Erik Baurdoux

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Last exit before an exponential time for spectrally negative Lévy processes

E.J. Baurdoux*

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Abstract

In [5], the Laplace transform was found of the last time a spectrally negative Lévy process, which drifts to infinity, is below some level. The main motivation for the study of this random time stems from risk theory: what is the last time the risk process, modeled by a spectrally negative Lévy process drifting to infinity, is zero? In this paper we extend the result found in [5] and we derive the Laplace transform of the last time before an independent, exponentially distributed time, that a spectrally negative Lévy process (without any further conditions) exceeds (upwards or downwards) or hits a certain level. As an application we extend a result found by Doney in [6].

Key words: Spectrally negative Lévy processes; exit problems; fluctuation theory; last passage time; risk theory.
MSC 2000 subject classifications: 60G99, 91B30.

1 Introduction

The classical risk process, as introduced in [15], consists of a deterministic, positive drift c plus a compound Poisson process which has only negative jumps. We denote by $\lambda > 0$ the rate of the Poisson process and by μ the expected jump size. A main quantity of interest is the moment of ruin, i.e. the first time the risk process becomes negative. To ensure the moment of ruin is not almost surely finite, the net profit condition

$$\frac{\lambda\mu}{c} < 1 \tag{1}$$

is imposed. This condition ensures that the risk process drifts to $+\infty$. Recently, various authors (see for example [4], [10], [11],[12]) have replaced the classical risk process by a general spectrally negative Lévy process, which we shall denote

*Department of Statistics, London School of Economics. Houghton street, London, WC2A 2AE, United Kingdom. E-mail: e.j.baurdoux@lse.ac.uk

by X. Also, in some cases, the moment of ruin may not be the most important quantity concerning the risk process. Indeed, consider the following scenario. Instead of going bankrupt when the risk process becomes negative, the firm has other funds which it can use to support the negative surplus for a while. For this reason, in [8], the Laplace transform was found for the last passage time at a certain level for the classical risk process. This was extended to the case of a general spectrally negative Lévy process in [5]. However, a more realistic quantity for study might be the last passage time below zero before a fixed time t, i.e.

$$S_t^- := \sup\{0 \le u \le t : X_u \le 0\} \qquad \text{for } t \ge 0 \tag{2}$$

As is often the case, it turns out that it is easier to replace the fixed, deterministic time horizon by an independent, exponentially distributed random time. For $\theta \ge 0$, we denote by $\tilde{\mathbf{e}}_\theta$ an exponentially distributed random variable with parameter θ. Here, we use the convention that an exponential random variable with parameter zero is taken to be infinite with probability one. A Lévy process starting from $x \in \mathbb{R}$ (with respect to some probability space $(\Omega, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P}_x)$) is said to be spectrally negative when it has no positive jumps and when it does not have monotone paths. We suppress the subscript in $\mathbb{P}_x$ when $x = 0$. Now, define the random time

$$\sigma_\theta^- = S_{\tilde{\mathbf{e}}_\theta}^- = \sup\{0 \le t \le \tilde{\mathbf{e}}_\theta : X_t \le 0\},$$

with the convention that $\sup \emptyset = 0$. In the main result of this paper, Theorem 2, we give the Laplace transform of σ_θ^-. Using similar techniques, we also find the Laplace transform of

$$\sigma_\theta^+ = \sup\{0 \le t \le \tilde{\mathbf{e}}_\theta : X_t \ge 0\}$$

and of

$$T_\theta = \sup\{0 \le t \le \tilde{\mathbf{e}}_\theta : X_t = 0\}.$$

For convenience, we suppress the subscript when $\theta = 0$. For spectrally negative Lévy processes drifting to ∞, the Laplace transform of σ^- was found in [5]. Trivially, in this case it holds that $T = \sigma^-$.
As an application of Theorem 2, we extend a result from [6]. In that paper it was proved that for a spectrally negative stable process with index α ,

$$\mathbb{P}(X_t = \overline{X}_t = t \text{ for some } 0 < t < \infty) = \frac{1}{\alpha},$$

where $\overline{X}_t$ is the running supremum of X, i.e. $\overline{X}_t = \sup_{0\le s\le t} X_s$. We find in the final section of this paper the Laplace transform of $\sup\{t \ge 0 : X_t = \overline{X}_t = t\}$ for a general spectrally negative Lévy process.

Remark 1. The random times introduced above are not stopping times, as they depend on the future of the process $\{X_t\}_{t\ge 0}$.

2 Preliminaries

In this section we review some important properties of spectrally negative Lévy processes. For further details we refer to the books [2] and [13]. For a spectrally negative Lévy process $\{X_t\}_{t\geq 0}$ it holds that the Laplace exponent

$$\psi(\lambda) := \log \mathbb{E}[e^{\lambda X_1}] \quad \lambda \geq 0$$

is well defined, convex and infinitely differentiable on $(0, \infty)$. Furthermore, when X is of bounded variation, we can express the Laplace exponent as

$$\psi(\lambda) = \mathrm{d}\lambda + \int_{(-\infty,0)} (e^{\lambda x} - 1)\Pi(dx),$$

where Π is the jump measure of X and $\mathrm{d} > 0$ is called the drift.

For $q \geq 0$ the scale function $W^{(q)}(x)$ is defined as the continuous function on $[0, \infty)$ such that

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)\, dx = \frac{1}{\psi(\lambda) - q} \quad \text{for any } \lambda > \Phi(q).$$

Here Φ denotes the right inverse of ψ. See for example Section VII.2 in [2] or Chapter 8 in [13] for a detailed study of the scale function. When $q = 0$ we omit the superscript and write $W(x)$ instead. The function $W^{(q)}$ is extended to the negative half line by putting $W^{(q)}(x) = 0$ when $x < 0$. Note that $W^{(q)}$ is not necessarily continuous in 0. In fact, it is not difficult to show that $W^{(q)}(0) = 0$ when X is of unbounded variation and $W^{(q)}(0) = 1/\mathrm{d}$ when X is of bounded variation with drift d. Furthermore, for $q \geq 0$ we define the function $Z^{(q)}$ by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)\, dy.$$

Note that $Z^{(q)}(x) = 1$ when $x \leq 0$. Integration by parts yields

$$\int_0^\infty e^{-\lambda x} Z^{(q)}(x)\, dx = \frac{1}{\lambda} + \frac{q}{\lambda} \int_0^\infty e^{-\lambda x} W^{(q)}(x)\, dx = \frac{1}{\lambda} + \frac{q}{\lambda(\psi(\lambda) - q)}. \tag{3}$$

For $a, b \in \mathbb{R}$, denote first passage times by

$$\tau_a^- := \inf\{t > 0 : X_t \leq a\} \tag{4}$$

and

$$\tau_b^+ := \inf\{t > 0 : X_t \geq b\}. \tag{5}$$

Also, we denote the first hitting time by

$$T(a) := \inf\{t > 0 : X_t = a\}. \tag{6}$$

Scale functions play a vital role in exit problems. For example, it holds that

$$\mathbb{E}_x[e^{-q\tau_0^-} \mathbf{1}_{\{\tau_0^- < \infty\}}] = Z^{(q)}(x) - W^{(q)}(x) \frac{q}{\Phi(q)}, \tag{7}$$

where for the case $q = 0$ the fraction $q/\Phi(q)$ is to be understood in the limiting sense. Expression (7) first appeared in the form of its Fourier transform in [7]. To derive our results concerning the last exit times we also make use of potential measures. For spectrally negative Lévy processes, the q-potential measure $U^{(q)}(dy)$, defined by

$$\int_0^\infty e^{-qt}\mathbb{P}(X_t \in dy)\,dt,$$

is absolutely continuous with respect to Lebesgue measure and a version of its density is given by

$$u^q(y) = \Phi'(q)e^{-\Phi(q)y} - W^{(q)}(-y), \tag{8}$$

see [3].

