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# The allocation of shared fixed costs

## Working paper

# THE ALLOCATION OF SHARED FIXED COSTS

**by**

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**Abstract**

We consider the problem of sharing the fixed costs of facilities among a number of users. Although the problem can be formulated and solved as an Integer Programme this provides limited accounting information. Ways of overcoming this are suggested. In addition we consider the issue of *fairness* among different possible cost allocations and how such 'fair' costs may be derived

**Keywords:** *Fixed Cost Allocation, Duality, Fairness*

## 1. INTRODUCTION

In [ 2 ] we considered the problem of allocating the fixed costs of shared facilities among the users in a manner which was both *efficient* (in the sense of leading to the most profitable or economic overall solution) and *fair* (in a sense discussed in that paper). The problems which we considered could all be solved as *Linear Programmes (*LPs) which automatically gave *integer* solutions. They therefore had well defined *duals.* The dual values could be interpreted as cost allocations which led to the optimal solution (whether 'optimal' was defined using an efficiency or fairness criteria).

This problem has been considered by a number of authors. See, for example, Biddle and Steinberg[ 1 ], Shubik[ 7 ], Williams[ 8 ] and [9] and Young[ 13 ].

In general however such facility location models are not LPs but Integer Programmes (IPs). Solving them as LPs (the *LP Relaxation)* yields a fractional (and therefore largely meaningless) solution. We illustrate this by an example, in section 3, below.

First, however, we present the basic model, in section 2, and give an early, economically motivated, dual discussing its merits and inadequacies. We show that this dual is a specialisation of using *price functions* in place of prices. In section 4 we adopt an alternative approach of solving the linear programme associated with the optimal solution. This is solved for the numerical example where it is shown how it leads to a cost allocation.

## 2. THE BASIC PROBLEM

We have a set of Facilities $\mathbf{F} = \{1,2,..,m\}$ serving a set of 'customers' $\mathbf{C} = \{1,2,..,n\}$.

Customer $\mathbf{j}$ requires one of each of facility $\boldsymbol{i} \ \varepsilon \ \boldsymbol{F}_j^k \subset \mathbf{F}$ for each $\mathbf{k} \ \varepsilon \ \mathbf{K}_j$, ie customer $\mathbf{j}$ requires one of $\boldsymbol{F}_j^1$, one of $\boldsymbol{F}_j^2$ etc.

The fixed cost of $\mathbf{i} \ \varepsilon \ \mathbf{F}$ is $\mathbf{f}_i$.

Customer $\mathbf{j}$ produces a benefit (revenue) $\mathbf{b}_j$.

It may not be desirable to cater for all the customers or provide all the facilities.

If a facility **i** is provided its cost $\mathbf{f}_i$ must be split up among the customers **j** that use it in an acceptable or desirable way.

We formulate this problem as a 0-1 Integer Programming model (known as the *Primal* model)

**Variables**

$\delta_i$ = 1 if facility **i** is provided
= 0 otherwise

$\gamma_j$ = 1 if customer **j** is catered for
= 0 otherwise

**Objective**

$$\textbf{Maximise} \quad \sum_j \mathbf{b}_j \gamma_j - \sum_i \mathbf{f}_i \delta_i \tag{1}$$

**Constraints**

$$\gamma_j - \sum_{i \varepsilon F_j^k} \delta_i \leq \mathbf{0} \quad \text{all } j \ \varepsilon \ C, \text{ all } k \ \varepsilon \ K \tag{2}$$

$$\gamma_j \leq 1 \quad \text{all } j \ \varepsilon \ C \tag{3}$$

$$\delta_i \geq \mathbf{0} \quad \text{all } i \ \varepsilon \ F \tag{4}$$

We refer to the above model as **P.**

Constraints (2) force at least one of $F_j^k$ to be provided, for each **j** and **k**, if customer **j** is to be served. It is not necessary to impose non-negativity conditions on the $\gamma$ variables or append upper bounds of 1 on the $\delta$ variables. These conditions are guaranteed by the structure of the model.

There is no guarantee that this general model will produce integer solutions if solved as a Linear Programme. In certain special cases, however, integral solutions to the LP are guaranteed. For example if for all **j** and **k** , $| F_j^k | = 1$ then this is the case. This is discussed in, for example, Rhys[ 6 ] and Williams [10] .

