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Danny Quah*
LSE and FMG

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ABSTRACT

Much macroeconometric discussion has recently emphasized the economic signi cance of the size of the permanent component in GNP. Consequently, a large literature has developed that tries to estimate this magnitude | measured, essentially, as the spectral density of increments in GNP at frequency zero. This paper shows that unless the permanent component is a random walk this attention has been misplaced: in general, that quantity does not identify the magnitude of the permanent component. Further, by developing bounds on reasonable measures of this magnitude, the paper shows that a random walk speci cation is biased towards establishing the permanent component as important.

Communications to: D. T. Quah, LSE, Houghton Street, London WC2A 2AE.
[Tel: +44-171-955-7535, Email: `dquah@exz.lse.ac.uk`]
[(URL) `http://econ.lse.ac.uk/~dquah/`]

1. Introduction

A large literature has recently developed that purports to estimate the magnitude of a time series's permanent component. For GNP, this literature includes the influential papers by Nelson and Plosser (1982), Watson (1986), Campbell and Mankiw (1987), and Cochrane (1988). Using non-parametric reasoning Cochrane (1988) has, further, developed a measure that allows arbitrary serial correlation in the transitory components. All this research, however, has assumed either that there is only one disturbance perturbing the time series under study, or that the underlying permanent component has very special structure—for instance, that it has serially uncorrelated increments.

More recently, other researchers, e.g., Shapiro and Watson (1988) and Blanchard and Quah (1989), have argued that the economic forces underlying GNP movements imply that multiple disturbances perturb GNP and that the underlying permanent component has rich dynamics. In this paper I show that under these circumstances the measures that had been earlier proposed, in fact, cannot identify the magnitude of the permanent component. In particular, I prove that the underlying permanent component in every integrated time series can be taken to be arbitrarily smooth, so that at all finite horizons it is the transitory component that dominates that series's fluctuations—these permanent and transitory components can, further, be chosen to be uncorrelated at all leads and lags. This arbitrary smoothness is achievable regardless of the values taken by Campbell and Mankiw's "long-run effect of a shock" or Cochrane's or Watson's "size of the random walk component" for that integrated time series. This proposition therefore casts serious doubt on the usefulness of those measures for assessing the magnitude of a time series's permanent component. Without explicitly identifying the underlying economic disturbances, a researcher cannot quantify the relative importance of permanent and transitory components.

To see the implications of this arbitrary smoothness result, recall some well-known assertions in the literature: Nelson and Plosser (1982) and Campbell and Mankiw (1987) have criticized traditional models of economic fluctuations by observing that US GNP might be better characterized as integrated rather than trend stationary. According to these investigators' reasoning, traditional macroeconomic models predict at most transitory effects of disturbances to output. In contrast, Nelson and Plosser (1982) and Campbell and Mankiw (1987), separately, put forward two claims—both claims based on univariate characterization of GNP's time series properties: one, that GNP's permanent component is highly volatile and, two, that disturbances to GNP have significant and long-lived effects. The accuracy of the univariate time series models that these investigators used remains controversial (see, e.g., Watson (1986), Cochrane (1988), Perron (1989), and Christiano and Eichenbaum (1990)). But, according to the results in the current paper the accuracy of those measurements turns out to be irrelevant for whether permanent disturbances are important for GNP fluctuations and for whether most disturbances to GNP have long-lived effects. The analysis below shows that a time series can be integrated and can show significant persistence in its innovations, but nevertheless still have its fluctuations dominated by transitory disturbances. Thus this paper makes explicit an important general message: Because studying the univariate time series characterizations of a variable leaves unidentified the sources of that variable's fluctuations, without additional ad hoc restrictions those characterizations are completely uninformative for the relative importance of the underlying permanent and transitory components.

To sharpen understanding of the arbitrary smoothness property, I derive below explicit lower bounds for two natural measures of the importance of a permanent component when that permanent component is restricted to be an $ARIMA$ sequence. Choosing the permanent component to be a random walk—i.e., to have serially uncorrelated increments—turns out to maximize both these lower bounds. This therefore makes precise a sense in which a random walk specification for the permanent component biases the analysis towards finding the permanent component to be important.

