STUDENTIZATION IN EDGECWORTH EXPANSIONS
FOR ESTIMATES OF
SEMIPARAMETRIC INDEX MODELS

by

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Abstract

We establish valid theoretical and empirical Edgeworth expansions for density-weighted averaged derivative estimates of semiparametric index models.

Keywords: Edgeworth expansions; semiparametric estimates; averaged derivatives.

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1. INTRODUCTION

During the 1970’s, Takeshi Amemiya considerably advanced the asymptotic theory of estimation of parametric econometric models for cross-sectional data. Previously, most work had concerned closed form estimates, such as generalized least squares and instrumental variable estimates of linear regressions, or two- or three-stage least squares estimates of linear-in-equations-and parameters simultaneous equations systems. Prompted by Jennrich’s (1969) work on strong consistency and asymptotic normality of nonlinear least squares, Amemiya developed asymptotic theory for implicitly-defined extremum estimates of a variety of econometric models.

Let \( Y_i, X_i, \ i = 1, 2, \ldots \), be sequences of, respectively, scalar and \( d \times 1 \) vector observables, and define

\[
Y_i = (\beta'X_i + \varepsilon_i) \ 1 (\beta'X_i + \varepsilon_i > 0) , \ i = 1, 2, \ldots ,
\]

where \( \varepsilon_i, \ i = 1, 2, \ldots \), is a sequence of unobservable zero-mean random variables, \( \beta \) is a \( d \times 1 \) unknown vector, \( \tau \) denotes transposition, and \( 1 (\cdot) \) is the indicator function. (1.1) is called a Tobit model. Least squares regression of \( Y_i \) on \( X_i \), using either all observations or all observations such that \( Y_i > 0 \), inconsistently estimates \( \beta \). Assuming the \( \varepsilon_i \) are independent and identically distributed (iid) normal variates, maximum likelihood (ML) estimates based on (1.1) can be consistent. These, however, are only implicitly defined. Amemiya (1973) established their strong consistency and asymptotic normality, later extending these results (Amemiya (1974a)) to a multivariate version of (1.1).

Another model of econometric interest is

\[
\left( \frac{Y_i^\lambda - 1}{\lambda} \right) 1 (\lambda > 0) + (\log Y_i) 1 (\lambda = 0) = \beta'X_i + \varepsilon_i , \quad (1.2)
\]

where the scalar \( \lambda \) is unknown. This is called a Box-Cox transformation model. If \( \lambda \) is specified incorrectly, least squares regression inconsistently estimates \( \beta \). Thus, methods have been proposed for estimating \( \beta \) and \( \lambda \) simultaneously. One such purports to be ML, based on normal \( \varepsilon_i \), but unless \( \lambda = 0 \) or \( 1/\lambda \) is odd the left hand side of (1.2) cannot possibly be
conditionally normal. Alternative, logically consistent, distributions have been proposed, e.g., Amemiya and Powell (1981), but if the distribution is misspecified inconsistent estimates again result. Amemiya and Powell (1981) also applied non-linear two-stage least squares estimation, which applies to a general class of models including (1.2), and whose asymptotic theory was earlier developed by Amemiya (1974b). This estimate, which again is only implicitly defined, is consistent over a wide class of \( \varepsilon_i \). Amemiya (1977) also developed asymptotic theory for non-linear three-stage least squares and ML estimates of non-linear simultaneous equations, to provide an extension to vector dependent variables.

Both models (1.1) and (1.2) are of the single linear index type
\[
E(Y_i | X_i) = G(\beta^\top X_i), \quad i = 1, 2, \ldots
\]
almost surely (a.s.), for a function \( G : \mathbb{R}^r \to \mathbb{R} \). Let \( F \) be the distribution function of \( \varepsilon_i \). In (1.1),
\[
G(u) = u \int_{-\infty}^{\infty} dF(v) - \int_{-\infty}^{u} vdF(v).
\]
If \( F \) is an unknown, nonparametric function, then so is \( G \). Then \( \beta \) can be identified only up to scale. But if we can estimate \( \beta \) up to scale in (1.3), with unknown \( G \), we have a form of robustness with respect to \( F \). In (1.2),
\[
G(u) = \int \{ 1 + \lambda(u + v) \}^{1/\lambda} dF(v) 1(\lambda > 0) + e^{-u} \int e^v dF(v) 1(\lambda = 0),
\]
so the same considerations arise. As already noted, we can robustly estimate \( \beta \) (and also \( \lambda \)) in (1.2) using nonlinear two-stage least squares. However, the general index form (1.3) indicates that we may be able to estimate \( \beta \) up to scale whether or not the transformation of \( Y_i \) is of Box-Cox type. Note that \( \beta \) can be identifiable on the basis of objective functions used in other semiparametric methods such as LAD (Powell, 1984), symmetrically trimmed least squares (Powell, 1986), semiparametric M-estimation (Horowitz, 1988), and semiparametric least squares (Horowitz, 1986; Lee, 1992).

We can estimate \( \beta \) up to scale by the density-weighted averaged derivative statistic
\[
U = \left( \frac{n}{2} \right)^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} U_{ij},
\]
where
\[ U_{ij} = h^{-d-1}K'( \frac{X_i - X_j}{h} ) (Y_i - Y_j), \]
such that \( K'(u) = (\partial u/\partial u)K(u), \) where \( K : R^d \rightarrow R \) is a differentiable (kernel) function such that \( \int_{R^d} K(u) \, du = 1 \), and \( h = h_n \) is a positive (bandwidth or smoothing) sequence which tends to zero slowly as \( n \rightarrow \infty \). For an unknown scalar \( c \), \( n^{1/2} (U - c\beta) \) was shown to be asymptotically normal when the \( Y_i, X_i \) are iid (Powell, Stock and Stoker (1989)) and when they are weakly dependent (Robinson (1989)), and to be possibly asymptotically non-normal in case of an element of long range dependence (Cheng and Robinson (1994)).

Thus, in case of the Tobit model (1.1), for example, \( U \) achieves the same rate of convergence as that of the ML estimate established by Amemiya (1973) where the \( \varepsilon_i \) are normal, and by Robinson (1982) where the \( \varepsilon_i \) are normal but actually weakly dependent. (Robinson (1982) also established consistency when the \( \varepsilon_i \) are long range dependent normal).

On the other hand, the smoothing entailed in \( U \) might be expected to produce inferior higher-order asymptotic properties, since these more closely approximate the finite sample situation. We know of no explicit treatment of higher-order properties of the Tobit MLE (or of the Box-Cox estimates we have mentioned), but general results of Pfanzagl (1971, 1973), Bhattacharya and Ghosh (1978), Linton (1996) suggest that, under suitable conditions, they are likely to have an \( O(n^{-1/2}) \) Berry-Esseen bound (uniform rate of convergence to normality) and valid Edgeworth expansion in powers of \( n^{-1/2} \), while Robinson (1991) established a Berry-Esseen bound for an optimal version of Amemiya’s (1977) nonlinear three-stage least squares estimate. Robinson (1995) showed that while in general \( U \) has a Berry-Esseen bound of order greater than \( n^{-1/2} \), it can be implemented (using suitable \( h \) and \( K \)) to have an \( O(n^{-1/2}) \) bound. Correspondingly, Nishiyama and Robinson (1998) (hereafter NR) established that the leading Edgeworth expansion term is \( O(n^{-1/2}) \) or larger.