When the LP solution is *integer* then there is a well defined *dual* LP model (see eg Dantzig [ 3 ]). We consider the dual of the LP relaxation of the above model. (For convenience we have reversed the direction of some of the constraints in the formal dual model)

**Dual Model (of LP Relaxation)**

**Minimise** $\sum_{j \varepsilon C} \mathbf{u}_j$ (5)

**Subject to:** $\mathbf{u}_j + \sum_{k \varepsilon K_j} \mathbf{v}_j^k = \mathbf{b}_j$ all j $\varepsilon$ C (6)

$$\sum_{j:\, j \varepsilon F_i^k} \mathbf{v}_j^k \leq \mathbf{f}_i \quad \text{all } i\ \varepsilon\ F \tag{7}$$

$$\mathbf{u}_j,\ \mathbf{v}_j^k \geq \mathbf{0} \quad \text{all } j\ \varepsilon\ C,\ \varepsilon\ k\ \varepsilon\ K \tag{8}$$

We refer to the above model as **D.**

$\mathbf{v}_j^k$ can be interpreted as the portion of the fixed cost $\mathbf{f}_i$ of each of the facilities i $\varepsilon$ F (for k $\varepsilon$ K ) that is allocated to j. In the case that a customer is not catered for $\mathbf{u}_j$ will be zero and the $\mathbf{v}_j^k$, if positive, can be ignored since, in this case, the corresponding facilities will not be built (this results from orthogonality in LP). $\mathbf{u}_j$ can be interpreted as the excess benefit (revenue) which **j** obtains after contributing all the required costs. The objective is to minimise the total excess.

Constraints (7) split the cost **f** between the customers using the facility. Should the full cost not be met (the constraint is non-binding) the orthogonality result of LP guarantees that $\delta_i = 0$, ie that the facility not be built. Constraints (6) split the benefits to customers between the imputed costs and excesses. Generally there will be a number of alternate optimal dual solutions.

When, however, **P** does not yield an integer optimal solution to the LP relaxation no feasible set of solutions to **D** can lead to the optimal *integer* solution to P. This is a result of the Duality Theorem of Linear Programming.

If, however, we append *cutting planes* in order to obtain the optimal integer solution then this produces a ( superadditive) *price function* in place of a set of prices (see Williams[11]. Unfortunately it is generally not possible to associate the multipliers or the rounding operation directly with individual coefficients (see, for example Wolsey[12].) Nor is it possible (owing to the rounding operation) to 'balance' costs of facilities with charges to customers. The price function does, however, (in the absence of degeneracy) allow one to 'price' economic activities and price out uneconomic ones analogous to that done for LP.

If, however, we ignore the rounding operation then we can associate prices directly with constraints, analogously to dual values in the LP case. This is due to Gomory and Baumol [5]. Each of the cutting planes arises as a linear combination of the original constraints (together with the intermediate rounding operations). Ignoring the rounding operations we can consider how each of the original constraints contributes to the final, integer, solution both directly and indirectly through the cutting planes. We also have to include some of the non-negativity constraints 'x >= 0' in this analysis. Gomory and Baumol apply Gomory cuts derived in the course of Gomory's All-Integer algorithm [4]. This results in some arbitrariness. In order to give the method a uniqueness we apply only the *facet defining* constraints, necessary to obtain the integer optimum, Each of the original constraints and cutting planes will have associated dual values in the (integer) LP solution. The dual values associated with the cutting planes are imputed back to the original constraints according to the linear multiples of these constraints from which they arise. This procedure is illustrated by a numerical example in section 3. The major defect of this method is that *positive* dual values may also have to be applied to constraints which are not satisfied as equalities. In particular this can apply to non-negativity constraints 'x>=0' when the associated variable 'x' takes a positive value. This breaks the *orthogonality* result which applies in the LP case. There a positive dual value on a non-negativity constraint would be interpreted, in the LP case, as a positive *reduced cost* indicating that the variable takes the value 0 (ie the associated activity should not be carried out). Here we can interpret the dual value, on such a constraint, as a *subsidy* which should be applied to the activity to compensate for its integral nature and therefore allow it to be carried out. The drawback of the procedure is that it is no longer always possible to decide, on the basis of dual values alone, which activities should not be carried out. The resultant dual values may also depend on which cutting planes are used in the derivation of the optimal solution. While that is analogous to the alternate dual solutions which arise in the LP case from degeneracy, in practice it is more serious.