Section 2 provides rigorous statements of our theoretical decomposition results in a general setting; Section 3 does the same for permanent components a priori restricted to be *ARIMA* processes. While those two sections consider the lack of identifiability of arbitrary permanent and transitory components, some positive results are in fact available. Section 4 considers permanent and transitory components subject to certain orthogonality and informational restrictions. I show that under those restrictions these components are unique; further, whether such components exist can be tested for by a Granger-causality characterization. The paper then concludes with Section 5.

The reader will notice that the analysis of Sections 2 through 4 use moment-matching reasoning to construct permanent and transitory components. Sometimes a researcher might wish to go beyond this, i.e., to construct what—in the terminology of stochastic differential equations—are called strong sense solutions rather than only weak sense ones. The Appendix provides calculations that accomplish this. The Technical Appendix contains all the proofs.

2. General Results

This section shows that every integrated sequence admits a decomposition into permanent and transitory components with the increments of the permanent component having arbitrarily small variance—this decomposition is possible even when the permanent and transitory components are uncorrelated at all leads and lags.

First, establish notation: A random sequence

$$W = \{W(t);\ \text{non-negative integer } t\}$$

is **integrated** or **difference stationary** when its first difference or increment $\Delta W(t) \stackrel{\text{def}}{=} W(t) - W(t-1)$ is covariance stationary, but W itself is not. We use the method in Doob (1953) pp. 461-463 to always extend definition of covariance stationary sequences over all the integers, even if those sequences are initially defined only for integer $t \geq 1$. For the purposes of this paper, we call an integrated sequence a **random walk** if its increments are serially uncorrelated (not necessarily iid). Elements of a sequence, stochastic or otherwise, are denoted by integer arguments in parentheses; subscripts indicate either distinct sequences or the elements of a matrix. Thus, for example, Y_1 and Y_0 are different stochastic sequences with the t-th element of each written as $Y_1(t)$ and $Y_0(t)$. Without loss of generality, all covariance stationary sequences are taken to have mean zero. Since there is some arbitrariness in a 2π normalization, we explicitly specify the spectral density matrix to be the fourier transform of the covariogram matrix sequence: When W is a jointly covariance stationary vector sequence, its spectral density matrix is $S_W(\omega) \stackrel{\text{def}}{=} \sum_{j=-\infty}^{\infty} E\left[W(j)W(0)'\right] e^{-i\omega j}$. Finally, all integrals below are taken from $-\pi$ to π.

Next, make precise the decompositions that we are investigating:

Definition 2.1: *Let Y be an integrated sequence. A* **permanent-transitory (PT) decomposition** *for Y is a pair (Y_1, Y_0) such that: (i) Y_1 is integrated and Y_0 is covariance stationary; (ii)* $\mathrm{Var}(\Delta Y_1(t))$ *and* $\mathrm{Var}(\Delta Y_0(t))$ *are strictly positive; and (iii) $Y(t) = Y_1(t) + Y_0(t)$. Further, if (iv) ΔY_1 is uncorrelated with Y_0 at all leads and lags, then the PT decomposition is said to be* **orthogonal.**

Given a PT decomposition (Y_1, Y_0) for Y, call Y_1 a **permanent component** for Y; similarly, call Y_0 a **transitory component**. Permanent in this context indicates only that disturbances to Y_1 have long run effects on Y, not that the increments of Y_1 are serially uncorrelated. We will also say that (Y_1, Y_0) decomposes Y when (i)-(iii) of **2.1** hold, and that (Y_1, Y_0) orthogonally decomposes Y when in addition (iv) is true.

Condition (ii) rules out trivial cases. For instance, when Y is a random walk, it might be natural to set Y_1 to Y. But if so, there is no transitory component; the definition then sensibly asserts that $(Y_1, Y - Y_1) = (Y, 0)$ is not a PT decomposition for Y.