Theorems 1 and 2 of NR established valid theoretical and empirical Edgeworth expansions of \( Z = n^{1/2} \alpha_{\nu}^T (U - c\beta) \) for any \( d \times 1 \) vector \( \nu \), where \( \alpha_{\nu}^2 = \nu^T \Sigma \nu \) and \( \Sigma \).
is the asymptotic variance matrix of \( n^{1/2} (U - c\beta) \). Of course \( \Sigma \) is unknown so that these Edgeworth expansions fall short of being operational. For a consistent estimate, \( \hat{\Sigma} \), of \( \Sigma \), we are led to consideration of \( \hat{Z} = n^{1/2} \hat{\alpha}_U^{-1} \hat{\gamma}^\top (U - c\beta) \), where \( \hat{\alpha}_U^2 = \hat{\gamma}^\top \hat{\Sigma} \hat{\gamma} \). NR in fact proposed such a (jackknife) estimate \( \hat{\Sigma} \), and reported valid theoretical and empirical Edgeworth expansions for \( \hat{Z} \) in their Theorems 3 and 4. NR also derived a choice of \( h \) that is optimal in the sense of minimizing the maximal deviation of Edgeworth correction terms from the normal approximation, and proposed also a consistent estimate of the scale factor of this, leading to a feasible approximately optimal \( h \). NR also reported a Monte Carlo examination of their Edgeworth expansions, and of their bandwidth choice proposal. However, NR did not include the proofs of their Theorems 3 and 4, which entail additional regularity conditions and a considerable and lengthy development beyond that of their Theorems 1 and 2. By marked contrast with the routine application of Slutsky’s lemma which is all that is needed to deduce asymptotic normality of \( \hat{Z} \) from that of \( Z \), the Edgeworth expansions for \( \hat{Z} \) involve considerable extra work and actually differ from those for \( Z \). The present paper fills this gap, by providing the proofs of NR’s Theorems 3 and 4, while taking for granted the proofs of their Theorems 1 and 2. Callaert and Veraverbeke (1981), Helmers (1985, 1991) have established higher-order asymptotics for studentized versions of standard U-statistics. Though we follow their broad approach, our \( U \) is a U-statistic with an \( n \)-dependent "kernel" (through \( h \)) which significantly complicates matters, whereby we must also make substantial use of lemmas established by Robinson (1995) and NR.

The following section presents regularity conditions and theorem statements. Section 3 contains the main details of the proofs, with some detailed technical material left to appendices.

2. THEORETICAL AND EMPIRICAL EDGEWORTH EXPANSIONS
Our conditions below imply that $X$ has a probability density, $f(x)$, and existence of the conditional moments $g = E(Y|X)$, $q = E(Y^2|X)$, $r = E(Y^3|X)$, where, for a function $h : \mathbb{R}^d \to \mathbb{R}$, we write $h = h(X)$. For such a function, suitably smooth, we define $h' = (\partial/\partial X)h(X)$, $h'' = (\partial^2/\partial X^2)h(X)$, and $h''' = (\partial^3/\partial X^3)\text{vec}(h'(\cdot))$.

We define $e = fg$, $\mu = \mu(X, Y) = YE[f^E] - c$, $a = g^fx - E(g^fx)$, $\mu = E(\mu) = -E(g^fx)$ and $\Sigma = 4\text{Var}(\mu)$. We introduce the following assumptions.

(i) $E(Y^6) < \infty$.

(ii) $\Sigma$ is finite and positive definite.

(iii) The underlying measure of $(X^T, Y)$ can be written as $\mu_X \times \mu_Y$, where $\mu_X$ and $\mu_Y$ are Lebesgue measure on $\mathbb{R}^d$ and $\mathbb{R}$ respectively, and $(X^T_i, Y_i)$ are iid observations on $(X^T, Y)$.

(iv) $f$ is $(L+1)$ times differentiable, and $f$ and its first $(L+1)$ derivatives are bounded, for $2L > d+2$.

(v) $g$ is $(L+1)$ times differentiable, and $e$ and its first $(L+1)$ derivatives are bounded for $L \geq 1$.

(vi) $q$ is twice differentiable and $q', q'', g', g'', g'''$, $E(|Y|^3|X)f$ and $q^fx$ are bounded.

(vii) $f, g, f^E, q^f$ and $qf$ vanish on the boundaries of their convex (possibly infinite) supports.

(viii) $K(u)$ is even and differentiable,

$$\int_{\mathbb{R}^d} \left\{ (1 + \|u\|^2) \|K(u)\| + \|K'(u)\| \right\} du + \sup_{u \in \mathbb{R}^d} \|K'(u)\| < \infty,$$

and for the same $L$ as in (iv),

$$\int_{\mathbb{R}^d} u_1^{l_1} \cdots u_d^{l_d} K(u) du \begin{cases} 1, & \text{if } l_1 + \cdots + l_d = 0 \\ 0, & \text{if } 0 < l_1 + \cdots + l_d < L \\ \neq 0, & \text{if } l_1 + \cdots + l_d = L. \end{cases}$$

(ix) $\frac{(\log n)^9}{nh^{d+2}} + nh^{2L-2} \to 0$ as $n \to \infty$.

(x) $\sup_{\mathbb{V}} \limsup_{t \to \infty} |E \exp\{it\gamma_\mathbb{V} \mathbb{V}^T(\mu - \mu.)\}| < 1$.

These assumptions are the same as those of Theorem 1 of NR except (i) strengthens their third moment assumption to sixth moments, our treatment of studentization requiring finite third
moments of certain squared terms.

In our studentized statistic \(\hat{Z}\) we take

\[
\hat{\Sigma} = \frac{4}{(n-1)(n-2)^2} \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} (U_{ij} - U) \right\} \left\{ \sum_{k=1}^{n} (U_{ik} - U)^{2} \right\},
\]

(2.1)
a jackknife estimate of \(\Sigma\). We are concerned with approximating

\[
\hat{F}(z) = \Phi(z) - \varphi(z) \left[ n^{1/2}h^{-1}k_{1} - \frac{k_{2}}{nh^{d/2}} z - \frac{4}{3n^{1/2}} \left( 2z^{2} + 1 \right) k_{3} + 3 \left( z^{2} + 1 \right) k_{4} \right],
\]

where \(\Phi(z)\) and \(\varphi(z)\) are respectively the distribution function and density function of the standard normal, and, with,

\[
\Delta^{(i_{1}, \ldots, i_{d})} = \frac{\partial^{(i_{1} + \cdots + i_{d})}}{\partial x_{i_{1}} \cdots \partial x_{i_{d}}},
\]

\[
\begin{align*}
K_{1} &= \frac{2}{L} \sum_{i=1}^{L} \left\{ \prod_{i=1}^{d} u_{i}(u_{i}; a_{i} u_{i}) \right\} \left[ \Delta^{(i_{1}, \ldots, i_{d})} \nabla f \right] g,
\end{align*}
\]

\[
K_{2} = 2 \mathcal{C}_{\nu}^{2} \int \left\{ \nu^{\nu} K(u) \right\}^{2} du \left\{ (q - g^{2}) \right\},
\]

\[
K_{3} = \mathcal{C}_{\nu}^{3} E \left\{ \nu^{\nu} K(u) \left( \nu^{\nu} f^{2} \right)^{3} - 3 \left( q - g^{2} \right) \left( \nu^{\nu} f^{2} \right) \left( \nu^{\nu} a \right) - \left( \nu^{\nu} a \right)^{3} \right\},
\]

\[
K_{4} = -\mathcal{C}_{\nu}^{3} E \left\{ f(q - g^{2}) \left( \nu^{\nu} f^{2} \right) \left( \nu^{\nu} a \right) - f(\nu^{\nu} f^{2}) \left( \nu^{\nu} (g - 2g') \right) \left( \nu^{\nu} a \right) - f(q - g^{2}) \left( \nu^{\nu} a \right) \left( \nu^{\nu} f(\nu^{\nu} f^{2}) \right) + f(\nu^{\nu} g') \left( \nu^{\nu} a \right)^{2} \right\},
\]

