Rendezvous-evasion search in two boxes with incomplete information

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Abstract

An agent (who may or may not want to be found) is located in one of two boxes. At time 0 suppose that he is in box B. With probability p he wishes to be found, in which case he has been asked to stay in box B. With probability $1-p$ he tries to evade the searcher, in which case he may move between boxes A and B. The searcher looks into one of the boxes at times $1, 2, 3, \ldots$. Between each search the agent may change boxes if he wants. The searcher is trying to minimise the expected time to discovery. The agent is trying to minimise this time if he wants to be found, but trying to maximise it otherwise. This paper finds a solution to this game (in a sense defined in the paper), associated strategies for the searcher and each type of agent, and a continuous value function $v(p)$ giving the expected time for the agent to be discovered. The solution method (which is to solve an associated zero-sum game) would apply generally to this type of game of incomplete information.

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1 Introduction

Rendezvous search was presented within a formal mathematical framework by Alpern (1995). It has attracted much attention (see e.g., Alpern and Gal (2003), Anderson and Weber (1990) and Howard (1998)). A novel element which was introduced by Alpern and Gal (2002), involves an uncertainty regarding the motives of a lost agent: he may be a cooperating rendezvouser who shares the same aim as the searcher or he may want to evade her. (For clarity, we will always assume that the agent is male and the searcher is female.) We assume that the probability p of cooperation is known to the searcher and to the agent. In any given search context (search space and player motions) this gives rise to a continuous family $\Gamma(p)$ of search problems, $0 \leq p \leq 1$, where $\Gamma(0)$ is a search game and $\Gamma(1)$ is a rendezvous problem. As we shall show, the uncertainty regarding *a priori* agent motives affects both the strategies chosen by the searcher and the strategies chosen by the (cooperating or evading) agent.

Our formalization of the cooperative part of the problem follows that of Alpern and Gal (2002) who modelled this as an asymmetric rendezvous search, where the searcher and agent may agree in advance on what they will do in the event that the agent gets lost and wants to be found. For example a mother (the searcher) may tell her son (the agent) to stay in a specific location, knowing however that this instruction may be disregarded if the child does not want to be found. The child will follow this instruction if he wants to be found, and may use the knowledge of these instructions in deciding on an evasion strategy if he does not. The next section discusses this in more detail, and shows that the equilibria that interest us can be found by solving an associated zero-sum game.

The problem that we consider is deceptively simple. The search space consists of two boxes (A and B) and time is discrete. At times $1, 2, \ldots$ the searcher looks in one of the boxes, and if the agent is in the box he is found immediately (i.e. there are no overlook probabilities). The searcher's objective is to minimise the expected time to discovery. The agent wishes to be found with probability p, and with probability $1 - p$ he wishes to avoid capture. p is assumed to be common knowledge. In each time period the agent may hide in either of the boxes. The cooperative agent's objective is the same as the searcher's. The uncooperative agent's objective is the opposite, i.e. to maximise the expected time to discovery. We assume that before the game starts the agent has been told that if he wishes to be found he should stay in box B. We will seek solutions where a cooperating agent follows this advice, and then check at the end to see whether he has any reason to disregard it. For this simple problem, are there optimal strategies for the two players, and if so what are they, and what is the value of the game?

The problem was suggested by Alpern and Gal (2002), who suggested that for p near 1 the optimum strategy is for the searcher to begin by looking in box B, and if she fails to find him look equiprobably in either box thereafter. The agent should (if mobile) hide in box A in period 1, and thereafter hide equiprobably. This is correct. They also suggested that for p near 0 both players should use equal probabilities from the start. This is not quite correct. We show that for $0 < p < \frac{2}{3}$ the searcher should start by looking in box B with a probability greater than $\frac{1}{2}$. If she does look in B and does not find the hider, she should switch to looking equiprobably. If she looks in A and fails to find the hider, she repeats the procedure but this time with a higher probability of looking in B. If she continues to look in A, eventually the probability for looking in B will reach 1. The paper gives the sequence of probabilities that should be used by the searcher, the best response by the uncooperative agent, and the associated value function $v(p)$.

2 Associated zero-sum game

The problem is a non-zero-sum game, in fact a game of incomplete information (as defined by Harsanyi) in which one player has two types with totally opposite preferences. One equilibrium of this game would be for the searcher and both types of agent to choose a box with probability $\frac{1}{2}$ at each time period until the agent is found. (This gives a pooling equilibrium — one where both types of agent use the same strategy.) However we are interested in separating equilibria, which we take to be equilibria where the two types of agent have different strategies. (Since one of the agent strategies will in general be mixed, the difference in strategies will not always be shown by difference in behaviour.) We will show there is such an equilibrium, and if there is one there will be infinitely many. This follows because we could always stipulate that the players choose boxes equiprobably for n moves before switching to the old strategies. If the agent is not found in these first n moves, we are back to the starting situation, so the new strategy is also a separating equilibrium.