## 3. A NUMERICAL EXAMPLE (A)

We have six potential facilities {1,2,3,4,5,6} some of which are needed for three potential customers {A,B,C}.

Customer A requires 1 of {1,2,3} and 1 of {4,5,6}.

Customer B requires 1 of {1,4} and 1 of {2,5}.

Customer C requires 1 of {1,5} and 1 of {3,6}.

Customer A would derive a *benefit* of 8.

Customer B would derive a *benefit* of 11.

Customer C would derive a *benefit* of 19.

The (fixed) costs of the six investments are, respectively, 8, 7, 8, 9, 11, 10.

Model **P** is

Maximise:

$$8\gamma_a + 11\gamma_b + 19\gamma_c - 8\delta_1 - 7\delta_2 - 8\delta_3 - 9\delta_4 - 11\delta_5 - 10\delta_6 \qquad (9)$$

Subject to:

$$\begin{array}{lllllllllll}
\gamma_a & & & -\delta_1 & -\delta_2 & -\delta_3 & & & & \leq 0 & (10) \\
\gamma_a & & & & & & -\delta_4 & -\delta_5 & -\delta_6 & \leq 0 & (11) \\
 & \gamma_b & & -\delta_1 & & & -\delta_4 & & & \leq 0 & (12) \\
 & \gamma_b & & & -\delta_2 & & & -\delta_5 & & \leq 0 & (13) \\
 & & \gamma_c & -\delta_1 & & & & -\delta_5 & & \leq 0 & (14) \\
 & & \gamma_c & & & -\delta_3 & & & -\delta_6 & \leq 0 & (15) \\
\gamma_a & & & & & & & & & \leq 1 & (16) \\
 & \gamma_b & & & & & & & & \leq 1 & (17) \\
 & & \gamma_c & & & & & & & \leq 1 & (18)
\end{array}$$

The optimal solution is

$$\gamma_a = 1, \gamma_b = 1, \gamma_c = 1, \delta_1 = \delta_2 = 1, \delta_3 = \delta_4 = \delta_5 = 0, \delta_6 = 1 \quad , \text{ Objective} = 13$$

ie build only facilities 1, 2 and 6 and serve all customers.

Our (aspired) problem is to *split* the costs of the facilities up among A,B and C so as to:

1. Make facilities 3, 4 and 5 too expensive.
2. Pay for facilities 1, 2 and 6.

If **P** had an optimal *integer* solution to its LP Relaxation its dual model provides such a split in costs (if there is no degeneracy).

The optimal solution to the LP relaxation , in this case, is:

$$\gamma_a = 1,\ \gamma_b = \frac{1}{2},\ \gamma_c = 1, \delta_1 = \frac{1}{2}, \delta_2 = 0, \delta_3 = \frac{1}{2}, \delta_4 = 0, \delta_5 = \delta_6 = \frac{1}{2} \text{ , Objective} = 14$$

Clearly this *fractional* solution has little meaning, although there is a dual solution which would provide a split in costs which would produce it. This is:

$$v_a^1 = 0,\ v_a^2 = 2, v_b^1 = S, v_b^2 = 6,\ v_c^2 = 8,\ u_1 = 6,\ u_2 = 0,\ u_3 = 8$$

It is given in diagrammatic form in figure 1. Note that all the facilities to be built have the appropriate proportion of their fixed costs recompensed by the customers. Customers excess profits are indicated.

