Inheritance and the Distribution of Wealth

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Abstract

The theory of functional equations is used to clarify the relationship between equilibrium distributions of wealth and population parameters such as the distribution of families by size, marriage patterns, tax mechanisms and savings behaviour within a simple model of inheritance.

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1 Introduction

The shape of the distribution of wealth is a topic that has exerted a long fascination. The empirical regularities across different types of economies have often been remarked upon, and the literature contains several types of dynamic model that can be used, in part, to explain the characteristic shape of the distribution.[1] On the strength of this research some have gone so far as to suggest "laws of distribution" which societies must inexorably obey. By contrast this paper has a modest objective: it examines the effect on wealth distribution of an aspect of inheritance processes that has received relatively little attention. Using a simple framework that is consistent with standard models of savings and bequest behaviour it shows the way in which the equilibrium distribution can be determined over a specified wealth range. It also examines the relationship between equilibrium wealth inequality and the distribution of families by size within a broad class of models of the bequest process.

The principal result is that, under fairly weak conditions, parts of the equilibrium distribution of wealth must be characterised by a narrow class of functional forms. The parts of the distribution which can be captured in this way are delimited by regions in which specific behavioural characteristics of wealth-owners are assumed to hold. The conditions that are required for the main results are consistent with a number of models of wealth accumulation and bequests. The

[1] See for example Champernowne (1953, 1973), Eichhorn and Gleissner (1985), Lydall (1968), Mandelbrot (1960), Sargan (1957), Vaughan (1988), Wold and Whittle (1957).

family of functional forms includes the standard formulae that have been derived as equilibrium distributions in a variety of specific models of the wealth accumulation and distribution, and that are often utilised for ad hoc purposes such as curve-fitting for particular parts of empirical wealth distributions.

The approach that I use is to specify the conditions for equilibrium in a wealth model driven by inheritance. The theory of functional equations is then used to characterise the class of wealth distributions that are determined by the equilibrium conditions.

Section 2 outlines the fundamentals of the model. Section 3 proves the main result for a simplified version of the model, and shows how the equilibrium wealth distribution derived in the main result can be related to the parameters of the system. Then sections 4 and 5 demonstrate how the elementary model can easily be extended to a number of more interesting cases.

2 The Model

Consider a population that is made up of a sequence of generations. Let time be discrete and indexed by $t = ..., 0, 1, 2, ...$ and assume that each generation is uniquely associated with one contiguous pair of periods: those who are children at time t become adults at time $t + 1$. Assume that at any time the population consists of a number of families each of which has a determinate, finite number of children, and that there are no childless families:[2] apart from this the distribution of families in the population is arbitrary. Let the proportion of families with k children be $p_k \geq 0$, $k = 1, 2, ..., K$, and write the vector $(p_1, p_2, ...p_K)$ as $\mathbf{p}$. By definition:

$$\sum_{k=1}^{K} p_k = 1, \tag{1}$$

and for population stationarity $\mathbf{p}$ must satisfy:

$$\sum_{k=1}^{K} kp_k = 2. \tag{2}$$

In an infinite population this assumption can be relaxed.

Imagine the economy at any moment t: we can conceive of the population as being composed of families characterised by their joint wealth level and the number of children who will eventually inherit that wealth. The wealth distribution of such families depends upon the specific assumptions made about the way that wealth grows in each period, people's savings behaviour, the form of wealth taxation, the way in which new families are formed in each generation, and the way in which parents distribute their wealth. I shall assume the following:

Axiom 1 *Bequeathable wealth grows everywhere by an exogenous factor β during one generation.*

[2] Contrast this with, for example, the model of Wold and Whittle (1957) and Eichhorn and Gleissner (1985), in which the wealth of each vanishing household is distributed among n beneficiaries, where n is simply the average number of inheritors in the economy.

