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**Abstract**

Drawing on recent work concerning the statistical robustness of inequality statistics we examine the sensitivity of poverty indices to data contamination using the concept of the influence function. We show that poverty and inequality indices have fundamentally different robustness properties, and demonstrate that an important commonly used subclass of poverty measures will be robust under data contamination. We investigate both the case where the poverty line is exogenously fixed and where it must be estimated from the data.

**Keywords:** Poverty, inequality, robustness, influence function.
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# Introduction

Since Sen's (1976) pioneering article economists have been particularly interested in quantifying poverty in a manner consistent with principles that are commonly applied elsewhere in economic analysis. This literature has focused on a number of theoretical issues that are fundamental to an understanding of the extent and intensity of economic poverty, such as: What is the poverty line? Should incomes or expenditure be used for the identification of the poor? What function should be used for aggregating the incomes of the poor? At the same time it is commonly recognised that practical information about the poor is sometimes difficult to come by, difficult to quantify and prone to various sorts of error. Therefore the question naturally arises as to whether the problems with the data will be so great as to vitiate conclusions arrived at from careful application of theoretically appropriate poverty measures: will the contamination of the data drive the computation of the results? Our paper focuses on this topic and the way the statistical issues that it raises interact with some of the theoretical questions that we have just mentioned.

In doing this we can make use of insights that have been obtained in the inequality measurement literature. The formal similarities in economic terms between the measurement of poverty and the measurement of inequality have often been remarked upon: several of the same sort of properties are invoked in order to make an "appropriate" inequality index or an "appropriate" poverty index; for example it is common to find both sorts of indices making appeal to the transfer principle. Now we know that the transfer principle in conjunction with other standard properties of inequality measures gives rise to a problem of non-robustness of the statistics commonly used to estimate these inequality measures.[1] Will similar problems arise in the case of poverty-measurement statistics?

Because poverty indices typically incorporate a variety of issues based upon diverse principles it is useful to categorise poverty measures into a few broad classes of indices. One of the richest of these is discussed in the next section.

## A Characterisation of Poverty Measures

First we introduce some notation in order to describe the problem of poverty measurement formally:

Let $x \in \mathbb{R}$ denote an individual's *income* which is taken to be a comprehensive measure of that person's economic status (we do not attempt here to distinguish between, for example income and consumption for this purpose); and let $z \in \mathbb{R}$ be the *poverty*

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[1] See Cowell and Victoria-Feser (1993).

*line* also defined in terms of income. We assume that data is available on the interpersonal distribution of $x$ and that individual persons are effectively identical in every respect other than income. A person is said to be in poverty if he has an income $x<z$. An *income distribution* is interpreted as a conventional distribution function $F: \mathbb{R}\rightarrow[0,1]$

The *aggregate poverty index* is given as a functional $P$ defined on $\mathcal{F}$, the space of all income distributions; this functional has as one of its parameters the poverty line $z$, so we shall write a particular poverty index as $P(F\,;z)$. The aggregate poverty index is usually given economic meaning by invoking a set of axioms which induce a particular behaviour of the index in response to changes in the distribution of income. Unfortunately, because of the multifaceted nature of the concept of poverty there is a diversity of proposed axioms in the literature - each one of which makes some appeal to reasonableness. This diversity has led to a proliferation of proposed measures[2] and often results in mutually inconsistent criteria for poverty comparisons, as Kundu and Smith (1983) have shown. Such a situation is manifestly unsatisfactory for establishing general results on the statistical properties of poverty measures; a possible way forward is to focus upon

[2] See Callan and Nolan (1991), Foster (1984), Hagenaars (1986) and Seidl (1988) for surveys of this literature.

specific classes of poverty indices that include some of the most commonly-used specific measures and that represent the important properties of poverty measurement in a fairly general way.

As an important particular case the *Additively Separable Poverty* indices constitute a subclass of poverty measures $P$ which are defined by the formula:

$$P(F\,;z) = \int p(z,x)\mathrm{d}F(x) \qquad \textbf{(1)}$$

where $p: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a poverty evaluation function for individuals - cf Ravallion (1992) based on Atkinson (1987).[3] In effect the severity of poverty for a person with income $x$ is measured for each person, with reference to the fixed poverty line $z$; overall poverty is then computed as an aggregate of individual poverty levels. It is usually assumed that $p$ is continuous for $x<z$, nondecreasing in its first argument and nonincreasing in its second argument, and in most cases that $p$ is convex in the second argument. It is also usually assumed that $p(z, x)=0$ for $x \geq z$; we shall make this assumption here. Notice that the poverty measure (1) can be coherently decomposed as a function of the poverty attributable to arbitrary population subgroups.

[3] Note that Atkinson's presentation dictated that the individual poverty evaluation and function and the poverty index were defined as $-p$ and $-P$ respectively.

