Erik J. Baurdoux

Some excursion calculations for reflected Lévy processes

Article (Accepted version) (Refereed)

Original citation:
Baurdoux, Erik J. (2009) Some excursion calculations for reflected Lévy processes. ALEA: Latin American journal of probability and mathematical statistics, 6. pp. 149-162.

Some excursion calculations for reflected Lévy processes

E. J. Baurdoux

Department of Statistics, London School of Economics, Houghton Street, London, WC2A 2AE, United Kingdom
E-mail address: e.j.baurdoux@lse.ac.uk

Abstract. Using methods analogous to those introduced in Doney (2005), we express the resolvent density of a (killed) reflected Lévy process in terms of the resolvent density of the (killed) Lévy process. As an application we find a previously unknown identity for the potential density for killed reflected symmetric stable processes.

1. Introduction

Lévy processes reflected at their maximum or at their minimum appear in a wide variety of applications, such as the study of the water level in a dam, queueing (see Asmussen (1989); Borovkov (1976); Prabhu (1997)), optimal stopping (Baurdoux and Kyprianou (2007); Shepp and Shiryaev (1994)) and optimal control (De Finetti (1957); Gerber and Shiu (2004); Avram et al. (2007); Kyprianou and Palmowski (2007)). For example, in Shepp and Shiryaev (1994) it was shown that finding the value of the so-called Russian option (for a Brownian motion B) is equivalent to solving an optimal stopping problem of the form

$$\sup_{\tau} \mathbb{E}[e^{-\alpha\tau+Y^x_\tau}], \tag{1.1}$$

where α is some constant, where Y is the process B reflected at its infimum and where the supremum is taken over all stopping times with respect to the filtration generated by B. Recently, there have been various studies on (1.1) with the Brownian motion B replaced by a more general Lévy process X, see for example Asmussen et al. (2004); Avram et al. (2004), and also Baurdoux and Kyprianou (2007) for a two-player version of (1.1). It was found that for a broad class of Lévy processes, an optimal stopping time τ^* in (1.1) is given by the first time the reflected Lévy process exceeds a certain level, i.e.

$$\tau^* = \inf\{t \geq 0 : Y_t \geq b\}, \tag{1.2}$$

for a specific choice of b and where Y now is the process X reflected at its infimum. A similar strategy was proved to be optimal (under some conditions) for the optimal control problem considered for a Lévy process without positive jumps in De Finetti (1957). Hence, a further understanding of reflected Lévy processes

Key words and phrases. Reflected Lévy processes, symmetric stable processes, excursion theory, resolvent density.

killed at exceeding a certain level could be helpful for the study of certain optimal stopping and optimal control problems.
In Theorem 1, we express the resolvent density of a (killed) reflected Lévy process in terms of the resolvent density of the (killed) Lévy process. The proof of Corollary 1 indicates that the compensation formula allows us to find the joint law of the undershoot and the overshoot of Y in terms of the resolvent density of Y and the jump measure of the Lévy process, which, in turn, gives us information about the expressions involving the first passage time in (1.2).
As an application of Theorem 1, we find the potential density of a killed, reflected symmetric stable process. Possibly, the result in the symmetric stable case could lead to proving similar results for a broader class of reflected Lévy processes.
In Doney (2005), Doney introduced a new method based on excursion theory to find an expression for the resolvent density for reflected spectrally Lévy negative processes killed at exceeding a certain level. Previously, this density had been obtained in Pistorius (2004) using excursion theory, Itô calculus and martingale techniques (see also Nguyen-Ngoc and Yor (2005)). In this paper we extend the method introduced in Doney (2005) to general reflected Lévy processes. In Theorem 4.1 we express the resolvent density of a (killed) reflected Lévy processes in terms of the resolvent density of the (killed) Lévy process. As a new result and an application of Theorem 4.1, we find in Section 5 the potential density for the killed reflected symmetric stable process.

2. Preliminaries

Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process, starting from 0, with respect to some probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. To avoid trivialities, we exclude the case when X has monotone paths. We refer to the books Bertoin (1996) and Kyprianou (2006b) for a detailed description of Lévy processes. We denote by $\mathbb{P}_x$ the law of the Lévy process starting at x. Define the process $Y = \{Y_t\}_{t\geq 0}$ by

$$Y_t = X_t - \underline{X}_t,$$

where $\underline{X}_t = \inf_{0\leq s\leq t} X_s \wedge 0$. Denote by $L(t)$ a local time of Y at zero (note that the definition of L depends on the nature of the zero set of Y, see section IV.5 in Bertoin (1996)) and let n be the measure of excursions of Y away from zero, defined on the excursion space $\mathcal{E}$. Since a local time is only defined up to a (positive) multiplicative constant, most expressions involving local time also involve a multiplicative constant. However, this constant does not play a role in the results in this paper and hence we shall omit it. Define the inverse local time of Y by

$$L^{-1}(t) = \begin{cases} \inf\{s > 0 : L(s) > t\} & \text{when } t < L(\infty), \\ \infty & \text{otherwise.} \end{cases}$$

Furthermore, denote by $H = \{H_t\}_{t\geq 0}$ the downward ladder height process of X, i.e. $H_t = X_{L^{-1}(t)}$ when $0 \leq t < L(\infty)$ and $H_t = -\infty$ otherwise. The exit times for a generic excursion $\varepsilon \in \mathcal{E}$ we denote by

$$\rho_a = \inf\{t \geq 0 : \varepsilon(t) \geq a\}$$

and by ζ the length of an excursion. The renewal function $h : [0,\infty) \to [0,\infty)$ of H is defined by

$$h(x) = \int_0^\infty \mathbb{P}(H_t \geq -x)dt = \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{\underline{X}_t \geq -x\}} dL(t)\right].$$

Denote the first passage times for X by

$$\tau_b^- = \inf\{t \geq 0 : X_t \leq b\} \quad \text{and} \quad \tau_a^+ = \inf\{t \geq 0 : X_t \geq a\}$$

and by

$$T_a^+ = \inf\{t \geq 0 : Y_t \geq a\}$$

the first passage time for Y. For $q > 0$, let $\mathbf{e}_q$ be an exponentially distributed random variable with parameter q, independent of X. The function h can also be expressed in terms of the excursion measure as

$$h(x) = \lim_{q \downarrow 0} \frac{\mathbb{P}_x(\tau_0^- > \mathbf{e}_q)}{\eta q + n(\mathbf{e}_q < \zeta)},$$

where $\eta \geq 0$ is the drift of $L^{-1}(t)$. This is a consequence of the following result which we will use later.