Since, for $c \geq 0$, the process $\{e^{cX_t - \psi(c)t}\}$ is a martingale with mean 1, we can introduce the change of measure

$$\left.\frac{d\mathbb{P}^c}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{cX_t - \psi(c)t}.$$

The process $\{X_t\}_{t\geq 0}$ is still a spectrally negative Lévy process under $\mathbb{P}^c$ and we mark the Laplace exponent and scale functions of X under $\mathbb{P}^c$ with the subscript c. It is straightforward to check that

$$\psi_c(\lambda) = \psi(c+\lambda) - \psi(c) \tag{9}$$

for $\lambda \geq 0$ and, by taking Laplace transforms, we also find

$$W_c^{(q)}(x) = e^{-cx}W^{(q+\psi(c))}(x) \tag{10}$$

for $q \geq 0$. Furthermore, we readily check that for $c, p \geq 0$

$$\begin{aligned}
\Phi_c(p) &= \sup\{x : \psi_c(x) = p\} \\
&= \sup\{x : \psi(x+c) = p + \psi(c)\} \\
&= \Phi(p + \psi(c)) - c.
\end{aligned}$$

For future reference we also state the following result, the proof of which is given in the appendix.

Lemma 1. For $q > 0$ and $\lambda \geq -\Phi(q)$

$$\int_{[0,\infty)} e^{-\lambda x}W_{\Phi(q)}(dx) = \frac{\lambda}{\psi(\Phi(q)+\lambda) - q}, \tag{11}$$

where the right hand side is to be interpreted in the limiting sense as $\Phi'(q)$ for the case $\lambda = 0$.

Finally, we collect a couple of well-known expressions for first exit problems which we shall use throughout this paper.

Lemma 2. For $x > 0$ and $u, v \geq 0$

$$\mathbb{E}_x[e^{-u\tau_0^- + vX_{\tau_0^-}} \mathbf{1}_{\{\tau_0^- < \infty\}}] = e^{vx} \left(Z_v^{(p)}(x) - W_v^{(p)}(x) \frac{p}{\Phi_v(p)} \right), \tag{12}$$

where $p = u - \psi(v)$. For the case $u = \psi(v)$ the fraction $p/\Phi_v(p)$ is to be interpreted in the limiting sense as

$$\lim_{u \to \psi(v)} \frac{u - \psi(v)}{\Phi_v(u - \psi(v))} = \frac{1}{\Phi'(\psi(v))}. \tag{13}$$

For $x < 0$ and $q \geq 0$

$$\mathbb{E}_x[e^{-q\tau_0^+} \mathbf{1}_{\{\tau_0^+ < \infty\}}] = e^{\Phi(q)x}. \tag{14}$$

Finally for $x > 0$ and $q \geq 0$

$$\mathbb{E}_x[e^{-qT(0)} \mathbf{1}_{\{T(0) < \infty\}}] = e^{\Phi(q)x} - \psi'(\Phi(q)) W^{(q)}(x), \tag{15}$$

and the case $x = 0$ is given by $1 - (\mathrm{d}\Phi'(0))^{-1}$ when X has bounded variation with drift d.

Expression (12) follows after a change of measure and (7). From the fact that the process $\{e^{-qt + \Phi(q)X_t}\}_{t \geq 0}$ is a martingale one can deduce (14). Finally, (15) was established in the form of its Laplace transform in Theorem 1 in [6].

3 Main result

Not surprisingly, scale functions also play a predominant role when considering last exit times. The following result is Theorem 3.1 in [5].

Theorem 1. Suppose $\psi'(0) > 0$. Then for $q > 0$ and $x \in \mathbb{R}$

$$\mathbb{E}_x[e^{-q\sigma^-} \mathbf{1}_{\{\sigma^- > 0\}}] = \Phi'(q)\psi'(0)e^{\Phi(q)x} - \psi'(0)W^{(q)}(x).$$

In this paper we extend this result by considering last passage below a certain level before an independent, exponentially distributed time (as well as last passage above and last hit of a fixed level). We state the main result of this paper.

Theorem 2. For $q, \theta \geq 0$ and $x \in \mathbb{R}$

$$\begin{aligned} \mathbb{E}_x[e^{-q\sigma_\theta^-}] &= 1 + e^{\Phi(q+\theta)x}\Phi'(q+\theta) \left(\frac{\theta}{\Phi(\theta)} - \frac{\theta}{\Phi(q+\theta)} \right) \\ &\quad + \frac{\theta}{q+\theta} Z^{(q+\theta)}(x) - Z^{(\theta)}(x) + \frac{\theta}{\Phi(\theta)} \left(W^{(\theta)}(x) - W^{(q+\theta)}(x) \right) \end{aligned} \tag{16}$$

Furthermore,

$$\begin{aligned} \mathbb{E}_x[e^{-q\sigma_\theta^+}] &= \frac{q}{q+\theta} Z^{(q+\theta)}(x) - e^{\Phi(\theta)x} Z_{\Phi(\theta)}^{(q)}(x) \\ &\quad + \frac{\theta}{\theta+q} + e^{\Phi(q+\theta)x} \frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta) - \Phi(\theta))}. \end{aligned} \tag{17}$$

Finally,

$$\mathbb{E}_x[e^{-qT_\theta}] = 1 - e^{\Phi(\theta)x} + \frac{1}{\Phi'(\theta)}(W^{(\theta)}(x) - W^{(q+\theta)}(x)) + \frac{\Phi'(q+\theta)}{\Phi'(\theta)} e^{\Phi(q+\theta)x}. \quad (18)$$

Combined with the strong Markov property, Theorem 2 allows us to readily obtain expressions for the joint Laplace transform of first and last exit times.