For example, if there is an exogenous rate of growth of total wealth g, a uniform average propensity to consume out of wealth[3] c, and a tax on bequests τ, then

$$\beta = [1 + g][1 - c][1 - \tau]. \tag{3}$$

Axiom 2 *All parents whose individual wealth satisfies $W \in I$, where I is a proper interval that does not contain zero,[4] follow a policy of equal division amongst their k kids.[5]*

It is convenient to summarise the parameters characterising any specific implementation of these assumptions thus:

$$\pi := (\beta, \mathbf{p}) \tag{4}$$

Let the wealth distribution in any generation t be denoted by a distribution function $F_t: \Re \mapsto [0, 1]$. Because the distribution will be conditional upon the particular value of the set of parameters we shall write it as $F_t(W; \pi)$. For a given π equilibrium is defined as a situation where, for any t and for all $W \in I$:

$$F_{t+1}(W; \pi) = F_t(W; \pi) = F(W; \pi). \tag{5}$$

[3]This behaviour is consistent with utility maximisation where preferences are homothetic; see, for example, Becker and Tomes (1979).

[4]That is, I may be a closed interval, $[W_0, W_1]$, or an open interval, or may be unbounded above.

[5]This behaviour would be the consequence of utility maximisation under homothetic preferences - Becker and Tomes (1979). However we do not need to appeal to this specific assumption: in some some societies equal division may be imposed by law - see Kessler and Masson (1988).

We shall also assume:

Axiom 3 *F is continuous over I.*

The fundamental problem is to find the family of functions $F(W;\pi)$ given the set of parameters π. In principle a distribution function of wealth ought to be defined on the whole real line - people can have negative as well as positive net worth - but it would be a demanding and perhaps unilluminating task to try to specify every detail of the distribution function over its entire range. However, for theoretical and empirical reasons, it is often economically interesting to focus on certain parts of the distribution, for example the upper tail. So, what we will do is characterise the shape of the function F over the restricted domain I.

3 Assortative Mating

As a first step conside the case of positive assortative mating. This is a situation of strict "class marriage": a person at t with wealth W seeks marries another person with wealth W and forms a new family in $t+1$. Given the behavioural assumptions outlined above, the issue of the distribution of wealth then focuses on the distribution of families by size. In this case members of the population completely arecharacterised by the pair (W, k). A (W, k)-family consists of $2+k$ persons: two parents, each of whom possesses on marriage wealth W, and who divide their wealth equally among their k children.

3.1 Main result

Each child of a (W, k)-family inherits wealth $\frac{2\beta W}{k}$. So the equilibrium condition (5) requires

$$F(W) = \sum_{k=1}^{K} \frac{1}{2} k p_k F\left(\frac{kW}{2\beta}\right) \tag{6}$$

Equilibrium condition (6) implies that for any two distinct values W, $W' \in I$ the unknown function F must satisfy:

$$F(W) = \sum_{k=1}^{K} a_k \; F\left(W_k\right), \tag{7}$$

$$F(W') = \sum_{k=1}^{K} a_k \; F\left(W'_k\right), \tag{8}$$

where

$$a_k \; := \frac{1}{2} k p_k, \tag{9}$$

$$W_k := \frac{kW}{2\beta} \tag{10}$$

In view of (2), $0 \leq a_k < 1$ for non-trivial family structures. By definition of a distribution function $W' \geq W \iff F(W'; \pi) \geq F(W; \pi)$. There are two cases: (1) F is constant over I ; (2) there exist some $W, W' \in I$ such that $F(W'; \pi) > F(W; \pi)$. Case (1) is trivial since it means that there is no-one with wealth in I. In case (2), because of the assumed continuity of F over I, there must be an interval $I' \subseteq I$ for which F is increasing. For convenience introduce the following changes of variable:[6]

$$\gamma := \frac{1}{2}\left[\inf\left(I\right) + \sup\left(I\right)\right], \tag{11}$$

$$X := \left\{x : x = \frac{W}{\gamma},\ \forall W \in I\right\}, \tag{12}$$

$$G : X \mapsto [0,1],\ G\left(\frac{W}{\gamma}\right) = F(W, \pi),\ \forall W \in I. \tag{13}$$

It is then evident that (6) implies

$$x = G^{-1}\left(\sum_{k=1}^{K} a_k\ G\left(x_k\right)\right), \tag{14}$$

In other words the equilibrium condition for the wealth interval is equivalent to requiring that the younger generation's wealth be a quasilinear weighted mean[7] of K values of wealth in the older generation, where the quasilinear mean is constructed using the (transformed) distribution function. However we can further restrict the function G, and hence the distribution function F.