Lemma 2.1. *Let $q > 0$. Then*

$$\mathbb{P}_x(\mathbf{e}_q < \tau_0^-) = \mathbb{E}\left[\int_{[0,\infty)} e^{-qt}\mathbf{1}_{\{\underline{X}_t \geq -x\}}\, dL(t)\right](\eta q + n(\mathbf{e}_q < \zeta)). \tag{2.1}$$

Proof. The proof is based on excursion intervals and follows closely the proof of Lemma 8 in Section VI.2 of Bertoin (1996). Note that

$$\mathbb{P}_x(\tau_0^- > \mathbf{e}_q) = \mathbb{E}\left[\int_0^\infty qe^{-qt}\mathbf{1}_{\{\underline{X}_t \geq -x\}}\, dt\right].$$

By distinguishing between those times t for which $X_t = \underline{X}_t$ and those which lie in an excursion interval of the process $\{X_s - \underline{X}_s\}_{s \geq 0}$ we find that

$$\mathbb{P}_x(\tau_0^- > \mathbf{e}_q) = \mathbb{E}\left[\int_0^\infty qe^{-qt}\mathbf{1}_{\{\underline{X}_t \geq -x,\, \underline{X}_t = X_t\}}\, dt\right] + \mathbb{E}\left[\sum_g \mathbf{1}_{\{\underline{X}_g \geq -x\}} \int_g^d qe^{-qt}\, dt\right], \tag{2.2}$$

where the sum is taken over all left end points g of excursion intervals (g,d). Since $\mathbf{1}_{\{\underline{X}_t = X_t\}}dt = \eta dL(t)$ (see Theorem 6.8 in Kyprianou (2006b)), the first term on the right hand side of (2.2) is equal to

$$\eta q \mathbb{E}\left[\int_{[0,\infty)} e^{-qt}\mathbf{1}_{\{\underline{X}_t \geq -x\}}\, dL(t)\right].$$

From an application of the compensation formula it follows that the second term on the right hand side of (2.2) is equal to

$$\mathbb{E}\left[\sum_g e^{-qg}\mathbf{1}_{\{\underline{X}_g \geq -x\}}\mathbf{1}_{\{\mathbf{e}_q < d-g\}}\right] = n(\mathbf{e}_q < \zeta)\mathbb{E}\left[\int_{[0,\infty)} e^{-qt}\mathbf{1}_{\{\underline{X}_t \geq -x\}}\, dL(t)\right],$$

which completes the proof. □

We say that X drifts to ∞ $(-\infty)$ when $\lim_{t\to\infty} X_t = \infty$ $(-\infty)$. We say that X is regular upwards when the first hitting time of $(0,\infty)$ is almost surely equal to zero.

Whenever there exists some $\nu > 0$ (which is then called the Lundberg exponent of X) such that

$$\mathbb{E}[e^{\nu X_1}] = 1,$$

we can define the Laplace exponent ψ of X by

$$\psi(\lambda) = \log(\mathbb{E}[e^{\lambda X_1}]), \quad \text{for all } \lambda \in [0,\nu].$$

The function ψ is strictly convex on $[0,\nu]$ and $\psi(0) = \psi(\nu) = 0$, so we find that $\psi'(0+) < 0$, which implies that X drifts to $-\infty$. Furthermore, we can change measure by defining

$$\left.\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\nu X_t}.$$

Trivially, the Laplace exponent ψ_ν of X under $\mathbb{P}^\nu$ is given by

$$\psi_\nu(\lambda) = \log(\mathbb{E}^\nu[e^{\lambda X_1}]) = \psi(\lambda+\nu) \quad \text{for } \lambda \in [-\nu, 0].$$

In particular $\psi_\nu'(0-) = \psi'(\nu-) > 0$ and thus X drifts to $+\infty$ under $\mathbb{P}^\nu$.

3. Excursion measure in terms of renewal function

In this section we show that for a large class of Lévy processes, the excursion measure n can be expressed in terms of the renewal function h. We make use of various results obtained in Chaumont and Doney (2005) concerning Lévy processes conditioned to stay positive. The Lévy processes we consider are given in the following definition.

Definition 3.1. Let $\mathcal{H}$ be the class of those Lévy processes X such that X is not a compound Poisson process and X does not have monotone paths, and X has a Lundberg exponent if it drifts to $-\infty$.

Remark 3.2. For future reference we remark that $\mathcal{H}$ contains any (non-monotone, non compound Poisson) Lévy process for which its Lévy measure has support bounded from above. Indeed, when the support of the Lévy measure of X is bounded from above we know (e.g. Theorem 25.3 in Sato (1999)) that the Laplace exponent $\psi(\lambda)$ is finite for $\lambda \geq 0$. Furthermore it is not difficult to check that ψ is strictly convex and that $\lim_{\lambda\to\infty}\psi(\lambda) = \infty$. When X drifts to $-\infty$ it holds that $\psi'(0) < 0$ and thus the Lundberg exponent exists.

The following result indicates how, for processes in $\mathcal{H}$, the excursion measure n is related to the renewal function h.