Corollary 1. Let $p, q \geq 0$. When X does not oscillate

$$\mathbb{E}_x[e^{-pT(0)-qT}\mathbf{1}_{\{T(0)<\infty\}}] = \frac{\Phi'(q)}{\Phi'(0)} \left(e^{\Phi(p+q)x} - \frac{1}{\Phi'(p+q)} W^{(p+q)}(x) \right). \quad (19)$$

When X drifts to $-\infty$ and $x < 0$

$$\mathbb{E}_x[e^{-p\tau_0^+ - q\sigma^+}\mathbf{1}_{\{\tau_0^+<\infty\}}] = \frac{q\Phi(0)\Phi'(q)}{\Phi(q)(\Phi(q)-\Phi(0))} e^{\Phi(p+q)x}. \quad (20)$$

When X drifts to $+\infty$

$$\mathbb{E}_x[e^{-p\tau_0^- - q\sigma^-}\mathbf{1}_{\{\tau_0^-<\infty\}}] = \frac{\Phi'(q)}{\Phi'(0)} \left(e^{\Phi(q)x} Z^{(p)}_{\Phi(q)}(x) - \frac{p}{\Phi(p+q)-\Phi(q)} W^{(p+q)}(x) \right). \quad (21)$$

Proof of Corollary 1. The third equality was already obtained in [5]. We only prove (19), as the proofs of the other claims are similar. Suppose that X drifts to $+\infty$. Then $\Phi(0) = 0$ and from the strong Markov property applied at $T(0)$ and (15) we find

$$\begin{aligned} \mathbb{E}_x[e^{-pT(0)-qT}\mathbf{1}_{\{T(0)<\infty\}}] &= \mathbb{E}_x[e^{-(p+q)T(0)}\mathbf{1}_{\{T(0)<\infty\}}\mathbb{E}[e^{-qT}]] \\ &= \frac{\Phi'(q)}{\Phi'(0)} \left(e^{\Phi(p+q)x} - \frac{1}{\Phi'(p+q)} W^{(p+q)}(x) \right). \end{aligned}$$

□

Remark 2. Note that Theorem 1 follows by taking $p = 0$ in (21) (or in (19)).

When X is a stable process, we can invert the double Laplace transform in (18) (when $x = 0$) and retrieve the known result that, for each $t \geq 0$, the random variable defined, analogously to (2), by

$$S_t := \sup\{0 \leq u \leq t : X_u = 0\}, \qquad t \geq 0$$

is distributed according to the so-called generalized arcsine law. When $\alpha = 2$, this is the well-known arcsine law for Brownian motion (see eg. [14]). In fact, using the scaling property of stable processes, the following result can be shown to hold for any stable process with index $\alpha > 1$ (i.e. not only in the spectrally negative case). We refer to Theorem VIII.12 in [2] for the proof in the general case.

Corollary 2. Suppose X is a spectrally negative stable process with index $\alpha \in (1,2]$. Then for $0 \le s \le t$

$$\mathbb{P}(S_t \in ds) = \frac{\sin(\pi/\alpha)}{\pi} s^{-1/\alpha}(t-s)^{-1+1/\alpha} ds. \tag{22}$$

Also, the distribution of S_t^- is given by

$$\mathbb{P}(S_t^- \in ds) = \frac{1}{\alpha}\frac{\sin(\pi/\alpha)}{\pi} s^{-1/\alpha}(t-s)^{-1+1/\alpha} ds + (1-\frac{1}{\alpha})\delta_t(ds), \tag{23}$$

where δ_t is the Dirac measure at t.

Proof of Corollary 2. When X is a spectrally negative stable process of index α, it holds that (without loss of generality) $\psi(\lambda) = \lambda^\alpha$ for $\alpha \ge 0$ and thus $\Phi(q) = q^{1/\alpha}$ for $q \ge 0$. It is straightforward to check that

$$\int_0^\infty \int_s^\infty e^{-qs-\theta t} s^{-1/\alpha}(t-s)^{-1+1/\alpha} dt\, ds \quad = \quad \Gamma(1/\alpha)\Gamma(1-1/\alpha)\theta^{-1/\alpha}(\theta+q)^{-1+1/\alpha}.$$

From (18) we now deduce (22) and (23) follows in a similar way from (16). □

4 Proof of Theorem 2

For $q \ge 0$ we denote by $\mathbf{e}_q$ an exponentially distributed random variable with parameter q which is independent of X and $\tilde{\mathbf{e}}_\theta$. We split the proof of Theorem 2 in different parts.

Proof of (16). Let

$$A^+ = \{\tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q, X_{\mathbf{e}_q} > 0, X_s > 0 \text{ for all } s \in [\mathbf{e}_q, \tilde{\mathbf{e}}_\theta]\}.$$

We can then write the event $\{\sigma_\theta^- < \mathbf{e}_q\}$ as a disjoint union

$$\{\sigma_\theta^- < \mathbf{e}_q\} = \{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\} \cup A^+. \tag{24}$$

We thus have

$$\mathbb{E}_x[e^{-q\sigma_\theta^-}] = \mathbb{P}_x(\sigma_\theta^- < \mathbf{e}_q) = \mathbb{P}(\tilde{\mathbf{e}}_\theta < \mathbf{e}_q) + \mathbb{P}_x(A^+) = \frac{\theta}{\theta+q} + \mathbb{P}_x(A^+).$$

Now for $x \le 0$

$$\begin{aligned}
\mathbb{P}_x(A^+) &= q\mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbf{1}_{\{\tilde{\mathbf{e}}_\theta \ge t\}}\mathbf{1}_{\{X_s>0 \text{ for all } s\in[t,\tilde{\mathbf{e}}_\theta]\}}\,dt\right] \\
&= q\int_0^\infty e^{-(q+\theta)t}\int_{(0,\infty)}\mathbb{P}_x(X_t\in dy)\mathbb{P}_y(\tau_0^- > \tilde{\mathbf{e}}_\theta)\,dt \\
&= q\int_{(0,\infty)} u^{(q+\theta)}(y-x)(1-\mathbb{E}_y[e^{-\theta\tau_0^-}])\,dy \\
&= q\int_0^\infty \Phi'(q+\theta)e^{-\Phi(q+\theta)(y-x)}(1-Z^{(\theta)}(y)+W^{(\theta)}(y)\frac{\theta}{\Phi(\theta)})\,dy \\
&= q\Phi'(q+\theta)e^{\Phi(q+\theta)x}\left(\frac{1}{\Phi(q+\theta)}-\frac{1}{\Phi(q+\theta)}-\frac{\theta}{q\Phi(q+\theta)}+\frac{\theta}{\Phi(\theta)}\frac{1}{q}\right) \\
&= \Phi'(q+\theta)e^{\Phi(q+\theta)x}\left(\frac{\theta}{\Phi(\theta)}-\frac{\theta}{\Phi(q+\theta)}\right), \qquad (25)
\end{aligned}$$

where the second equality follows from the Markov property and lack of memory of the exponential distribution, the fourth equality from (8) and (12) and the fifth one from (3) and the definition of $W^{(q)}$. Hence,

$$\mathbb{E}_x[e^{-q\sigma_\theta^-}] = \frac{\theta}{\theta+q} + \Phi'(q+\theta)e^{\Phi(q+\theta)x}\left(\frac{\theta}{\Phi(\theta)}-\frac{\theta}{\Phi(q+\theta)}\right) \quad \text{for } x\le 0.$$

Next, let $x>0$. In this case, σ_θ^- is equal to zero whenever X does not become negative before $\tilde{\mathbf{e}}_\theta$. Taking this into account, we refine (24) and write the event $\{\sigma_\theta^- < \mathbf{e}_q\}$ as a disjoint union

$$\begin{aligned}
\{\sigma_\theta^- < \mathbf{e}_q\} &= \{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\}\cup\{\sigma_\theta^- = 0, \tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q\}\cup\{\sigma_\theta^-\in(0,\mathbf{e}_q), \tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q\} \\
&= \{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\}\cup\{\tau_0^- > \tilde{\mathbf{e}}_\theta, \tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q\}\cup(\{\tau_0^- < \tilde{\mathbf{e}}_\theta\}\cap A^+).
\end{aligned}$$