THEOREM A: Under assumptions (i)-(x), as \(n \to \infty\)

\[
\sup_{\nu} \sup_{z} |\hat{F}(z) - F^{*}(z)| = o \left( n^{-1/2} + n^{-1} h^{-d/2} + n^{1/2} h^{-2} \right).
\]

The correction terms in \(F^{*}(z)\) are of the same orders as those in the unstudentized case (see Theorem 1 of NR), though their coefficients are different.
The $\kappa_i$ are unknown, but a feasible, empirical Edgeworth expansion is

$$F^*(z) = \Phi(z) - \phi(z) \left[ n^{1/2} h^{1/2} \tilde{\kappa}_1 - \frac{\tilde{\kappa}_2}{nh^{d+2} z} + \frac{4}{3n^{1/2}} (2z^2 + 1) \tilde{\kappa}_3 + 3 (z^2 + 1) \tilde{\kappa}_4 \right],$$

where

$$\tilde{\kappa}_1 = \frac{2 (-1)^j \partial^j}{L!} \sum_{a_1, \ldots, a_d = 1} \left\{ \int \prod_{i=1}^d K(u) \, du \right\} \frac{1}{n!} \sum_{i=1}^n \left\{ \Delta \left( l_1, \ldots, l_d \right) \right\}^{1/2} Y_i,$$

$$\tilde{\kappa}_2 = \partial_v^2 \left( \frac{n}{2} \right) \sum_{i=1}^n \sum_{j=1}^n h^{d+2} \tilde{W}_{ij}^2,$$

$$\tilde{\kappa}_3 = \frac{\partial_v^3}{n} \sum_{i=1}^n \tilde{V}_i^3,$$

$$\tilde{\kappa}_4 = \frac{\partial_v^4}{n (n-1)} \sum_{i=1}^n \sum_{j=1}^n \left( \nabla^2 U_{ij} \tilde{V}_i \tilde{V}_j \right),$$

where for positive $b$ and a function $H : \mathbb{R}^d \to \mathbb{R}$

$$F(X_i) = \frac{1}{(n-1) b^d} \sum_{j=1}^n H \left( \frac{X_j - X_i}{b} \right),$$

and

$$\tilde{U}_i = \frac{1}{n-1} \sum_{j \neq i} U_{ij}, \quad \tilde{V}_i = \nabla^2 (\tilde{U}_i - U), \quad \tilde{W}_{ij} = \nabla^2 (U_{ij} - \tilde{U}_i - \tilde{U}_j + U). \quad \text{(2.2)}$$

We impose the following additional assumptions, which are identical to those of Theorem 2 of NR.

(iv)’ $f$ is $(L+2)$ times differentiable, and $f$ and its first $(L+2)$ derivatives are bounded, where $2L > d + 2$.

(v)’ $g$ is $(L+2)$ times differentiable, and $e$ and its first $(L+2)$ derivatives are bounded.

(ix)’ $(\log n)^9 / nh^{d+3} + nh^{2L} \to 0$ as $n \to \infty$.

(xi)’ $H(u)$ is even and $(L+1)$ times differentiable,

$$\int_{\mathbb{R}^d} H(u) \, du = 1,$$

$$\int_{\mathbb{R}^d} \| \Delta \left( l_1, \ldots, l_d \right) \| \, du + \sup_{u \in \mathbb{R}^d} \| \Delta \left( l_1, \ldots, l_d \right) H'(u) \| < \infty$$

for any integers $l_1, \ldots, l_d$ satisfying $0 \leq l_1 + \cdots + l_d \leq L$ and $0 \leq l_i \leq L$.

(xii)’ $b = o$ and $\frac{(\log n)^2}{nb^{d+2+2L}} = O(1)$ as $n \to \infty$. 

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THEOREM B : Under assumptions (i)-(iii), (iv)', (v)', (vi) - (viii), (ix) and (x)-(xii),
\[
\sup_{v: \forall v = 1} \sup_{z} |F(z) - F^*(z)| = o(n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L) \quad a.s.
\]

3. PROOF OF THEOREMS A AND B

Proof of Theorem A.

In the sequel, C denotes a generic, finite, positive constant and the qualification "for sufficiently large \( n \)" may be omitted.

As is standard in U-statistic theory, we write
\[
n^{1/2} \sigma_\nu^{-1} \mathcal{V}^\top(U - \mu.) = \frac{2}{\sqrt{n}} \sum_{i=1}^{n} V_i + n^{1/2} \left( \frac{n}{2} \right)^{-1} \sum_{i=1}^{n} \sum_{j>i+1}^{n} W_{ij} + n^{1/2} \sigma_\nu^{-1} \mathcal{V}^\top(EU - \mu.) = \bar{V} + \bar{W} + \Delta. \quad (3.1)
\]

where \( U_i = E(U_{ij} | i) \), \( V_i = \sigma_\nu^{-1} \mathcal{V}^\top(U_i - EU) \), \( W_{ij} = \sigma_\nu^{-1} \mathcal{V}^\top(U_{ij} - EU) - V_i - V_j \), such that \( E(\cdot | i_1, \ldots, i_r) = E(\cdot | (X_{i_j}, Y_{i_j}), j = 1, \ldots, r) \). Writing \( S = 4 \text{Var}(U_i) \), \( s^2 = \sigma_\nu^{-2} \mathcal{V}^\top S^\top \mathcal{V} \), Taylor’s theorem gives
\[
\sigma_\nu \hat{\sigma}_\nu^{-1} = s^{-1} - \frac{s^{-3}}{2} (\hat{\sigma}_\nu^2 - s^2) + \frac{3}{4} \left( s^2 + \Theta (\hat{\sigma}_\nu^2 - s^2) \right)^{-5/2} (\hat{\sigma}_\nu^2 - s^2)^2
\]
\[
= s^{-1} + \tilde{R} + \tilde{R} \quad (3.2)
\]

for some \( \Theta \in [0, 1] \). Similarly to Callaert and Veraverbeke (1981), we expand \( \tilde{R} \) as follows. With \( \mathcal{V}_i = E(V_{ij} \mathcal{W}_{ij} | i) \), \( \mathcal{W}_{jk} = E(\mathcal{W}_{ij} \mathcal{W}_{jk} | j, k) \), we have
\[
\tilde{R} = T + Q + R, \quad T = T_1 + T_2 + T_3, \quad Q = Q_1 + Q_2, \quad R = R_1 + R_2 + R_3 + R_4 + R_5
\]
where
\[
T_1 = \frac{4 \delta n}{(n-2)^2} E(\mathcal{W}_{12}^2), \quad T_2 = \frac{\delta}{n} \sum_{i=1}^{n} \left( 4 V_i^2 s^2 + 8 V_i \right), \quad T_3 = 4 \delta \left( \frac{n-1}{2} \right)^{-1} \sum_{i<j}^{n} \mathcal{W}_{ij},
\]
\[
Q_1 = 4 \delta \left( \frac{n}{2} \right)^{-1} \sum_{i<j}^{n} \left( V_i + V_j \right) \mathcal{W}_{ij} - V_i - V_j, \quad Q_2 = -\frac{8 \delta}{n} \left( \frac{n-1}{2} \right)^{-1} \sum_{i=1}^{n} \sum_{k<m}^{n} \mathcal{W}_{km},
\]