We are interested in one particular separating equilibrium. We imagine that before the game starts the searcher announces instructions for the agent to follow. We seek solutions where the cooperating agent will *choose* to follow the instructions (because doing this gives him the minimum expected time to discovery). However, we can imagine another game in which with probability p the agent *must* follow the instructions, and with probability $1-p$ he can move freely. This is effectively the same game, but the agent can now be assumed always to want to evade the searcher — although he may be constrained to move (against his wishes) in order to comply with the searcher's instructions. The game becomes zero-sum. The instructions could be a pure strategy for the agent (for example, 'start in box A, and then change boxes every time period') or a mixed strategy ('hide in box B with probability 0.7 independently each period'). Intuitively, issuing instructions in the form of a mixed strategy will make it harder for the searcher to find a cooperating agent. On the other hand, it might make it more difficult for a non-cooperating agent to use his knowledge of the instructions against the searcher. We will shortly show that the searcher in fact gains no advantage by giving instructions to use a random strategy.

Let $\mathcal{S}$ be the set of all infinite sequences of A's and B's. A pure set of instructions given to the cooperating agent is a member z of $\mathcal{S}$. Once the instructions have been issued, the agent and the searcher can choose (each knowing z) pure strategies x and y in $\mathcal{S}$ The players are not restricted to pure strategies, so let $\mathcal{S}^*$ be the set of probability measures on the σ-field generated by the cylinder sets of $\mathcal{S}$. The searcher wishes, if possible, to find a strategy (y, z) which guarantees

$$v(p) = \inf_{y,z \in \mathcal{S}^*} \sup_{\{x | x:\mathcal{S}^* \to \mathcal{S}^*\}} \left[pT\left(z, y\right) + \left(1-p\right) T\left(x(z), y\right) \right]$$

where $T\left(x, y\right)$ is the expected time to discovery if the agent uses strategy x and the searcher uses strategy y ($x, y \in \mathcal{S}^*$) in the simple search and evasion game in two boxes.

Consider first the situation after the searcher has announced a particular pure set of instructions $z \in \mathcal{S}$. Although the remaining sub-game still has infinitely many pure strategies, for any fixed p consider the game G_n in which the agent is forced to follow the instructions from time n onwards, whether he is cooperating or not (and hence can always be found by time n). The agent has only a finite number of pure strategies in G_n, and the searcher need consider only a finite number of pure strategies. So G_n has a value v_n and there is a strategy $x_n \in \mathcal{S}^*$ for the agent which guarantees him at least

v_n in expectation whatever the searcher does, and also a strategy $y_n \in \mathcal{S}^*$ for the searcher which guarantees her at most v_n in expectation whatever the agent does. Because the searcher has the option of looking at random from the start

$$1 \leq v_n \leq 2$$

and clearly

$$v_n \leq v_{n+1}$$

so $v_n \to v \leq 2$ as $n \to \infty$.

Now $\mathcal{S}^*$ is compact under the topology of weak convergence, so there will be probability measures x and y on $\mathcal{S}$ which guarantee the agent at least v and hold the searcher to at most v in expectation in the unbounded time zero-sum game. So for each pure set of instructions z the game has a solution with value say $v(z)$. Moreover, because $\mathcal{S}$ is compact, there will be a particular set of instructions which achieves the infimum of $\{v(z) \mid z \in \mathcal{S}\}$. The following lemma shows the searcher could not gain by issuing instructions to the agent to use a mixed strategy.

Lemma 1 *Let G be a two-person zero-sum game in strategic form. G is modified to give a new game G^*. Before G is played, the column player can issue instructions telling the row player which pure or mixed strategy to use when G is played. Before G is played, Nature secretly tells the row player whether he has to obey the instructions. He has to obey with probability p, and the value of p is common knowledge. The payoffs in G^* are the same as the payoffs in the game G that forms part of G^*. Suppose that for each pure set of instructions G^* has a solution, and the infimum of the values of these games is achievable. Then in the modified game G^*, the column player need not consider mixed strategies as instructions to the row player.*

Proof The column player solves the game for each possible instruction — i.e. for each pure strategy for the row player in game G. She chooses the instruction (i say) which gives the lowest game value (v say). By assumption, if she chooses any instruction j, the row player has a mixed strategy which guarantees him at least v. The effect of the instruction to choose strategy j is that he will have to choose strategy j at least with probability p (because he always has to choose it when he has to obey the instruction). So the maximin problem for the original game G is modified in just one respect: if p_j is the probability with which he will choose the j'th strategy, he now has a constraint $p_j \geq p$. So for each j he has a strategy s_j for G such that the strategy

$$p\,\{\text{strategy } j\} + (1-p)\,s_j$$

guarantees v in G.

Suppose now that the column player issues instructions to use a mixed strategy q. Then the row player is constrained to use q with probability p. Suppose that with probability $(1-p)$ the row player uses the q mixture of the s_j's. Then the row player is just using a mixture of strategies in G all of which guarantee at least v, so the mixture will also guarantee v in G. Hence the instruction to use a mixed strategy gives no improvement for the column player. We may assume that the column player will issue instructions to choose a particular pure strategy. ∎

One example of an application of the lemma would occur when giving instructions to an agent who might actually be a double agent (with probability $1-p$). It could never be advantageous to instruct him to use a mixed strategy to coordinate his actions with our own. In our game, the instruction to use a particular pure strategy for the agent is the instruction to hide in the boxes in a particular sequence of A's and B's. However, any instructions which specify a particular sequence of boxes in which to hide will be equivalent (by relabelling) to hiding in box B in every time period, so henceforth we assume these are the instructions.

