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Endogenous State Prices, Liquidity, Default, and the Yield Curve

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Abstract

We show, in an exchange economy with default, liquidity constraints and no aggregate uncertainty, that state prices in a complete markets general equilibrium are a function of the supply of liquidity by the Central Bank. Our model is derived along the lines of Dubey and Geanakoplos (1992). Two agents trade goods and nominal assets (Arrow-Debreu (AD) securities) to smooth consumption across periods and future states, in the presence of cash-in-advance financing costs. We show that, with Von Neumann-Morgenstern logarithmic utility functions, the price of AD securities, are inversely related to liquidity. The upshot of our argument is that agents' expectations computed using risk-neutral probabilities give more weight in the states with higher interest rates. This result cannot be found in a Lucas-type representative agent general equilibrium model where there is neither trade or money nor default. Hence, an upward yield curve can be supported in equilibrium, even though short-term interest rates are fairly stable. The risk-premium in the term structure is therefore a pure *default risk premium*.

Keywords: cash-in-advance constraints; risk-neutral probabilities; state prices; term structure of interest rates

JEL Classification: E43; G12

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1 Introduction

Many financial economists have been puzzled by the fact that historical forward interest rates are on average higher than future spot interest rates. According to the expectation hypothesis, forward interest rates should reflect expectations of future spot rates, and this forms the basis of the efficiency market hypothesis. However, in the words of Shiller (1990), the forward term premium (i.e. the difference between the forward rate and the expectation of the corresponding future spot rate) has empirically been positive. Therefore, the puzzle is that forward rates have usually been higher than the historically stable interest rates, a situation that is evident in figure 1 in the period 1983-1984.

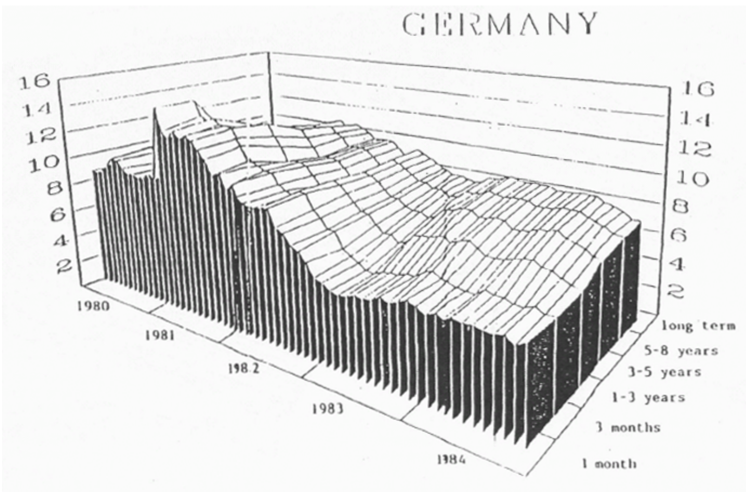

Figure 1: The term structure of interest rate in Germany, source: McCulloch data (Shiller and McCulloch (1987))

In the absence of arbitrage, with complete markets, no transaction costs, unsegmented markets, and equal tax treatment[1], an increasing term structure is possible if and only if instantaneous forward rates are increasing[2]. The history of upward sloping term structure implies increasing forward rates, and therefore, assuming the expectation hypothesis with risk-neutrality, in-

[1]The McCulloch data shown in the picture correct for differences in taxation

[2]At time t, the yield of a long-term bond maturing at T is equal to the simple averages of the instantaneous forward rates (see Shiller (1990) p.640).

creasing expected future spot rates - something that does not correspond to the historical evolution of short-term interest rates.

Are rational expectations failing and bond markets inefficient then? The early literature on the term structure had tried to explain this puzzle by appealing to liquidity or risk. Hicks (1946) emphasized that a risk-averse investor would prefer to lend short-term, if he was not given any premium on long-term lending, because there is a higher risk that the prices of long-term bonds change. Lutz (1940) suggested that long-term securities are less liquid than short-term ones, where the most liquid asset is money. Finally, Modigliani and Sutch (1966) proposed the preferred habitat hypothesis - a theory that has been very influential - arguing that agents prefer to trade bonds to match asset and liability maturities. Hence, the markets for long-term and short-term bonds would somehow be segmented, and therefore the link between long and short-term interest rates breaks down.

The modern literature has emphasized the importance of aggregate risk, following Lucas (1978). In these representative agent models, the forward interest rate is higher than the expected spot rate because of risk-aversion. The risk-premium is shown to be proportional to the correlation between marginal utility and the payoff of the asset (Breeden, 1979). Since high interest rates tend to depress activity, the correlation between the payoff of a bond and marginal utility is likely to be significant. Put differently, risk-neutral probabilities - which should be used instead of subjective probabilities to price any asset - are proportional to future marginal utilities (Breeden and Litzenberger, 1978). The application of this model to the term structure is due to Cox, Ingersoll and Ross (1985a and 1985b). Other well-known applications of this representative agent model are the Consumption CAPM and the attempt to explain the equity-premium puzzle using the Breeden formula (Mehra and Prescott, 1985).

In the present paper, we argue that aggregate consumption risk is not the only source of risk-premia in asset prices. An additional risk-premium exists because of the effect of financing costs on marginal utilities. This risk-premium cannot be captured by representative agent models because this premium exists even in absence of aggregate uncertainty (*i.e.* endowments and aggregate consumption are constant). We model financing costs with cash-in-advance constraints. The risk-premium exists for any asset price, but we focus as an application on the term structure and show that it can be upward-sloping in equilibrium, even when aggregate real uncertainty is null.