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\[
R_1 = -4\delta \left( \frac{n}{2} \right)^{-1} \sum_{i<j} V_i V_j \ , \quad R_2 = \frac{4\delta}{n-2} \left( \frac{n-1}{2} \right)^{-1} \sum_{i=1}^{n} \sum_{k<m}^{(i)} (W_{ik} W_{im} - W_{km}) ,
\]
\[
R_3 = \frac{4\delta n}{(n-2)} \left( \frac{n}{2} \right)^{-1} \sum_{i<j} \{ W_{ij}^2 - \bar{W}_{1i}^2 - \bar{W}_{jj}^2 + E(W_{1i}^2) \} ,
\]
\[
R_4 = \frac{8\delta}{(n-2)} \sum_{i=1}^{n} \{ \bar{W}_{1i}^2 - E(W_{1i}^2) \} , \quad R_5 = -\frac{4\delta n (n-1)}{(n-2)^2} \left\{ \left( \frac{n-1}{2} \right)^{-1} \sum_{i<j} W_{ij}^2 \right\}^2 ,
\]
where \( \delta = -s^{-3}/2 \) and \( \sum_{k<m}^{(i)} \) denotes summation with respect to \( k \) and \( m \) for \( 1 \leq k < m \leq n \) excluding \( k=i \) and \( m=i \). Because
\[
\hat{Z} = (s^{-1} + R + R) (\bar{V} + \bar{W} + \Delta) ,
\]
by a standard inequality
\[
sup_x |\hat{F}(z) - F'(z)| \leq sup_x |\hat{F}(s^{-1} + T + Q) (\bar{V} + \bar{W}) + s^{-3} \Delta \leq z) - F'(z)|
\]
\[
+ P \left\{ |(R+R)(\bar{V} + \bar{W} + \Delta) + (T+Q)\Delta| \geq a_n \right\} + O(a_n) \quad (3.3)
\]
for \( a_n > 0 \), where here and subsequently we drop reference to \( sup \). Taking
\[
a_n = \frac{1}{\log n} \max (n^{-1/2}, n^{-1/2}h^{-d-2}, n^{1/2}h^{-1}) ,
\]
we bound the second term on the right of (3.3) by
\[
P \left\{ |(R+R)(\bar{V} + \bar{W} + \Delta)| \geq \frac{a_n}{2} \right\} + P \left\{ |(T+Q)\Delta| \geq \frac{n^{1/2}h}{21\log n} \right\}
\]
\[
\leq P \left\{ |R+R| \geq \frac{a_n}{2\log n} \right\} + P \left\{ |\bar{V} + \bar{W} + \Delta| \geq \log n \right\} + P \left\{ |(T+Q)\Delta| \geq \frac{n^{1/2}h}{21\log n} \right\} \quad (3.4)
\]
The first term in (3.4) is, by an elementary inequality, bounded by
\[
P \left\{ |R| \geq \frac{a_n}{4\log n} \right\} + P \left\{ \frac{|\bar{V}|}{R} \geq C_0 \right\} + P \left\{ \frac{v}{R} \geq \frac{a_n}{4C_0\log n} \right\} \quad (3.5)
\]
for a constant \( C_0 \) determined later. The third term of (3.5) is bounded by
\[
P \left\{ T_2^2 \geq \frac{a_n}{12C_0\log n} \right\} + P \left\{ |T_1 + T_3| \geq \frac{a_n}{12C_0\log n} \right\} + P \left\{ |Q + R|^2 \geq \frac{a_n}{12C_0\log n} \right\}
\]
\[
= (a) + (b) + (c) .
\]
Lemmas 10-19 and Markov’s inequality give, for \( \zeta > 0 \),

\[
(a) \quad \frac{E|T_2|^2 (1+\zeta)}{\frac{a_n}{12\zeta \log n}} \leq \frac{Cn^{-1+\zeta} (\log n)^2 (1+\zeta)}{n^{-\frac{1}{2}} (1+\zeta)} = o\left(n^{-\frac{1}{2}}\right),
\]

\[
(b) \quad \frac{E|T_1+T_3|^2 (1+\zeta)}{\frac{a_n}{12\zeta \log n}} \leq \frac{C (n^{-1}h^{-d-2}) (1+\zeta)}{(n^{-1}h^{-d-2}) (1+\zeta)} = o\left(n^{-1}h^{-d-2}\right),
\]

\[
(c) \quad \frac{E|R+Q|^2}{\frac{a_n}{12\zeta \log n}} \leq \frac{Cn^{-2}h^{-d-2} (\log n)^2}{n^{-1/2}} = o\left(n^{-1}h^{-d-2}\right),
\]

where \( \zeta = \frac{2}{7} \) suffices in (b), and \( \zeta \) arbitrarily small suffices in (a).

The first term of (3.5) is, using Markov’s inequality, (ix) and Lemmas 15-19, bounded by

\[
\frac{16 E(R^2) (\log n)^2}{a_n^2} \leq C (n^{-1} + n^{-2}h^{-2-\delta}) (\log n)^4 = o\left(n^{-1/2} + n^{-1}h^{-d-2}\right).
\]

Now, in view of (3.2), \( R = 3s(1-2\Theta sR)^{-5/2}R \) so that because \( R \geq 0 \) and \( 0 \leq \Theta \leq 1 \),

\[
P \left( \frac{R}{R} \geq C_0 \right) = P \left( 3s(1-2\Theta sR)^{-5/2} \geq C_0 \right) \leq P \left( \frac{|R|}{2s} \geq \left( \frac{3s}{C_0} \right)^{2/5} \right). \tag{3.6}
\]

Taylor’s expansion of \( s^\zeta \) around \( s^2 = 1 \) and Lemma 2 of Robinson(1995) give for integer \( \tau \),

\[
s^\zeta = 1 + O\left( \zeta^2 s \right) (S-\Sigma) s^\tau = 1 + O\left( h^\tau \right) \tag{3.7}
\]

so that we can choose \( C_0 \) such that \( C_0 > 3s \) for sufficiently large \( n \) by (ii). Then by (3.7) and

Markov’s inequality, (3.6) is bounded by a constant times \( E|T+Q+R|^3 \)

\( = O\left( n^{-3/2} + n^{-3}h^{-3d-6} \right) \) from Lemmas 10-19, so that the second term of (3.5) is \( O\left( n^{-3/2} + n^{-3}h^{-3d-6} \right) \). Therefore,

\[
P \left( \frac{|R+R|}{a_n} \geq \frac{a_n}{\log n} \right) = o\left(n^{-2} + n^{-1}h^{-d-2}\right). \tag{3.8}
\]

Put \( F(z) = P\left[ n^{1/2} \Theta^1 (U - \mu) \leq z \right] \). Then

\[
P (|\tilde{V} + \tilde{W} + \Delta| \geq \log n) = 1 - F(\log n) + F(-\log n). \tag{3.9}
\]

10
NR proved in Theorem 1 that

\[ \sup_z | F(z) - \tilde{F}(z) | = o (n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L) \]

where

\[ \tilde{F}(z) = \Phi(z) - \varphi(z) \left\{ n^{1/2}h^Lk_1 + \frac{k_2}{nh^{d/2}} z + \frac{4(k_3 + 3k_4)}{3n^{1/2}} (z^2 - 1) \right\} \]  \hspace{1cm} (3.10)

which implies that for any \( z \)

\[ 1 - F(z) + F(-z) = 1 - \tilde{F}(z) + \tilde{F}(-z) + o (n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L) . \]  \hspace{1cm} (3.11)

Now by (3.10),

\[ 1 - \tilde{F}(z) + \tilde{F}(-z) = 1 - \Phi(z) + \Phi(-z) + \varphi(z) \frac{2\kappa_2}{nh^{d/2}} z = 2 - 2\Phi(z) + \varphi(z) \frac{2\kappa_2}{nh^{d/2}} z . \]  \hspace{1cm} (3.12)

Substituting (3.12) into (3.11) and putting \( z = \log n \), because \( 1 - \Phi(\log n) = o (n^{-1/2}) \) and \( \varphi(\log n) \log n = o (n^{-1/2}) \), we have

\[ 1 - F(\log n) + F(-\log n) = o (n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L) . \]  \hspace{1cm} (3.13)

By (3.9) and (3.13),

\[ P \left( | \tilde{V} + \tilde{W} + A | \geq \log n \right) = o (n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L) . \]  \hspace{1cm} (3.14)