Lemma 3.3. *Let $X \in \mathcal{H}$ and A a Borel subset of $\mathbb{R}_+$ satisfying* $\inf A > 0$. *For $q \geq 0$ it holds that*

$$\int_0^\infty e^{-qt} n(\varepsilon(t) \in A, t < \zeta \wedge \rho_a) dt = \lim_{z\downarrow 0} \frac{\int_0^\infty e^{-qt}\mathbb{P}_z(X_t \in A, t < \tau_a^+ \wedge \tau_0^-) dt}{h(z)} \tag{3.1}$$

Furthermore,

$$n(\rho_a \leq \zeta \wedge \mathbf{e}_q) = \lim_{x\downarrow 0} \frac{1}{h(x)} \mathbb{P}_x(\tau_a^+ < \tau_0^- \wedge \mathbf{e}_q), \tag{3.2}$$

and, when $q > 0$,

$$n(\mathbf{e}_q < \rho_a \wedge \zeta) = \lim_{z\downarrow 0} \frac{\mathbb{P}_z(\mathbf{e}_q < \tau_a^+ \wedge \tau_0^-)}{h(z)}. \tag{3.3}$$

Proof. Let $X \in \mathcal{H}$ and suppose for the moment that X does not drift to $-\infty$. According to Lemma 1 in Chaumont and Doney (2005) we can then introduce the family of probability measures by

$$\mathbb{P}_x^\uparrow(X_t \in dy) = \frac{h(y)}{h(x)} \mathbb{P}_x(X_t \in dy, t < \tau_0^-) \quad \text{for } x, y > 0.$$

Proposition 1 in Chaumont and Doney (2005) provides the justification for calling $\mathbb{P}_x^\uparrow$ the law of X conditioned to stay positive. When X is regular upwards we know from Theorem 2 in Chaumont and Doney (2005) that the laws $(\mathbb{P}_x^\uparrow)$ converge in the Skorokhod topology as $x \downarrow 0$ to a probability measure denoted by $\mathbb{P}^\uparrow$. In Chaumont (1996) Theorem 3, under the assumptions that 0 is regular downwards for X, X does not drift to $-\infty$ and its semigroup is absolutely continuous, it is shown that this measure is related to the excursion measure n in the following way:

$$n(B, t < \zeta) = \mathbb{E}^\uparrow[(h(X_t))^{-1}\mathbf{1}_B] \quad \text{for any } B \in \mathcal{F}_t \text{ such that } n(\partial B) = 0, \tag{3.4}$$

where ∂B denotes the boundary of B with respect to the Skorokhod topology. From Theorem 1 in Chaumont and Doney (2005) it follows that (3.4) still holds whenever X is not a Poisson process. By Fubini's theorem and (3.1) we then find for any Borel subset A of $\mathbb{R}_+$ satisfying $\inf A > 0$ that

$$n\left(\int_0^{\zeta\wedge\rho_a} e^{-qt}\mathbf{1}_{\{\varepsilon_t \in A\}}dt\right) = \mathbb{E}^\uparrow\left[\int_0^{\tau_a^+} e^{-qt}\mathbf{1}_{\{X_t\in A\}}(h(X_t))^{-1}dt\right].$$

Let $T \in \mathbb{R}_+$. We show that $\omega \to \int_0^T F(\omega_t)dt$ is a continuous functional of the paths $\omega \in D$ whenever F is bounded and continuous. Denote by d a metric which induces the Skorokhod topology. and let $\omega^n \in D$ and $\omega \in D$ be such that $d(\omega^n, \omega) \to 0$ as $n \to \infty$. Define the countable set

$$C := \cup_{n\in\mathbb{N}}\{t : \omega_t^n \neq \omega_{t-}^n\} \cup \{t : \omega_t \neq \omega_{t-}\}.$$

Since Skorokhod convergence implies pointwise convergence at points of continuity we can invoke the bounded convergence theorem to deduce that

$$\int_0^T (F(\omega_t^n) - F(\omega_t))\, dt = \int_0^T (F(\omega_t^n) - F(\omega_t))\mathbf{1}_{\{C^c\}}dt \to 0$$

as $n \to \infty$. Since h is an increasing function and since $\inf A > 0$, we deduce by a monotone class argument and (3.4) that

$$\begin{aligned} n\left(\int_0^{\zeta\wedge\rho_a} e^{-qt}\mathbf{1}_{\{\varepsilon_t\in A\}}dt\right) &= \lim_{x\downarrow 0}\mathbb{E}_x^\uparrow\left[\int_0^\infty e^{-qt}\mathbf{1}_{\{X_t\in A, t<\tau_a^+\}}(h(X_t))^{-1}dt\right] \\ &= \lim_{x\downarrow 0}\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbf{1}_{\{X_t\in A, t<\tau_a^+\wedge\tau_0^-\}}dt\right], \end{aligned}$$

which is (3.1). The proof of (3.2) is similar, since $h(X_{\tau_a^+})$ is bounded away from zero.

Next, we show (3.3) (still under the assumption that X drifts to $+\infty$). Let $q>0$. Similar to the reasoning above, we can show that for any $\delta>0$ it holds that

$$\lim_{x\downarrow 0}\mathbb{E}_x^{\uparrow}\left[\int_0^{\infty}e^{-qt}(h(X_t))^{-1}\mathbf{1}_{\{X_t>\delta\}}dt\right]=\mathbb{E}^{\uparrow}\left[\int_0^{\infty}e^{-qt}(h(X_t))^{-1}\mathbf{1}_{\{X_t>\delta\}}dt\right]. \tag{3.5}$$

Also, as $q>0$ it follows from the definition of h that

$$\mathbb{E}\left[\int_0^{\infty}e^{-qt}\mathbf{1}_{\{\underline{X}_t\geq -x\}}dL(t)\right]\leq h(x).$$