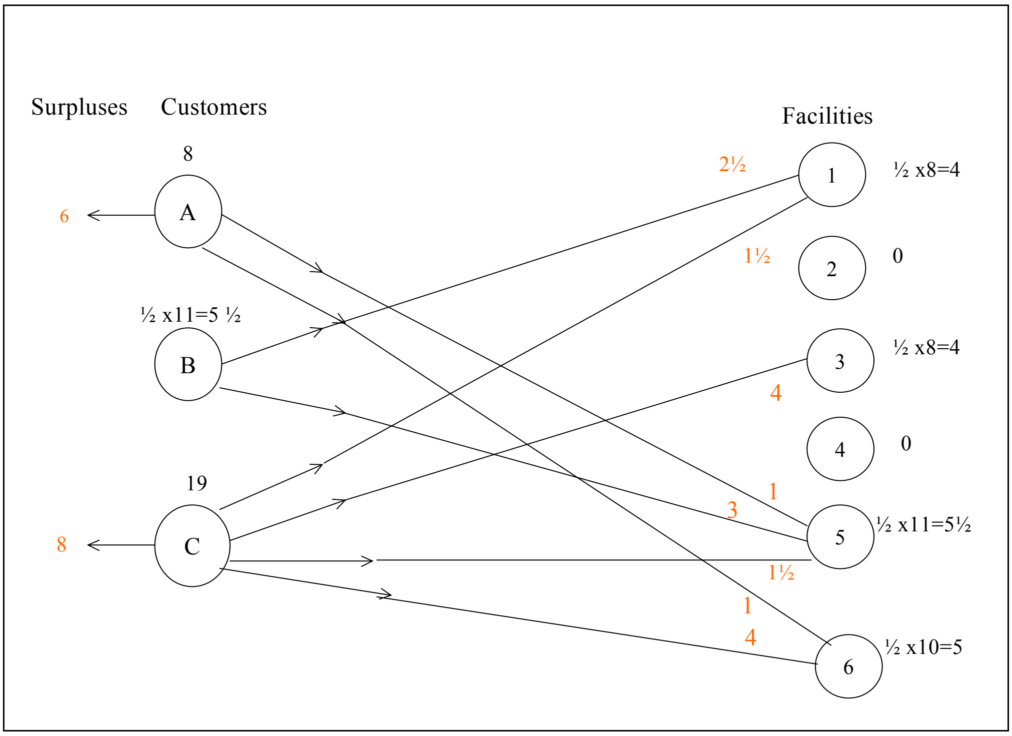


Figure 1
<u>Linear Programming (Fractional) Solution and Cost Allocation</u>

If, however, we append the following (facet) cutting plane to the model above the optimal solution to the LP relaxation is the optimal integer solution given above.

$$\gamma_b + \gamma_c - \delta_1 - \delta_2 - \delta_4 - \delta_5 \leq 0 \qquad (19)$$

This cutting plane arises as the following linear combination of the original constraints (including the non-negativity constraints).

$$(12) + (13) + (14) + (18) + (-\delta_2 \leq 0) + (-\delta_4 \leq 0) \qquad (20)$$

The result of applying these multipliers and rounding down the resultant right-hand-side gives the cutting plane (19).

These operations produce the following *price function* on the coefficients $\alpha$ in the constraints (10) to (15), and the coefficients $\beta$ in the constraints (16) to (18) for each column of the model **P**.

$$\frac{1}{2}(5\alpha_2 + \alpha_4 + 3\alpha_5 + 15\alpha_6 + 11\beta_1 + 8\beta_2 + 7\beta_3 + 13\left\lfloor \frac{1}{2}(\alpha_3 + \alpha_4 + \alpha_5 + \beta_3) \right\rfloor) \qquad (21)$$

Applying this price function to the coefficients of each column in **P** customers A, B and C each have an associated price equal to their benefit (the prices on (16), (17), (18) are the surpluses they each gain). Facilities 1,2 and 6 are recompensed their full costs . Facility 3 is not fully recompensed and therefore not built. Facilities 4 and 5 are exactly paid for but are not built as the resultant solution is degenerate. (Degeneracy is likely to happen in a model with the structure of **P**).

If we ignore the rounding operations in (21) then we can aggregate the multipliers to obtain the following amended dual values for the constraints. We also give the dual values which apply to the non-negativity constraints used in the derivation of the cutting plane.

| Constraint | Amended Dual Value |
|---|---|
| (10) | 0 |
| (11) | 5/2 |
| (12) | 13/4 |
| (13) | 15/4 |
| (14) | 19/4 |
| (15) | 15/2 |
| (16) | 11/2 |
| (17) | 4 |
| (18) | 27/4 |
| $\delta_2 \geq 0$ | 13/4 |
| $\delta_4 \geq 0$ | 13/4 |

Note that constraint '$\delta_2 \geq 0$' is not satisfied as an equality in the optimal IP solution although it has a positive 'dual value'. Constraint '$\delta_2 \geq 0$' is not redundant in the absence of constraint (10) and therefore has a positive 'economic value' in that sense.