So with the instructions, $\mathbf{b}$, to the cooperating hider being to stay in box B until found, $v(p)$ can be expressed as

$$\begin{aligned} v(p) &= \inf_{y\in\mathcal{S}^*}\sup_{x\in\mathcal{S}^*}\left[pT\left(\mathbf{b},y\right)+(1-p)\,T\left(x,y\right)\right] \\ &= \min_{y\in\mathcal{S}^*}\max_{x\in\mathcal{S}^*}\left[pT\left(\mathbf{b},y\right)+(1-p)\,T\left(x,y\right)\right] \\ &= \min_{y\in\mathcal{S}^*}\left[pT\left(\mathbf{b},y\right)+(1-p)\max_{x\in\mathcal{S}^*}T\left(x,y\right)\right]. \end{aligned}$$

This formulation follows Alpern and Gal (2002).

We summarise the argument so far in the following theorem.

Theorem 2 *Suppose that at discrete times* $1, 2, \ldots$ *a searcher looks in one of two boxes* A *and* B *to try to find a hider.* $\mathcal{S}$ *is the set of all infinite* *sequences of A's and B's, and* $\mathcal{S}^*$ *is the set of probability measures on the* σ*-field generated by the cylinder sets of* $\mathcal{S}$*. Before the search starts, the searcher announces instructions for the hider to follow in the form of* $z \in \mathcal{S}^*$*. The searcher then searches using a mixed strategy* $y \in \mathcal{S}^*$*. With probability* p *the hider is constrained to follow these instructions, and otherwise he is free to use a sequence of boxes using any probability mixture* $x \in \mathcal{S}^*$*. The hider is found as soon as the searcher looks in the box in which he is hiding.* $T(x, y)$ *is the expected time to discovery if a hider uses strategy* x *and a searcher uses strategy* y $(x, y \in \mathcal{S}^*)$ *in the simple two box search and evasion game.*

Then the searcher can guarantee getting

$$v(p) = \inf_{y,z \in \mathcal{S}^*} \sup_{\{x | x : \mathcal{S}^* \to \mathcal{S}^*\}} [pT(z, y) + (1-p) T(x(z), y)]$$

by taking z *as the infinite sequence* $\mathbf{b} = (B, B, B, \ldots)$*. The game will have a value which will be* $v(p)$ $(1 \leq v(p) \leq 2)$ *and*

$$\begin{aligned} v(p) &= \inf_{y \in \mathcal{S}^*} \sup_{x \in \mathcal{S}^*} [pT(\mathbf{b}, y) + (1-p) T(x, y)] \\ &= \min_{y \in \mathcal{S}^*} \max_{x \in \mathcal{S}^*} [pT(\mathbf{b}, y) + (1-p) T(x, y)] \\ &= \max_{x \in \mathcal{S}^*} \min_{y \in \mathcal{S}^*} [pT(\mathbf{b}, y) + (1-p) T(x, y)], \end{aligned}$$

so optimum mixed strategies exist for both the hider and the searcher. ∎

So we consider a game in which the agent always wishes to evade capture, but a chance move by nature at the beginning of the game determines whether the agent can move or not. (Perhaps the searcher shoots at the agent and wounds him, but does not know for certain whether he is still mobile.) So from now on we will always refer to the agent as the 'hider'.

Definition 3 *The* two box evasion game with immobility probability p *is a zero-sum two player game with discrete time periods* $1, 2, 3, \ldots$*. In each time period, Player 2 (the searcher) looks in one of two boxes* A *and* B*. Player 1 (the hider) is immobile with probability* p*, in which case he must occupy box* B *at all times. He is mobile with probability* $1-p$*, in which case he can hide in either box and switch boxes whenever he wishes.* p *is common knowledge. There are no overlook probabilities. The searcher's objective is to minimise the expected discovery time, the hider's is to maximise it.*

3 Searcher strategies

Once the searcher has looked in box B and not found the hider, she knows that he is mobile (and he knows that she knows, etc.), and so both will choose at random between the two boxes from that time onwards. Hence the searcher can restrict herself to the strategies

- BR (box B, thereafter random)
- ABR (box A, then B, thereafter random)
- $AABR$
- etc.

Note that the strategy, R, of 'random from the start' is available as the probability mixture $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ of the above strategies.

Similarly, the hider can restrict himself to the set: $AR, BAR, BBAR, \ldots$ when he is mobile, (and has no choice when he is immobile).

The payoff table, showing payoffs to the hider when mobile and when immobile is:

Table 1
Payoffs to the mobile and immobile hider

		Searcher				
		BR	ABR	$AABR$	$AAABR$	$\ldots$
Immobile hider		1	2	3	4	
	AR	3	1	1	1	
Mobile	BAR	1	4	2	2	
hider	$BBAR$	1	2	5	3	
	$BBBAR$	1	2	3	6	
	$BBBBAR$	1	2	3	4	
	$\ldots$			$\ldots$		

We will call a mixture of the first i columns of the table (with non-zero probability for the i'th column) a type i strategy for the searcher. We will call a mixture of all the columns (with non-zero probabilities given to indefinitely large column numbers) a type ∞ strategy.