We thus have that

$$\mathbb{E}_x[e^{-q\sigma_\theta^-}] = \frac{\theta}{\theta+q} + \mathbb{P}_x(\tau_0^- > \tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q) + \mathbb{P}_x(\tau_0^- < \tilde{\mathbf{e}}_\theta, A^+)$$

and deduce

$$\begin{aligned}
\mathbb{P}_x(\tau_0^- > \tilde{\mathbf{e}}_\theta, \tilde{\mathbf{e}}_\theta \ge \mathbf{e}_q) &= \mathbb{E}_x\left[\int_0^\infty \theta e^{-\theta y}\mathbf{1}_{\{\tau_0^- > y > \mathbf{e}_q\}}\,dy\right] \\
&= \mathbb{E}_x[(e^{-\theta\mathbf{e}_q} - e^{-\theta\tau_0^-})\mathbf{1}_{\{\tau_0^- > \mathbf{e}_q\}}] \\
&= \mathbb{E}_x\left[\int_0^\infty (e^{-\theta z} - e^{-\theta\tau_0^-})qe^{-qz}\mathbf{1}_{\{\tau_0^- > z\}}\,dz\right] \\
&= \frac{q}{q+\theta} + \frac{\theta}{q+\theta}\mathbb{E}_x[e^{-(q+\theta)\tau_0^-}] - \mathbb{E}_x[e^{-\theta\tau_0^-}] \\
&= \frac{q}{q+\theta} + \frac{\theta}{q+\theta}\left(Z^{(q+\theta)}(x) - \frac{q+\theta}{\Phi(q+\theta)}W^{(q+\theta)}(x)\right) \\
&\quad -Z^{(\theta)}(x) + \frac{\theta}{\Phi(\theta)}W^{(\theta)}(x),
\end{aligned}$$

where (7) was used for the last equality. For $\theta, q \geq 0$ define

$$\lambda(\theta, q) := \Phi'(q+\theta)\left(\frac{\theta}{\Phi(\theta)} - \frac{\theta}{\Phi(q+\theta)}\right).$$

From the strong Markov property applied at τ_0^-, the memoryless property of the exponential distribution, (12) and (25) we deduce that

$$\begin{aligned}
\mathbb{P}_x(\tau_0^- < \tilde{\mathbf{e}}_\theta, A^+) &= \mathbb{E}_x[\mathbf{1}_{\{\tau_0^- < \mathbf{e}_q \wedge \tilde{\mathbf{e}}_\theta\}} \mathbb{P}_{X_{\tau_0^-}}(A^+)] \\
&= \lambda(\theta, q)\mathbb{E}_x[e^{\Phi(q+\theta)X_{\tau_0^-}} \mathbf{1}_{\{\tau_0^- < \mathbf{e}_q \wedge \tilde{\mathbf{e}}_\theta\}}] \\
&= \lambda(\theta, q)\mathbb{E}_x[\, e^{-(q+\theta)\tau_0^- + \Phi(q+\theta)X_{\tau_0^-}} \mathbf{1}_{\{\tau_0^- < \infty\}}] \\
&= \lambda(\theta, q) e^{\Phi(q+\theta)x}\left(1 - W_{\Phi(q+\theta)}(x)\frac{1}{\Phi'(q+\theta)}\right),
\end{aligned}$$

where we also used the fact that $\mathbf{e}_q \wedge \tilde{\mathbf{e}}_\theta$ is exponentially distributed with parameter $q+\theta$ for the third equality. From (10) we know that $e^{\Phi(p+q)x}W_{\Phi(p+q)}(x) = W^{(p+q)}(x)$ and thus (16) follows. □

Proof of (17). We can write the event $\{\sigma_\theta^+ < \mathbf{e}_q\}$ as a disjoint union

$$\{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\} \cup A^-,$$

where

$$A^- = \{\tilde{\mathbf{e}}_\theta > \mathbf{e}_q, X_s < 0 \text{ for all } s \in [\mathbf{e}_q, \tilde{\mathbf{e}}_\theta]\}.$$

We thus have

$$\mathbb{E}_x[e^{-q\sigma_\theta^+}] = \mathbb{P}_x(\sigma_\theta^+ < \mathbf{e}_q) = \frac{\theta}{\theta+q} + \mathbb{P}_x(A^-). \tag{26}$$

Let $x \geq 0$. Then

$$\begin{aligned}
\mathbb{P}_x(A^-) &= q\mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbf{1}_{\{\tilde{\mathbf{e}}_\theta > t\}}\mathbf{1}_{\{X_s<0 \text{ for all } s\in[t,\tilde{\mathbf{e}}_\theta]\}}\, dt\right] \\
&= q\int_0^\infty e^{-(q+\theta)t}\int_{(-\infty,0)} \mathbb{P}_x(X_t \in dy)\mathbb{P}_y(\tau_0^+ > \tilde{\mathbf{e}}_\theta)\, dt \\
&= q\int_{(-\infty,0)} u^{(q+\theta)}(y-x)\mathbb{P}_y(\tau_0^+ > \tilde{\mathbf{e}}_\theta)\, dy \\
&= q\int_{(-\infty,0)} \left(\Phi'(q+\theta)e^{\Phi(q+\theta)(x-y)} - W^{(q+\theta)}(x-y)\right)(1-e^{\Phi(\theta)y})\, dy \\
&= q\int_{(0,\infty)} \left(\Phi'(q+\theta) - W_{\Phi(q+\theta)}(x+y)\right)(e^{\Phi(q+\theta)(x+y)}(1-e^{-\Phi(\theta)y})\, dy \\
&= qe^{\Phi(q+\theta)x}\int_{(0,\infty)}\int_{x+y}^\infty W'_{\Phi(q+\theta)}(z)(e^{\Phi(q+\theta)y} - e^{(\Phi(q+\theta)-\Phi(\theta))y})\, dz\, dy \\
&= qe^{\Phi(q+\theta)x}\int_{(x,\infty)}\int_0^{z-x} W'_{\Phi(q+\theta)}(z)(e^{\Phi(q+\theta)y} - e^{(\Phi(q+\theta)-\Phi(\theta))y})\, dy\, dz,
\end{aligned} \tag{27}$$

where the second line follows from the Markov property and the lack of memory property of the exponential distribution, the fourth line from (8) and (14), the fifth one from (10), the penultimate equality from $W_{\Phi(q+\theta)}(\infty) = \Phi'(q+\theta)$ (see (38) in the Appendix) and the last equality from an application of Fubini's theorem. Denote

$$f(x,z) := \frac{e^{\Phi(q+\theta)(z-x)}}{\Phi(q+\theta)} - \frac{e^{(\Phi(q+\theta)-\Phi(\theta))(z-x)}}{\Phi(q+\theta)-\Phi(\theta)} + \frac{\Phi(\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))}.$$