From Beveridge and Nelson (1981), we know that every integrated sequence admits a decomposition into perfectly correlated permanent and transitory components where further the permanent component has serially uncorrelated increments. Watson (1986) and Cochrane (1988) have considered models where the permanent component remains a random walk, but has increments that might be imperfectly correlated with the transitory component. In all these, the variance of increments in the permanent component can be identified from the spectral density of increments in the original sequence (see, e.g., Watson (1986) or Cochrane (1988)).

By contrast, Shapiro and Watson (1988) and Blanchard and Quah (1989) have considered models where permanent components have richer dynamics than those in a random walk. In these more general specifications, permanent components turn out to have variances that can no longer be identified from just the second moments of the original sequence. The extent of this lack of identification can be seen in the following result.

Theorem 2.2: *Fix S, a spectral density satisfying:*

$$\int |1-\exp(i\omega)|^{-2}\;|S_{\Delta Y}(\omega)-S_{\Delta Y}(0)|\,d\omega<\infty \text{ and } 0<S_{\Delta Y}(0)<\infty .$$

Let be an arbitrary non-negative function on $[-\pi,\pi]$, symmetric about 0, and such that:

(i) $0 < \;(\omega) < 1$ *for* $\omega \neq 0$*; and*
(ii) $\int |1-\exp(i\omega)|^{-2}(1-\;(\omega))\,d\omega<\infty$.

Suppose that ΔX_1 and ΔX_0 are stochastic sequences orthogonal at all leads and lags, and have spectral densities $S_{\Delta X_1} = \; S$ and $S_{\Delta X_0} = (1-\;)S$, respectively. Then (X_1, X_0) is an orthogonal PT decomposition for an integrated sequence whose increments have the given spectral density S.

Theorem **2.2** asserts that under regularity conditions the second moments of an arbitrary integrated sequence are consistent with a wide range of dynamics in the underlying permanent and transitory components.[1] Since the sum of orthogonal sequences has spectral density equal to the sum of the spectral densities

[1] The alert reader will notice that Theorem **2.2** only gives a pair (X_1, X_0) whose second moments sum correctly to match a given spectral density S. The Theorem

of the underlying sequences, it is obvious that $\Delta X_1 + \Delta X_0$ has spectral density $S = \ \ S + (1 - \ \ )S$. Thus, the only subtlety in Theorem **2.2** is whether X_0 (a sequence with increments ΔX_0) could be covariance stationary. But this follows from noting that:

$$
\begin{aligned}
\mathrm{Var}(X_0) &= \int |1 - \exp(i\omega)|^{-2} S_{\Delta X_0}(\omega)\, d\omega \\
&= \int |1 - \exp(i\omega)|^{-2} S_{\Delta Y}(\omega)(1 - \ \ (\omega))\, d\omega \\
&= S_{\Delta Y}(0) \int |1 - \exp(i\omega)|^{-2} (1 - \ \ (\omega))\, d\omega \\
&\quad + \int |1 - \exp(i\omega)|^{-2} (S_{\Delta Y}(\omega) - S_{\Delta Y}(0))(1 - \ \ (\omega))\, d\omega < \infty.
\end{aligned}
$$

Finiteness results from the first summand's being finite by (ii), and the second summand's being bounded from above by:

$$
\sup_{-\pi \leq \lambda \leq \pi} |1 - \ \ (\ \ )| \int |1 - \exp(i\omega)|^{-2} (S_{\Delta Y}(\omega) - S_{\Delta Y}(0))\, d\omega < \infty.
$$

Thus, X_0 can in fact be chosen to be covariance stationary.