Definition 4 *In the two box evasion game with immobility probability* p*, a* type i strategy for the searcher *is to look in* B *with successive probabilities* $q_i, q_{i-1}, \ldots, q_1$ *with* $q_j < 1$ *for* $1 < j \leq i$ *and* $q_1 = 1$ *until she does look in* B *and to randomise* $\left(\frac{1}{2} : \frac{1}{2}\right)$ *thereafter.*

A type ∞ strategy for the searcher *is to look in* B *with successive probabilities* $q_1, q_2, \ldots$ *with* $q_j < 1$ *for all* j *until she does (if ever) look in* B *and to randomise* $\left(\frac{1}{2} : \frac{1}{2}\right)$ *thereafter.*

Theorem 5 *In the two box evasion game with immobility probability* p*,* $\frac{2}{3} \leq p \leq 1$*, the type 1 searcher strategy* BR *is optimal for the searcher. An optimal strategy for the mobile hider is* AR*.*

Proof A and then any policy, and specifically AR, is a best response of the hider to BR.

If the mobile hider plays AR and the searcher starts by looking in A, the expected meeting time is at least $2p + (1-p) = 1 + p$, while if the searcher starts by looking in B (and then uses any policy), the expected discovery time is $p + 3(1-p) = 3 - 2p \leq 1 + p$ for $p \geq \frac{2}{3}$. Thus for $\frac{2}{3} \leq p$ the strategy pair (AR, BR) for the hider and searcher is a solution to the game. ∎

Lemma 6 *In the two box evasion game with immobility probability* p*,* $p < \frac{2}{3}$*, any optimal strategy for either player will use a proper mixture of* A *and* B *on the first move.*

Proof If the searcher strategy starts with her looking in A for certain on the first move, then the hider would have dominant strategies hiding in B on the first move. But then the searcher would do better searching B first, so this will never be part of a solution. On the other hand, if the searcher definitely looks in B on the first move, the hider will hide in A on that move, and can hide at random thereafter. But against this hider strategy the searcher need consider only BR and ABR. For $p > \frac{2}{3}$ she will use BR, and for $p < \frac{2}{3}$ she will use AR (so she will not look in B on the first move as was assumed). This shows that for $p < \frac{2}{3}$ it is optimal for the searcher to make a random choice where to search on the first move.

We have seen that a solution strategy for the hider could not involve him hiding in B for certain on the first move. If the hider hides in A for certain on the first move and $p < \frac{2}{3}$, we saw that the searcher does best to use AR, so this also could not be part of a pair of solution strategies. So the mobile hider also will make a random choice where to hide on the first move. ∎

Theorem 7 *In the two box evasion game with immobility probability p, for each i there is at most one type i strategy, S_i, which is a possible optimum strategy (for some p). This S_i uses the sequence $q_i, q_{i-1}, \ldots, q_1$ of conditional probabilities for looking in B defined by*

$$q_1 = 1$$
$$q_{j+1} = \frac{1 + 2q_j}{3 + 2q_j},$$

until the searcher does look in B and to randomise $\left(\frac{1}{2} : \frac{1}{2}\right)$ thereafter.

Against S_i the mobile hider can get at most $u_i = 1 + 2q_i$, which he can obtain by playing (until he is found) any sequence of A's and B's except for those sequences which start with i B's.

Against S_i the immobile hider gets w_i defined by

$$w_1 = 1$$
$$w_{j+1} = 1 + (1 - q_{j+1})\, w_j .$$

Proof We prove this by induction. The statement is true for $i = 1$ because there is only one type 1 strategy. Suppose it is true for i. Consider a general type $i+1$ strategy with successive probabilities $s_{i+1}, s_i, \ldots, s_1 = 1$. If the searcher looks in A on the first move and the original strategy was optimal, the remaining sequence $s_i, s_{i-1}, \ldots, s_1$ must be an optimal strategy of type i by the induction hypothesis. (We have now got a problem of the same form as we started with, but with a different value for the probability of cooperation, p. Once the game is solved, we will be able to calculate the new value using Bayes' theorem.) Hence the only type $i+1$ strategies we need consider are of the form $q, q_i, q_{i-1}, \ldots, q_1$ and we can assume that if the mobile hider hides in B on the first move and is not found, all his pure strategies thereafter, except those which hide in B on every one of the next i moves, will give him u_i.

For the strategy to be optimal, we must have $p < \frac{2}{3}$ by Lemma 6, and so the mobile hider must be indifferent between hiding in A (then any sequence) or hiding in B (and then if not found getting u_i in expectation thereafter). If he hides in A he expects

$$(1-q)+3q=1+2q$$

and if he hides in B and then A (then any sequence) he expects

$$\begin{aligned} & q+(1-q)\left[2\left(1-q_i\right)+4q_i\right] \\ & =2\left(1+q_i\right)-q\left(1+2q_i\right). \end{aligned}$$

Equating these we find

$$\begin{aligned} 1+2q &= 2\left(1+q_i\right)-q\left(1+2q_i\right) \\ q\left(3+2q_i\right) &= 1+2q_i \\ q &= \frac{1+2q_i}{3+2q_i}. \end{aligned}$$

Hence there is only one possibility for a strategy of type $i+1$, and it is of the form stated in the induction hypothesis. If the searcher uses this strategy and the mobile hider hides in A the hider expects

$$u_{i+1}=1+2q_{i+1}$$

and if he hides in B and then A he expects the same (by the construction of q_{i+1}). If he hides in B and continues with any other strategy, except those which hide in B on every one of the next i moves, he also expects the same by the induction hypothesis.