Finally, Markov's inequality, (ix), Lemma 1 of Robinson (1995), and Lemmas 10-14 bound the last term of (3.4) by

\[ \frac{\Delta^2 E \left( T + Q \right)^2 (2\log n)^2}{nh^{2L}} \leq C \left( n^{-1} + n^{-2}h^{-2d-4} \right) \left( \log n \right)^2 = o \left( n^{-1/2} + n^{-1}h^{-d-2} \right) . \]  \hspace{1cm} (3.15)

Substituting (3.8), (3.14) and (3.15) into (3.4),

\[ P \left( \left| (R + R) \left( \tilde{V} + \tilde{W} + \Delta \right) + (T + Q) \Delta \right| \geq a_n \right) = o \left( n^{-1/2} + n^{-1}h^{-d-2} + n^{1/2}h^L \right) . \]  \hspace{1cm} (3.16)

To deal with the first term on the right of (3.3), write \( b_2 = s^{-1}\tilde{V} \), \( b_3 = s^{-1}\tilde{W} \), \( \tilde{b}_2 = (T + Q) \tilde{V} \), \( \tilde{b}_3 = (T + Q) \tilde{W} \), \( b_4 = b_2 + b_3 \), \( \tilde{b}_4 = \tilde{b}_2 + \tilde{b}_3 \), \( B = b_4 + \tilde{b}_4 \), and define

\[ \chi'(t) = \int e^{itz} dF'(z) \]

\[ = e^{-\frac{s^2t^2}{2}} \left[ 1 + \left\{ n^{1/2}h^Lk_1 - \frac{4(k_3 + 2k_4)}{n^{1/2}} \right\} (it) \right] \]
Esseen’s smoothing lemma gives for \( N_0 = \log \min (n^{1/2}, nh^{d/2}) \), \( n = (E|s^{-1}V_1|^3)^{-1} \)

\[
\sup_z \left| \left( s^{-1} + T + Q \right) \left( \tilde{V} + \tilde{W} \right) + s^{-1} \Delta \right| - F^*(z) \right| \\
\leq \int_{-N_0}^{N_0} \frac{Ee^{i t (B + s^{-1} \Delta)}}{t} \left| \chi^*(t) \right| dt + O \left( N_0^{-1} \right),
\]

which, for \( p = \min (\log n, \varepsilon n^{1/2}, nh^{d/2}) \), is bounded by

\[
\int_{-p}^{p} \frac{Ee^{i t (B + s^{-1} \Delta)}}{t} \left| \chi^*(t) \right| dt + \int_{p}^{t} \left| \chi^*(t) \right| dt + o \left( n^{-1/2} + n^{-1}h^{-d-2} \right)
\]

\[
= (I) + (II) + (III) + o \left( n^{-1/2} + n^{-1}h^{-d-2} \right).
\]

Here we can set \( \varepsilon \in (0, \eta] \) for sufficiently large \( n \) as discussed in the proof of the Theorem of Robinson (1995) using (3.7) also so that \( p \leq N_0 \). We first mention an inequality frequently used hereafter:

\[
\left| e^{ix} - 1 - ix - \cdots - \frac{(ix)^{k-1}}{(k-1)!} \right| \leq \frac{|x|^k}{k!} \quad \text{for integer } k \text{ and real } x. \tag{3.17}
\]

**Estimation of (I)**

Since \( s^{-1} \Delta \) is nonstochastic,

\[
E \left\{ e^{it(B + s^{-1} \Delta)} \right\} = e^{its^{-1} \Delta} E \left( e^{itB} \right), \tag{3.18}
\]

where (3.7) and (3.18) yield

\[
e^{its^{-1} \Delta} = 1 + it \Delta + O \left( t^2 \Delta^2 + |t| h L \Delta \right). \tag{3.19}
\]

Writing \( \tilde{B}_2 = \tilde{B}_2' + \tilde{B}_2'' \) where \( \tilde{B}_2' = T \tilde{V} \) and \( \tilde{B}_2'' = Q \tilde{V} \), and applying (3.17) repeatedly, we have

\[
E(e^{itb}) = E(e^{itb_1}) + E(e^{itB} - e^{itb_1})
\]
\[
E(e^{itb_1}) + \left\{ Ee^{it(b_1+\mathbf{E}_b \cdot \mathbf{E}_b)} - Ee^{itb_1} \right\} + \left\{ Ee^{itb_2} - E(e^{itb_1}) \right\} \\
= E(e^{itb_1}) + O(|t|E|\mathbf{E}_b|^2 + \mathbf{E}_b|) + \left\{ Ee^{it(b_1+\mathbf{E}_b')} - E(e^{itb_1}) - itE(\mathbf{E}_b'e^{itb_1}) \right\} \\
\quad + itE(\mathbf{E}_b'e^{itb_1} - \mathbf{E}_b'e^{itb_2}) + itE(\mathbf{E}_b'e^{itb_2}) \\
= E(e^{itb_1}) + itE(\mathbf{E}_b'e^{itb_2}) + O\left( |t|E|\mathbf{E}_b'^2 + \mathbf{E}_b'| + t^2(E|\mathbf{E}_b'^2 + E|\mathbf{E}_b'b_3|) \right). \\
(3.20)
\]

Write
\[
E(e^{itb_1}) = E\left[e^{itb_1} + \frac{(it)^2}{2} b_3^2 \right] + O\left( |t|^3 E|b_3|^3 \right), \\
(3.21)
\]
and put \( \gamma(t) = E(e^{\frac{it}{\sqrt{n}} V_i}) \). As in Appendix A of NR,
\[
E(e^{itb_2}) = \{ \gamma(t) \}^n, \\
(3.22)
\]
\[
E(b_3 e^{itb_1}) = \{ \gamma(t) \}^{n-2} \left[ \frac{4(it)^2}{n^{1/2}} E(W_{12} V_1 V_2) \\
\quad + O\left( \frac{t^2 h}{n^{1/2}} + (\frac{t^4}{n^{3/2}} + \frac{|t|^3}{n}) h^{-2-d-1} \right) \right], \\
(3.23)
\]
\[
E(b_3^2 e^{itb_1}) = \frac{2}{n-1} \{ \gamma(t) \}^{n-2} \left[ E(W_{12}^2) + O\left( \frac{h L}{nh^{d-2}} + |t|n^{-1/2} h^{-\frac{4}{3}d-2} \right) \\
\quad + \{ \gamma(t) \}^{n-3} O\left( |t|n^{-3/2} h^{-\frac{4}{3}d-2} \right) \\
\quad + \{ \gamma(t) \}^{n-4} O\left( t^4 n^{-1} + t^8 n^{-3} h^{-\frac{4}{3}d-2} + t^6 n^{-2} h^{-\frac{4}{3}d-2} \right) \right]. \\
(3.24)
\]