We set out a monetary general equilibrium model with cash-in-advance constraints built along the lines of Dubey and Geanakoplos (1992), Geanakoplos and Tsomocos (2002), Tsomocos (2003, 2007) and Goodhart *et al.*(2006). We need a general equilibrium model because we want to endogenise all demands for money in order to construct the risk-neutral probabilities and the yield curve. Provided the existence of outside money, these models are able to generate proper demand for liquidity and unique positive nominal interest rates. We assume that all states are equiprobable both in reality and as understood by the investors (i.e. subjective probabilities are uniform and beliefs are correct and homogeneous across agents). This way, we exclude the interesting issue of agents' utility heterogeneity (an issue covered, for instance, in Fan, 2006). We price nominal Arrow-Debreu securities (AD securities). The strong assumption of complete markets is needed here because we want to solve for all AD securities' prices. If the prices of AD securities were constant through all states of nature, then the historical average of spot interest rates that would proxy rational expectations without risk-premium $\mathbb{E}_t[r_{t,t+s}]$ would be equal to the expected interest rate $\mathbb{E}_\pi[r_{t,t+s}]$ using risk-neutral probabilities π. However, we will show that this is not the case in our model, even though there is no real uncertainty.

The main result of the paper is that states with higher interest rates (lower liquidity supplied by the Central Bank) have higher state prices. Intuitively, since we model consumer's utility with a Cobb-Douglas specification with equal weights on all states of nature, the cost of consumption is constant across all states. This cost of consumption is equal to the opportunity cost to transfer money from period 0 to period 1 (*i.e.* the AD security price) multiplied by the value of trade in period 1(*i.e.* the price of the good multiplied by the volume traded).

In the first paper in our study of liquidity, Espinoza and Tsomocos (2007), we used a cash-in-advance model in conjunction to exogenous inside and outside money stocks. Such a cash-in-advance model is widely considered to capture the effects of liquidity constraints in an analytically tractable way, and allows the quantity theory of money to hold. As a result, in such models, the value of trade is equal to overall supply of liquidity (i.e. money supply from the Central Bank plus outside money). If state 1 has more liquidity than state 2, the value of trade in state 1 has to be higher than the value of trade in state 2. With the Cobb-Douglas utility function assumption, this is possible only if the cost of transferring money in state 1 (*i.e.* the price of a claim with state-contingent payoff, also called the state price) is lower than the cost of transferring money in state 2. Therefore, a state with lower interest rate (higher liquidity) is also a state with lower state price (and therefore lower risk-neutral probability).

But there is an unresolved lacuna (puzzle), which is why in representative agent models, complete markets, and a transversality condition, in which everyone always pays off their debts in full with certainty, there is any need for money at all? Why cannot all exchanges simply be undertaken via book-keeping, with no prior need for an exchange of money? Why is there a cash-in-advance constraint at all? And what exactly are these costs of financing?

Our answer to this is that the most egregious (worst) assumption is the transversality condition. Once one allows for the possibility of default by the buyer of the good, who becomes a debtor to the seller, the seller will no longer be prepared to accept the buyer's IOU. She will want, instead, a safe asset, which will be more generally acceptable in subsequent transactions, *i.e.* money. Moreover, financing costs involve assessing the credit-worthiness of the buyer of a good, or of an asset, and of the IOU which he may be preferring as a counterpart to the purchase (n.b., even cash needs inspection to avoid forgery, and bank drafts and cheques may be defaulted).

Incorporating positive probabilities of default (PD) into a model is complex. Not only is the event of default non-linear, but the existence of positive PD is hardly consistent with many of the elements of the kind of theoretical models which are commonly in use, involving complete markets, no aggregate uncertainty and representative agents (meeting the transversality conditions). In order to move from the standard models towards the 'real world' in the simplest, smallest step possible, we have in this paper included the assumption of 'exogenous default', whereby each agent fails, and is expected to fail, to meet all his/her financial committments in asset markets. Otherwise all the trappings of the standard formal model, e.g. complete markets, a representative agent model, would remain in place. Of course, incomplete markets and endogenous default and ultimately essential for the actual phenomenon of monetary and financial dealings that we observe. At least, we are aiming here to make one small step towards reality.

We need to emphasize that our assumption of 'exogenous default' is compatible with earlier work in this class of models with endogenous default. Put differently, we could have employed Goodhart *et al.* (2006) and arrive at the same conclusion, however at a significant computational and analytical cost. The essence of the argument rests on the presence of positive default that is turn generates positive nominal interests that ultimately affect the overall liquidity in the economy. The upshot of studying liquidity, more generally and as in Goodhart *et al.* (2006), is its interaction with default. Liquidity affects and is affected by default.

The lesson of the model is that uncertainty in aggregate production or in aggregate consumption is only one part of uncertainty in agents' marginal utilities. Bansal and Coleman II (1996) produce a representative agent general equilibrium model with transaction costs that capture partly this effect on bond prices. However, in their model, trade is forced, since the representative agent sells all of his endowment and subsequently buys it back, and the transaction cost function is exogenously specified. In particular, transaction services are generated only from bond holdings and not from asset holdings, and this is how they show that the equity premium may be large.

In our model, uncertainty in future financing costs and trade volumes matter, and are endogenously derived. Therefore, any model of risk-premium that attempts to proxy welfare by production or consumption will underestimate risk-premium. This is especially important for the term structure risk premium since the spot interest rate has both an effect on the asset price and on the inter-temporal financing cost. In that case, the correlation between the marginal utilities and the asset price is likely to be high. The error will be an under-estimation of the risk-premium.

The model is an exchange economy with cash-in-advance constraints where larger money supply has the only effect of lowering short-term financing costs and has no other effect on production or endowments. More money supply allows for more efficient trade since financing costs are lower, provided that nominal interest rates are positive. Efficiency is established when money supply is infinite. These cash-in-advance models have several drawbacks since money supply is exogenously given, and the value of the interest rate depends on the existence of "outside money" (money endowed and free and clear of any liability, see Dubey and Geanakoplos, 1992), a somehow controversial assumption. However, the advantage of the cash-in-advance constraint model we are using here is that it allows us to incorporate markets for money even in a finite-horizon model and markets for AD securities. More fundamentally, the cash-in-advance constraints allow money to be non-neutral, although money does not affect consumption.