Lemma 1 *The function G must satisfy either*

$$G(x) = A\log(x) + B,\ x \in X \tag{15}$$

[6]Notice that, by definition, $1 \in X$.
[7]See Aczél (1966) page 240, Dhombres (1984).

or

$$G(x) = Ax^{\theta} + B, \; x \in X \tag{16}$$

where A is nonzero.

Proof. See Appendix.

Lemma 1 leads immediately to a result that has particularly interesting economic implications.

Theorem 2 *In the case of strict class marriage, the equilibrium wealth distribution must belong to the extended Pareto Type I family, throughout the region where the equal-division inheritance rule applies. In other words F must satisfy*

$$F(W;\pi) = a + b\frac{W^{-\alpha(\pi)} - 1}{\alpha(\pi)} \tag{17}$$

where a and b are constants and $\alpha(\pi) \in \Re$.

Proof. Using (11)-(13) Lemma 1 implies that the equilibrium distribution must take either the form

$$F(W;\pi) = A \log\left(\frac{W}{\gamma}\right) + B \tag{18}$$

or

$$F(W;\pi) = A\left[\frac{W}{\gamma}\right]^{\theta} + B \tag{19}$$

$\forall W \in I$ where $W = \gamma x$. Using the standard result

$$\lim_{\alpha \to 0} \frac{[x^\alpha - 1]}{\alpha} = \log(x) \tag{20}$$

equations (18) and (19) can each be written in the form (17) for a suitable specification of $\alpha(\pi)$. ■

The class of equilibrium distributions (17) includes not only the conventional Pareto curve, but also the rectangular distribution and the "reverse Pareto" for which $\alpha < -1$.[8] The particular member of the class that is appropriate will depend on the parameters and the relevant domain I.

To see the intuition behind this result consider the following argument using the equilibrium condition (6). In view of the population stationarity condition (2) equation (6) implies

$$\sum_{k=1}^{K} k p_k F(W;\pi) = \sum_{k=1}^{K} k p_k F\left(\frac{kW}{2\beta};\pi\right) \tag{21}$$

for all W. In other words F must satisfy:

$$\sum_{k=1}^{K} k p_k \left[F(W;\pi) - F\left(\frac{kW}{2\beta};\pi\right)\right] = 0 \tag{22}$$

[8] See, for example, Champernowne (1953); see also Champernowne (1952), Fisk (1961).

The solution to (22) will characterise the shape that the wealth distribution must adopt in equilibrium. Equation (22) can only hold for arbitrary W, if an appropriate separability result holds. Specifically there must be functions $g : I \mapsto \Re$, $h : \Re \mapsto \Re$ such that

$$F(\xi W; \pi) - F(W; \pi) = g(W)h(\xi) \tag{23}$$

for all W and for $\xi = \frac{k}{2\beta}$. In this case we see immediately from (23) that either $g(W) = 0$ for all $W \in I$, or else we have $h(1) = 0$.