Because of (i), $\ \ $ provides a cleaving of the given spectral density S into two non-negative pieces $\ \ S$ and $(1 - \ \ )S$. From (ii), $\ \ (0) = 1$, so that the spectral density of ΔX_1 coincides with S at the origin; everywhere else, $S_{\Delta X_1}$ is strictly smaller than S. In the Theorem, condition (ii) and its analogue for $|S(\omega) - S(0)|$ impose smoothness on $\ \ $ and S at frequency zero: the conclusion of the Theorem

---

does *not* show how to construct processes (X_1, X_0) that will sum to a given process Y, where the last has increments with spectral density S. In the terminology of stochastic differential equations, Theorem **2.2** gives only a solution in the weak sense. The strong sense solution is given below in Theorem **A.1** in the Appendix.

can be shown to remain true if certain time domain restrictions replace these frequency domain conditions. In particular, from Solo (1989) Lemma 1, we can instead use “ is the spectral density of a random sequence that has 1=2-summable Wold moving average coefficients, with (0) = 1” and “S is the spectral density of a random sequence that has 1=2-summable Wold moving average coefficients.”[2]

Although we are interested in obtaining general PT decompositions, the formal discussion considers only the orthogonal case. The reasons for doing so are two-fold: first, if we can find an orthogonal PT decomposition satisfying certain properties, then we will always be able to find a non-orthogonal PT decomposition otherwise satisfying the same properties. Second, for certain applications (e.g. Quah (1990)), that the PT decomposition is orthogonal is essential.

Theorem **2.2** provides, for an integrated sequence with given second moments, a range of feasible orthogonal PT decompositions. Significantly, that range always includes a decomposition where the permanent component is arbitrarily smooth, i.e, where the permanent component has increments with arbitrarily small variance.

Corollary 2.3: *Fix a spectral density S*

Corollary **2.3** implies that without restricting further the dynamics of the permanent component X_1 the lower bound on $\mathrm{Var}(\Delta X_1)$ is simply zero. Therefore, the results here highlight a similarity between integrated and trend stationary sequences: both can have their stochastic dynamics dominated by transitory components. This observation has been raised elsewhere in the literature but our reasoning here differs significantly from those other arguments: (1) The results require only weak regularity assumptions on the univariate dynamics of the integrated sequence of interest. Thus, contrast the analysis here with, e.g., Clark (1988), Diebold and Rudebusch (1988), or West (1988), who argue that for certain parameter values a trend-stationary sequence is close to a difference-stationary one. (2) The results here rely neither on dynamics being only imprecisely estimated, nor on a researcher's misspecifying a Wold representation. Thus, our analysis differs from Cochrane's (1988) and Christiano and Eichenbaum's (1990) criticisms of Campbell and Mankiw (1987). Arguments about imprecision and possible misspecification can also be subsumed in the following: (3) The statements here apply to the underlying population probability model and are not conclusions due to an investigator's having only finite samples. (4) What is not obvious from the above is that the transitory component X_0 can have its autoregressive roots bounded away from 1 as $\mathrm{Var}(\Delta X_1)$ decreases. The truth of this is shown in the numerical calculations in Quah (1990) where the transitory component X_0 has fixed autoregressive roots despite $\mathrm{Var}(\Delta X_1)$ growing arbitrarily small.

That the lower bound on the importance of the permanent component is zero may at first seem puzzling. We can get some intuition for this zero lower bound property by studying more restricted and more explicit decompositions where the permanent components are constrained to be $ARIMA$ sequences—less trivial bounds then result. That analysis forms the content of the next section.

3. Finite ARIMA Components

This section specializes the analysis to *ARIMA* permanent components; doing so allows explicit formulas for the lower bounds on $\mathrm{Var}(\Delta Y_1)$. Some additional notation will be needed: If W is covariance stationary, let $innov(W)$ denote its innovation, i.e., the residual in the minimum mean square error linear predictor of W based on its own lagged values.

Theorem 3.1: *Suppose* $(Y_1; Y_0)$ *decomposes* Y, *with* ΔY_1 *a moving average sequence of order* q. *Then (i)* $\mathrm{Var}(innov(\Delta Y_1)) \quad 4^{-q} \quad S_{\Delta Y}(0)$; *and (ii)* $\mathrm{Var}(\Delta Y_1) \quad (q+1)^{-1} \quad S_{\Delta Y}(0)$. *Further, there exist (di erent) PT decompositions with* ΔY_1 *moving average of order* q *having innovation variances and variances arbitrarily close to the bounds in (i) and (ii).*

The lower bounds in Theorem **3.1** are strictly decreasing in the moving average order of ΔY_1, and apply regardless of the correlation between the components. Thus, letting ΔY_1 be a random walk must maximize the theoretical lower bound on Y_1's contribution to Y.