Integration by parts yields

$$\begin{aligned}
&\int_{(x,\infty)}\int_0^{z-x} W'_{\Phi(q+\theta)}(z)(e^{\Phi(q+\theta)y} - e^{(\Phi(q+\theta)-\Phi(\theta))y})\,dy\,dz \\
&\quad= \int_{[0,\infty)} f(x,z)\,W_{\Phi(q+\theta)}(dz) - \int_{[0,x]} f(x,z)\,W_{\Phi(q+\theta)}(dz) \\
&\quad= \int_{[0,\infty)} f(x,z)\,W_{\Phi(q+\theta)}(dz) \\
&\qquad + \int_0^x W_{\Phi(q+\theta)}(z)(e^{\Phi(q+\theta)(z-x)} - e^{(\Phi(q+\theta)-\Phi(\theta))(z-x)})\,dz \\
&\quad= \int_{[0,\infty)} f(x,z)\,W_{\Phi(q+\theta)}(dz) \\
&\qquad + e^{-\Phi(q+\theta)x}\frac{Z^{(q+\theta)}(x)-1}{q+\theta} - e^{(\Phi(\theta)-\Phi(q+\theta))x}\frac{Z^{(q)}_{\Phi(\theta)}(x)-1}{q}.
\end{aligned}$$

Using this expression and invoking (26) and (27) allows us to conclude that

$$\begin{aligned}
\mathbb{E}_x[e^{-q\sigma_\theta^+}] &= \frac{\theta}{q+\theta} + \mathbb{P}_x(A^-) \\
&= \frac{\theta}{q+\theta} + qe^{\Phi(q+\theta)x}\int_{[0,\infty)} f(x,z)\,W_{\Phi(q+\theta)}(dz) \\
&\quad + \frac{q}{q+\theta}\left(Z^{(q+\theta)}(x)-1\right) - e^{\Phi(\theta)x}\left(Z^{(q)}_{\Phi(\theta)}(x)-1\right) \\
&= \frac{\theta}{q+\theta} + \frac{q}{q+\theta} - \frac{q}{\Phi(q+\theta)-\Phi(\theta)}\frac{\Phi(\theta)-\Phi(q+\theta)}{\psi(\Phi(\theta))-(q+\theta)}e^{\Phi(\theta)x} \\
&\quad + \frac{q\Phi(\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))}W_{\Phi(q+\theta)}(\infty)e^{\Phi(q+\theta)x} \\
&\quad + \frac{q}{q+\theta}\left(Z^{(q+\theta)}(x)-1\right) - e^{\Phi(\theta)x}\left(Z^{(q)}_{\Phi(\theta)}(x)-1\right) \\
&= \frac{q}{q+\theta}Z^{(q+\theta)}(x) - e^{\Phi(\theta)x}Z^{(q)}_{\Phi(\theta)}(x) \\
&\quad + \frac{\theta}{\theta+q} + e^{\Phi(q+\theta)x}\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))},
\end{aligned} \tag{28}$$

where the third equality follows from an application of Lemma 1 and where the last equality is again a consequence of $W_{\Phi(q+\theta)}(\infty) = \Phi'(q+\theta)$. Note that (28) implies that

$$\mathbb{P}(A^-) = \frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))} - \frac{\theta}{\theta+q}. \tag{29}$$

Next, let $x < 0$. We decompose $\{\mathbf{e}_q > \sigma_\theta^+\}$ as

$$\{\sigma_\theta^+ < \mathbf{e}_q\} \quad = \quad \{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\} \cup \{\tau_0^+ > \tilde{\mathbf{e}}_\theta, \tilde{\mathbf{e}}_\theta \geq \mathbf{e}_q\} \cup (\{\tau_0^+ < \tilde{\mathbf{e}}_\theta\} \cap A^-).$$

As before, we deduce from the strong Markov property, the memoryless property of the exponential distribution, (14) and (29) that

$$\begin{aligned}
\mathbb{E}_x[e^{-q\sigma_\theta^+}] \quad &= \quad \frac{\theta}{\theta+q} + \mathbb{P}_x(\tau_0^+ > \tilde{\mathbf{e}}_\theta > \mathbf{e}_q) + \mathbb{P}_x(\sigma_\theta^+ \in (0,\mathbf{e}_q), \tilde{\mathbf{e}}_\theta \geq \mathbf{e}_q) \\
&= \quad \frac{\theta}{\theta+q} + \mathbb{E}_x\left[\int_0^\infty \theta e^{-\theta y}\mathbf{1}_{\{\tau_0^+ > y > \mathbf{e}_q\}}\, dy\right] + \mathbb{E}_x\left[\mathbf{1}_{\{\tau_0^+ < \mathbf{e}_q \wedge \tilde{\mathbf{e}}_\theta\}}\mathbb{P}(A^-)\right] \\
&= \quad \frac{\theta}{\theta+q} + \mathbb{E}_x[(e^{-\theta \mathbf{e}_q} - e^{-\theta\tau_0^+})\mathbf{1}_{\{\tau_0^- > \mathbf{e}_q\}}] \\
&\quad\quad + \left(\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))} - \frac{\theta}{\theta+q}\right)\mathbb{P}_x(\tau_0^+ < \mathbf{e}_q \wedge \tilde{\mathbf{e}}_\theta) \\
&= \quad \frac{\theta}{\theta+q} + \mathbb{E}_x\left[\int_0^\infty qe^{-qt}(e^{-\theta t} - e^{-\theta\tau_0^+})\mathbf{1}_{\{\tau_0^+ > t\}}\, dt\right] \\
&\quad\quad + \left(\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))} - \frac{\theta}{\theta+q}\right)\mathbb{E}_x[e^{-(q+\theta)\tau_0^+}] \\
&= \quad 1 + \frac{\theta}{q+\theta}\mathbb{E}_x[e^{-(q+\theta)\tau_0^+}] - \mathbb{E}_x[e^{-\theta\tau_0^+}] \\
&\quad\quad + \left(\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))} - \frac{\theta}{\theta+q}\right)e^{\Phi(q+\theta)x} \\
&= \quad 1 - e^{\Phi(\theta)x} + \frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))}e^{\Phi(q+\theta)x},
\end{aligned}$$

which is (17), since $Z_\nu^{(r)} = 1$ for all $x \leq 0$ and $\nu, r \geq 0$. □

Proof of (18). We can write the event $\{T_\theta < \mathbf{e}_q\}$ as a disjoint union as

$$\{\tilde{\mathbf{e}}_\theta < \mathbf{e}_q\} \cup A^- \cup A,$$

where

$$A = \{\tilde{\mathbf{e}}_\theta > \mathbf{e}_q, X_{\mathbf{e}_q} > 0, X_s \neq 0 \text{ for all } s \in [\mathbf{e}_q, \tilde{\mathbf{e}}_\theta]\}$$

and where

$$A^- = \{\tilde{\mathbf{e}}_\theta > \mathbf{e}_q, X_s < 0 \text{ for all } s \in [\mathbf{e}_q, \tilde{\mathbf{e}}_\theta]\}.$$

Since we already have an expression for $\mathbb{P}_x(A^-) = \mathbb{E}_x[e^{-q\sigma_\theta^+}] - \theta/(q+\theta)$ we need only to consider A.