Five important examples of this type of function are as follows (in each case the relevant *p*-function is easily found by inspection):[4]

■ 1 The *Head-count ratio*

$$P_{HC}(F\,;z) = \int \iota(z,x)\ \mathrm{d}F(x) \tag{2}$$

where ι is the *poverty indicator function,* taking the value 1 if $x < z$ and 0 otherwise: obviously (2) is simply $F(z)$.

■ 2 The *Normalised Poverty Deficit*

$$P_{NPD}(F\,;z) = \int \iota(z,x)\ \frac{z-x}{z}\ \mathrm{d}F(x) \tag{3}$$

■ 3 The *Foster-Greer-Thorbecke class* of poverty measures:

$$P_{FGT}(F\,;z) = \int \iota(z,x) \left[\frac{z-x}{z}\right]^{\alpha} \mathrm{d}F(x) \tag{4}$$

where α is an indicator of sensitivity to poverty gaps. To make sense as a poverty index we must have α≥0. The subclass that satisfies the requirement of

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[4] See Clark *et el.* (1981), Foster *et al.* (1984), Watts (1968).

convexity has the more stringent requirement $\alpha \geq 1$.

■ 4 The *Clark-Hemming-Ulph class* of poverty measures:

$$P_{\mathrm{CHU}}(F;z) = \int \iota(z,x)\, \frac{1 - [x/z]^{\beta}}{\beta}\, \mathrm{d}F(x) \tag{5}$$

where β is a sensitivity index akin to the inequality aversion parameter commonly used in inequality analysis; in order to satisfy the transfer principle one must have $\beta < 1$.

■ 5 The *Watts* index:

$$P_{\mathrm{W}}(F;z) = -\int \iota(z,x)\, \log\left(\frac{x}{z}\right) \mathrm{d}F(x) \tag{6}$$

As we will see, the implied *p*-function in each case is crucial in determining the robustness property of the poverty measure.

## Contamination and The Influence Function

Data contamination can be seen as endemic to the problem of measuring the incomes of the poor. We approach the problem by stylising the data contamination in a fashion that makes the analysis tractable.

Suppose $F$ is the "true" model of income distribution so that $P(F; z)$ is the true amount of poverty in the population. Let $H^{(y)}$ be an elementary perturbation distribution which consists of a point mass at income $y$. This has the probability distribution:

$$dH^{(y)}(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

and from (7) we may define the following mixture distribution:

$$G_{\varepsilon}^{(y)} = [1-\varepsilon]\, F + \varepsilon\, H^{(y)} \tag{8}$$

To quantify the importance of mixing the perturbation with the true model upon the statistic under consideration we may use the concept of the *influence function*.[5] In the present case this is given by

$$\text{IF}(y;P,F) = \lim_{\varepsilon \to 0} \frac{P(G_{\varepsilon}^{(y)}; z) - P(F; z)}{\varepsilon} \tag{9}$$

or, where the derivative exists, by

[5] See Hampel (1974), and Hampel *et al.* (1986).

$$\mathrm{IF}(y;P,F) = \left.\frac{\partial\, P(G_\varepsilon^{(y)};z)}{\partial \varepsilon}\right|_{\varepsilon=0} \qquad \textbf{(10)}$$

The influence function is a statistical tool to assess the influence of an infinitesimal amount of contamination upon the value of a statistic. Here the statistic is the estimator of the poverty measure and (10) indicates to what extent the poverty measure is stable in the presence of a small proportion of arbitrary extreme observations. If the influence function can take on large values then this implies that a single observation - if sufficiently extreme - could drive the poverty measure by itself. The influence function will also carry information about the bias of the estimate of the poverty index: it can be shown (see Hampel *et al.*, 1986) that the influence function is the first-order term in the linear expansion of the asymptotic bias of the estimator. So it is obviously of central importance to know how the influence function will behave for various types of data contamination for a wide class of poverty measures. In particular it is interesting to know whether the influence function can actually be unbounded.

## The ASP Class: Results with Fixed Poverty Line

Applying the formula for the influence function (9) to the ASP class of poverty indices (1) we get:

$$\mathbb{IF}(y;P,F) = \lim_{\varepsilon \to 0} \frac{\int p(z,x) \, \mathrm{d}\left[ G_{\varepsilon}^{(y)}(x) - F(x) \right]}{\varepsilon} \tag{11}$$
$$= \int p(z,x) \, \mathrm{d}\,[H^{(y)}(x) - F(x)] \, ,$$

which implies

$$\mathbb{IF}(y;P,F) = p(z,y) - P(F\,;z) \quad . \tag{12}$$

So the importance of data contamination in this model is determined by a very simple rule: evaluate the poverty contribution of an extra observation at the point of contamination ($p$) and subtract from it the estimate of aggregate poverty ($P$); this ($p,P$)-rule makes it easy to understand the impact of data-contamination. The second term on the right-hand side, $P(F;\,z)$, is constant under contamination (it does not depend on $y$); however, whether the first term is bounded depends on (i) the form of the individual poverty-evaluation function $p$ and (ii) the point $y$ at which the contamination is assumed to be present. In particular the behaviour of $p(z,y)$ as $y \to 0$ is especially important. Notice that for cases where income cannot be negative[6] and for poverty measures defined in terms of *income gaps* of the poor the problem of unboundedness of the