The presence of exogenous default plays the same role as the existence of outside money. Arguably, liquidity in any way defined becomes important and has real effects whenever default in the economy is present. We use therefore default in this paper to ensure a positive interest rate and a positive value for money. Money has then an effect on asset prices and payoffs through the nominal and the real channels. This is summarised in figure 5 (last page). Let us see the nominal channel first. A positive interest rate ensures that there exists a unique money demand, equal to money supply, thus leading to a unique interest rate that clears the credit market. Consequently, the price level is pinned down. This is in stark contrast with dichotomous models where, since money has no value, any money demand (and therefore any price level) can constitute equilibrium. However, when the indeterminacy on price levels is removed, asset prices are also uniquely determined (we show indeed that they are inversely related to liquidity, which answers our term structure puzzle). Money also has an effect through the real channel, because positive interest rates create a wedge between buying and selling prices and, therefore, distort marginal rates of substitution. This is the source of non-neutrality of money. In particular, trade and prices are typically higher with more liquidity. This in turn has an effect on the real asset payoffs.

INSERT FIGURE 5 HERE (SEE LAST PAGE)

This allows us to show the existence of a 'liquidity-premium'. However, there is a definition issue here. One can think of three effects of liquidity on bond prices. One liquidity premium would come from the costs incurred in a market where volumes and trade in an asset are small so that transaction costs are larger. A second cost is the one described by Hicks (1946):

> "the imperfect 'moneyness' of those bills which are not money is due to their lack of general acceptability which causes the trouble of investing in them, and causes them to stand at a discount"

The third effect, the liquidity premium we have here, comes from the the additional cost incurred by investors (and priced in the term structure) that an uncertain money supply will generate when liquidity is restricted (i.e. when the constraint binds, which is the assumption behind a cash-in-advance constraint model). Note that the level of money supply does not really matter: in the long run, if prices adjust to the money supply, constraints on liquidity do not have real effects - although this is not captured in our cash-in-advance constraint where the optimal supply of money would be infinite[3]. However, what still has effects is the variance (or risk) of liquidity. This is exactly what is captured in the model, where we show that larger liquidity risks generate higher long-term interest rates. *Stricto senso*, our model is therefore a model of the "*liquidity-risk* premium". This liquidity risk-premium is deduced immediately from the previous section, since risk-neutral probabilities are high when the interest rates are high. The existence of the "liquidity-risk premium" has at least two consequences. First, the term structure is upward sloping above what is expected from the pure expectation hypothesis even if the Lucas-type risk-premium is incorporated. Second, stability of monetary policy matters.

[3] We acknowledge that a deterministic decrease in money supply may also have a short-term liquidity cost if there is some inertia

2 The Model

The model is an exchange economy without production. Trade takes place between two agents who want to trade across periods (for consumption smoothing purposes) and across states (because of risk-aversion). Because cash is needed before commodity transactions, and because receipts of sells cannot be used immediately to buy commodities (the timing of the markets' meetings is represented in figure 2), agents require cash as a derived demand due to their transaction needs. Liquidity is supplied exogenously by the Central Bank who can diminish short-term financing costs by increasing money supply. With lower financing costs, more trade (i.e. more activity in this exchange economy) takes places and agents are closer to the standard General Equilibrium Pareto optimum.

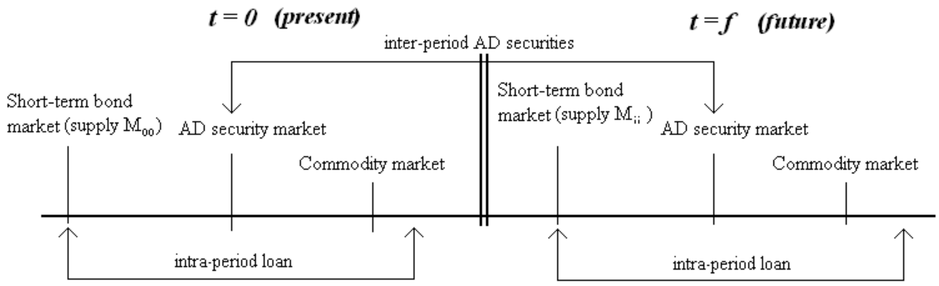

Figure 2: Timing of commodity and money markets

2.1 Structure of the Model

The model is built around two periods, period 0 (now) and period f (future). Periods are divided into sub-periods at which the different commodity and money markets meet, as pictured in figure 2. The supply of money by the Central Bank is the only source of randomness in the future. There are n states of nature possible, indexed by $i \in \mathscr{N} = \{1, \ldots, n\}$ and with subjective probabilities all equal to $\frac{1}{n}$. The Central Bank thus provides money for:

- the short-run period-0 money market, with money supply M_{00}, interest rate r_{00} and bond price $\eta_{00} = \dfrac{1}{1 + r_{00}}$

- $\forall i \in \mathscr{N}$ the state-i money market, with money supply M_i, interest rate r_i and bond price $\eta_i = \dfrac{1}{1+r_i}$