Finally the immobile hider will be found in expected time

$$\begin{aligned} w_{i+1} &= q_{i+1}+\left(1-q_{i+1}\right)\left(1+w_i\right) \\ &= 1+\left(1-q_{i+1}\right)w_i. \end{aligned}$$

The induction step is complete. ∎

Note that if the searcher uses S_i and the hider responds optimally, the searcher will find the hider in expected time

$$v_i(p)=pw_i+(1-p)\,u_i.$$

Note also that if the searcher uses S_i the mobile hider will be indifferent between all his pure strategies (i.e. all sequences of A's and B's) except those which hide in B on every one of the first i moves. The included strategies all give him u_i in expectation whilst the excluded strategies all give w_i in expectation. Since $w_i < u_i$ as the mobile hider does not wish to be found, he would be content with this. Also, if the immobile hider is staying in B because he wants to be found, he would be happy to remain immobile and get w_i instead of u_i.

Theorem 8 *In the two box evasion game with immobility probability p, there is at most one type ∞ strategy, S_∞, which is a possible optimum strategy for some p. This strategy has $q_j = \frac{1}{2}$ for all j, and finds the hider (mobile or immobile) in expected time 2.*

Proof By Theorem 5 and Lemma 6 a type ∞ strategy could be optimal only for $p < \frac{2}{3}$. So the mobile hider should also be using a proper probability mixture of A and B on the first move. Supposing the strategy is given by $s_1, s_2, \ldots$. The mobile hider must be indifferent between starting with A and starting with B. Further, if the hider hides in B and is not found we are back to the same position, so once again he must be indifferent between continuing with A or with B. And so on.

So the mobile hider must be indifferent at the start between starting with A (then random) or starting with B followed by A (then random). Hence

$$\begin{aligned} 1 + 2s_1 &= s_1 + (1 - s_1)\left[2(1 - s_2) + 4s_2\right] \\ 1 + s_1 &= 2(1 - s_1)(1 + s_2) \\ s_2 &= \frac{1 + s_1}{2(1 - s_1)} - 1 \\ &= \frac{3s_1 - 1}{2(1 - s_1)}. \end{aligned}$$

If the mobile hider starts by hiding in B and is not found he will then be indifferent between continuing with A or with B followed by A. This gives

$$s_3 = \frac{3s_2 - 1}{2(1 - s_2)}$$

and in general

$$s_{j+1} = \frac{3s_j - 1}{2(1 - s_j)}. \tag{1}$$

Note that (1) implies that if $\frac{1}{2} < s_j < 1$ then $s_{j+1} > s_j$. Thus, $s_1 > \frac{1}{2}$ implies that we will obtain an increasing sequence of s_j's. There can be no limit, $s \le 1$, to that sequence because then s would have to satisfy $s(1 + 2s) = 1$ which is impossible because $s > \frac{1}{2}$. Thus for some j, $s_j \ge 1$. Similarly if $0 < s_j < \frac{1}{2}$ then $s_{j+1} < s_j$. Thus, $s_1 < \frac{1}{2}$ implies that we will obtain a decreasing sequence of s_j's. There can be no limit, $s \ge 0$, to that sequence because then s would have to satisfy $s(1+2s) = 1$ which is impossible because $0 \le s < \frac{1}{2}$. Thus for some j, $s_j < 0$.

It is clear that in general starting with an arbitrary s_1, after some iterations we would get a s_j which was negative or more than 1, or that we find $s_j = 1$ having found the S_j strategy. The only other possibility is $s_j = \frac{1}{2}$ for all j. This means that whatever the hider does he will be found in expected time 2. ∎

S_∞ is the strategy R, S_1 is BR, and S_2 is the mixture $\left(\frac{3}{5}, \frac{2}{5}\right)$ of BR and ABR. The payoff table now becomes:

Table 2

Payoffs to the hider with searcher strategies S_i

		Searcher					
		S_∞	S_1	S_2	S_3	S_4	...
Immobile hider		2	1	$\frac{7}{5}$	w_3	w_4	
	AR	2	3	$\frac{11}{5}$	u_3	u_4	
Mobile	BAR	2	1	$\frac{11}{5}$	u_3	u_4	
hider	$BBAR$	2	1	$\frac{7}{5}$	u_3	u_4	
	$BBBAR$	2	1	$\frac{7}{5}$	w_3	u_4	
	$BBBBAR$	2	1	$\frac{7}{5}$	w_3	w_4	
	...			...			

When S_i is used against an immobile hider the expected finding time is w_i; when it is used against a mobile hider using the n'th listed strategy for the mobile hider, the expected finding time is u_i for $n \le i$ and w_i for $n > i$.

Now we can find explicit formulas for q_i, u_i, and w_i as

$$u_i = 2 + \frac{3}{4^i - 1}$$

$$q_i = \frac{4^i + 2}{2\,(4^i - 1)}$$

$$
\begin{aligned}
w_i &= 2 - \frac{3\,(2^i - 1)}{4^i - 1} \\
&= \frac{2^{i+1} - 1}{2^i + 1}.
\end{aligned}
$$

(It is easy to check that these formulas are correct.)

If the searcher uses strategy S_i her maximum expected loss is

$$
\begin{aligned}
v_i(p) &= pw_i + (1-p)u_i \\
&= 2 + c_i - p\,(u_i - w_i) \\
&= 2 + c_i - pd_i
\end{aligned}
$$

where

$$
c_i = \frac{3}{4^i - 1}
$$

and

$$
\begin{aligned}
d_i &= u_i - w_i \\
&= \frac{3 \cdot 2^i}{4^i - 1}.
\end{aligned}
$$

The figure shows $v_1(p), \ldots, v_3(p)$ and the line $v = 2$ which the searcher can obtain with S_∞.