However, as in LP, there are alternate 'dual values' (arising from alternative cutting planes) corresponding to a degenerate solution. In the presence of constraint (10) the constraint '$\delta_2 \geq 0$' would be redundant having a zero 'dual value' (and constraint (10) a positive 'dual value').

Facility 1 would be recompensed a total cost of 5/2 + 11/2 = 8 making it worthwhile.

Facility 2 would be recompensed a total cost of 15/4 which, together with the subsidy of 13/4, gives 7 making it worthwhile.

Facility 3 would be recompensed a total cost of 15/2 making it not worthwhile.

Facility 4 would be recompensed a total cost of $5/2 + 13/4 = 23/4$ which, together with a subsidy of 13/4 gives 9 making it appear worthwhile although it is not built. This is an example of degeneracy.

Facility 5 would be recompensed a total cost of $5/2 + 15/4 + 19/4 = 11$ making it appear worthwhile although it is not built. Again this is an example of degeneracy.

Facility 6 would be recompensed a total of $5/2 + 15/2 = 10$ making it worthwhile.

Customer A would be charged a total cost of 5/2 allowing it to be served with a surplus of 11/2.

Customer B would be charged a total of 7 allowing it to be served with a surplus of 4.

Customer C would be charged a total of 49/4 allowing it to be served with a surplus of 27/4.

This allocation of costs has a number of unsatisfactory features.

(i) Facilities 4 and 5 would have all of their costs recompensed but neither is in the optimal solution.

In the LP case this would indicate alternate solutions each involving facilities 4 and 5. This is not, however, the case here.

(ii) The allocation of subsidies to facilities 2 and 4 seems somewhat arbitrary and the reason for this is not transparent.

(iii) The total payment of 21 ¾ by the customers falls short of the total cost of 25 for the facilities.

## 5. AN ALTERNATIVE APPROACH

Since it is, in general, impossible to have a set of prices which

(i) allows one to discriminate between economic and uneconomic activities

and

(ii) satisfactorily splits costs of facilities between users

we will abandon the requirement of (i) and concentrate on (ii).

In order to do this we first obtain the optimal solution and then modify the model to only include those facilities which should be built. If the resultant model is solved as an LP the optimal solution (ignoring possible alternatives) will be the optimal integer solution. The dual values corresponding to this solution then give a satisfactory allocation of costs of the facilities to the customers. We illustrate this by the numerical example.

For model **P** we only include variables $\gamma_a, \gamma_b, \gamma_c, \delta_1$, $\delta_2$ and $\delta_6$ corresponding to the optimal solution. The LP relaxation gives the solution presented above and dual values

$$v_a^1 = 0,\ v_a^2 = 0,\ v_b^1 = 0, v_b^2 = 7,\ v_c^1 = 8, v_c^2 = 10,\ u_1 = 8,\ u_2 = 4,\ u_3 = 1$$

This solution, interpreted as a cost allocation is given in Figure 2.