First assume that $x \leq 0$. In a similar fashion as in the proof of (25) we deduce from (8), (15), the lack of memory property of the exponential distribution and the Markov property that

$$
\begin{aligned}
\mathbb{P}_x(A) &= q\mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbf{1}_{\{\tilde{\mathbf{e}}_\theta \geq t\}}\mathbf{1}_{\{X_t>0, X_s\neq 0 \text{ for all } s\in[t,\tilde{\mathbf{e}}_\theta]\}}\,dt\right] \\
&= q\int_0^\infty e^{-(q+\theta)t}\int_0^\infty \mathbb{P}_x(X_t\in dy)\mathbb{P}_y(T(0)>\tilde{\mathbf{e}}_\theta)\,dt \\
&= q\int_0^\infty u^{(q+\theta)}(y-x)\mathbb{P}_y(T(0)>\tilde{\mathbf{e}}_\theta)\,dy \\
&= q\int_0^\infty \Phi'(q+\theta)e^{-\Phi(q+\theta)(y-x)}(1-e^{\Phi(\theta)y}+\psi'(\Phi(\theta))W^{(\theta)}(y))\,dy \\
&= q\Phi'(q+\theta)e^{\Phi(q+\theta)x}\left(\frac{1}{\Phi(q+\theta)}-\frac{1}{\Phi(q+\theta)-\Phi(\theta)}+\frac{1}{q\Phi'(\theta)}\right).
\end{aligned}
$$

We find that

$$
\begin{aligned}
\mathbb{E}_x[e^{-qT_\theta}] &= \mathbb{P}_x(A)+\mathbb{P}_x(A^-)+\frac{\theta}{\theta+q} \\
&= q\Phi'(q+\theta)e^{\Phi(q+\theta)x}\left(\frac{1}{\Phi(q+\theta)}-\frac{1}{\Phi(q+\theta)-\Phi(\theta)}+\frac{1}{q\Phi'(\theta)}\right) \\
&\quad +1-e^{\Phi(\theta)x}+e^{\Phi(q+\theta)x}\frac{q\Phi(\theta)\Phi'(q+\theta)}{\Phi(q+\theta)(\Phi(q+\theta)-\Phi(\theta))} \\
&= 1-e^{\Phi(\theta)x}+\frac{\Phi'(q+\theta)}{\Phi'(\theta)}e^{\Phi(q+\theta)x}.
\end{aligned}
$$

Finally, let $x>0$. As before, we find

$$
\{T_\theta<\mathbf{e}_q\} = \{\tilde{\mathbf{e}}_\theta<\mathbf{e}_q\}\cup\{T(0)>\tilde{\mathbf{e}}_\theta, \tilde{\mathbf{e}}_\theta\geq\mathbf{e}_q\}\cup\left(\{T(0)<\tilde{\mathbf{e}}_\theta\}\cap(A\cup A^-)\right).
$$

An application of the strong Markov property at $T(0)$, the memoryless property of the exponential distribution, (12), (13) and (15) imply that

$$
\begin{aligned}
\mathbb{E}_x[e^{-qT_\theta}] &= \frac{\theta}{\theta+q}+\mathbb{E}_x[(e^{-\theta\mathbf{e}_q}-e^{-\theta T(0)})\mathbf{1}_{\{T(0)>\mathbf{e}_q\}}] \\
&\quad +\mathbb{P}_x(T(0)<\mathbf{e}_q\wedge\tilde{\mathbf{e}}_\theta)\mathbb{P}(T_\theta<\mathbf{e}_q<\tilde{\mathbf{e}}_\theta) \\
&= 1+\frac{\theta}{\theta+q}\mathbb{E}_x[e^{-(q+\theta)T(0)}]-\mathbb{E}_x[e^{-\theta T(0)}] \\
&\quad +\mathbb{E}_x[e^{-(q+\theta)T(0)}]\left(\frac{\Phi'(q+\theta)}{\Phi'(\theta)}-\frac{\theta}{\theta+q}\right) \\
&= 1-e^{\Phi(\theta)x}+\frac{1}{\Phi'(\theta)}(W^{(\theta)}(x)-W^{(q+\theta)}(x))+\frac{\Phi'(q+\theta)}{\Phi'(\theta)}e^{\Phi(q+\theta)x},
\end{aligned}
$$

which completes the proof of Theorem 2. $\square$

Remark 3. Two of the main ingredients in the proof of Theorem 2 are the q-potential measure of X and the Laplace transform of the first passage time above or below a given level. These quantities are also known for certain Lévy processes which do have positive jumps. Proposition 2 in [1] indicates that results similar to (16) and (17) can be obtained for so-called phase-type Lévy processes. Similarly, as mentioned before Corollary 2, we can use the scaling property to find the Laplace exponent of the last hitting time of zero for any stable process with index $\alpha > 1$. See the proof of Lemma VIII.13 in [2] for details.

Remark 4. As mentioned in the introduction, result (16) could be useful in risk theory, since it gives information about the last time when the risk process is negative before an independent, exponentially distributed time. Indeed, the last passage of X below zero before a fixed time horizon can be found by inverting the double Laplace transform in (16). Unfortunately, this seems to be tractable analytically only in very specific cases. An additional complication is that the scale function is not always available explicitly. We refer to [9] for examples of explicit examples of scale functions. Furthermore, scale functions can be evaluated numerically and we refer to [17] and [18] for such numerical schemes.

5 Application: an extension of a result of Doney

Doney showed in Corollary 3 in [6] that for a spectrally negative stable process with index α it holds that

$$\mathbb{P}(X_t = \overline{X}_t = t \text{ for some } 0 < t < \infty) = \frac{1}{\alpha}. \tag{30}$$

In this section we extend this result and, in particular, we find (for a general spectrally negative Lévy process) the Laplace exponent of the random time τ_1 defined by

$$\tau_1 := \sup\{t \geq 0 : X_t = \overline{X}_t = t\},$$

recalling the convention that $\sup \emptyset = 0$. Similarly, we define

$$\begin{aligned} \tau_2 &= \sup\{t \geq 0 : X_t = t\}, \\ \tau_3 &= \sup\{t \geq 0 : X_t \geq t\}, \\ \tau_4 &= \sup\{t \geq 0 : \overline{X}_t \geq t\}. \end{aligned}$$

Since

$$\{t \geq 0 : X_t = \overline{X}_t = t\} \subseteq \{t \geq 0 : X_t = t\} \subseteq \{t \geq 0 : X_t \geq t\} \subseteq \{t \geq 0 : \overline{X}_t \geq t\}$$

we have that

$$\tau_1 \leq \tau_2 \leq \tau_3 \leq \tau_4.$$

These random times are trivial when X is of bounded variation with drift $\mathrm{d} \leq 1$ (since they are all equal to the first jump time when $\mathrm{d} = 1$ and all equal to zero when $\mathrm{d} < 1$) and hence we assume throughout this section that

$$\mathrm{d} > 1 \quad \text{whenever } X \text{ is of bounded variation.}$$

Let $q > 0$. Since

$$\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\lambda} = \begin{cases} \infty & \text{when } X \text{ is of unbounded variation,} \\ \mathrm{d} & \text{when } X \text{ is of bounded variation with drift d,} \end{cases}$$

we see that there exists a unique $y_q > 0$ such that

$$\psi(y_q) = q + y_q.$$

Now let $z_q := \psi(y_q)$. Then $\Phi(z_q) = \Phi(\psi(y_q)) = y_q = \psi(y_q) - q = z_q - q$. Finally, set

$$y_0 := \begin{cases} 0 & \text{when } \psi'(0) \geq 1, \\ y & \text{when } \psi'(0) < 1, \end{cases} \tag{31}$$

where y is the unique solution on $(0, \infty)$ of $\psi(\lambda) = \lambda$ when $\psi'(0) < 1$. We can use Theorem 2 to establish the following result.