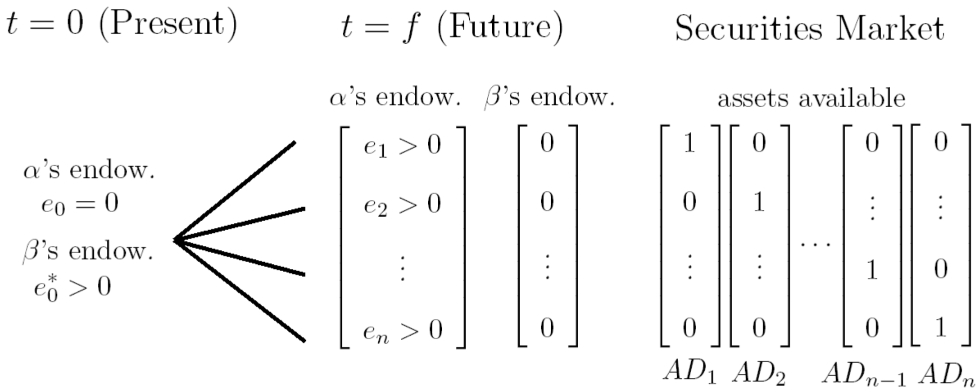

Figure 3: Uncertainty Tree

The money supplies M_i in the different states of nature are exogenous. Although the interest rate in each of these money markets will be deduced from demand and supply, the form of the model ensures in fact that the interest rates are just inversely proportional to the Central Bank money supply (see below). Therefore, the interest rates for all money markets are almost exogenous to the model. In addition to these $n+1$ money markets, the two agents can trade n Arrow-Debreu securities $(AD_i)_{1 \leq i \leq n}$ that give 1 in state i and 0 in all other states $j \neq i$. All Arrow-Debreu securities are available for trade (but in zero net supply) and therefore financial markets are complete with this structure. Figure 2.1 summarizes all endowments and assets in the model. Knowing the exogenous interest rates, we will compute the AD security prices and show how they are related to interest rates. The first step is however to ensure a positive value for money and nominal determinacy using a cash-in-advance model.

2.2 Cash-in-Advance Models and the Value of Money

Cash-in-advance models[4] aim at capturing the importance of liquidity for transactions. To ensure a positive nominal interest rate, a sufficient re-

[4]The modern treatment of cash-in-advance models dates as far as Clower (1967).

quirement is that agents hold some exogenous endowment of money (called outside money), and this has been a common modern treatment of cash-in-advance constraints as in Dubey and Genanakoplos (1992, 2006). In these models, when M is the supply of money by the Central Bank, if outside money endowed to the agents is m, the nominal interest is $r = \frac{m}{M}$. Although an exogenous endowment of money can be justified in a one-period model[5], this assumption is harder to explain in a multi-period setting. In fact, one should think of outside money as a compact simplification for a more general nominal friction that pins down the price of money. Default on the money market can play, for instance, the same role as outside money to ensure existence of positive interest rate. Shubik and Tsomocos (1992) model endogenous default, and generate positive interests, but we draw here a simpler model with exogenous default, without loss of generality, and show that the interest rate is equal to $r = \frac{d}{M}$ (see below)[6].

Positive interest rates, as we already suggested in the introduction, are key to the model because they ensure nominal determinacy, which is required for a theory of the term structure, and create financing costs which affect real variables. Positive interest rates are therefore determinant for both the yield curve and for the real payoffs of assets.

2.3 Budget Set for Agent α

There are two agents in the model. For any period or state of nature, each agent can pay b units of money to buy b/p units of good, or can sell q units of good and receive pq units of money. Hence, consumption in each period or state is

$$c = e - q + \frac{b}{p}$$

keeping in mind that either $q = 0$ (if the agent wants to buy) or $b = 0$ (if the agent wants to sell).

Agent α does not own any good in period 0 but owns $e > 0$ units of the

[5]Outside money may be inherited from previous periods and free from any debt requirement outstanding.

[6]We could have allowed for endogenous default and still get the same results at a significant computational cost. The focus of the present paper is primarily, however, on liquidity

consumption good in the future, where e is non-random. (Variables without supscript refer to agent α - who will be the borrower- while variables with supscript $*$ will refer to agent β -who will be the lender). Agent α maximises an inter-temporal Von Neumann-Morgenstern utility function with discount factor 1, equal weights between the n-states (since the states are assumed to be equi-probable) and logarithmic felicity function u.

In period 0, agent α sells q_{AD_i} securities at price θ_i to finance consumption at time 0. This is equivalent to say that agent α borrows with repayments conditional on the state of nature i.

In the future, in state i, agent α has to to give q_{AD_i} (the state-contingent repayment) to agent β who - as we will see later - has bought the AD securities. Since agent α cannot use yet the receipts of the goods he just sold, he borrows $\dfrac{\mu_i}{1+r_i}$ to the Central Bank (*i.e.* rolls over his debt) to pay agent β the q_{AD_i} he owed him from the AD_i-security. He can then use the receipts of his sells ($p_i q_i$) to repay the short-term loan μ_i that he had contracted with the Central Bank. However, agent α defaults of d_i units of account on his repayments to the Central Bank and does not repay the whole loan. He repays therefore only $\mu_i - d_i$ to the Central Bank. This default is assumed non-random, without loss of generality: $\forall i \in \mathscr{N}\ d_i = d > 0$.

To summarise, agent α's maximisation programme is (in brackets are the lagrangian multipliers used in the Annexes)[7]

$$\max_{b_0,(q_i,\mu_i,q_{AD_i})_{i\in\mathscr{N}}} U\left(b_0,(q_i)_{i\in\mathscr{N}}\right) = ln\left(\frac{b_0}{p_0}\right) + \frac{1}{n}\sum_{i\in\mathscr{N}} ln(e_i - q_i) \tag{1}$$

$$\text{s.t.} \quad b_0 \leq \sum_{1\leq i\leq n} \theta_i q_{AD_i} \qquad (\varphi) \tag{2}$$

$$\forall i \in \mathscr{N} \begin{cases} q_{AD_i} \leq \eta_i \mu_i & (\Psi_i) \\ \mu_i - d_i \leq p_i q_i & (\chi_i) \end{cases} \tag{3}$$

[7]We do not make explicit in these equations that agent α can carry money over from period 0 to period f because he will in fact never choose to do so, as all constraints are binding - something we will see later.