Evidently one trivial solution of (23) is $\beta = \frac{1}{2}k$ (the case $\xi = 1$) and $g(W)$ arbitrary. In this case there is only one size of family and, as long as the parameters determining α are appropriately set, an arbitrary wealth distribution reproduces itself. A second solution can be found for the case where g is a constant: this yields

$$F(W; \pi) = a \log(W) + b \,. \tag{24}$$

Finally, if g is not constant, write:

$$F(W; \pi) - F\left(\frac{W}{\xi}; \pi\right) = g\left(\frac{W}{\xi}\right) h(\xi) \,. \tag{25}$$

Combining (23) and (25) we have

$$F(\xi W; \pi) - F\left(\frac{W}{\xi}; \pi\right) = \left[g\left(W\right) + g\left(\frac{W}{\xi}\right)\right] h(\xi) \,. \tag{26}$$

for all $W, \frac{W}{\xi} \in I$. However the left-hand side of (26) equals $g\left(\frac{W}{\xi}\right) h(\xi^2)$. So

$$g\left(W\right) = g\left(\frac{W}{\xi}\right)\left[\frac{h(\xi^2)}{h(\xi)} - 1\right] \tag{27}$$

which implies

$$g(\xi W) = g(W)g(\xi) \tag{28}$$

Therefore $g\left(W\right) = AW^{\theta}$ for some $A \in \Re_{+}$, $\theta \in \Re$.

One can generate the entire class of equilibrium distributions $F(W;\pi)$ by allowing π to range over all possible parameter values.

3.2 Determination of α

To illustrate the way an equilibrium distribution is determined, take the situation in which $I = [W_0, \infty)$ where W_0 is a specified strictly positive level of wealth. From Theorem 2 the equilibrium distribution of wealth is given by the Paretian distribution

$$F(W;\pi) = 1 - AW^{-\alpha(\pi)} \tag{29}$$

where A is a constant and $\alpha(\pi)$ is the largest root of the following implicit equation in α:

$$\sum_{k=1}^{K} p_k \left[\frac{k}{2}\right]^{1-\alpha} = \beta^{-\alpha} \tag{30}$$

	p_1	p_2	p_3	p_4	p_5	p_6
case (a)	0.30	0.45	0.20	0.05	0	0
case (b)	0.35	0.45	0.10	0.06	0.03	0.01

Table 1: Two family structures

	case (a)		**case (b)**	
β	α	Gini	α	Gini
0.95	1.55	0.476	1.43	0.543
0.90	2.11	0.311	1.90	0.357
0.85	2.73	0.224	2.44	0.257
0.80	3.43	0.171	3.07	0.196
0.75	4.28	0.132	3.86	0.148

Table 2: Equilibrium values of α for two distributions of family size

To see the implications of this for wealth inequality take the two cases of the family-size distributions in Table 1: in case (b) there is a wider spread of families by size in comparison to case (a).

The equilibrium distribution of wealth will depend on the rate of inheritance tax and the other components of the autonomous growth factor of net wealth β as well as the structure of families by size. Table 2 gives the Pareto coefficient α and also the implied Gini coefficient of the resulting equilibrium distribution (which equals $\frac{1}{2\alpha-1}$ in this case) using equation (30), for various values of β and the two cases in Table 1. Looking down any one column of Table 2 it is clear that α decreases and equilibrium inequality increases as β increases. Using the interpretation (3) this implies that equilibrium inequality decreases with the rate of inherit tax. Looking across any row Table 2 we see that increasing the spread of family sizes increases the inequality of the wealth distribution.

4 Marriage out of Class

In this and section 5 we reconsider some of the restrictive assumptions used in the basic model, and we investigate to what extent they may be relaxed.

The assumption that people only marry those who have wealth equal to their own is perhaps one of the most restrictive features of the simple framework that has been used so far. However, the basic model can be adapted in a way that permits a simple modification of Theorem 2. In this variant of the model families are characterised by three parameters (W, δ, k), where W is now the wealth of the poorer of the two marriage partners, and $\delta \geq 0$ is a "class-disloyalty" parameter. A person with wealth W marries someone with wealth $W[1+\delta]$, for W, $W[1+\delta] \in I$; so the wealth received by a child in a (W, δ, k)-family is $\frac{[2+\delta]W}{k}$.

To make this version of the model operational we require an assumption about the distribution of marriage partners - the distribution of δ. An additional equilibrium condition is also required since those who marry above their station must be matched by those who marry beneath them. This induces a constraint on the admissible class of distributions of δ. we shall return to this below. The key assumption is as follows.