Corollary 3. Suppose X is a spectrally negative Lévy process which is of unbounded variation or of bounded variation with drift $\mathrm{d} > 1$. Then

$$\mathbb{E}[e^{-q\tau_1}] = \frac{\psi'(y_q)}{\psi'(y_0)} \frac{\psi'(y_0) - 1}{\psi'(y_q) - 1}, \tag{32}$$

$$\mathbb{E}[e^{-q\tau_2}] = \frac{\psi'(y_0) - 1}{\psi'(y_q) - 1} \quad \text{and} \tag{33}$$

$$\mathbb{E}[e^{-q\tau_3}] = \frac{q y_0}{y_q(y_q - y_0)(\psi'(y_q) - 1)}. \tag{34}$$

Finally,

$$\mathbb{E}[e^{-q\tau_4}] = \frac{q y_0 \psi'(y_q)}{\psi(y_q)(\psi(y_q) - y_0)(\psi'(y_q) - 1)}. \tag{35}$$

Proof. First, suppose that X does not drift to $-\infty$. Introduce the processes $Y_t = X_t - t$ and $Z_t = t - \tau_t^+$, which are both spectrally negative Lévy processes. The assumption that X does not drift to $-\infty$ is used here to ensure that $\mathbb{P}(\tau_t^+ < \infty) = 1$. Note that, since

$$\{t \geq 0 : X_t = \overline{X}_t = t\} = \{t \geq 0 : \tau_t^+ = t\},$$

the random times τ_1 and τ_2 are, respectively, the last hitting times T of zero for Z and Y and that τ_3 and τ_4 are the last passage times above 0 of Y and Z, respectively. Using obvious notation, it holds that $\psi^Y(\lambda) = \psi(\lambda) - \lambda$ and $\psi^Z(\lambda) = \lambda - \Phi(\lambda)$, hence

$$\Phi^Y(q) = y_q \quad \text{and } \Phi^Z(q) = z_q.$$

From the implicit function theorem we find that

$$\frac{d}{dq} y_q = \frac{1}{\psi'(y_q) - 1}$$

and that

$$\frac{d}{dq}z_q = \frac{1}{1-\Phi'(z_q)} = \frac{\psi'(y_q)}{\psi'(y_q)-1}.$$

The result now follows by taking $\theta = 0$ and $x = 0$ in Theorem 2.
When X does drift to $-\infty$, (33) and (34) still hold, but in this case τ_t^+ is a subordinator killed at exponential rate $\Phi(0)$, which is strictly positive as $\psi'(0) < 0$. Hence, we are now looking for the last passage times before $\mathbf{e}_{\Phi(0)}$ of a Lévy process with Laplace exponent given by $\lambda - \Phi(\lambda) + \Phi(0)$. Statements (32) and (35) now follow by an application of Theorem 2 with $\theta = \Phi(0)$ and $x = 0$. $\square$

Define for $s \geq 0$

$$A_s := \{\text{there exists some } t > s : X_t = \overline{X}_t = t\}$$

and denote $A = A_0$. Equation (30) is contained in the following Corollary.

Corollary 4. For a spectrally negative Lévy process it holds that

$$\mathbb{P}(A) = \begin{cases} \frac{1}{\psi'(y_0)} & \text{when} \quad X \text{ has unbounded variation,} \\ \frac{\mathrm{d}-\psi'(y_0)}{\mathrm{d}\psi'(y_0)-\psi'(y_0)} & \text{when} \quad X \text{ has bounded variation with drift } \mathrm{d} > 1, \end{cases}$$

where y_0 was defined in (31). In particular $\mathbb{P}(A) = \frac{1}{\alpha}$ for a spectrally negative stable process of index α. Also

$$\mathbb{P}(A) = 1 \Leftrightarrow \psi'(0) = 1. \tag{36}$$

In fact, when $\psi'(0) = 1$,

$$\mathbb{P}(A_s) = 1$$

for all $s \geq 0$.

Proof. Since

$$\lim_{\lambda\to\infty} \psi'(\lambda) = \begin{cases} \infty & \text{when } X \text{ is of unbounded variation,} \\ \mathrm{d} & \text{when } X \text{ is of bounded variation with drift d,} \end{cases}$$

it follows from Corollary 3 that

$$\begin{aligned} \mathbb{P}(A) &= 1 - \mathbb{P}(\tau_1 = 0) \\ &= 1 - \lim_{q\to\infty} \mathbb{E}[e^{-q\tau_1}] \\ &= 1 - \lim_{q\to\infty} \frac{\psi'(y_q)}{\psi'(y_0)} \frac{\psi'(y_0)-1}{\psi'(y_q)-1} \\ &= \begin{cases} \frac{1}{\psi'(y_0)} & \text{when} \quad X \text{ is of unbounded variation,} \\ \frac{\mathrm{d}-\psi'(y_0)}{\mathrm{d}\psi'(y_0)-\psi'(y_0)} & \text{when} \quad X \text{ is of bounded variation with drift } \mathrm{d} > 1. \end{cases} \end{aligned}$$

When X is a stable process of index $\alpha \in (1, 2]$, we have $y_0 = 1$ and thus

$$\mathbb{P}(A) = \frac{1}{\psi'(1)} = 1/\alpha.$$

To show (36), suppose that $\psi'(0) = 1$. It then holds that $y_0 = 1$ and hence $\mathbb{P}(A) = 1$.

For the other direction we remark that $\psi'(0) > 1$ implies that $\psi'(y_0) = \psi'(0) > 1$. Also, when $\psi'(0) < 1$ we have that $\psi'(y_0) > 1$ as y_0 is the unique solution to $\psi(y) = y$ on $(0, \infty)$ and because ψ is a strictly convex function on $[0, \infty)$. We conclude that whenever $\psi'(0) \neq 1$ we have that $\psi'(y_0) > 1$ from which it follows that $\mathbb{P}(A) < 1$.