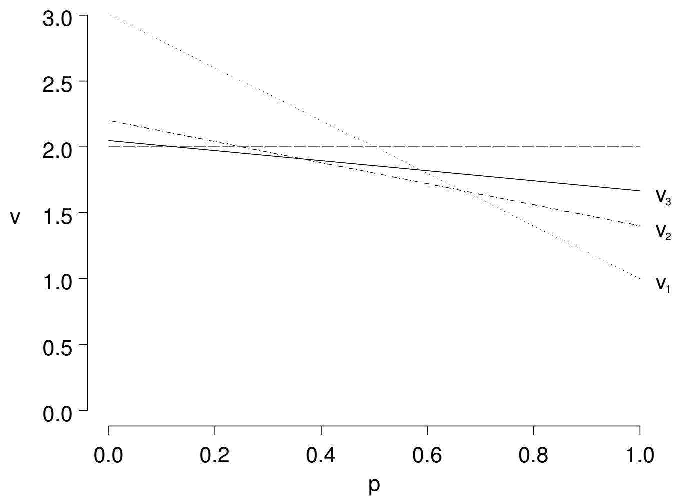

Figure 1. *Value functions* v_1, v_2, v_3, *and the line* $v = 2$

Now $v_i(p) = v_{i-1}(p)$ when

$$p = p_i = \frac{3 \cdot 2^i}{4^i + 2}.$$

Since the w_i are increasing and the u_i are decreasing, v_i is better than v_{i-1} for the searcher when $p < p_i$. Since we have shown that the searcher must choose a strategy from the set S_i for $i = 1, 2, 3, \ldots, \infty$, he should use S_i when p is in the range

$$p_{i+1} < p < p_i$$

and should use S_∞ only when $p = 0$. Since Theorem 2 implies that there exists an optimal search strategy for any $p, 0 \leq p \leq 1$, we have now shown:

Theorem 9 *Let*

$$p_i = \frac{3 \cdot 2^i}{4^i + 2} \qquad i = 1, 2, \ldots$$

and

$$q_i = \frac{4^i + 2}{2\,(4^i - 1)}.$$

Then in the two box evasion game with immobility probability p with $p_{i+1} < p < p_i$, the unique optimum strategy for the searcher is to look in box B with successive probabilities $q_i, q_{i-1}, \ldots, q_1 = 1$ until B is actually searched, and then switch to looking equiprobably thereafter. This strategy guarantees for the searcher (in expectation)

$$v_i(p) = p w_i + (1 - p) u_i$$

where

$$w_i = \frac{2^{i+1} - 1}{2^i + 1}$$

and

$$u_i = 2 + \frac{3}{4^i - 1}.$$

Against this searcher strategy, the mobile hider will obtain u_i in expectation from all his pure strategies (i.e. all sequences of A's and B's) except those which hide in B on every one of the first i moves. The excluded strategies all give $w_i < u_i$ in expectation. The immobile hider is found in expected time w_i. ∎

The table shows the calculations for the first few intervals.

Table 3

Data for S_1 to S_5

i	Strategy	p_{i+1}	p_i	q_i	u_i	w_i
1	S_1	0.667	1	1	3	1
2	S_2	0.364	0.667	0.600	2.200	1.400
3	S_3	0.186	0.364	0.524	2.048	1.667
4	S_4	0.094	0.186	0.506	2.012	1.824
5	S_5	0.047	0.094	0.501	2.003	1.909
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Notice that $p_i \to 0$ as $i \to \infty$.

4 Behavioural approach

The game is specified by one parameter, p, the probability of immobility, which both players know at the start of the game. Given p, suppose that with optimal play the searcher will look in cell B with probability $q(p)$ and the hider will (if mobile) hide in cell A with probability $r(p)$. If the searcher looks in A and does not find the hider she will change her value for p using Bayes' theorem. However, the hider will know she is updating her probability (because he knows that she has looked in A and not found him), and he can also calculate her new value p' of p, so we return to the same game with a different starting parameter. If the searcher looks in B and the hider hides in A they both switch to random thereafter. In any other case the game ends. So all we need are the functions q and r and the Bayesian updating formula to play the game. What are these functions?

We have already found the searcher's q function:

$$q(p) = q_i = \frac{4^i + 2}{2\,(4^i - 1)} \text{ for } p_{i+1} < p < p_i.$$

The hider's r function can be found because the mobile hider must randomise between A and B so that the searcher is indifferent between the strategy BR and the strategy AS_{i-1} when $p_{i+1} < p < p_i$ with $i > 1$. The strategy BR gives

$$p \cdot 1 + (1-p)\,[(1-r) \cdot 1 + r \cdot 3]$$

and AS_{i-1} gives

$$p\,(1+w_{i-1})+(1-p)\,[r\cdot 1+(1-r)\,(1+u_{i-1})]\,.$$

Equating these gives

$$r(p)=\frac{v_{i-1}(p)}{(1-p)\,(2+u_{i-1})}\text{ for }p_{i+1}<p<p_i.$$

$r(p)$ can be re-expressed as

$$\begin{aligned}r(p)&=\frac{v_i(p)-1}{2(1-p)}\text{ for }p_{i+1}<p<p_i\\&=\frac{2+4^i-3\cdot 2^ip}{2\,(4^i-1)\,(1-p)}.\end{aligned}$$