Since, for \( m=0,1,2,3, \)
\[
\{ \gamma(t) \}^{n-m} = e^{-\frac{it}{\sqrt{n}}} \left\{ 1 + \frac{E(V_1)^3}{6n^{1/2} s^3} (it)^3 \right\} + O\left( n^{-1/2} (|t|^3 + t^6) e^{-\frac{4}{3}} \right), \\
(3.25)
\]
by Lemma 1 of Robinson (1995), Appendix B-(a) and (3.18)-(3.25),
\[
E(e^{it(B + s^{-1} \Delta)}) = \left\{ 1 + it \Delta + O\left( t^2 nh^{2L} + |t|n^{1/2} h^{2L} \right) \right\} \\
\times \left\{ e^{-\frac{it}{\sqrt{n}}} \left\{ 1 + \frac{4E(V_1)^3}{3n^{1/2} s^3} (it)^3 \right\} + O\left( n^{-1/2} (|t|^3 + t^6) e^{-\frac{4}{3}} \right) \right\} \\
\times 1 + \frac{4(it)^3}{n^{1/2}} E(W_{12} V_1 V_2) + \frac{(it)^2}{n} E(W_{12}^2) - \frac{2(it)^2}{n} E(W_{12}^2)
\]
\[
- \frac{(it)^3 + (it)}{n^{1/2}} \left\{ 4E(V_1^3) + 8E(W_{12} V_1 V_2) \right\} + O(A_n) \] \\
\left( t |E| \tilde{B}_2'' + \tilde{B}_3 | + t^2 (E| \tilde{B}_2' |^2 + E| \tilde{B}_2 B_3 |) \right) \right) \}
\]

where

\[
A_n = \frac{|t|^3 h^L}{n^{1/2}} + \frac{|t|^5}{n^{3/2}} + \frac{t^4}{n} \frac{h^{-2d+1}}{n^{2h^{d^2}}} + \frac{t^2}{n^{2h^{d^2}}} + \frac{t^2 h^L}{n^{2h^{d^2}}} + \frac{|t|^3}{n^{3/2} h^{d^2}} + \frac{t}{n}
\]

\[
+ \frac{t^{10}}{n^3 h^{5d^2}} + \frac{t^8}{n^2 h^{4d^2}} + \frac{|t|^3}{(nh^{d^2})^{3/2}} \frac{2}{n^{3/2} h^{d^2}} + \frac{t^2}{n^{2h^{d^2}}} + \frac{|t|^3}{n^2 h^{d^2}} + \frac{|t|^7}{n^3 h^{d^2}} + \frac{t^6}{n^{5/2 h^{d^2}}} + \frac{|t|^3 + t^4}{n^2 h^{d^2}}
\]

\[
= O \left( \frac{t^2 + t^{10}}{nh^{d^2}} + \frac{t^2 + t^6}{n^{1/2}} \right).
\]

Expanding (3.26), we have

\[
E(e^{it(B + s^{-1}A)}) = e^{-\frac{t^2}{2}} \left[ \frac{4E(V_1^3) + 8E(W_{12} V_1 V_2)}{3n^{1/2}} (it) - \frac{E(W_{12}^2)}{n} (it)^2 \right]
\]

\[
4 \left( \frac{2E(V_1^3) + 3E(W_{12} V_1 V_2)}{3n^{1/2}} (it)^3 \right) + D_n,
\]

where

\[
D_n = O \left\{ \left\{ e^{-\frac{t^2}{2}} \frac{|t|^3}{n^{1/2}} + o(n^{-1/2} t^6 + |t|^3) e^{-\frac{t^2}{4}} \right\} \left\{ \frac{t^2}{n^{2h^{d^2}}} + \frac{|t|^3}{n^{1/2}} + A_n \right\} \right\}
\]
\[
+ e^{-\frac{t^2}{2}} \left( \frac{t}{n^{1/2}} h^L + t^2 h^{2L} + |t| n^{1/2} h^{2L} \right) \left( \frac{t^2}{n^{2h^{d^2}}} + \frac{|t|^3}{n^{1/2}} + A_n \right) + o(n^{-1/2} t^6 + |t|^3) e^{-\frac{t^2}{4}} \right\}
\]

\[
+ \left( \frac{|t|^3}{n^{1/2}} + t^2 nh^{2L} \right) e^{-\frac{t^2}{2}} \left( \frac{|t|^3}{n^{1/2}} + o(n^{-1/2} t^6 + |t|^3) e^{-\frac{t^2}{4}} \right)
\]

\[
+ \left( \frac{|t|^3}{n^{1/2}} + t^2 nh^{2L} \right) e^{-\frac{t^2}{2}} \left( \frac{|t|^3}{n^{1/2}} + o(n^{-1/2} t^6 + |t|^3) e^{-\frac{t^2}{4}} \right)
\]
\[ + (|t| + t^2 n^{1/2} h^L + |t|^3 n h^{2L}) E|\tilde{B}_2'' + \tilde{B}_3| \]
\[ + (t^2 + |t|^3 n^{1/2} h^L + t^4 n h^{2L}) (E|\tilde{B}_2'|^2 + E|\tilde{B}_2'B_3|) \]  
(3.28)

By Hölder's inequality, equation (14) of Robinson (1995) and Lemmas 9-14,
\[ E|\tilde{B}_2'| = E|Q \tilde{V}| \leq (E|Q|^2 E|\tilde{V}|^2)^{1/2} = O\left(n^{-1} h^{-\frac{d-2}{2}}\right) \]  
(3.29)
\[ E|\tilde{B}_3| = E|T + Q| \tilde{W}| \leq (E|T + Q|^2 E|\tilde{W}|^2)^{1/2} \]
\[ = O\left((n^{-1/2} + n^{-1} h^{-d-2}) (n^{-1} h^{-d-2})^{1/2}\right). \]  
(3.30)

Writing \( E|\tilde{B}_2'|^2 \leq C \left(|T_1|^2 E|\tilde{V}|^2 + E|T_2\tilde{V}|^2 + E|T_3\tilde{V}|^2 \right), \) Lemmas 9, 10, 12 and Hölder's inequality give
\[ |T_1|^2 E|\tilde{V}|^2 + E|T_3\tilde{V}|^2 \leq |T_1|^2 E|\tilde{V}|^2 + (E|T_3|^4 E|\tilde{V}|^4)^{1/2} = O\left(n^{-2} h^{-2d-4}\right), \] and (3.7), (i), (iii), Lemma 1-(d) of NR and (A.1) give
\[ E|T_2\tilde{V}|^2 \leq \frac{C}{n^3} E \left| \sum_{i=1}^{n} (4V_i^2 - s^2 + 8V_i) \sum_{j=1}^{n} V_j \right|^2 \]
\[ = \frac{C}{n^3} \left\{ n E| (4V_i^2 - s^2 + 8V_i) V_i |^2 + n (n-1) E| (4V_i^2 - s^2 + 8V_i) V_i^2 \right\} \]
\[ = O\left(n^{-1}\right). \]  
Thus
\[ E|\tilde{B}_2'|^2 = E|T \tilde{V}|^2 = O\left(n^{-1} + n^{-2} h^{-2d-4}\right). \]  
(3.31)

Hölder's inequality, (3.31) and equation (14) of Robinson (1995) yield
\[ E|\tilde{B}_2'B_3| = (E|\tilde{B}_2'|^2 E|B_3|^2)^{1/2} = O\left((n^{-1/2} + n^{-1} h^{-d-2}) (n^{-1} h^{-d-2})^{1/2}\right). \]  
(3.32)

It is straightforward due to (C.1) of NR that \( E(V_i) = E(v_i) + o(1) \) and \( E(W_i V_i V_i) \)
\[ = E(W_i v_i v_i) + o(1) \] where \( v_i = \sigma_i^{-1} \nu_i \{ \mu(X_i, Y_i) - \mu. \} \). Therefore, using (3.27)-(3.32) and Lemmas 11-13 of NR,
\[ (I) \leq \int_{-\log n}^{\log n} \frac{E e^{it(B + s^\Lambda(t))} - \tilde{\chi}(t)}{t} dt = o\left(n^{-1/2} + n^{-1} h^{-d-2} + n^{1/2} h^L\right). \]

Estimation of (II)

Put \( \tilde{B}_3' = T \tilde{W}, \tilde{B}_3'' = Q \tilde{W} \), then noting that \( \tilde{B}_3 = \tilde{B}_3' + \tilde{B}_3'' \) and \( B = b_1 + \tilde{B}_2 + \)
\[ (\tilde{E}_3' + \tilde{E}_3'' ) \]

