned}
\gamma_0(\boldsymbol{\theta}_t) &= 1,\\
\gamma_1(\boldsymbol{\theta}_t) &= \frac{-a(\boldsymbol{\theta}_t)}{b(\boldsymbol{\theta}_t)},\\
\gamma_2(\boldsymbol{\theta}_t) &= \frac{a^2(\boldsymbol{\theta}_t) - b^2(\boldsymbol{\theta}_t) + 1}{b^2(\boldsymbol{\theta}_t)\sqrt{2}},\\
\gamma_3(\boldsymbol{\theta}_t) &= \frac{sk_t - a^3(\boldsymbol{\theta}_t) - 3a(\boldsymbol{\theta}_t) + 3a(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t)}{b^3(\boldsymbol{\theta}_t)\sqrt{3!}},\\
\gamma_4(\boldsymbol{\theta}_t) &= \frac{6a^2(\boldsymbol{\theta}_t) - 6a^2(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 6b^2(\boldsymbol{\theta}_t) + 3b^4(\boldsymbol{\theta}_t) + 3}{b^4(\boldsymbol{\theta}_t)\sqrt{4!}}\\
&\quad + \frac{6a^2(\boldsymbol{\theta}_t) - 6a^2(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 6b^2(\boldsymbol{\theta}_t) + 3b^4(\boldsymbol{\theta}_t) + 3}{b^4(\boldsymbol{\theta}_t)\sqrt{4!}}
\end{aligned}$$

Next, if we use the property that $o(n^p)o(n^q) = o(n^{p+q})$ (see Davidson and MacKinnon, 1993), we will have

$$\exp\left(\sigma_{t,\tau}^{\mathbb{Q}} a(\boldsymbol{\theta}_t) + \frac{1}{2}\sigma_{t,\tau}^{\mathbb{Q}\,2}\left(b^2(\boldsymbol{\theta}_t) - 1\right)\right)\sum_{k=0}^{2m}\gamma_k(\boldsymbol{\theta}_t)\frac{(\sigma_{t,\tau}^{\mathbb{Q}} b(\boldsymbol{\theta}_t))^k}{\sqrt{k!}} = \left[\sum_{k=0}^{4}\gamma_k(\boldsymbol{\theta}_t)\frac{(\sigma_{t,\tau}^{\mathbb{Q}} b(\boldsymbol{\theta}_t))^k}{\sqrt{k!}}\right]$$
$$\times\left[1 + a(\boldsymbol{\theta}_t)\sigma_{t,\tau}^{\mathbb{Q}} + \frac{a^2(\boldsymbol{\theta}_t) + b^2(\boldsymbol{\theta}_t) - 1}{2}\sigma_{t,\tau}^{\mathbb{Q}\,2} + \frac{a^3(\boldsymbol{\theta}_t) + 3a(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 3a(\boldsymbol{\theta}_t)}{6}\sigma_{t,\tau}^{\mathbb{Q}\,3}\right.$$
$$\left.+\frac{3b^4(\boldsymbol{\theta}_t) - 6b^2(\boldsymbol{\theta}_t) + 3 + 6a^2(\boldsymbol{\theta}_t)b^2(\boldsymbol{\theta}_t) - 6a^2(\boldsymbol{\theta}_t) + a^4(\boldsymbol{\theta}_t)}{24}\sigma_{t,\tau}^{\mathbb{Q}\,4}\right] + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right).$$

Finally, we can use tedious but otherwise straightforward algebraic operations to show that a Taylor expansion of the argument in the logarithm of (C34) around $\sigma_{t,\tau}^{\mathbb{Q}} = 0$ yields the proposed result.

Proposition 12

We can rewrite C_t^{SNP} in Proposition 11 as

$$\begin{aligned}
C_t^{SNP} &= S_t e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau}\Phi\left(d_{1t}^*\right)\left[1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t - 3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}\right] - Ke^{-r_t\tau}\Phi\left(d_{1t}^* - \sigma_{t,\tau}^{\mathbb{Q}}\right)\\
&\quad + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}} S_t e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau}\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}^*\right)\phi\left(d_{1t}^*\right)\\
&\quad + \frac{(ku_t - 3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}} S_t e^{\left(\mu_t^{\mathbb{Q}} - r_t\right)\tau}\left(3\sigma_{t,\tau}^{\mathbb{Q}\,2} - 3d_{1t}^*\sigma_{t,\tau}^{\mathbb{Q}} + d_{1t}^{*2} - 1\right)\phi\left(d_{1t}^*\right),
\end{aligned} \tag{C35}$$

where we have neglected ζ_t. From lemma 2, we finally have that

$$\begin{aligned}\exp[(\mu_t^{\mathbb{Q}} - r_t)\tau] &= \frac{1}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4} + o(\sigma_{t,\tau}^{\mathbb{Q}\,4})} \\ &= \frac{1}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right)\end{aligned}$$

because as $o(n^0) + o(n^p) = o(n^0)$ (see Davidson and MacKinnon, 1993), which, substituted into (C35), gives

$$\begin{aligned}C_t^{SNP} &= S_t\Phi\left(d_{1t}^*\right) - Ke^{-r_t\tau}\Phi\left(d_{1t}^* - \sigma_{t,\tau}^{\mathbb{Q}}\right) \\ &+\frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\frac{\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}^*\right)\phi\left(d_{1t}^*\right)}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}} \\ &+\frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\frac{\left(3\sigma_{t,\tau}^{\mathbb{Q}\,2} - 3d_{1t}^*\sigma_{t,\tau}^{\mathbb{Q}} + d_{1t}^{*2} - 1\right)\phi\left(d_{1t}^*\right)}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}}. \qquad \text{(C36)}\end{aligned}$$