Since X is regular upwards, the drift η of $L^{-1}(t)$ is equal to zero and thus we deduce from (2.1) that for $\delta, x>0$

$$\begin{aligned}
&\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_0^-}e^{-qt}\mathbf{1}_{\{X_t\leq\delta\}}dt\right]+\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_0^-}e^{-qt}\mathbf{1}_{\{X_t>\delta\}}dt\right] && (3.6)\\
&\quad= \frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_0^-}e^{-qt}dt\right]\\
&\quad= \frac{1}{h(x)}\mathbb{E}\left[\int_0^{\infty}e^{-qt}\mathbf{1}_{\{\underline{X}_t\geq -x\}}dL(t)\right]n\left(\int_0^{\zeta}e^{-qt}dt\right)\\
&\quad\leq n\left(\int_0^{\zeta}e^{-qt}dt\right)\\
&\quad= n\left(\int_0^{\zeta}e^{-qt}\mathbf{1}_{\{\varepsilon(t)\leq\delta\}}dt\right)+n\left(\int_0^{\zeta}e^{-qt}\mathbf{1}_{\{\varepsilon(t)>\delta\}}dt\right). && (3.7)
\end{aligned}$$

It follows from (3.5) that the second term in (3.6) converges to the second term in (3.7) as $x\downarrow 0$. The fact that $n(\int_0^{\zeta}\mathbf{1}_{\{\varepsilon(t)=0\}}dt)=0$ implies that for any $\xi>0$ there exists some $\delta_0>0$ such that for $0<\delta<\delta_0$

$$\limsup_{x\downarrow 0}\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_0^-}e^{-qt}\mathbf{1}_{\{X_t\leq\delta\}}dt\right]\leq n\left(\int_0^{\zeta}e^{-qt}\mathbf{1}_{\{\varepsilon(t)\leq\delta\}}dt\right)\leq\xi.$$

It now readily follows that

$$\begin{aligned}
\lim_{x\downarrow 0}\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_a^+\wedge\tau_0^-}e^{-qt}dt\right] &= \lim_{\delta\downarrow 0}\lim_{x\downarrow 0}\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_a^+\wedge\tau_0^-}e^{-qt}\mathbf{1}_{\{X_t>\delta\}}dt\right]\\
&= \lim_{\delta\downarrow 0}n\left(\int_0^{\rho_a\wedge\zeta}e^{-qt}\mathbf{1}_{\{\varepsilon(t)>\delta\}}dt\right)\\
&= n\left(\int_0^{\rho_a\wedge\zeta}e^{-qt}dt\right),
\end{aligned}$$

which is (3.3).

When X is irregular upwards (and still does not drift to $-\infty$), the drift η of $L^{-1}(t)$ is strictly positive. We now deduce from (2.1) and reasoning similar to the above that for $0 < r \leq q$ and $\delta > 0$

$$\begin{aligned}\limsup_{x\downarrow 0}\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_0^-} e^{-qt}\mathbf{1}_{\{X_t\leq\delta\}}\,dt\right] &\leq \limsup_{x\downarrow 0}\frac{1}{h(x)}\mathbb{E}_x\left[\int_0^{\tau_0^-} e^{-rt}\mathbf{1}_{\{X_t\leq\delta\}}\,dt\right]\\ &\leq n\left(\int_0^{\zeta} e^{-rt}\mathbf{1}_{\{\varepsilon(t)\leq\delta\}}\,dt\right)+r\eta,\end{aligned}$$

which can be made arbitrarily small by taking r and δ close to zero. The proof of (3.3) now follows similarly to the regular case above.

Finally, let X be a process in $\mathcal{H}$ which drifts to $-\infty$ and denote by ν its Lundberg exponent. We denote by n^ν the excursion measure of $X_t - \underline{X}_t$ under $\mathbb{P}^\nu$. Then we claim that the excursion measure n^ν can be expressed in terms of n by

$$n^\nu(\varepsilon(t)\in dy, t<\zeta) = e^{\nu y} n(\varepsilon(t)\in dy, t<\zeta). \tag{3.8}$$

In order to prove this, we show that the left hand side and the right hand side of (3.8) have the same double Laplace transform. Denote by κ ($\hat{\kappa}$) the Laplace exponent of the downward (upward) ladder height. We use the obvious notation for κ_ν and $\hat{\kappa}_\nu$ of the analogous objects under $\mathbb{P}^\nu$. Then from equation (7) on p. 120 in Bertoin (1996), the duality principle, which implies that $X_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q}$ has the same distribution as $\overline{X}_{\mathbf{e}_q} := \sup_{0\leq s\leq \mathbf{e}_q} X_s$, and the Wiener Hopf factorisation we deduce that for any $q, \lambda \geq 0$

$$\begin{aligned} n^\nu\left(\int_0^\zeta\int_0^\infty e^{-qt-(\nu+\lambda)y}\mathbf{1}_{\{\varepsilon(t)\in dy\}}dt\right) &= \frac{\kappa_\nu(q,0)}{q}\mathbb{E}^\nu[e^{-(\nu+\lambda)(X_{\mathbf{e}_q}-\underline{X}_{\mathbf{e}_q})}]\\ &= \frac{\kappa_\nu(q,0)}{q}\mathbb{E}^\nu[e^{-(\nu+\lambda)\overline{X}_{\mathbf{e}_q}}]\\ &= \frac{\kappa_\nu(q,0)}{q}\frac{\hat{\kappa}_\nu(q,0)}{\hat{\kappa}_\nu(q,\nu+\lambda)}\\ &= c\frac{1}{\hat{\kappa}_\nu(q,\nu+\lambda)}\end{aligned}$$

for some constant $c > 0$. We also have that (with $(\hat{L}_t, \hat{H}_t)$ the upward ladder process)

$$\begin{aligned}\hat{\kappa}_\nu(q,\nu+\lambda) &= -\log\left(\mathbb{E}^\nu[e^{-q\hat{L}_1^{-1}-(\nu+\lambda)\hat{H}_1}\mathbf{1}_{\{\hat{L}(\infty)>1\}}]\right)\\ &= -\log\left(\mathbb{E}[e^{-q\hat{L}_1^{-1}-(\nu+\lambda)\hat{H}_1+\nu X_{\hat{L}_1^{-1}}}\mathbf{1}_{\{\hat{L}(\infty)>1\}}]\right)\\ &= \hat{\kappa}(q,\lambda).\end{aligned}$$

This concludes the proof of (3.8).