From (32) we see that $\psi'(0) = 1$ implies that $\mathbb{E}[e^{-q\tau_1}] = 0$ for any $q > 0$. The final statement in Corollary 4 now follows. □

Remark 5. As an example we consider a standard Brownian motion. Due to its continuous paths we have $\tau_2 = \tau_3$. Its Laplace exponent is given by $\psi(\lambda) = \lambda^2/2$, so $\psi_Y(\lambda) = \lambda^2/2 - \lambda$ and $\psi_Z(\lambda) = \lambda - \sqrt{2\lambda}$, hence

$$y_q = 1 + \sqrt{2q+1} \quad \text{and } z_q = 1 + q + \sqrt{2q+1}.$$

From Corollary 3 we readily deduce that

$$\begin{aligned}
\mathbb{E}[e^{-q\tau_1}] &= \frac{a_q + 1}{2a_q}, \\
\mathbb{E}[e^{-q\tau_2}] &= E[e^{-q\tau_3}] = \frac{1}{a_q} \quad \text{and} \\
\mathbb{E}[e^{-q\tau_4}] &= \frac{2a_q + 2}{(q+2)a_q + 4q + 2},
\end{aligned}$$

where $a_q = \sqrt{2q+1}$.

Appendix

Here we prove Lemma 1.

Proof of Lemma 1. Suppose $q > 0$. First, let $\lambda > 0$. Then (11) follows by integration by parts. Indeed, in this case

$$\begin{aligned}
\int_{[0,\infty)} e^{-\lambda x}\, W_{\Phi(q)}(dx) &= \lambda \int_0^\infty e^{-\lambda x}\, W_{\Phi(q)}(x)\, dx \\
&= \frac{\lambda}{\psi_{\Phi(q)}(\lambda)} \\
&= \frac{\lambda}{\psi(\Phi(q) + \lambda) - q}. \qquad (37)
\end{aligned}$$

Under $\mathbb{P}_{\Phi(q)}$, the process $\{X_t\}_{t \geq 0}$ drifts to ∞ and now from equation (8.15) in [13] we deduce that

$$W_{\Phi(q)}(x) = \frac{1}{\psi'_{\Phi(q)}(0+)} \mathbb{P}_x(\inf_{t \geq 0} X_t \geq 0).$$

It follows that

$$\lim_{x\to\infty} W_{\Phi(q)}(x) = \Phi'(q) \tag{38}$$

and hence (11) holds for $\lambda = 0$ as well.

Next, we show that (11) holds for $\lambda = -\Phi(q)$. We make use of the resolvent measure for the reflected process $\{V_t\}_{t\geq 0}$ defined by

$$V_t = \sup_{0\leq s\leq t} (X_s \vee 0) - X_t.$$

In Theorem 1 (ii) in [16], the resolvent measure

$$R_a^q(x, dy) = \int_0^\infty e^{-qt}\, \mathbb{P}_x(V_t \in dy, \sup_{0\leq s\leq t} V_s \leq a)$$

of V killed at exceeding a certain level $a > 0$ was found. In particular, for $x = 0$ it holds that

$$R_a^q(0, dy) = \left(W^{(q)}(a) \frac{W^{(q)\prime}(y+)}{W^{(q)\prime}(a+)} - W^{(q)}(y) \right) dy \quad \text{for } y \in (0, a]$$

and $R_a^q(0, \{0\}) = W^{(q)}(a) W^{(q)}(0)/W^{(q)\prime}(a+)$. Using the fact that $W_{\Phi(q)}(\infty) < \infty$ and (10), we can take $a \to \infty$ and deduce that the resolvent measure

$$R^q(0, dy) = \int_0^\infty e^{-qt}\, \mathbb{P}_0(V_t \in dy)$$

of the unkilled reflected process is given by

$$R^q(0, dy) = \left(\frac{1}{\Phi(q)} W^{(q)'}(y+) - W^{(q)}(y) \right) dy = \frac{1}{\Phi(q)} e^{\Phi(q)y} W_{\Phi(q)}(dy) \quad \text{for } y \geq 0.$$

An application of Fubini's theorem yields

$$\begin{aligned}
\int_{[0,\infty)} e^{\Phi(q)x}\, W_{\Phi(q)}(dx) &= \Phi(q) \int_{[0,\infty)} R^{(q)}(0, dx) \\
&= \Phi(q) \int_{[0,\infty)} \int_0^\infty e^{-qt}\, \mathbb{P}(V_t \in dx)\, dt \\
&= \Phi(q) \int_0^\infty e^{-qt} \mathbb{P}(V_t \in [0, \infty))\, dt \\
&= \frac{\Phi(q)}{q},
\end{aligned} \tag{39}$$

which is (11) for $\lambda = -\Phi(q)$.

Finally, for the case $-\Phi(q) < \lambda < 0$ we make use of analytic extension. We can extend the Laplace exponent ψ to those $z \in \mathbb{C}$ for which $\Re(z) > 0$ and we denote this extension by Ψ. Define the function $g : A \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{z}{\Psi(z+\Phi(q))-q} & \text{when } z \neq 0 \text{ and } \Re(z) > -\Phi(q), \\ \Phi'(q) & \text{when } z = 0, \end{cases}$$

where A is an open set in $\mathbb{C}$ such that $\Re(z) > -\Phi(q)$ for all $z \in A$ and such that $\Psi(z+\Phi(q)) \neq q$ on $A\backslash\{0\}$. Since the Laplace exponent Ψ is analytic when $\Re(z) > 0$, we can write

$$\Psi(z + \Phi(q)) = q + \sum_{k=1}^{\infty} \frac{z^k}{k!} \Psi^{(k)}(\Phi(q)) \quad \text{when } \Re(z) > -\Phi(q),$$

where $\Psi^{(k)}$ denotes the k-th derivative of Ψ. The fact that $\psi'(\Phi(q)) > 0$ implies that g is bounded in some (complex) neighbourhood of 0, and we can use the Riemann removable singularity theorem to deduce that $g(\lambda)$ is real analytic for $\lambda > -\Phi(q)$. The coefficients c_n in the power series of g are given in terms of the n-th (right) derivative at zero of the left hand side of (37). Specifically, because of (39),

$$c_n = \int_{[0,\infty)} \frac{(-x)^n}{n!} W_{\Phi(q)}(dx) \quad \text{for } n \in \mathbb{N}.$$

In particular, for $\lambda \in (-\Phi(q), 0)$

$$g(\lambda) = \sum_{n=0}^{\infty} \lambda^n \int_{[0,\infty)} \frac{(-x)^n}{n!} W_{\Phi(q)}(dx) = \sum_{n=0}^{\infty} \int_{[0,\infty)} \frac{|\lambda x|^n}{n!} W_{\Phi(q)}(dx).$$

From another application of Fubini's theorem it follows that for any $|\lambda| < \Phi(q)$

$$\begin{aligned} g(\lambda) &= \sum_{n=0}^{\infty} c_n \lambda^n \\ &= \sum_{n=0}^{\infty} \lambda^n \int_{[0,\infty)} \frac{(-x)^n}{n!} W_{\Phi(q)}(dx) \\ &= \int_{[0,\infty)} \sum_{n=0}^{\infty} \frac{(-\lambda x)^n}{n!} W_{\Phi(q)}(dx) \\ &= \int_{[0,\infty)} e^{-\lambda x} W_{\Phi(q)}(dx). \end{aligned}$$

This completes the proof of Lemma 1. $\square$

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