Axiom 4 *The distribution of δ is independent of W for all $W \in I$.*

This means that class disloyalty is independent of wealth. Let Φ be the distribution function of δ. Then the equilibrium condition (6) can be modified to read:

$$F(W)=\sum_{k=1}^{K}\frac{1}{2}kp_k\int F\left(\frac{kW}{[2+\delta]\,\beta}\right)d\Phi(\delta) \tag{31}$$

On replacing the weights a_k in (9) by $\frac{1}{2}kp_k d\Phi(\delta)$ it is clear that (31) will again yield a weighted quasilinear mean, similar to (14). Therefore, in view of Theorem 2 we have:

Corollary 3 *Given the wealth-independence of the distribution of the class-disloyalty parameter δ, the equilibrium distribution of wealth in the interval must belong to the extended Pareto class, as specified in Theorem 2.*

To see the way in which the marriage-out-of-class model works take a specific example in which the pattern of families is the same again as it was in Case (a) of Table 1. above, and in which the distribution of the disloyalty parameter is very simple - only one value is possible. Now, instead of the two partners having equal wealth before marriage, suppose that one partner is just twice as wealthy as the other ($\delta = 1$), and consider the effect that this would have on the degree of inequality of the equilibrium wealth distribution in the model. Although this important modification to the model will not change the resulting function form of the wealth distribution it will change the particular member of the class of equilibrium wealth distributions that corresponds to a specific instance of the family-size distribution $\mathbf{p}$. For example, take the bequests made to children of families with a combined wealth-level of 2 units. In the model of section 3, 15 percent of their descendants leave 2β, 45 percent leave β, 30 percent leave $\frac{2}{3}\beta$

and 10 percent leave $\frac{1}{2}\beta$. But now, in each of these four groups of descendants a proportion λ will marry partners twice as wealthy, and so the average wealth will be 1.5 times the climber's pre-marriage wealth: the remaining $1-\lambda$ will marry partners half as wealthy, and for them the average wealth after marriage will be only three-quarters of their own pre-marital wealth.

The value λ of that is consistent with the assumed marriage rule in equilibrium will depend on the distribution of wealth. Since corollary 3 implies that wealth has a Pareto distribution with parameter α, and since there must be $[1+\delta]^\alpha$ times as many marriages of spouses in the range $[1 \pm \epsilon]$ with spouses in the range $\frac{[1\pm\epsilon]}{[1+\delta]}$ as with spouses in the range $[1 \pm \epsilon][1+\delta]$, this is only possible if

$$\lambda = \frac{1}{1+[1+\delta]^\alpha} \tag{32}$$

Given the class-disloyalty assumptions and a specific value of the disloyalty parameter δ, we may write the required parameter set π as a triple:

$$\pi := (\beta, \delta, \mathbf{p}) \tag{33}$$

If the family size-distribution parameters $\mathbf{p}$ satisfy the stationarity condition (2) the general implicit equation giving α as a function of β and δ may be written:

$$\sum_{k=1}^{K} p_k k^{1-\alpha} = \frac{1+[1+\delta]^\alpha}{\beta^\alpha[2+\delta]^\alpha} \tag{34}$$

β	δ 0	0.20	0.5	1.0	1.5	2.0	2.5
0.95	1.43	1.44	1.52	1.80	2.52	3.66	4.79
0.90	1.90	1.94	2.10	2.72	3.93	5.44	7.06
0.85	2.44	2.50	2.78	3.87	5.80	8.32	11.82
0.80	3.07	3.17	3.66	5.57	9.23	16.27	37.20
0.75	3.86	4.03	4.89	8.88	21.50		

Table 3: Effects of growth factor β and class disloyalty δ on equilibrium Pareto coefficient α