, we have, using (3.17),

\[
| \mathbb{E} e^{it\theta} | \leq | \mathbb{E} e^{it\theta} e^{-it(b_i+\tilde{E}_3')} - it\mathbb{E} \tilde{E}_3'' e^{it(b_i+\tilde{E}_3')} | + | \mathbb{E} e^{it(b_i+\tilde{E}_3') |} \\
+ | t | \mathbb{E} \tilde{E}_3'' e^{it(b_i+\tilde{E}_3')} | \\
\leq | t |^2 \mathbb{E} | \tilde{E}_3'' |^2 + | \mathbb{E} e^{it(b_i+\tilde{E}_3')} | + | t | \mathbb{E} \tilde{E}_3'' e^{it(b_i+\tilde{E}_3')} | .
\] (3.33)

Writing \( E|b_i| \leq C \) (\( E|Q_1\tilde{W}|^2 + E|Q_2\tilde{W}|^2 \)) , Hölder's inequality, equation (14) of Robinson (1995) and Lemma 14 give

\[
E|Q_2\tilde{W}|^2 \leq (E|Q_2|^4)^{1/2} (E|\tilde{W}|^4)^{1/2} = O\left((n^{-4}h^{-3d-4})^{1/2}n^{-1}h^{-d-2}\right)
\] (3.34)

and

\[
E|Q_1\tilde{W}|^2 \leq \frac{C}{n^7} E| \sum_{i<j} \{ (V_i+V_j) \ W_{ij} - \tilde{V}_i - \tilde{V}_j \} \sum_k \sum_{k<i} \mathbb{W}_{k1} |^2 \\
\leq \frac{C}{n^7} E| \sum_{i<j<k<i} \sum_j \{ (V_i+V_j) \ W_{ij} - \tilde{V}_i - \tilde{V}_j \} \mathbb{W}_{k1} |^2 \\
+ \frac{C}{n^7} E| \sum_{i<j} \sum_k \sum_{k<i} \{ (V_i+V_j) \ W_{ij} - \tilde{V}_i - \tilde{V}_j \} \mathbb{W}_{k1} |^2 \\
+ \frac{C}{n^7} E| \sum_{i<j} \sum_k \sum_{k<i} \{ (V_i+V_j) \ W_{ij} - \tilde{V}_i - \tilde{V}_j \} \mathbb{W}_{k1} |^2 , \\
\leq \frac{C}{n^7} \sum_{i<j<k<i} \sum_j E| \{ (V_i+V_j) \ W_{ij} - \tilde{V}_i - \tilde{V}_j \} \mathbb{W}_{k1} |^2 \\
+ \frac{C}{n^7} \sum_{i<j<k<i} \sum_j n^2 E| \{ (V_i+V_j) \ W_{ij} - \tilde{V}_i - \tilde{V}_j \} \mathbb{W}_{k1} |^2 \\
+ \frac{C}{n^7} n^4 E| \{ (V_i+V_j) \ W_{ij} - \tilde{V}_i - \tilde{V}_j \} \mathbb{W}_{k1} |^2 ,
\] (3.35)

where the third inequality uses the Theorem of Dharmadhikari et.al.(1968; abbreviated to DFJ hereafter), and the equality uses nested conditional expectation, Lemmas 1-(d), 4 of NR, Lemma 4 of Robinson (1995) and Lemma 2. Therefore by (3.34) and (3.35),

\[
E|\tilde{E}_3'|^2 = E|Q \tilde{W}|^2 = O\left(n^{-3}h^{-3d-4}\right)
\] (3.36)

To investigate the second term of (3.33), let

\[
d_i = (4V_i^2 - s^2) + 8V_i , \ e_{ij} = 4\left\{ (V_i+V_j) W_{ij} - \tilde{V}_i - \tilde{V}_j + \frac{n}{(n-2)} \tilde{V}_i \right\}
\] (3.37)
then

\[ \tilde{\beta}_2 = -\frac{s^{-3}}{2} \left\{ \frac{4n}{(n-2)^2} E(W_{12}^2) + \frac{1}{n} \sum_{i=1}^n d_i + \left( \frac{n}{2} \right)^{-1} \sum_{j=1}^{n-1} \sum_{j'=1}^{n} e_{ij} \right\} - 4 \left( \frac{n-1}{2} \right)^{-1} \sum_{i=1}^n \sum_{k=1}^{n-1(i)} \sum_{l=1}^{n-1(i)} V_{i}W_{kl} \left( \tilde{V} \right), \]

\[ \tilde{\beta}_3' = -\frac{s^{-3}}{2} \left\{ \frac{4n}{(n-2)^2} E(W_{12}^2) + \frac{1}{n} \sum_{i=1}^n d_i + \left( \frac{n}{2} \right)^{-1} \sum_{j=1}^{n-1} \sum_{k=1}^{n} 4W_{jk} \left( \tilde{W} \right) \right\}. \]

Define

\[ b_{3m} = s^{-1} n^{-1} \left( \frac{n}{2} \right)^{-1} \sum_{i=1}^m \sum_{j=1}^n W_{ij}, \]

\[ \tilde{\beta}_{2m} = -\frac{s^{-3}}{2} \left\{ \frac{8n^{1/2}}{(n-2)^2} E(W_{12}^2) \sum_{i=1}^m V_{i} + \frac{2}{n^{3/2}} \left( \sum_{i=1}^m \sum_{j=1}^n d_i V_{ij} + \sum_{i=1}^m \sum_{s=1}^n \sum_{m=1}^n \sum_{l=1}^n e_{ij} V_{s} \right) \right\} + \frac{2}{n^{1/2}} \left( \frac{n-1}{2} \right)^{-1} \left( \sum_{i=1}^m \sum_{j=1}^n \sum_{s=1}^n e_{ij} V_{s} + \sum_{i=1}^m \sum_{j=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{l=1}^n e_{ij} V_{s} \right) \]

\[ - \frac{8}{n^{3/2}} \left( \frac{n-1}{2} \right)^{-1} \left( \sum_{i=1}^m \sum_{j=1}^n \sum_{s=1}^n e_{ij} W_{kl} V_{s} + \sum_{i=1}^m \sum_{j=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{l=1}^n e_{ij} W_{kl} V_{s} \right) \]

\[ + \sum_{i=1}^m \sum_{j=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{l=1}^n e_{ij} W_{kl} V_{s} \left( \tilde{V} \right) \left( \tilde{W} \right) \right\}, \]

\[ \tilde{\beta}_{3m}' = -\frac{s^{-3}}{2} \left\{ \frac{4n^{3/2}}{(n-2)^2} E(W_{12}^2) \left( \frac{n}{2} \right)^{-1} \sum_{i=1}^m \sum_{j=1}^n W_{ij} \right\} + \frac{1}{\sqrt{n}} \left( \frac{n}{2} \right)^{-1} \left( \sum_{i=1}^m \sum_{l=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{l=1}^n d_i W_{ls} \right) \]

\[ + n^{1/2} \left( \frac{n}{2} \right)^{-2} \left( \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \sum_{s=1}^n 4W_{jk} W_{ls} + \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \sum_{s=1}^n 4W_{jk} \right) \]
for $m=1,\ldots,n-1$. Note that $b_1-b_{3m}$, $\tilde{B}_2-\tilde{B}_2$ and $\tilde{B}_3-\tilde{B}_3$ are independent of $(X_1^T, Y_1), \ldots, (X_m^T, Y_m)$. Putting $\bar{B}_m = (b_1-b_{3m}) + (\tilde{B}_2-\tilde{B}_2) + (\tilde{B}_3-\tilde{B}_3)$, and using (3.17) repeatedly, we have

$$|Ee^{it(b_1+\tilde{B}_2+\tilde{B}_2)}| \leq \frac{t^2}{2}E|\tilde{B}_{3m}|^2 + |Ee^{it(\tilde{B}_2+\tilde{B}_3)}| + |Ee^{it(\tilde{B}_2+\tilde{B}_3)}|$$

$$\leq \frac{t^2}{2}E|\tilde{B}_{3m}|^2 + \left[ \frac{|t|^3}{6}E|b_{3m}|^3 + |Ee^{it\tilde{B}_2}(1 + itb_{3m} + \frac{(it)^2}{2}b_{3m})| \right]$$

$$+ \left[ t^2E|b_{3m}||\tilde{B}_2 + \tilde{B}_3| + |t||Ee^{it\tilde{B}_2}(\tilde{B}_2 + \tilde{B}_3)| \right]$$

$$= \left[ \frac{t^2}{2}E|\tilde{B}_{3m}|^2 + t^2E|b_{3m}||\tilde{B}_2 + \tilde{B}_3| \right]$$

$$+ \left[ \frac{|t|^3}{6}E|b_{3m}|^3 + |Ee^{it\tilde{B}_2}(1 + itb_{3m} + \frac{(it)^2}{2}b_{3m})| \right]$$

$$+ \left[ |t||Ee^{it\tilde{B}_2}(\tilde{B}_2 + \tilde{B}_3)| \right]. \quad (3.38)$$