2.4 Budget Set for Agent β

Agent β is endowed with e_0^* units of the good in period 0, but has nothing in the future. He has the same preferences than agent α.

In period 0, he sells q_0^* to agent α and wants to invest it for next period, lending to agent α with repayments conditional on the state of nature (*i.e.* he buys AD securities $b_{AD_i}^*$).[8] However, he will receive the cash only at the end of the period, after the securities market meets. Hence, he first borrows $\frac{\mu_{00}^*}{1+r_{00}}$ to the Central Bank. He will repay the loan with the receipts of his sells $p_0 q_0^*$. Since agent β also defaults by $d_0^* > 0$, he in fact will repay only $\mu_{00}^* - d_0^*$.

In state i, he receives the state-contingent repayments for agent α (*i.e.* he receives $\frac{b_{AD_i}^*}{\theta_i}$ from each AD_i-security) and he uses this to buy $\frac{b_i^*}{p_i}$ units of the consumption good (at total cost b_i^*).

To summarise, agent β's maximisation programme is

$$\max_{q_0^*,(b_i^*,b_{AD_i}^*,\mu_0^*)_{1\leq i\leq n}} U\left(q_0^*,(b_i^*)_{i\in\mathscr{N}}\right) = ln(e_0^* - q_0^*) + \frac{1}{n}\sum_{1\leq i\leq n} ln\left(\frac{b_i^*}{p_i}\right) \tag{4}$$

$$\text{s.t.} \qquad \sum_{1\leq i\leq n} b_{AD_i}^* \leq \eta_{00}\mu_{00}^* \qquad (\varphi^*) \tag{5}$$

$$\mu_{00}^* - d_{00}^* \leq p_0 q_0^* \qquad (\xi^*) \tag{6}$$

$$\forall i \in \mathscr{N} \quad b_i^* \leq \frac{b_{AD_i}^*}{\theta_i} \qquad (\chi_i^*) \tag{7}$$

2.5 Financial General Equilibrium

The Financial General Equilibrium is reached when:

(i) Agents maximise utility, as explicited above

(ii) Commodity markets clear, i.e.

[8] We do not make explicit the possibility for agent β to carry money over since he will never do so, since we will show that all constraints are binding. As a result, all the receipts from the sells of good 0 are invested in AD securities, and this means that agent β borrows short-term as much as he will be able to repay

$$p_0 = \frac{b_0}{q_0^*} \Longleftrightarrow p_0 q_0^* = b_0$$

$$\forall i \in \mathscr{N} \quad p_i = \frac{b_i^*}{q_i} \Longleftrightarrow p_i q_i = b_i^*$$

(iii) Money and AD security markets clear, i.e.

$$\mu_{00}^* = (1 + r_{00}) M_{00} = M_{00}/\eta_{00}$$

$$\forall i \in \mathscr{N} \quad \mu_i = (1 + r_i) M_i = M_i/\eta_i$$

$$\forall i \in \mathscr{N} \quad \theta_i q_{AD_i} = b_{AD_i}^*$$

3 Interest Rates and the Quantity Theory of Money

The following results, proved in Annex A[9], determine the value of money and recall the quantity theory of money in a cash-in-advance model with default. Proposition 1 shows that short-term interest rates are inversely related to the supply of money by the Central Bank. Proposition 2 shows that the Quantity Theory of Money holds in a liquidity-constrained economy, *i.e.* the nominal activity is equal to the supply of money.

Proposition 1: Term Structure

$$r_{00} = \frac{d_{00}^*}{M_{00}}$$

$$\forall i \in \mathscr{N} \quad r_i = \frac{d_i}{M_i}$$

Proposition 2: Quantity theory of money in period 0

$$p_0 q_0^* = b_0 = M_0$$

$$\forall i \quad p_i q_i = b_i^* = M_i$$

4 Financing Costs and State Prices

This section shows how the marginal utilities and the financing costs affect equilibrium state prices. The proofs are available in Annex B, and for more

[9] see Geanakoplos and Tsomocos (2002) for similar proofs using outside money

general utility functions in Espinoza and Tsomocos (2007). First, note that all constraints are binding, *i.e.* no money is carried over. This is a result of the completeness of markets which ensures that the strategy of carrying money is dominated by holding a riskless debt instrument.[10] The same argument applies for agent β.

Theorem 1: Endogenous State Prices

In an equilibrium, states with higher interest rates correspond to higher state prices (i.e. bigger risk-neutral probabilities).

Proof

The first order conditions for agent β give (see Annex B)

$$\theta_i b_i^* = \theta_j b_j^* \quad . \tag{8}$$

Since we know from the section on the term structure that $b_j^* = M_j$ and $M_i = b_i^*$ and that interest rates are inversely related to liquidity. Therefore

$$r_j < r_i \Longleftrightarrow M_j > M_i \Longleftrightarrow b_j^* > b_i^* \Longleftrightarrow \theta_j < \theta_i$$

The intuition of the result is straightforward. Since we model consumer's utility with a Cobb-Douglas specification with equal weights on all states of nature, the cost of consumption is constant through all states (see equation 8). This cost of consumption is equal to the cost to transfer money from period 0 to period 1 (i.e. the AD security price) multiplied by the value of trade in period 1 (i.e. the price of the good multiplied by the volume traded). The value of trade is equal to the overall supply of liquidity because the quantity theory of money holds. For example, if there is more liquidity in state 1 than in state 2, the value of trade in state 1 has to be higher than the value of trade in state 2. However, according to equation 8 , this is possible only if the cost of financing consumption in state 1 is lower than the financing cost in state 2. Therefore, a state with lower interest rate (higher liquidity) is also a state with lower inter-temporal financing cost (i.e. lower state price and therefore lower risk-neutral probability).