The analysis for autoregressive models for ΔY_1 is even simpler. A first order autoregressive model for ΔY_1 suffices to obtain a theoretical lower bound of zero on both its variance and innovation variance. To see this, apply the arguments in the proof of Theorem **3.1** to

$$S_{\Delta Y_1}(0) = |1 - C(1)|^{-2} \quad \mathrm{Var}(innov(\Delta Y_1)) = S_{\Delta Y}(0)$$

$$\Longrightarrow \quad \mathrm{Var}(innov(\Delta Y_1)) = |1 - C(1)|^2 \quad S_{\Delta Y}(0);$$

and

$$\mathrm{Var}(\Delta Y_1) = \quad (1 - C(1)) \quad (1 + C(1))^{-1} \quad S_{\Delta Y}(0);$$

where now $C(1)$ is the projection coefficient in a first order autoregression. Then simply let $C(1) \uparrow 1$. The same conclusion obviously applies to higher order autoregressive models. But choosing the permanent component in this way does make the transitory component more like an integrated sequence, in that the largest autoregressive root in its $ARMA$ representation approaches unity. On the other hand, this does not happen for the moving average cases in Theorem **3.1**.

Finally, since a purely autoregressive model is simply a restriction of a mixed moving average autoregressive model, the result for a first order autoregression applies directly to general $ARMA$ models for ΔY_1.

4. Identification under Orthogonality and Informational Restrictions

In certain applications, e.g., Shapiro and Watson (1988) and Blanchard and Quah (1989), a researcher wishes to explicitly construct permanent and transitory components, where these components are somehow restricted so that they are uniquely identified. For instance, the researcher might be interested in those orthogonal permanent and transitory components contained in the history of output and unemployment or in that of income and consumption. Such an informational restriction will be made precise below.

This section answers two questions: (1) Can one test whether there exist any such permanent and transitory components, and (2) If such PT decompositions do exist, how rich is their class? We will see that a Granger-causality characterization is available for the existence property—thus, the usual exclusion tests in a vector autoregression can be used to establish if a particular orthogonal PT decomposition exists. Further, under certain conditions, such an orthogonal PT decomposition is unique and can be straightforwardly obtained from the Wold representation of the variables of interest. This result drives the analysis in Blanchard and Quah (1989), and is one way of allowing a researcher to uniquely identify permanent and transitory components. But Blanchard and Quah (1989) never confronted the orthogonal PT decomposition. First, following Rozanov (1967), let $H_{\xi}^{-}(t)$ denote the space spanned by square-summable linear combinations of $\{\xi(t), \xi(t-1), \xi(t-2), \ldots\}$, complete under mean square norm. The convolution operator $*$ is such that for b and X sequences with X covariance stationary and defined over all the integers, the t-th element of $b * X$ is $\sum_j b(j)X(t-j)$. Also, the discussion will be more transparent if we use the frequency-zero smoothness conditions in time domain form, as discussed above in Section 2. The following result gives an equivalence between existence and uniqueness of a specific orthogonal PT decomposition for Y and the failure of ΔY to be Granger-causally prior in some system.

Theorem 4.1: *Let Y be integrated and x W, a stochastic sequence. Call $\xi = (\Delta Y, W)'$, and assume: (i) ξ is jointly covariance stationary and linearly regular with spectral density matrix full rank at the origin; and (ii) in Wold representation, $\xi = C * \epsilon$, each entry in C is $1/2$-summable. Then there exists (Y_1, Y_0) orthogonally decomposing Y with both $\Delta Y_1(t)$ and $Y_0(t)$ contained in $H_{\xi}^{-}(t)$ if and only if ΔY is not Granger causally prior to W. If such a decomposition exists, then it is unique.*

The implications of Theorem **4.1** can be understood as follows: Let W be a literature—for instance, Futia (1981), Hansen and Sargent (1991), Lippi and Reichlin (1990), Quah (1990), and Townsend (1983)—questions the appropriateness of this identifying restriction: explicit economic models can be constructed where the representations that result from this informational restriction bear no relation to the true underlying dynamics. While this difficulty is not central to the current discussion, the applied researcher should note the potential problem in more general contexts.