This graph shows the two functions:

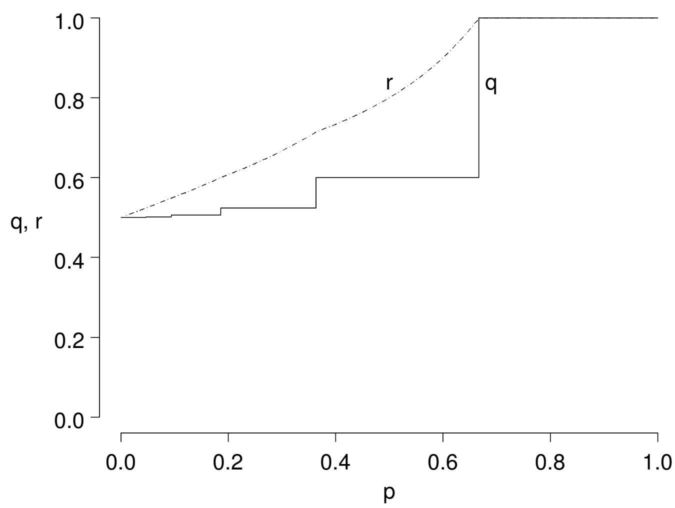

Figure 2. *q and r as functions of p*

With optimal play, the value of the game is $v(p)$ where

$$v(p)=v_i(p)\text{ for }p_{i+1}<p<p_i$$

and this function is the lower envelope of the family of lines whose first three members were graphed in Figure 1.

After one move, if the searcher looks in A and the hider hides in B, p is updated using Bayes' theorem. The new p is

$$p'(p) = \frac{p}{1-(1-p)\,r(p)}.$$

It is easy to check that

$$p'(p_{i+1}) = p_i$$

so that until the hider is found or the searcher looks in B, the value of p moves from one interval to the next higher in each time period. We have now proved:

Theorem 10 *Let*

$$p_i = \frac{3 \cdot 2^i}{4^i + 2}.$$

Then in the two box evasion game with immobility probability p, $p_{i+1} < p < p_i$, the unique optimum first move for the mobile hider is to hide in box A with probability $r(p)$, where

$$r(p) = \frac{2 + 4^i - 3 \cdot 2^i p}{2\,(4^i - 1)\,(1-p)}$$

and to switch to equiprobable hiding if he does hide in A and is not found. If he hides in B and is not found, he should update p to

$$p'(p) = \frac{p}{1-(1-p)\,r(p)}$$

where $p'(p)$ will satisfy

$$p_i < p'(p) < p_{i-1}.$$

He then iterates with p' and $i-1$ instead of p and i. Eventually he will actually hide in A or he will reach the situation $p' > \frac{2}{3}$ and $i = 1$ when he will hide in A with probability 1. ∎

5 Behavioural derivation

It is of interest to give an alternative derivation of our results based entirely on the behavioural approach. Given p, let $v(p)$ be the value of the game. Let $u(p)$ be the value of the game given that the hider is actually mobile, and $w(p)$

be the value of the game given that the hider is actually stationary. Suppose that the searcher (playing optimally) searches box B with probability $q(p)$, and the hider (playing optimally) hides in box A with probability $r(p)$. Once the searcher looks in cell B she will switch to $q = \frac{1}{2}$ until the hider is found, but until then she will use some sequence $q(p), q'(p), q''(p), \ldots$. $u(p)$ must be the value that can be obtained by a mobile hider making a best response to this searcher strategy. If the hider hides in B and is not found, the searcher will then use the strategy $q'(p), q''(p), q'''(p), \ldots$ which must be a best strategy for her updated p, say p'. Let the game starting with $p = p'$ have value v' and new values u' and w' for u and w.

Theorem 5 established for $\frac{2}{3} \leq p \leq 1$

$$\begin{aligned} q(p) &= r(p) = 1 \\ w(p) &= 1 \\ u(p) &= 3 \\ v(p) &= 3\,(1-p) + p = 3 - 2p. \end{aligned}$$

By Lemma 6 if $p < \frac{2}{3}$ both players will make a random choice where to hide or search on the first move. If the hider discovers he is mobile, and then chooses, whichever choice he makes will give him the same value $u(p)$. So for $p < \frac{2}{3}$ we have

$$\begin{aligned} u &= (1-q) + 3q = 1 + 2q && \text{if mobile hider chooses } A \\ u &= q + (1-q)\,[1 + u'] = 1 + (1-q)u' && \text{if mobile hider chooses } B \end{aligned}$$

Now in fact the first equation

$$u(p) = 1 + 2q(p)$$

holds for all p, since it holds for $\frac{2}{3} \leq p \leq 1$. So for $p < \frac{2}{3}$ the second equation gives

$$\begin{aligned} 1 + 2q &= 1 + (1-q)\,[1 + 2q'] \\ q &= \frac{1 + 2q'}{3 + 2q'}. \end{aligned}$$

Alternatively

$$q' = \frac{3q - 1}{2\,(1-q)}.$$

It is clear that in general starting with an arbitrary $q(p)$, after some iterations we would get a q which was negative or more than 1. The only way out is $q = \frac{1}{2}$ or that the sequence of ascending q's ends with 1. This implies we can iterate backwards from 1 to get a magic sequence of q's — $1, \frac{3}{5}, \frac{11}{21}, \ldots$.