Then, using again lemma 2, we can obtain the relationship

$$d_{1t}^* = d_t^* + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right),$$

which, once introduced in (C36), yields the Corrado-Su modified formula after neglecting the terms $o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right)$.

Proposition 13

Expanding d_{1t}^* around d_{1t}, we have

$$\begin{aligned}d_{1t}^* &= d_{1t} - \frac{1}{\sigma_t^{\mathbb{Q}}\sqrt{\tau}}\log\left(1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,4}\right)\right) \\ &= d_{1t} - \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,2} - \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,3}\right),\end{aligned}$$

$$\begin{aligned}\Phi\left(d_{1t}^*\right) &= \Phi\left(d_{1t}\right) - \phi\left(d_{1t}\right)\frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,2} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right) \\ \Phi\left(d_{1t}^* - \sigma_{t,\tau}^{\mathbb{Q}}\right) &= \Phi\left(d_{1t} - \sigma_{t,\tau}^{\mathbb{Q}}\right) - \phi\left(d_{1t} - \sigma_{t,\tau}^{\mathbb{Q}}\right)\frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,2} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right) \\ &= \Phi\left(d_{1t} - \sigma_{t,\tau}^{\mathbb{Q}}\right) - \phi\left(d_{1t}\right)\frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,2} + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right),\end{aligned}$$

$$\begin{aligned}\frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\frac{\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}^*\right)\phi\left(d_{1t}^*\right)}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}} &= \frac{sk_t}{3!}\frac{\sigma_{t,\tau}^{\mathbb{Q}}S_t\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}\right)\phi\left(d_{1t}\right) + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right)}{1 + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right)} \\ &= \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\left(2\sigma_{t,\tau}^{\mathbb{Q}} - d_{1t}\right)\phi\left(d_{1t}\right) + o\left(\sigma_{t,\tau}^{\mathbb{Q}\,2}\right),\end{aligned}$$

and

$$\frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\frac{\left(3\sigma_{t,\tau}^{\mathbb{Q}\,2} - 3d_{1t}^*\sigma_{t,\tau}^{\mathbb{Q}} + d_{1t}^{*2} - 1\right)\phi\left(d_{1t}^*\right)}{1 + \frac{sk_t}{3!}\sigma_{t,\tau}^{\mathbb{Q}\,3} + \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}\,4}} = \frac{(ku_t-3)}{4!}\sigma_{t,\tau}^{\mathbb{Q}}S_t\left(d_{1t}^2 - 3d_{1t}\sigma_{t,\tau}^{\mathbb{Q}} - 1\right)\phi\left(d_{1t}\right).$$

Then, we can easily take a Taylor series expansion of (C36) around $\sigma_{t,\tau}^{\mathbb{Q}} = 0$. If we only retain the terms in $\sigma_{t,\tau}^{\mathbb{Q}\,k}$, for $k = 0, 1, 2$, we finally obtain the desired result.

Proposition 14

Ψ_t is the implied volatility that equates the call market price C_t to the Black-Scholes formula, i.e. $C_t = C_t^{BS}(\Psi)$ where $C_t^{BS}(.)$ is the Black-Scholes formula. Following Jurczenko, Maillet, and Negrea (2002a), we can take a linear approximation of the Black-Scholes formula around the true volatility $\sigma_{t,\tau}^{\mathbb{Q}}$ of the underlying asset

$$C_t = C_t^{BS}(\Psi_t) = C_t^{BS}\left(\sigma_{t,\tau}^{\mathbb{Q}}\right) + \left.\frac{\partial C_t^{BS}(x)}{\partial x}\right|_{x=\sigma_{t,\tau}^{\mathbb{Q}}} \left(\Psi_t - \sigma_{t,\tau}^{\mathbb{Q}}\right)$$

Since

$$\left.\frac{\partial C_t^{BS}(x)}{\partial x}\right|_{x=\sigma_{t,\tau}^{\mathbb{Q}}} = K\phi\left[d_{1t} - \sigma_{t,\tau}^{\mathbb{Q}}\right] = S_t e^{r_t \tau}\phi\left[d_{1t}\right],$$

then

$$C_t \simeq C_t^{BS}\left(\sigma_{t,\tau}^{\mathbb{Q}}\right) + S_t\phi\left[d_{1t}\right]\left(\Psi_t - \sigma_{t,\tau}^{\mathbb{Q}}\right). \tag{C37}$$

Finally, if the call market price follows the SNP model, i.e. $C_t = C_t^{SNP}$, we can equate (A2) and (C37) to obtain the approximation to Ψ_t given in (A3).