[6] The argument can be extended to any case where the support of $F$ is bounded below.

influence function is unlikely to arise. The intuition here is straightforward: $x$ is confined to a bounded subset of $\mathbb{R}$, namely $[0,z]$, and so $z$-$x$ is bounded everywhere; the only potential difficulty could arise in the neighbourhood of $z-x = 0$; but if $p$ is continuous in this region we can be sure that $p(z,x) \to 0$ at this point; moreover indices of the ASP class do not use information about the incomes of the rich (for example they do not depend on the population mean) and so contamination of very high-value incomes is irrelevant. However, by contrast, for poverty measures of the form $P_{CHU}$, which are expressed in terms of power functions of income itself, the influence function will become unbounded as $y \to 0$. This will occur for all $\beta<0$; it will also occur in the case of $P_W$, which is based upon the logarithm of income.

## The qASP Class

The insight gained from the discussion of the ASP class suggests that similar robustness properties may apply to indices included in an extension of the ASP class. We consider a subclass of poverty measures $P$ that are defined by the formula:

$$P(F\,;z) = \int p(z,x,F(x))\ \mathrm{d}F(x) \tag{13}$$

Apart from the fact that the poverty evaluation function $p$ now has an extra term reflecting the individual's rank in the population, $F(x)$, the basic idea is the same and we shall retain the assumptions about the behaviour of $p$; we shall also assume that the derivative $p_F(z,x,F(x)) := \partial p(z,x,F(x))/\partial F(x)$ exists. A slight rearrangement makes the properties of the class (13) clearer. Use the quantile function associated with the distribution function $F$: this is in effect the inverse of $F$ and may be written as $Q$:[0,1]→$\mathbb{R}$ such that $Q(t) = \min\{x: F(x) \geq t\}$. Then equation (13) becomes

$$P(F;z) = \int_0^1 p(z,\ Q(t),\ t)\ \mathrm{d}t \tag{14}$$

In either of the equivalent forms (13) and (14) it is clear that the poverty measure could be coherently decomposed into population subgroups that form compact subintervals of [0,1] but not into arbitrary subgroups. For this reason we shall refer to the class as quasi-additively separable poverty measures the qASP class. The *Sen index* belongs to this class: it can be written

$$P_S(F;z) = \int \imath(z,x)\ \frac{z - w(x;F,z)\,x}{z}\ \mathrm{d}F(x)$$
$$\text{where}\quad w(x;F,z) := 2\left[1 - \frac{F(x)}{F(z)}\right] \tag{15}$$

Now consider the influence function for this class and for the same model of contamination as in (7) and (8) we find:

$$\mathrm{IF}(y;P,F) = \lim_{\varepsilon\to 0} \frac{\int p\left(z,x,G_\varepsilon^{(y)}(x)\right) \mathrm{d}\, G_\varepsilon^{(y)}(x) - \int p(z,x,F(x))\, \mathrm{d}F(x)}{\varepsilon} \tag{16}$$

$$= p(z,y,F(y)) - P(F;z) + \int \frac{\partial p(z,x,F(x))}{\partial F(x)} \left[ H^{(y)}(x) - F(x) \right] \mathrm{d}F(x)$$

This shows that the result for the ASP class can be extended to the qASP class with only a slight modification. Equation (16) becomes

$$\mathrm{IF}(y;P,F) = p(z,y,F(y)) - P(F;z) + \left[ \int_y^z p_F(z,x,F(x))\mathrm{d}F(x) - \int_0^z p_F(z,x,F(x))F(x)\, \mathrm{d}F(x) \right] \tag{17}$$

The terms in [] on the right-hand side of (17) will be bounded as long as $p_F(z,x,F(x)$ is bounded for all $x$: given that $p$ is nonincreasing in $x$ and continuous it is clear that if $p$ is bounded over a region, then so too is $p_F$; a small change in people's rankings does not *per se* have an unlimited effect upon the individual poverty evaluation function. Hence we can see that the boundedness properties of the influence function again depends upon the behaviour of the individual poverty evaluation function $p$. as in the case of the ASP class. [7]

In view of this it is immediate that the Sen measure $P_S$ will

[7] Notice that once again we do not have to concern ourselves about the second term on the right-hand side of (17), because this is independent of the contamination at $y$.

have a bounded influence function, because the individual poverty evaluation function in this case is given by $1\text{-}w(z,x,F(x))x/z$ for all $x<z$.