To eliminate the possibility of $q = q' = \frac{1}{2}$ (which implies the searcher continues to look in 2 with probability $\frac{1}{2}$, so $u = v = w = 2$) look at the choices for the searcher. We have

$$v = p + (1-p)\left[(1-r) + 3r\right] = 1 + 2(1-p)\,r \quad \text{if searcher chooses } B.$$

But then $(1-p)\,r = \frac{1}{2}(v-1) = \frac{1}{2}$ and the Bayesian update is

$$\begin{aligned} p' &= \frac{p}{1-(1-p)\,r} \\ &= 2p \end{aligned}$$

which must lead (for $p > 0$) to some $p' > \frac{2}{3}$ in which case $q' \neq \frac{1}{2}$.

Hence for each $p < \frac{2}{3}$ the searcher must use one of the discrete sequence of q's, $q_1, q_2, \ldots$ (i.e. $1, \frac{3}{5}, \frac{11}{21}, \ldots$) and then follow the sequence backwards until she looks in B or she reaches $q_1 = 1$.

Suppose for a given p, a best strategy for the searcher is to use strategy S_i, i.e. to start with $q = q_i$ where $i = i(p)$ with corresponding $u(p) = u_i$ and $w(p) = w_i$. We have found the sequence of q_i's, and we have that

$$u_i = 1 + 2q_i,$$

and so the sequence of u_i's is determined as $3, \frac{11}{5}, \frac{43}{21}, \ldots$.

The recursion for w is

$$\begin{aligned} w &= q + (1-q)\,(1+w') \\ &= 1 + (1-q)\,w' \end{aligned}$$

so

$$w_i = 1 + (1-q_i)\,w_{i-1}.$$

This gives the sequence of w_i's starting $1, \frac{7}{5}, \frac{5}{3}, \ldots$.

When the searcher uses strategy S_i, and the probability of immobility is p, the value she expects is

$$v_i(p) = pw_i + (1-p)\,u_i$$

and this will be the value for the game if S_i is a best strategy for p, i.e.

$$v(p) = v_{i(p)}(p).$$

It remains to find the values of p for which the searcher should use q_i. We could argue (as in section 3) that the searcher must choose

$$i(p) = \text{argmin}_i \{v_i(p)\}.$$

So that $v(p)$ will be the lower envelope of the set of lines $\{v_i(p)\}$.

Alternatively, note that q_1 is used when $p > \frac{2}{3}$ and is not used when $p < \frac{2}{3}$. Let $p_2 = \frac{2}{3}$: then p_2 solves

$$v_2(p) = v_1(p).$$

Also

$$\begin{aligned} p'(p_2) &= \frac{2p_2}{3 - v(p_2)} \\ &= p_1 \end{aligned}$$

Let p_3 be the value of p which solves

$$v_3(p) = v_2(p),$$

then if $p_3 < p < p_2$ since v is a continuous function of p, we must have $q(p) = q_2$ and $v(p) = v_2(p)$. We also find that

$$\begin{aligned} p'(p_3) &= \frac{2p_3}{3 - v(p_3)} \\ &= p_2 \end{aligned}$$

so if we had $q(p) = q_2$ for some $p < p_3$ the searcher would not follow this q_2 with q_1. Hence q_2 is used within (p_3, p_2) and never used outside $[p_3, p_2]$.

Continuing in this way we find the same intervals as before. ∎

Also note that from the equations

$$u_i = 1 + (1 - q_i)u_{i-1}$$

and

$$w_i = 1 + (1 - q_i)\, w_{i-1}$$

we obtain

$$v_i(p) = 1 + (1 - q_i)\, v_{i-1}(p).$$

However at $p = p_i$ we have $v_i(p_i) = v_{i-1}(p_i)$. Hence we must have

$$v_i(p_i) = \frac{1}{q_i}.$$

6 Related problems

6.1 More than 2 boxes

Suppose there are 3 boxes with an immobile hider forced to stay in box 3. At some stage of the game the searcher might have looked in box 3 and found nobody. She will then know that the hider is mobile. But, unlike the two box case, the hider will not know that she knows that. Similarly, if the mobile hider hides in box 3 on the first move and is not found, he will know that the searcher does not know that he is mobile, but the searcher will not know that he knows that. The situation seems more complicated than the two box case.

It seems obvious that, for any time period, if the searcher decides to not to look in the agreed meeting box, say box n, then he should look in any of the boxes $1, 2, \ldots, n-1$ with an equal probability, $\frac{1}{n-1}$, and similarly for the hider. Thus, if both the searcher and the evader are *not* in cell n, then the probability of capture is $\frac{1}{n-1}$. This leads to the following more general formulation. Assume that within the rendezvous-evasion problem there are two boxes A and B, as in the original problem. However, for any time period, if both the searcher and the evader are in A, then there is a probability α of capture, where $0 < \alpha \leq 1$ is a known constant, and if capture does not occur, neither will know which box the other was in. If they are both in B, then the hider is discovered at once. If they are in different boxes, the hider is not caught. The solution to this problem seems quite complex even for the three box problem in which $\alpha = 1/2$.

6.2 Continuous time

If the searcher looks in box A, and the hider is there, she finds him in a Poisson process with rate 1. Similarly for box B. How should the two players behave if they can switch boxes at zero cost at any time?

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