5. Conclusion

It is now well-known that integrated and trend stationary time series produce different implications for classical econometric inference. How does this difference extend to the observable dynamics of economic variables? What are the implications for economic theorizing?

In this paper, I have approached these issues by considering an integrated time series as the sum of permanent and transitory components. I have characterized the range of such decompositions for arbitrary integrated time series. That range turns out to always include a smooth permanent component, i.e., a permanent component with increments having arbitrarily small variance. Further, the associated transitory component need not have high persistence. Thus, the observable dynamics of an arbitrary integrated sequence are similar to those of *some* trend stationary sequence. While there are infinitely many permanent-transitory decompositions, all share the same long run effect of a disturbance in the permanent component in that their increments all have the same spectral density value at frequency zero. That value is therefore uninformative for the importance of permanent components in a time series.

In summary, the attention that has been devoted to measuring the size of the permanent component in GNP, as the spectral density at frequency zero of its increments, is unwarranted—without explicitly identifying the underlying economic disturbances, it is simply not possible to gauge the magnitude of the permanent component in a time series.

Finally, I have provided exact lower bounds on the magnitude of permanent components that are restricted to be *ARIMA* processes. Those bounds imply that restricting the permanent component to be a random walk maximizes its theoretical minimum importance.

Appendix: Explicit Construction of a Strong Sense Solution

This appendix shows how one can, beginning with the PT decomposition of Section 4, explicitly compute other orthogonal PT decompositions—decompositions having the properties specified in Theorem **2.2**. We use the notation for ˜ and ˜ established below in the proof of Theorem 4.1.

Theorem A.1: *Suppose that Y is a given integrated sequence, with the spectral density of ΔY satisfying:*

$$\int |1 - \exp(i\omega)|^{-2} |S_{\Delta Y}(\omega) - S_{\Delta Y}(0)| \, d\omega < \infty \text{ and } 0 < S_{\Delta Y}(0) < \infty;$$

and that for some given W, (Y_1, Y_0) is the orthogonal PT decomposition of Y given by **A.1**. *Suppose further that $0 < S_{\Delta Y_1}(\omega) < S_{\Delta Y}(\omega)$ for all ω in $(0, \pi]$. Let be a non-negative function on $[-\pi, \pi]$, symmetric about 0, and such that:*

(i) $0 < (\omega) < 1$ *for* $\omega \neq 0$*; and*
(ii) $\int |1 - \exp(i\omega)|^{-2}(1 - (\omega))\, d\omega < \infty$.

Define $\stackrel{\text{def}}{=} S_{\Delta Y_1}/S_{\Delta Y}$ *and choose sequences and so that*

$$|\tilde{\ }|^2 = \begin{cases} (1 - \ )\ ^{-1}(1 - \ )^{-1}, & \text{for } \omega \neq 0; \\ 0 & \text{for } \omega = 0 \end{cases}$$

and

$$\tilde{\ } = \ - \tilde{\ } .$$

If $\Delta X_1 \stackrel{\text{def}}{=} \ \Delta Y + \ \Delta Y_1$ *and* $\Delta X_0 \stackrel{\text{def}}{=} \Delta Y - \Delta X_1$*, then* (X_1, X_0) *is an orthogonal PT decomposition for* Y *with* $S_{\Delta X_1} = \ S_{\Delta Y}$.

The restrictions on $S_{\Delta Y}$ and in Theorem **A.1** are unchanged from Theorem **2.2**, but the result here gives an explicit construction for an orthogonal PT decomposition for Y.