By denoting h^ν the renewal function under P^ν we can use (3.1) to deduce that for any $t, y > 0$

$$\begin{aligned} n(\varepsilon(t)\in dy, t<\zeta) &= e^{\nu y} n^\nu(\varepsilon(t)\in dy, t<\zeta)\\ &= e^{\nu y} k\lim_{x\downarrow 0}\frac{\mathbb{P}_x^\nu(X_t\in dy, t<\tau_0^-)}{h^\nu(x)}\\ &= k\lim_{x\downarrow 0}\frac{\mathbb{P}_x(X_t\in dy, t<\tau_0^-)}{h(x)},\end{aligned}$$

since
$$\lim_{x\downarrow 0}\frac{h^\nu(x)}{h(x)}=1,$$
which is a consequence of
$$e^{-\nu x}\mathbb{P}(X_{L^{-1}(t)}\geq -x)\leq \mathbb{E}[e^{\nu X_{L^{-1}(t)}}\mathbf{1}_{\{X_{L^{-1}(t)}\geq -x\}}]\leq \mathbb{P}(X_{L^{-1}(t)}\geq -x).$$
The results in Lemma 3.3 now follow. □

4. Resolvent measure of the killed reflected process

The q-resolvent measure of X killed at exiting $[0,a]$ is given by
$$U^{(q)}(x,dy)=\int_0^\infty e^{-qt}\mathbb{P}_x(X_t\in dy, t<\tau_0^-\wedge\tau_a^+)dt.$$
We assume throughout this paper that $U^{(q)}(x,dy)$ is absolutely continuous with respect to Lebesgue measure and we denote a version of its density by $u^{(q)}(x,y)$. We also assume that X is regular upwards. These assumptions are not strictly necessary, but suffice for the application we consider in Section 5. We refer to Remark 6.2 for a discussion about how these assumptions can be weakened.

Similarly, denote by $R^{(q)}(x,dy)$ the q-resolvent measure of the process $\{Y_t\}_{t\geq 0}$ killed at exceeding a, i.e.
$$R^{(q)}(x,dy)=\int_0^\infty e^{-qt}\mathbb{P}_x(Y_t\in dy, t<T_a^+)dt.$$
By the strong Markov property applied at τ_0^- we have for any $y\geq 0$
$$R^{(q)}(x,dy)=u^{(q)}(x,y)dy+\mathbb{E}_x[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}]R^{(q)}(0,dy), \tag{4.1}$$
and thus the problem of finding $R^{(q)}(x,dy)$ reduces to finding an expression for $R^{(q)}(0,dy)$, provided of course that we have an expression for the two-sided exit problem. The main result of this paper shows that (under the conditions above) $R^{(q)}(x,dy)$ is absolutely continuous with respect to Lebesgue measure and that a version of its density is given in terms of $u^{(q)}(x,y)$ and the two-sided exit problem.

Theorem 4.1. *Suppose X is a Lévy process satisfying the conditions mentioned above. Let $0<x\leq a$, $0\leq y\leq a$ and $q\geq 0$.*

The resolvent measure of the killed reflected process has a density, which can be expressed in terms of $u^{(q)}$ and the two sided exit problem as
$$r^{(q)}(x,y)=u^{(q)}(x,y)+\mathbb{E}_x[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}]r^{(q)}(0,y), \tag{4.2}$$
where
$$r^{(q)}(0,y)=\lim_{z\downarrow 0}\frac{u^{(q)}(z,y)}{1-\mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^-<\tau_a^+\}}]}. \tag{4.3}$$
Similarly,
$$r(x,y):=r^{(0)}(x,y)=u(x,y)+\mathbb{P}_x(\tau_0^-<\tau_a^+)r(0,y), \tag{4.4}$$
where
$$r(0,y):=r^{(0)}(0,y)=\lim_{z\downarrow 0}\frac{u(z,y)}{\mathbb{P}_z(\tau_a^+<\tau_0^-)}. \tag{4.5}$$

Before proving Theorem 4.1 we obtain a couple of auxiliary results. Since $R^{(q)}$ depends only on the behavior of Y until the first time Y exceeds the level a, we can replace all jumps of X greater than a by jumps of size a without affecting $R^{(q)}$. Hence, recalling Remark 3.2, we may assume without loss of generality that $X \in \mathcal{H}$.

Denote by $\overline{\varepsilon}$ the height of the excursion, i.e.

$$\overline{\varepsilon} = \sup\{\varepsilon(s) : 0 \le s \le \zeta\}.$$

Recall that ρ_a is the first time an excursion exceeds the level a. Now, for any $q > 0$, define the event $A_q = B_q \cup C_q$, where

$$B_q = \{\varepsilon \in \mathcal{E} : \rho_a \le \zeta \wedge \mathbf{e}_q\} \text{ and } C_q = \{\varepsilon \in \mathcal{E} : \mathbf{e}_q < \rho_a \wedge \zeta\}.$$

Hence an excursion is in A_q if and only if its height is at least a or if its length is at least $\mathbf{e}_q$. Similarly, we define

$$A := \{\varepsilon \in \mathcal{E} : \rho_a \le \zeta\}.$$

In the following lemma we find an expression for the excursion measure of the set A_q.