## The Endogenous Poverty Line

So far we have assumed that we have the informational luxury of knowing the poverty line $z$ in terms of dollars or whatever. In many cases this assumption is inappropriate, either because the official poverty line is unsatisfactory and has to be estimated from other sources of information, or because the official concept of the poverty line itself incorporates an explicit dependence upon the income distribution to which the poverty line is being applied. For example Eurostat conventionally takes as a poverty line for a particular country 50% of median income; as a second example the recent approach to the presentation of data on the incomes of the poor adopted by the UK's Department of Social Security[8] focuses upon an estimate of the mean as a crucial step in identifying the low-income population: groups are specifically characterised as being those with incomes less than 40%, 50%, 60%... of the mean.

We can incorporate either of these approaches within our

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[8] See Department of Social Security (1992).

model by introducing a functional $\zeta$: $\mathscr{F}\to\mathbb{R}$ such that $z = \zeta(F)$. This specification means that the poverty line itself is being estimated from the data, and it thus implies that there is an additional channel by means of which data contamination may bias the estimates of poverty.

It is straightforward to extend the analysis to cover this case also. Substituting the explicit dependence of $z$ on $F$ we now find

$$\begin{aligned} \mathrm{IF}(y;P,F) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int p\left(\zeta(G_\varepsilon^{(y)}),x\right)\mathrm{d}G_\varepsilon^{(y)}(x) - \int p(\zeta(F),x)\,\mathrm{d}F(x)\right] \\ &= \int p(z,x)\;\mathrm{d}\left[H^{(y)}(x)-F(x)\right] + \int\frac{\partial p(\zeta(F),x)}{\partial z}\,\mathrm{d}F(x)\;\frac{\partial\zeta(G_\varepsilon^{(y)})}{\partial\varepsilon}\bigg|_{\varepsilon=0} \end{aligned} \quad \mathbf{(18)}$$

Simplifying, this gives:

$$\mathrm{IF}(y;P,F) = p(z,y) - P(F;\zeta(F)) + \frac{\partial P(F;\zeta(F))}{\partial z}\,\frac{\partial\zeta(G_\varepsilon^{(y)})}{\partial\varepsilon}\bigg|_{\varepsilon=0} \quad \mathbf{(19)}$$

It is evident from (19), (21) that the boundedness or otherwise of the influence function now depends on two things: (1) the determinants of the influence function in the exogenous-poverty-line case (see the first two terms on the right-hand side), and (2) the sensitivity of the poverty line to contamination. When considering an infinitesimal amount of contamination, either of these two effects can make the influence function unbounded: $p(z,y)$ can become large for abnormally small values of $y$, while $\partial\zeta(G_\varepsilon^{(y)})/\partial\varepsilon\,|_{\varepsilon=0}$ will typically become large for abnormally high values of $y$.

Let us consider the second point a little further. If the dependence of $z$ on $F$ can be represented as a function of the mean, thus

$$z = \phi(\mu(F)) \tag{20}$$

where $\phi$ is a monotonic function, then serious problems may arise, since it is well-known that the mean is not robust in the presence of contamination at the top of the distribution. In the present case we find that this property implies that the influence function for poverty line itself is unbounded, because (20) yields

$$\mathrm{IF}(y;z,F) = \frac{\partial\phi(\mu(F))}{\partial\mu(F)} \left[ y - \mu(F) \right] \tag{21}$$

It is immediate that (21) is unbounded as $y \to \infty$. By contrast if the dependence of $z$ on $F$ were to be represented as a function of the median or some other quantile, the estimate of the poverty line would be robust under data contamination.

## Conclusions

We have demonstrated that there is a fundamental difference between the statistical properties of inequality and poverty indices in the presence of contaminated data. The reason is that the commonly accepted axiomatic structure of poverty indices automatically restricts the class of admissible functions in an interesting way. The restriction ensures that, if the poverty line is exogenous, the poverty measures are not sensitive to the values (real or contaminated) of the incomes of the rich; and that they are well-behaved in the neighbourhood of the poverty line. However, not all poverty measures will have bounded influence functions. Problems can arise both with the specification of the poverty line (the identification issue) and with the function used to pick up the sensitivity to the income distribution amongst the poor (the aggregation issue). A poverty line that is related to a function of mean income exposes the resulting poverty indices to undue influence from spurious information on the incomes of the very rich. Poverty evaluation functions that have a singularity at some point (such as the point $x=0$ in our examples) will also run into trouble: in such cases rogue data on the poorest of the poor may completely distort the estimate of aggregate poverty. However poverty measures that take as their primitive concept poverty *gaps* rather than the incomes of the poor will almost certainly avoid this particular problem.

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