Proofs

Proof of Theorem 2.2: *Trivial, following discussion in the text.* *Q.E.D.*

Proof of Corollary 2.3: *Clearly, in* **2.1** *can always be selected so that* $\int S_{\Delta Y}(\omega)\ (\omega)\,d\omega < 2$. *Q.E.D.*

Proof of Theorem 3.1: *Since* ΔY_1 *is a moving average process of order* q*,*

$$\Delta Y_1(t) = \sum_{j=0} C(j)\,innov(\Delta Y_1)(t-j); \qquad \text{for all } t;$$

with $C(0) = 1$ *and* $\sum_{j=0}^{q} C(j)z^j \neq 0$ *for* $|z| < 1$*. But because* Y_1 *is a permanent component for* Y*, both* ΔY_1 *and* ΔY *have the same spectral density at frequency zero:*

$$\left(\sum_{j=0} C(j)\right)^2 \mathrm{Var}(innov(\Delta Y_1)) = S_{\Delta Y}(0):$$

Thus the lower bound on $\mathrm{Var}(innov(\Delta Y_1))$ *is obtained by solving:*

$$\sup_C \left(\sum_{j=0} C(j)\right)^2 = \left(\sum_{j=0} C(j)z^j\right)^2_{z=1}$$

$$\text{subject to } C(0) = 1 \text{ and } \sum_{j=0} C(j)z^j \neq 0 \text{ for } |z| < 1:$$

Recall that any such polynomial $\sum_{j=0}^{q} C(j)z^j$ *can be written as the product of* q *monomials:* $\sum_{j=0}^{q} C(j)z^j = \prod_{j=1}^{q}(1 + D(j)z)$*, with* $|D(j)|\ 1;\ j = 1; 2; \ldots; q$*, and* $D(j)$ *appearing in complex conjugate pairs if not real. Since*

$$\left|\sum C(j)z^j\right|^2 = \left|\prod_{j=1}(1 + D(j)z)\right|^2 = \prod_{j=1}|1 + D(j)z|^2;$$

its maximization at $z = 1$ is equivalent to the maximization of $|1 + D(j)|^2$, for each $j = 1, 2, \ldots, q$. This occurs at $D(j) = 1$ for each j. Therefore, the solution to the optimization problem attains the value 4^q. The lower bound on the innovation variance is then $4^{-q} S_{\Delta Y}(0)$, and results when the moving average representation for ΔY_1 is $(1+L)^q innov(\Delta Y_1)$, where L is the lag operator. Next, the lower bound on $\mathrm{Var}(\Delta Y_1)$ is obtained by solving:

$$\inf_{(C,\sigma^2)} \sigma^2 \sum_{j=0}^{q} C(j)^2$$

$$\text{subject to } C(0) = 1; \quad \sum_{j=0}^{q} C(j) z^j \neq 0 \text{ for } |z| < 1; \text{ and } \left(\sum_{j=0}^{q} C(j) \right)^2 \sigma^2 = S_{\Delta Y}(0).$$

Substituting out for σ^2, we need to minimize $\sum_{j=0}^{q} C(j)^2 / \left(\sum_{j=0}^{q} C(j) \right)^2$ subject to the boundary conditions above. First notice that

$$\left(\sum_{j=0}^{q} C(j) \right)^2 \leq \left(\sum_{j=0}^{q} |C(j)| \right)^2.$$

Next apply the triangle and Cauchy-Schwarz inequalities:

$$\left(\sum_{j=0}^{q} C(j) \right)^2 \leq \left(\sum_{j=0}^{q} |C(j)| \cdot |1| \right)^2 \leq \sum_{j=0}^{q} |C(j)|^2 \sum_{j=0}^{q} 1^2 = \sum_{j=0}^{q} C(j)^2 \cdot (q+1);$$

so that:

$$\sum_{j=0}^{q} C(j)^2 \Big/ \left(\sum_{j=0}^{q} C(j) \right)^2 \geq (q+1)^{-1}.$$