Lemma 4.2. *For $q > 0$*

$$n(A_q) = \lim_{z \downarrow 0} \frac{1 - \mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}]}{h(z)}$$

and

$$n(A) = \lim_{z \downarrow 0} \frac{\mathbb{P}_z(\tau_a^+ \le \tau_0^-)}{h(z)}.$$

Proof of Lemma 4.2. Let $q > 0$. Conditional on $\rho_a < \infty$, $\{\varepsilon(t + \rho_a)\}_{t \ge 0}$ is equal in law to the process $\{X_t\}_{t\ge0}$, started at $\varepsilon(\rho_a)$ and killed at entering $(-\infty, 0]$. Using this observation in combination with an application of the strong Markov property at time ρ_a and the assumption that X is regular upwards allows us to deduce that $n(\overline{\varepsilon} = a) = 0$. From the definition of A_q and B_q it then follows that $n(\partial A_q) = n(\partial B_q) = 0$ and thus we can apply Lemma 3.3 to deduce that

$$\begin{aligned} n(A_q) &= n(B_q) + n(C_q) \\ &= n(\rho_a \le \zeta \wedge \mathbf{e}_q) + n(\mathbf{e}_q < \rho_a \wedge \zeta) \\ &= \lim_{z\downarrow 0} \frac{1}{h(z)} \left(\mathbb{E}_z \left[e^{-q\tau_a^+} \mathbf{1}_{\{\tau_a^+ < \tau_0^-\}} \right] + \mathbb{P}_z(\mathbf{e}_q < \tau_0^- \wedge \tau_a^+) \right) \\ &= \lim_{z\downarrow 0} \frac{1}{h(z)} \left(1 - \mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}] \right) \end{aligned}$$

The expression for $n(A)$ follows similarly. □

Next we show that $R^{(q)}(0, dy)$ can be expressed as a quotient involving excursion measures.

Lemma 4.3. *For $q > 0$*

$$R^{(q)}(0, dy) = \frac{n(\varepsilon \in \mathcal{E} : \mathbf{e}_q < \zeta, \varepsilon(\mathbf{e}_q) \in dy, \overline{\varepsilon}(\mathbf{e}_q) \le a)}{qn(A_q)}. \tag{4.6}$$

Also

$$R(0, dy) = \frac{\int_0^\infty n(\varepsilon \in \mathcal{E} : \varepsilon(t) \in dy, t < \rho_a \wedge \zeta)dt}{n(A)}. \tag{4.7}$$

Proof of Lemma 4.3. Let $q > 0$. We have

$$R^{(q)}(0,dy) = \int_0^\infty e^{-qt}\mathbb{P}(Y_t \in dy, \overline{Y}_t \le a)dt = \frac{\mathbb{P}(Y_{\mathbf{e}_q} \in dy, \overline{Y}_{\mathbf{e}_q} \le a)}{q}.$$

We denote by $\mathcal{T}$ the (countable) set of times t such that $L^{-1}(t-) < L^{-1}(t)$ and note that excursions away from zero of Y always start at time $L^{-1}(t-)$ for some $t \in \mathcal{T}$. We introduce the family $\{\mathbf{e}_q^t\}_{t\in\mathcal{T}}$ of independent copies of the exponential random variable $\mathbf{e}_q$ and we assume this family is independent of X as well. Since $\{\varepsilon_t\}_{t\in\mathcal{T}}$ is a Poisson point process with characteristic measure n, the random variable σ_q defined by

$$\sigma_q = \inf\{t \in \mathcal{T} : \varepsilon_t \in A_q\}$$

is exponentially distributed with parameter $n(A_q)$.

The memoryless property of the exponential distribution allows us to use the compensation formula in excursion theory to deduce that

$$\begin{aligned}
&\mathbb{P}(Y_{\mathbf{e}_q} \in dy, \overline{Y}_{\mathbf{e}_q} \le a) \\
&\quad = \mathbb{E}\left[\sum_{t\in\mathcal{T}} \mathbf{1}_{\{\varepsilon_t(\mathbf{e}_q^t)\in dy, \mathbf{e}_q^t \in (L^{-1}(t-),L^{-1}(t)), \mathbf{e}_q^t < \rho_a(\varepsilon_t), \sup_{s<t, s\in\mathcal{T}} \overline{\varepsilon}_s \le a\}}\right] \\
&\quad = \mathbb{E}[\sigma_q] n(\varepsilon \in \mathcal{E} : \mathbf{e}_q < \zeta, \varepsilon(\mathbf{e}_q) \in dy, \overline{\varepsilon}(\mathbf{e}_q) \le a),
\end{aligned} \tag{4.8}$$

from which (4.6) follows.

For (4.7) we use similar reasoning to deduce that

$$\int_0^\infty \mathbb{P}(Y_t \in dy, \overline{Y}_t \le a)dt = \mathbb{E}[\sigma]\int_0^\infty n(\varepsilon \in \mathcal{E} : \varepsilon(t) \in dy, t < \rho_a \wedge \zeta)dt,$$

where the random variable σ defined by

$$\sigma = \inf\{t \in \mathcal{T} : \varepsilon_t \in A\}$$

has an exponential distribution with parameter $n(A)$. □

Proof of Theorem 4.1: By the strong Markov property, it suffices to show (4.3) and (4.5). For (4.3) we use (4.6) and Lemmas 3.3 and 4.2 to find

$$\begin{aligned}
R^{(q)}(0,dy) &= \frac{n(\mathbf{e}_q < \zeta, \varepsilon(\mathbf{e}_q) \in dy, \overline{\varepsilon}(\mathbf{e}_q) \le a)}{qn(A_q)} \\
&= \lim_{z\downarrow 0} \frac{u^{(q)}(z,y)}{1 - \mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}]}dy,
\end{aligned}$$

where the limit is understood in the weak sense. For (4.5) we use Lemma 3.3 and (4.7) to find

$$\begin{aligned}
R(0,dy) &= \frac{\int_0^\infty n(\varepsilon \in \mathcal{E} : \varepsilon(t) \in dy, t < T_a(\varepsilon) \wedge \zeta)dt}{n(A)} \\
&= \lim_{z\downarrow 0} \frac{u(z,y)}{\mathbb{P}(\tau_a^+ \le \tau_0^-)}dy,
\end{aligned}$$

where, again, the limit is understood in the weak sense. This completes the proof of Theorem 4.1. □

5. Resolvent density for reflected symmetric stable process killed at exceeding a

In this section, as an application of Theorem 4.1, we find the resolvent density for reflected symmetric stable processes killed at exceeding a. A Lévy process X is called strictly stable with index α when for each $k > 0$ the process $\{k^{-1/\alpha} X_{kt}\}_{t\geq 0}$ has the same finite dimensional distributions as $\{X_t\}_{t\geq 0}$. From the Lévy-Khintchine formula it follows that $\alpha \in (0, 2]$. The characteristic exponent of X is of the form

$$\Psi(\theta) = \begin{cases} c|\theta|^\alpha (1 - i\beta \tan \frac{\pi\alpha}{2} \operatorname{sgn} \theta) & \text{when } \alpha \neq 1, \\ c|\theta| + i\eta\theta & \text{when } \alpha = 1, \end{cases}$$

where $\beta \in [-1, 1]$, $c > 0$ and $\eta \in \mathbb{R}$. A strictly stable process is symmetric when $\alpha = 1$ and $\eta = 0$ or when $\alpha \neq 1$ and $\beta = 0$. We refer to the books Bertoin (1996), Sato (1999) and Zolotarev (1986) for further details on stable processes.