[10]proof: since the marginal utility of consuming b_0 is never null (this is the case with the logarithmic utility function used here) and is equal to φ, $\varphi \neq 0$. Similarly, $\chi_i \neq 0$. Finally, $\eta_i \Psi_i = \chi_i$ and this ensures that the Ψ_i are non-null.

We can also work with the risk-neutral probabilities

$$\pi_i = \frac{\theta_i}{\sum_{1 \leq k \leq n} \theta_k} \tag{9}$$

and show that

$$\forall i \neq j \quad r_j > r_i \Longleftrightarrow \pi_j > \pi_i \tag{10}$$

This is the main result of our model. It first shows that risk-neutral probabilities are endogenous in a cash-in-advance general equilibrium model, in contrast with arbitrage models whereby prices follow various stochastic processes and thus risk-neutral probabilities are not demand and supply driven. Furthermore, the fact that states with higher interest rates are given higher weights yields important results for the yield curve and resolves the paradox we stated in the introduction without violating the rational expectation hypothesis. A comparative statics version of this result is possible, with a logarithmic version.

Theorem 2: Comparative statics

With this logarithmic utility function, an increase in state-i interest rate increases the risk-neutral probability associated to this state.

This requires a closed-form solution for the state price, something we do easily with this logarithmic model. We show in fact that the state price is $\theta_i = \dfrac{M_{00}}{nM_i}$ (See proof in Annex C), from which the theorem follows.

5 Interpretation and Real Asset Payoffs

The model excludes aggregate endowment uncertainty by setting $e_i = e$. Therefore, our result is not simply a version of the risk-premium found in pure exchange general equilibrium models with heterogeneous agents or in a representative agent model (Lucas, 1978; Breeden, 1979 ; Cox, Ingersoll and Ross, 1985). Indeed, the endowment risk-premium has been removed in our model, and state prices are only a function of money. However, the model still exhibits a risk-premium, since risk-neutral probabilities are higher in states of nature with higher spot interest rate. As we are going to see, the additional risk-premium has both a nominal and a real component. When r_i is high, activity (i.e. q_i) is low, something showed in Annex C and in Espinoza and Tsomocos (2007) for any Von Neumann-Morgenstern concave utility function. The fact that the state price is simply a ratio of money supplies, *i.e.*

$$\theta_i = \frac{1}{n}\frac{M_{00}}{M_i} = \frac{1}{n}\frac{r_i M_{00}}{d_i} \tag{11}$$

(proved in Annex C) makes clear why the risk-premium is to a large extent a nominal risk-premium: state prices are higher for states with low money supply simply because the value of money increases. But this is only part of the story. The other part is that real variables and marginal utilities are also affected by changes in money supply.

It is however not possible to link directly state prices and the effect of money on marginal utilities, because the model is not a representative agent model. When r_i is low, because q_i is high, agent α's marginal utility is low, but agent β's marginal utility is high. This is why we looked at demand and supply of assets to solve for state prices, and we proved in theorem 1 that the nominal state price θ_i is low when r_i is low. One can also show that $\frac{1}{p_i\theta_i}$, the real payoff of an Arrow-Debreu security is decreasing when the spot interest rate r_i increases. This is easily seen by computing the price level (the proof is in Annex C)

$$p_i = \frac{M_i(2 + r_i)}{e_i} \tag{12}$$

Therefore, the real payoff of the i^{th} Arrow-Debreu security is:

$$\frac{1}{p_i\theta_i} = \frac{e_i}{(2 + r_i)M_i\theta_i} = \frac{e_i}{(2 + r_i)M_{00}/n}$$

and is clearly decreasing in r_i. The fact that the asset's real payoff is lower when the interest rate is higher confirms that the risk-premium also includes a real component.

A representative agent model is unable to reproduce such a result if there is no aggregate real uncertainty (aggregate endowment or aggregate consumption is $e_i + e_i^* = e_i = e = q_i + (e_i - q_i)$).The upshot of our argument is that uncertainty in aggregate production or in aggregate consumption is only one part of uncertainty in agents' marginal utilities. Financing costs also generate variability of marginal utilities (and therefore of asset demands) in the future. Therefore, any model of risk-premium that attempts to proxy welfare by production or consumption will underestimate risk-premium. This is especially important for the term-structure risk premium since the spot interest rate has both an effect on the asset price (inter-temporal financing cost) and on the short-term financing cost. In that case, the correlation between the marginal utilities and the asset price is likely to be high. The error will be an under-estimation of the risk-premium.

6 The Term Structure of Interest Rates

Let B a bond that would be bought in the first period at the time when the money market clears and that would mature at the time the intra-period bond of second period matures, as shown in the thick arrow of figure 6.

A no arbitrate argument ensures that the price of such a bond is:

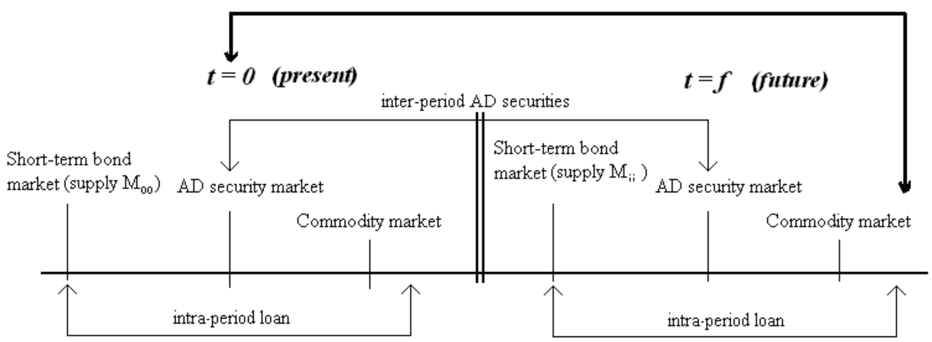

Figure 4: Long-term bond

$$P_B = \eta_{00} \left(\sum_{i=1}^{n} \theta_i \eta_i \right) \tag{13}$$

or equivalently

$$P_B = \eta_{00}\eta_0 \left(\sum_{i=1}^{n} \pi_i \eta_i \right) \tag{14}$$

Note that no ones in our model needs such a bond. A more relevant bond may be the one that is bought at the time the AD market meets and would mature at the time the intra-period bond of second period matures. For this bond b, a no-arbitrage arguments ensure that its price is