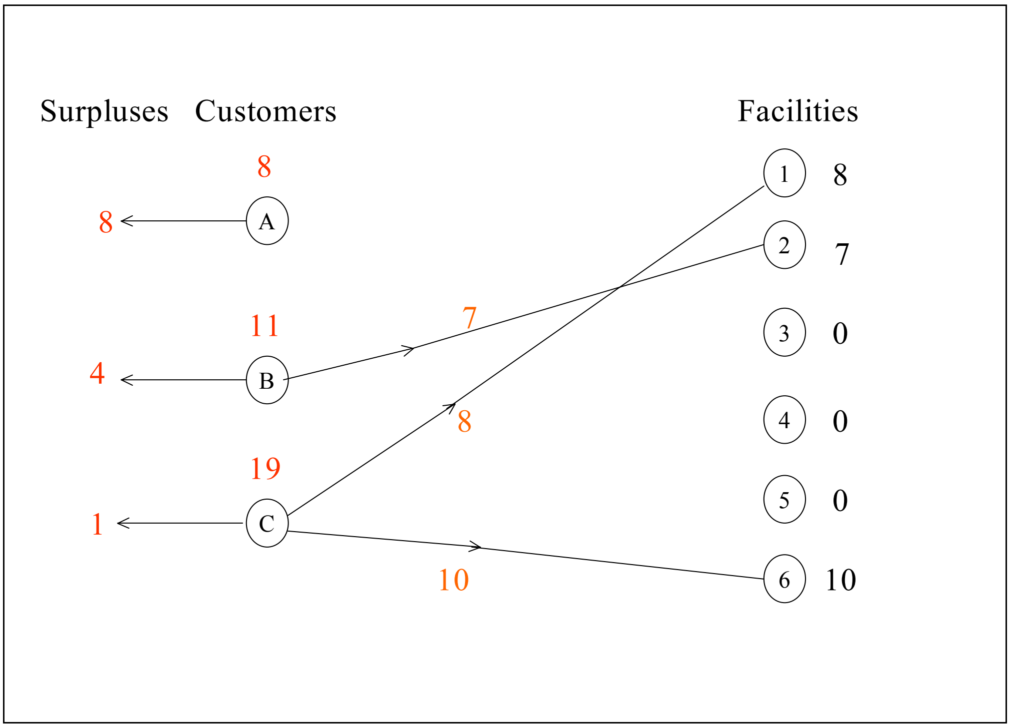


Figure 2

Integer Programming Solution and Cost Allocation

## 6. A FAIR SOLUTION

There are alternative dual solutions to the model above. Following the discussion in [2] some resulting splits in the costs of the facilities among the customers can be regarded as 'fairer' than others. To illustrate this we will try to equalise the contribution each customer makes to the facilities that serve it by the following objective applied to the dual of the restricted model above.

$$\text{Minimise } \left(\text{Maximum}(u_1, u_2, u_3)\right) \qquad (22)$$

This can be dealt with by

$$\text{Minimise} \quad Z \qquad (23)$$
$$\text{Subject to} \quad Z \geq u_1, u_2, u_3 \qquad (24)$$

It results in the allocation of costs given in figure 3.

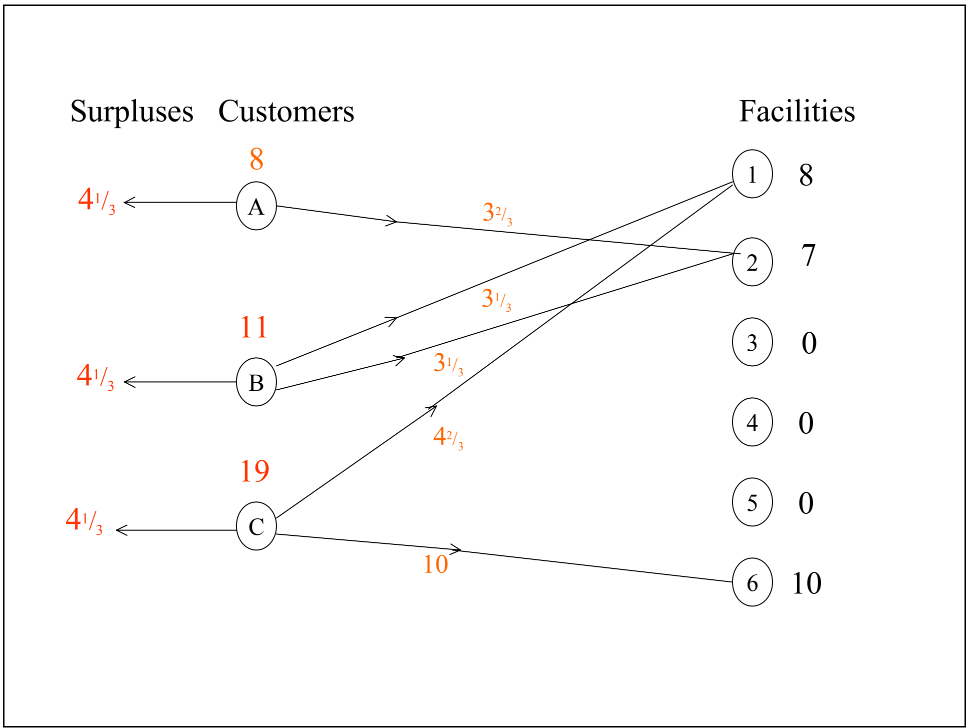


Figure 3
<u>A Fair Allocation</u>

Clearly the distribution of surpluses is as equitable as it can be. In practice it will not always be possible to make them all exactly equal.

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