For a killed symmetric stable process we have the following expression for the potential density, which follows after rescaling of the formula in Corollary 4 in Blumenthal et al. (1961).

Theorem 5.1. *The potential measure for a symmetric stable process killed at exiting* $[0, a]$ *has a density given by*

$$u^{(0)}(x, y) = \frac{1}{2^\alpha \Gamma^2(\alpha/2)} \mid x - y \mid^{\alpha - 1} \int_0^{s(x,y)} \frac{u^{\alpha/2-1}}{\sqrt{u+1}} du$$

where

$$s(x, y) = \frac{4xy(a - x)(a - y)}{a^2(x - y)^2}. \tag{5.1}$$

Furthermore

$$\mathbb{P}_x(\tau_a^+ < \tau_0^-) = \frac{2^{1-\alpha}\Gamma(\alpha)}{\Gamma^2(\alpha/2)} \int_{-1}^{-1+2x/a} (1 - u^2)^{\alpha/2-1} du.$$

We can apply Theorem 4.1 to establish the following result.

Theorem 5.2. *The potential measure for a reflected symmetric stable process killed at exceeding a has a density given by*

$$r(0, y) = \frac{y^{\alpha/2-1}(a - y)^{\alpha/2}}{\Gamma(\alpha)} \quad \text{for } y \in [0, a] \tag{5.2}$$

and thus for any $x, y \in [0, a]$

$$\begin{aligned} r(x, y) \quad = \quad & \frac{1}{2^\alpha \Gamma^2(\alpha/2)} \mid x - y \mid^{\alpha-1} \int_0^{4xy(a-x)(a-y)/(a(x-y))^2} \frac{u^{\alpha/2-1}}{\sqrt{u+1}} du \\ & + \frac{y^{\alpha/2-1}(a - y)^{\alpha/2}}{\Gamma(\alpha)} \left(1 - \frac{2^{1-\alpha}\Gamma(\alpha)}{\Gamma^2(\alpha/2)} \int_{-1}^{-1+x/2a} (1 - u^2)^{A/2-1} du\right) \end{aligned} \tag{5.3}$$

Proof. Any non-monotone stable process is regular upwards (in the case of bounded variation, this follows from the fact that the Lévy measure of such a process satisfies the integral test in Bertoin (1997)) and thus we are within the scope of Theorem 4.1. Let s be defined as in (5.1). A quick calculation shows that

$$\lim_{z \downarrow 0} \frac{\partial s(z, y)}{\partial z} = \frac{4(a - y)}{ay}.$$

For (5.2) we deduce from Theorem 4.1 and 5.1 that

$$\begin{aligned} r(0,y) &= \lim_{z\downarrow 0}\frac{u(z,y)}{\mathbb{P}_z(\tau_a^+<\tau_0^-)} \\ &= \frac{1}{2\Gamma(\alpha)}\lim_{z\downarrow 0}\frac{|\, z-y \,|^{\alpha-1}\int_0^{s(z,y)} u^{\alpha/2-1}(u+1)^{-1/2}du}{\int_{-1}^{-1+2z/a}(1-u^2)^{\alpha/2-1}du} \\ &= \frac{y^{\alpha-1}}{2\Gamma(\alpha)}\lim_{z\downarrow 0}\frac{s(z,y)^{\alpha/2-1}(s(z,y)+1)^{-1/2}\frac{\partial s(z,y)}{\partial z}}{(1-(2z/a-1)^2)^{\alpha/2-1}2/a} \\ &= \frac{y^{\alpha/2-1}(a-y)^{\alpha/2}}{\Gamma(\alpha)}. \end{aligned}$$

Formula (5.3) now follows directly from (4.1). □

As a corollary we find the joint law of the undershoot and the overshoot at level a of the reflected symmetric stable process Y.

Corollary 5.3. *For* $0\le z\le a\le y$

$$\mathbb{P}(Y_{T_a^+-}\in dz, Y_{T_a^+}\in dy)=\frac{\alpha\sin(\alpha\pi/2)}{\pi}(y-z)^{-\alpha-1}z^{\alpha/2-1}(a-z)^{\alpha/2}dy\,dz.$$

Proof. The ladder height process of a stable process is again stable and hence it has no drift. It follows that X does not creep upwards, which implies $\mathbb{P}(Y_{T_a}^+=a)=0$, and thus Y exceeds the level a by a jump. By the compensation formula we find that for any $0\le z\le a\le y$

$$\mathbb{P}(Y_{T_a^+-}\in dz, Y_{T_a^+}\in dy)=r(0,z)\Pi(y-z)dz\,dy. \tag{5.4}$$

The result now follows from (5.2) and from taking into account that the right hand side of (5.4) has unit mass on $[0,a]\times[a,\infty)$. □

Remark 5.4. When we integrate both sides of the equation in Corollary 5.3 over z, we deduce the result in Theorem 2 in Kyprianou (2006a) for the special case when the stable process is symmetric.

6. Concluding remarks

Remark 6.1. When considering reflected processes, excursion theory is not only useful for finding the resolvent density. For example, a reasoning analogous to the proof of Lemma 4.3 leads to the expression for the overshoot of any reflected strictly stable process, as was first found in Kyprianou (2006a) using martingale techniques. Similarly, we can retrieve Theorem 1 in Avram et al. (2004), which gives the joint Laplace transform of the first passage time and overshoot of a spectrally negative process reflected at its maximum.