$$P_b = \sum_{i=1}^{n} \theta_i \eta_i \tag{15}$$

or equivalently

$$P_b = \eta_0 \sum_{i=1}^{n} \pi_i \eta_i \tag{16}$$

By approximating, we find

$$r_b \approx r_0 + \sum_{i=1}^{n} (\pi_i(r_i) r_i) \tag{17}$$

It is clear here how the long-term interest rate depends on a convex function[11] of r_i. The first consequence of this is that the long-term interest rate is above the average of spot rates, something we define as the "liquidity-risk premium". The second result is that a larger variance in spot rates will generate a higher "liquidity-risk premium" and long-term interest rate, so that stability of monetary policy matters in determining the equilibrium value of long-term interest rates.

7 Concluding Remarks

In a state with low liquidity, trade has to be low, and, in order to induce consumers to have trade at a low level, the opportunity cost of transferring money to this state must be high. This inter-temporal financing cost is equal to the state price. Therefore, state prices and risk-neutral probabilities are higher in states with higher interest rates. It is important to stress that the result is due to the interaction of the money market (the quantity theory of money) with the exchange economy (the maximisation problem) and there-

[11] almost a quadratic function in fact

fore cannot be found in a pure financial model. Ultimately, it is the risk in the supply of money that matters to determine the risk in trade values ; and because of the Von Neumann-Morgenstern Cobb-Douglas utility function, it is trade values (i.e. nominal trade as opposed to real trade) that are equalised through states. Our result is therefore more general and can be found in any model with complete markets, Von Neumann-Morgenstern logarithmic utility, some form of the quantity theory of money, and some transaction cost effect on trade. Other shocks different from liquidity shocks may of course also affect the transaction technology, and hence risk-neutral probabilities. Liquidity shocks are however crucial to understand the upward sloping term structure because two phenomena push in the same direction: first, the futures spot interest rates are affected ; second the risk-neutral probabilities are modified. The interaction of these two effects pushes long-term rates above the historical average of future spot rates, even with nonexistent aggregate real risk. And the more uncertainty in the future spot rates, the higher the long-term rates. Stability of monetary policy is, therefore, required to maintain flat yield curves.

Appendix A - Value of Money and the Quantity Theory of Money

First, note that all budget constraints are binding. This is proved in Appendix B, and means that no money is carried over, an intuitive result with complete markets.

Proof of Proposition 1

Money and AD security markets clear, i.e.

$$\mu_{00}^* = (1 + r_{00})M_{00} = M_{00}/\eta_{00}$$

$$\forall i \in \mathscr{N} \quad \mu_i = (1 + r_i)M_i = M_i/\eta_i$$

$$\forall i \in \mathscr{N} \quad \theta_i q_{AD_i} = b_{AD_i}^*$$

As a result, we have the existence of a positive interest rate

$$(1 + r_{00})M_{00} = \mu_{00}^* = p_0 q_0^* + d_{00}^* = b_0 + d_{00}^* = \sum_{1 \le i \le n} \theta_i q_{AD_i} + d_{00}^*$$

$$= \sum_{1 \le i \le n} b_{AD_i}^* + d_{00}^* = \eta_{00}\mu_{00}^* + d_{00}^* = M_{00} + d_{00}^*$$

Therefore

$$r_{00} = \frac{d_{00}^*}{M_{00}}$$

Similarly, $\forall i \in \mathscr{N}$

$$(1 + r_i)M_i = \mu_i = p_i q_i + d_i = b_i^* + d_i = \frac{b_{AD_i}^*}{\theta_i} + d_i$$

$$= q_{AD_i} + d_i = \eta_i \mu_i + d_i = M_i + d_i$$

Therefore

$$r_i = \frac{d_i}{M_i}$$

Proof of Proposition 2

$$b_0 = \sum_{1 \le i \le n} \theta_i q_{AD_i} = \sum_{1 \le i \le n} b_{AD_i}^* = \eta_{00}\mu_{00}^* = M_{00}$$

Similarly

$$b_i^* = b_{AD_i}^*/\theta_i = q_{AD_i} = \eta_i \mu_i = M_i$$

Appendix B - First Order Conditions

Agent α First Order Conditions

The first order conditions are

$$\frac{1}{b_0} - \varphi = 0 \tag{18}$$

$$\forall i \in \mathscr{N} \ \frac{-1}{n(e_i - q_i)} + p_i \chi_i = 0 \tag{19}$$

$$\forall i \in \mathscr{N} \ \eta_i \Psi_i - \chi_i = 0 \tag{20}$$

$$\theta_i \varphi - \Psi_i = 0 \tag{21}$$

Agent β First Order Conditions

Denote $\mathscr{L}$ the lagragian formed from the maximisation problem. The first order conditions are:

$$\frac{\partial \mathscr{L}}{\partial q_0^*} = \frac{-1}{e_0^* - q_0^*} + p_0 \xi^* = 0 \tag{22}$$

$$\forall i \in \mathscr{N} \ \frac{\partial \mathscr{L}}{\partial b_i^*} = \frac{1}{n} \frac{1}{b_i^*} - \chi_i^* = 0 \tag{23}$$

$$\frac{\partial \mathscr{L}}{\partial \mu_{00}^*} = \eta_0 \varphi^* - \xi^* = 0 \tag{24}$$

$$\frac{\partial \mathscr{L}}{\partial b_{AD_i}^*} = -\varphi^* + \frac{\chi_i^*}{\theta_i} = 0 \tag{25}$$

Note that all constraints are binding. This is easy to prove since the marginal utility of consuming b_0 - which is never null with a well-defined utility function - is equal to φ : therefore, $\varphi \neq 0$. Similarly, $\chi_i \neq 0$. Finally, $\eta_i \Psi_i = \chi_i$ and this ensures that the Ψ_i are non-null. The consequence is that no money is carried over.