Consider the implications of the class-disloyalty model for the equilibrium distribution. Clearly the new assumptions of this model will change the relationship between the parameter β and the equilibrium inequality of wealth distribution (which is inversely related to the Pareto coefficient α). Take for example case (b) of the family-size distribution $\mathbf{p}$ in Table 1. By solving for $\alpha(\pi)$ from (34) for alternative values of and β, and δ, we obtain Tables 3 and 4. Quite modest increases in class-disloyalty have a considerable impact on equilibrium inequality. For example an increase in δ from 1.0 to 1.5 reduces the Gini coefficient by 30 to 45 percent. The table also implies the extent to which social forces may substitute for fiscal tools. Suppose that overall growth is just sufficient to finance lifetime consumption so that $\beta = 1 - \tau$ in (3) and consider the value of the inheritance tax that would yield a given equilibrium value of $\alpha = 2$ (a Gini coefficent of 0.333). If the conditions of strict assortative mating (the $\delta = 0$ column in Tables 3 and 4) were to be replaced by a state where everyone married some with twice (half) as much wealth ($\delta = 1$ in Tables 3 and 4) the tax rate required to support the given equilibrium distribution would fall from 11.0 percent to 6.2 percent.

β	δ 0	0.20	0.5	1.0	1.5	2.0	2.5
0.95	0.539	0.530	0.493	0.385	0.249	0.158	0.117
0.90	0.357	0.348	0.313	0.226	0.146	0.101	0.076
0.85	0.258	0.250	0.219	0.148	0.094	0.064	0.044
0.80	0.194	0.187	0.158	0.099	0.057	0.032	0.014
0.75	0.149	0.142	0.114	0.060	0.024		

Table 4: Effects of growth factor β and class disloyalty δ on equilibrium wealth inequality (Gini coefficient)

5 Extensions of the Model

5.1 Alternative Inheritance Rules

Suppose the equal-division rule for distributing one's bequests were replaced by some other inheritance principle. The principle may differ from one size of family to another but, as long as the rule is independent of the level of family wealth, the equilibrium condition (5) will once again lead to a quasilinear weighted mean of the form (14). The set of weights $a_1, ..., a_K$ will in general differ from those in section 3 and will depend on the precise division rule: for example the share of any one child might depend on its rank order in the family. However, the original results were established for arbitrary weights, and so again Lemma 1 holds in an appropriately modified form. Therefore we can extend Theorem 2 to cases in which some proportionate bequest rule other than equal division is consistently applied throughout the wealth interval I. Of course the equilibrium value of α will depend on the particular bequest rule that is employed.

5.2 Redistributive Taxation

As a further extension of the modification to the inheritance rule, consider the impact of redistributive wealth or inheritance taxation. So far we have just considered a proportionate rate of tax with the possibility that the proceeds are distributed somewhere outside the particular part of the economic system that we are examining. Now take a more comprehensive version of a redistributive tax:[9] we introduce a linear tax upon the joint bequest of the testators, a type of tax function that is widely used in simplified economic models, and is a reasonable approximation to many actual tax schedules. Under this linear redistributive tax-function assumption the inheritance of each child in a (W, k)-family is given by

$$\frac{\bar{W} + 2[1-\tau]\beta W}{k} \tag{35}$$

where τ and W are tax parameters and β is the growth factor for wealth before tax. Now, instead of equation (10) introduce the following in conjunction with equations (7) to (9):

$$W_k = \frac{k[W - \bar{W}]}{2[1-\tau]\beta} \ . \tag{36}$$

[9] There are other ways of generalsing the tax function - for example by introducing a constant residual progression tax function which would automatically leave the wealth distribution as a Pareto type I.

Use this modified definition of the wealth levels W_k and use the change of variables

$$\gamma := \frac{1}{2}\left[\inf(I) + \sup(I)\right] + \frac{\bar{W}}{2[1-\tau]\beta}$$

Then the quasilinear mean relationship given in equation (11, 13 and14) - holds once more, with modified weights. So Lemma 1 is valid in this modified version of the model also. On substituting back using (36) we find:

Theorem 4 *If private bequest rules are independent of wealth, and all inheritances are subject to a linear redistributive tax, then over the relevant wealth range the distribution of wealth follows a Pareto type II distribution:*

$$F(W;\pi) = a + b\frac{[c+W]^{-\alpha(\pi)} - 1}{\alpha(\pi)} \tag{37}$$

where a, b and c are constants and $\alpha(\pi) \in \Re$.