Remark 6.2. As mentioned before, the assumptions that X is regular upwards and that the resolvent measure $U^{(q)}(x,dy)$ has a density can be relaxed. When X is not regular upwards, the second part of Theorem 2 in Chaumont and Doney (2005) states that for any $\delta>0$ the process $(X\circ\theta_\delta,\mathbb{P}_x^\uparrow)$ converges weakly (as x goes to 0) towards $(X\circ\theta_\delta,\mathbb{P}^\uparrow)$, where θ denotes the shift operator. Reconsidering the proof of Theorem 4.1 and Lemma 4.2 in particular we find that $R^{(q)}(x,dy)$ is still given as in Theorem 4.1 when $x>0$, when the regularity condition is replaced by $n(\overline{\varepsilon}=a)=0$. The latter holds if Lévy measure Π of X does not have an atom at

a. When X is irregular upwards, the time Y spends at zero has positive Lebesgue measure and hence $R(x, dy)$ has an atom at zero in this case. We use the strong Markov property to derive

$$R^{(q)}(x, \{0\}) = \mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}]R^{(q)}(0, \{0\}).$$

Next we remark that

$$qR^{(q)}(0, \{0\}) \quad = \quad \mathbb{P}(\mathbf{e}_q < T_a^+) - q\int_0^a r^{(q)}(0, y)dy$$

and it now follows from Lemma 3.3, Theorem 4.1, Lemma 4.2 and (4.8) that

$$R^{(q)}(0, \{0\}) = \frac{1}{q}\lim_{z\downarrow 0}\frac{1 - \mathbb{E}_z[e^{-q(\tau_0^- \wedge \tau_a^+)}]}{1 - \mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}]} - \int_0^a \lim_{z\downarrow 0}\frac{u^{(q)}(z, y)}{1 - \mathbb{E}_z[e^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}]}dy. \tag{6.1}$$

When $U^{(q)}(x, dy)$ is not absolutely continuous with respect to Lebesgue measure, a version of Theorem 4.1 can be obtained in terms of measures.

When X is spectrally one-sided, $u^{(q)}(x, y)$ and the two-sided exit problem are given in terms of the so-called scale function and we find Theorem 1 of Pistorius (2004) (note that a bounded variation spectrally positive Lévy process is irregular upwards and thus the atom at zero of $R^{(q)}(x, dy)$ is given by (6.1)). This is essentially the method of proof as introduced in Doney (2005).

Acknowledgement: This paper was written partly during my stay at Heriot-Watt University in Edinburgh and at the University of Bath and I would like to express my gratitude for their hospitality and support. I'm grateful for various useful comments made by the referee on an earlier draft of this paper. I'd also like to thank Ron Doney for his valuable comments and Loïc Chaumont for assisting me with the proof of (3.3).

References

S. Asmussen. *Applied Probability and Queues.* Wiley (1989).

S. Asmussen, F. Avram and M. R. Pistorius. Russian and American put options under exponential phase-type Lévy models. *Stochastic Process. Appl.* **109**, 79–111 (2004).

F. Avram, A. E. Kyprianou and M. R. Pistorius. Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Probab.* **14**, 215–238 (2004).

F. Avram, Z. Palmowski and M. R. Pistorius. On the optimal dividend problem for a spectrally negative Lévy process. *Ann. Appl. Prob.* **17**, 156–180 (2007).

E. J. Baurdoux and A. E. Kyprianou. The Shepp–Shiryaev stochastic game driven by a spectrally negative Lévy process. *Submitted* (2007).

J. Bertoin. *Lévy Processes.* Cambridge University Press (1996).

J. Bertoin. Regularity of the half-line for Lévy processes. *Bull. Sci. Math.* **121**, 345–354 (1997).

R. M. Blumenthal, R. K. Getoor and D. B. Ray. On the distribution of first hits for the symmetric stable processes. *Trans. Amer. Math. Soc.* **99**, 540–554 (1961).

A. A. Borovkov. *Stochastic Processes in Queueing Theory.* Springer (1976).

L. Chaumont. Conditionings and path decompositions for Lévy processes. *Stochastic Process. Appl.* **64**, 39–54 (1996).

L. Chaumont and R. A. Doney. On Lévy processes conditioned to stay positive. *Electron. J. Probab.* **10**, 948–961 (2005).

B. De Finetti. Su un'impostazione alternativa della teoria colletiva del rischio. *Trans. XV Intern. Congress Act.* **2**, 433–443 (1957).

R. A. Doney. Some excursion calculations for spectrally one-sided Lévy processes. In *Séminaire de Probabilités, XXXVIII*, pages 5–15. Springer (2005).

H. U. Gerber and E. S. W. Shiu. Optimal dividends: analysis with Brownian motion. *N. Am. Actuar. J.* **8**, 1–20 (2004).

A. E. Kyprianou. First passage of reflected strictly stable processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **2**, 119–123 (2006a).

A. E. Kyprianou. *Introductory Lectures on Fluctuations of Lévy Processes with Applications.* Springer (2006b).

A. E. Kyprianou and Z. Palmowski. Distributional study of De Finetti's dividend problem for a general Lévy insurance risk process. *J. Appl. Probab.* **44**, 428–443 (2007).

L. Nguyen-Ngoc and M. Yor. Some martingales associated to reflected Lévy processes. In *Séminaire de Probabilités, XXXVIII*, pages 42–69. Springer (2005).

M. R. Pistorius. On exit and ergodicity of the spectrally one-sided Lévy process reflected at its infimum. *J. Theoret. Probab.* **17**, 183–220 (2004).

N. U. Prabhu. *Insurance, Queues and Dams.* Springer (1997).

K. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press (1999).

L. A. Shepp and A. N. Shiryaev. A new look at pricing the "Russian option". *Theory Probab. Appl.* **39**, 103–119 (1994).

V. M. Zolotarev. *One-dimensional Stable Distributions.* Amer. Math. Soc. (1986).