Notice that $C(j) = 1$ for $j = 0, 1, \ldots, q$, achieves this lower bound, and because

$$\sum_{j=0}^{q} z^j = \lim_{\lambda \uparrow 1} (1 - \lambda^{q+1} z^{q+1})/(1 - \lambda z);$$

this satis es the boundary conditions as well. Thus, $\mathrm{Var}(\Delta Y_1)$ $(q+1)^{-1}S_{\Delta Y}(0)$, *and this theoretical lower bound is evidently approached arbitrarily closely by nite moving average processes of the form* $\sum_{j=0}^{q}$ ^{j}*innov*$(\Delta Y_1)(t-j)$*, where* 1.
Q.E.D.

For the next result, it is convenient to establish a little more notation. If b is an absolutely summable sequence of numbers, $\tilde{b}$ denotes its fourier transform $\tilde{b}(\omega) \stackrel{\text{def}}{=} \sum_{j} b(j)e^{-i\omega j}$, and similarly for sequences of matrices.

Proof of Theorem 4.1: *In the Wold representation,* $(\Delta Y; W)' = C$ *, take* $C(0)$ *to be lower triangular with diagonal elements non-negative and to have the identity covariance matrix. Such a representation exists and is unique. Since* $\tilde{C}(0)$

Proof of Theorem A.1: *First, verify that ΔX_1 has the correct spectral density:*

$$\begin{aligned} S_{\Delta X_1} &= |\tilde{\ }|^2 S_{\Delta Y} + |\tilde{\ }|^2 S_{\Delta Y} + 2\mathrm{Re}[\tilde{\ }\tilde{\ }^* S_{\Delta Y}] \\ &= S_{\Delta Y}\left[|\tilde{\ }|^2 + |\tilde{\ }|^2 + 2\,\mathrm{Re}(\tilde{\ }\tilde{\ }^*)\right] . \end{aligned}$$

Completing the square,

$$\begin{aligned} S_{\Delta X_1} &= S_{\Delta Y}\left[|\tilde{\ } + \tilde{\ }|^2 + |\tilde{\ }|^2(\ \ - \ \ ^2)\right] \\ &= S_{\Delta Y}\ \ ^2 + \ \ (1 - \ \ ) = S_{\Delta Y} . \end{aligned}$$

Next, check that ΔX_1 and ΔX_0 are uncorrelated at all leads and lags; by direct calculation, the cross-spectral density is:

$$S_{\Delta X_0 \Delta X_1} = S_{\Delta Y \Delta X_1} - S_{\Delta X_1} = S_{\Delta Y}\ \tilde{\ }^* + S_{\Delta Y}\ \tilde{\ }^* - S_{\Delta Y} = 0.$$

Finally, verify that ΔX_0 is the first difference of a covariance stationary sequence. From the uncorrelatedness just shown, the spectral density $S_{\Delta X_0} = S_{\Delta Y} - S_{\Delta X_1} = S_{\Delta Y}(1 - \ \ )$. Then,

$$\begin{aligned} &\int |1 - \exp(i\omega)|^{-2} S_{\Delta X_0}(\omega)\, d\omega \\ &\quad = \int |1 - \exp(i\omega)|^{-2} S_{\Delta Y}(\omega)(1 - \ \ (\omega))\, d\omega \\ &\quad = S_{\Delta Y}(0) \int |1 - \exp(i\omega)|^{-2}(1 - \ \ (\omega))\, d\omega \\ &\qquad + \int |1 - \exp(i\omega)|^{-2}(S_{\Delta Y}(\omega) - S_{\Delta Y}(0))(1 - \ \ (\omega))\, d\omega . \end{aligned}$$

From condition (ii), the first summand is finite. The result then follows from noting that the second summand is bounded from above by:

$$\sup_{-\pi \le \lambda \le \pi} |1 - \ \ (\ \ )| \int |1 - \exp(i\omega)|^{-2}(S_{\Delta Y}(\omega) - S_{\Delta Y}(0))\, d\omega < \infty ;$$

using (i). This completes the proof. *Q.E.D.*

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