5.3 Modified Savings Behaviour

The above results yield an insight into a further modification that may be made to the basic model of section 2. Thus far the assumption has been made that at every relevant wealth level bequests are proportional to wealth so that, of course, lifetime consumption is also proportional to wealth for all $W \in I$, with the same proportion applying to all families. Consider two extensions of the model in the direction of realism.

First, it is easy to see that the assumption that there is a single value of β can be replaced with an assumption that β is distributed in the population, according to differences in parents' tastes for their own (current) consumption as against the consumption of their offspring, or according to differences in tax-treatment of families according to some other personal circumstance of the families, or according to differences in exogenous growth rates of wealth. Once again we would need to assume that the distribution of β is independent of W, but that is all. Families would then be characterised by the quadruple (W, δ, β, k) specifying their pre-nuptial wealth level, class disloyalty on marriage, growth-factor of wealth and number of children. The modification to the basic result would follow essentially the lines of the modification for the class-disloyalty case discussed in section 4. A modified version of Corollary 1 will hold.

Second, we can generalise the form of the proportional savings function. Suppose instead that lifetime consumption takes the following form:

$$C = C_0 + c\beta^* W \tag{38}$$

where $C_0 \geq 0$, $c > 0$, $\beta^* > 0$ are parameters; β^* can be taken as the autonomous growth factor of after-tax wealth during a person's lifetime. Then once again the wealth of a child in a (W, k) family can be written

$$\frac{[1 - c\beta^*] W - C_0}{k} \tag{39}$$

Comparing (39) with (35), the following is immediate:

Corollary 5 *If lifetime consumption is an affine (linear) transformation of wealth in the basic inheritance model, then the equilibrium distribution of wealth is of Pareto Type II (Type I).*

6 Concluding Remarks

Pareto distributions pop up all over the literature on the distribution of wealth. It is interesting to know why this should be so.

The method of functional equation analysis suggests a simple reason why this pattern of the wealth distribution should be so persistent. This approach is undemanding in that it merely requires that the assumed equilibrium distribution be continuous, and that a particular system of marriage, saving and bequest rules apply over a wealth-interval that does not contain zero, and that is wide enough to accommodate the wealth-values of the offspring of some of the families with wealth in that interval. The method has the further advantage that the results go through merely as a consequence of the formal definition of equilibrium, and not with reference to any particular model of a process through time.

This is not to say that explicit modelling the process of wealth accumulation or the bequest decision is unimportant. Far from it. But it is useful to know that the equilibrium - if and when it is established - is bound to have a simple and familiar form.

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A Proof of Lemma 1

From (14) for some positive scalar λ we must have:

$$\lambda x = G^{-1}\left(\sum_{k=1}^{K} a_k \; G\left(\lambda x_k\right)\right), \tag{40}$$

or

$$x = H^{-1}\left(\sum_{k=1}^{K} a_k \; H\left(\lambda x_k\right)\right) \tag{41}$$

where $H(x) := G(\lambda x)$ for some given value of λ. Using (14) and (41) we have

$$G^{-1}\left(\sum_{k=1}^{K} a_k \; G\left(\lambda x_k\right)\right) = H^{-1}\left(\sum_{k=1}^{K} a_k \; H\left(\lambda x_k\right)\right). \tag{42}$$