Appendix C - Asset and Commodity Prices

We first compute state prices. From equation (5)

$$\sum_{1\leq k\leq n} b^*_{AD_k} = M_{00} \tag{26}$$

Since $\forall i,k \in \mathscr{N} \quad b^*_{AD_k} = \theta_k b^*_k = \theta_i b^*_i = b^*_{AD_i}$

$$\forall i \in \mathscr{N} \quad \sum_{1\leq k\leq n} b^*_{AD_k} = n\theta_i q_{AD_i} = M_{00} \tag{27}$$

Hence $\theta_i = \dfrac{M_{00}}{nq_{AD_i}} = \dfrac{M_{00}}{nM_i}$ because $q_{AD_i} = M_i$.

We compute now prices and trade volumes. From agent α first-order conditions:

$$p_i\chi_i = \frac{1}{n(e_i - q_i)} = p_i\eta_i\Psi_i = p_i\eta_i\theta_i\varphi = p_i\frac{\theta_i}{(1+r_i)b_0}$$

Furthermore, $q_i = M_i/p_i$. Hence, $p_i\theta_i n(e_i - M_i/p_i) = b_0(1+r_i)$. Therefore

$$p_i\theta_i n e_i = b_0(1+r_i) + M_i\theta_i n$$

Or,

$$p_i = \frac{b_0(1+r_i)/(n\theta_i) + M_i}{e_i} = \frac{M_i(1+r_i) + M_i}{e_i}$$

and

$$q_i = \frac{e_i}{1 + \frac{b_0(1+r_i)}{n\theta_i M_i}} = \frac{e_i}{2+r_i}$$

since $\theta_i = \dfrac{1}{n}\dfrac{M_{00}}{M_i}$. A higher M_i therefore increases q_i through a decrease in r_i. A lower interest rate ensures lower financing costs and therefore better consumption smoothing.

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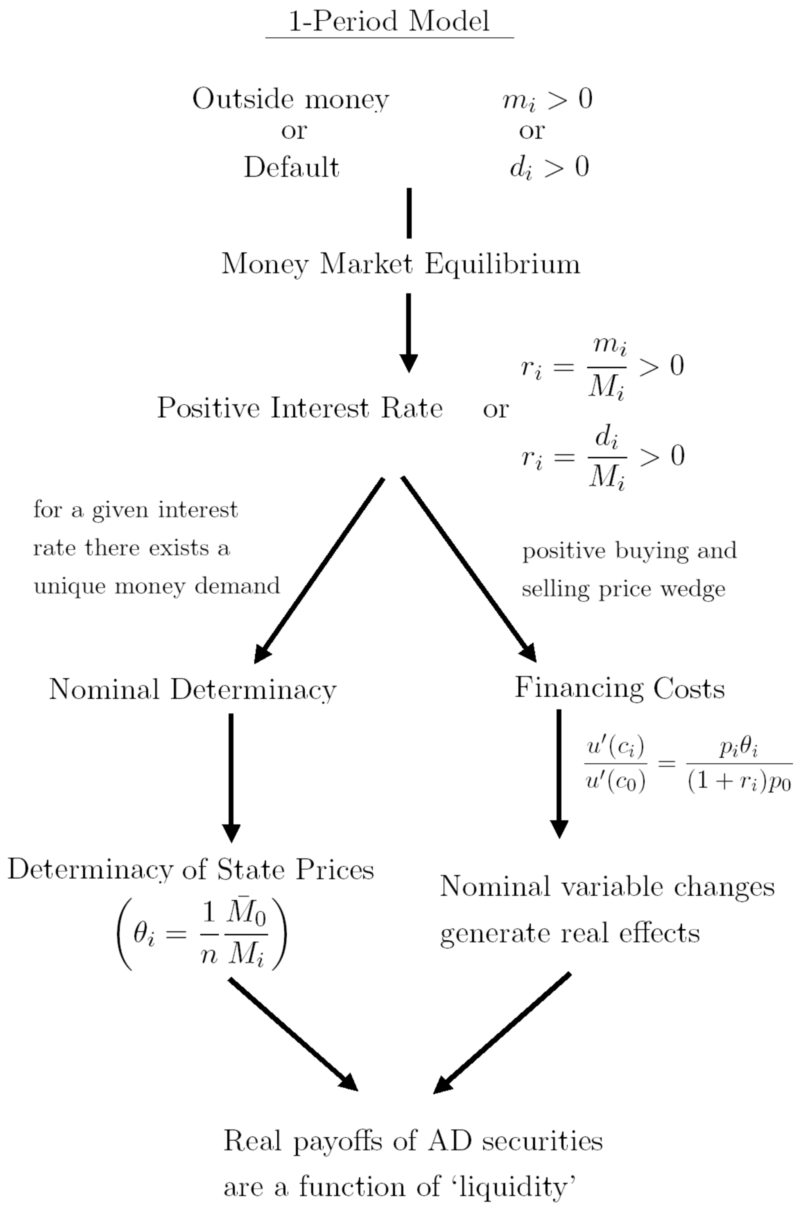

Figure 5: Summary of the Model