Without loss of generality we may take $a_1, a_2 > 0$, and set $x_3, ..., x_K$ equal to some arbitrary constants $\bar{x}_3, ..., \bar{x}_K$. Then introduce:

$$z_i := a_i G(x_i),\ i = 1, 2 \tag{43}$$

We may consider z_1, z_2 as varying over some open interval $J \subset [0, 1]$. Also define functions Ξ, Ψ, Φ:

$$\Xi : J \mapsto X,\ \Xi(z) := H\left(G^{-1}\left(z + \sum_{k=3}^{K} a_k \; G\left(\bar{x}_k\right)\right)\right) \tag{44}$$

$$\Psi : J \mapsto X,\ \Psi(z_1) := H\left(G^{-1}\left(\frac{z_1}{a_1}\right)\right) \tag{45}$$

$$\Phi : J \mapsto X,\ \Phi(z_2) := a_2 H\left(G^{-1}\left(\frac{z_2}{a_2}\right)\right) + \sum_{k=3}^{K} a_k\ H(\bar{x}_k) \tag{46}$$

Using these variables and functions we may rewrite (42) as:

$$\Xi(z_1 + z_2) = \Psi(z_1) + \Phi(z_2),\ z_1, z_2 \in Z \tag{47}$$

which is a restricted Pexider equation (Eichhorn 1978). The standard solution to this is

$$\Psi(z) = \phi(z) + c \tag{48}$$

where c is an arbitrary constant and ϕ is an arbitrary solution of the Cauchy equation

$$\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2),\ z_1, z_2 \in \Re \tag{49}$$

This implies

$$\Psi(z) = hz + c \tag{50}$$

where h is a positive constant. So, since G and H are strictly increasing functions, (45) and (50) yield:

$$H(x) = hG(x) + g$$

where $g := \frac{c}{a_1}$. In the present case it is clear from (40) and (41) that the values g and h may depend on the value of λ that had been chosen, so that we may write:

$$G(\lambda x) = h(\lambda)G(x) + g(\lambda), \forall \lambda, x, \lambda x \in X \tag{51}$$

There are two cases to consider in solving (51). First, if h is independent of λ (let us say $h(\lambda) = 1$ for all λ), then (51) implies

$$\bar{G}(\lambda x) = \bar{G}(x) + \bar{G}(\lambda), \tag{52}$$

where

$$\bar{G}(x) := G(x) - G(1). \tag{53}$$

In view of the assumed continuity of G over the interval X, equation (52) implies:

$$\bar{G}(x) = A \log(x), \;\; A \neq 0, \tag{54}$$

Alternatively, if h is not independent of λ we must have:[10]

$$h(\lambda x) = h(\lambda)h(x), \forall \lambda, x, \lambda x \in X \tag{55}$$

[10] Cf Aczél (1987) page 26.

which implies $h(x) > 0$ for all $x \in X$.[11] Take the following logarithmic transformations of the variables and of the function h

$$\left.\begin{array}{c} \phi(y) := h(\log(x)), \\ y := \log(x)\ z = \log(\lambda), \\ \forall y, z, \in Z\ , \forall \lambda, x, \lambda x \in X \end{array}\right\} \tag{56}$$

we then find that (55) becomes

$$\phi(y+z) = \phi(y) + \phi(z),\ \forall y, z, y+z \in Z \tag{57}$$

which is the standard Cauchy equation on a hexagon. Since $1 \in X$, we have $0 \in Z$, and under these conditions (57) has a unique extension from Z to $\Re$.[12] The solution to (57) is then:[13]

$$\phi(y) = \theta y,\ \theta \in \Re \backslash \{0\},\ \forall y \in Z \tag{58}$$

which implies

$$h(x) = x^{\theta}\ x \in X. \tag{59}$$

Taking the two cases together either (15) or (16) must be satisfied. ■

[11] The reason for this is that $1 \in X$, so that if $x \in X$, then (55) implies $\sqrt{x} \in X$ and, using (55) again, $h(x) = h(\sqrt{x})^2 \geq 0$. However, $h(x)$ cannot be zero for any $x \in X$ without violating the strict monotonicity of h over X.

[12] See, for example, Aczél and Dhombres (1989) Chapter 7, Theorem 5.

[13] See Aczél (1987), page 20.