By elementary inequalities, (ix), Appendix B-(d), (e) and equation (14) of Robinson (1995), the first bracketed term on the right of (3.38) is bounded by

$$Ct^2\left\{ E|\tilde{B}_{3m}|^2 + E|\tilde{B}_3|^4 + \left( E|b_{3m}|^2 \right)^{1/2}E|\tilde{B}_{3m}|^2 + E|\tilde{B}_3|^4 \right\}^{1/2}$$

$$\leq Cm\left\{ \frac{1}{n^4h^{2d+4}} + \frac{1}{n^2} + \frac{1}{n^6h^{3d+6}} \right\}$$

$$+ \frac{1}{(n^2h^{d+4})^{1/2}} \left( \frac{1}{n^2h^{2d+4}}^{1/2} + \frac{1}{n^4h^{3d+6}}^{1/2} \right)$$

$$\leq Cm\left\{ \frac{1}{n^2h^{d+2}} + \frac{1}{n^2h^{d+2}} \right\}. \quad (3.39)$$

The second bracketed term on the right of (3.38) is bounded by

$$C\left\{ \frac{|t|^3}{n^2h^{d+2}} \right\}^{3/2} + \left\{ \frac{1}{n^{1/2}h} + \frac{m|t|}{n^2h^{d+2}} + \frac{t^2m^2}{nh^2} \right\}$$

$$\left\{ V(t) \right\}^{m-4}, \quad (3.40)$$

which is verified as in equations (13)-(19) of Robinson (1995), because $s^{-1}$ is bounded due to (3.7) and $\tilde{B}_m$ is the sum of $\frac{1}{\sqrt{m}}\sum_{i=1}^{m} V\, \tilde{b}_i$ and $(\tilde{B}_2-\tilde{B}_2) + (\tilde{B}_3-\tilde{B}_3)$, the latter being
independent of \((X_1^T, Y_1), \ldots, (X_m^T, Y_m)\). Appendix B-(b), (c) bound the last term in (3.38) by

\[
\frac{C m |t|}{n^{1/2} h^2} |\gamma(t)|^{m-4}.
\]

(3.41)

Now we investigate the third term on the right of (3.33). Using elementary inequalities, (3.17), (3.36), equation (14) of Robinson (1995), Appendix B-(d), (e), (f),

\[
|E \tilde{E}_3^{iv} e^{it(b_1 + \tilde{E}_2 + \tilde{B}_3)}|
\]

\[
\leq |E \tilde{E}_3^{iv} e^{it(b_1 + \tilde{E}_2 + \tilde{B}_3)} - E \tilde{E}_3^{iv} e^{it(b_2 + b_3 + \tilde{E}_2 + \tilde{B}_2 + \tilde{E}_3 + \tilde{B}_3)}| + |E \tilde{E}_3^{iv} e^{it\tilde{B}_3}|
\]

\[
\leq C |t| \left( (E|B_3^{iv}|^2)^{1/2} + (E|\tilde{B}_2|^{1/2})^2 + (E|\tilde{B}_3|^{1/2})^2 \right) + |E \tilde{E}_3^{iv} e^{it\tilde{B}_3}|
\]

\[
\leq C \frac{|t| h}{(nh^{d+2})^{3/2}} \left( \frac{m}{n^2 h^{d+2}} \right)^{1/2} + \left( \frac{m}{n^3 h^{d+4}} \right)^{1/2} + \left( \frac{m}{n^4 h^{d+6}} \right)^{1/2}
\]

\[
+ \frac{C n^{1/2}}{h^2} |\gamma(t)|^{m-5}.
\]

(3.42)

Therefore, by (3.33), (3.36), (3.38)-(3.42),

\[
|E e^{it\tilde{B}}| \leq \frac{C t^2}{n^3 h^{3d+4}} + C m t^2 \left( \frac{1}{n^2 h^{d+2}} + \frac{1}{n^3 h^{d+4}} \right)
\]

\[
+ C \frac{|t|}{n^{1/2} h^3} \left( \frac{m}{n^2 h^{d+2}} \right)^{3/2} + \left( 1 + \frac{m|t|}{n^{1/2} h^2} + \frac{m|t|}{n^{2} h^{d+2}} + \frac{t^2 m^2}{n^2 h^2} \right) |\gamma(t)|^{m-4}
\]

\[
+ \frac{C m|t|}{n^{1/2} h^3} |\gamma(t)|^{m-4}
\]

\[
+ \frac{C h m^{1/2} t^2}{n^{1/2} (nh^{d+2})^2} + \frac{C n^{1/2} |t|}{h^2} |\gamma(t)|^{m-5}.
\]

(3.43)

Now divide (3.43) by \(|t|\) and integrate over \(p \leq |t| \leq N_0\), where we partition the range of integration into two parts, \(p \leq |t| \leq N_1\) and \(N_1 \leq |t| \leq N_0\), for \(N_1 = \min(\lceil n^{1/2}, nh^{d+2} \rceil)\).

(i) \(p \leq |t| \leq N_1\)

We can choose \(m = \lceil 9n\log n / t^2 \rceil\) to satisfy \(1 \leq m \leq n-1\) for large \(n\). For this \(m\), since \(E(2V_1/s) = 0\) and \(Var(2V_1/s) = 1\),

\[
|\gamma(t)|^{m-4} \leq \exp(-\frac{m-4}{3n} t^2) \leq C \exp(-3\log n) = \frac{C}{n^3}.
\]

(3.44)
By (3.43), (3.44) and (ix), we obtain

\[
\int_{x < |t| < N_0} \frac{E e^{itB}}{t} |dt| \\
\leq \frac{C}{n^3 h^{3d+4}} \int_{x < |t| < nh^d} |t| |dt| + C \left( \frac{n \log n}{n^2 h^{d+2}} + \frac{n \log n}{n^2 h^{d+4}} \right) \int_{x < |t| < \eta n^{1/2}} |t| |dt| \\
+ C \left( \frac{n \log n}{n^2 h^{d+2}} \right)^{3/2} \int_{x < |t| < \eta n^{1/2}} |t| |dt| \\
+ C \int_{x < |t| < \eta n^{1/2}} \left\{ \frac{1}{|t|} + \frac{n \log n}{n^{1/2} h t^2} + \frac{(n \log n)^2}{n h^2 |t|^3} \right\} |dt| \\
+ \frac{C n \log n}{n^{7/2} h^3} \int_{x < |t| < \eta n^{1/2}} \frac{|dt|}{t^2} \\
+ \frac{C h (n \log n)^{1/2}}{n^{1/2} (nh^{d+2})^2} \int_{x < |t| < nh^d} |dt| + \frac{C}{n^{5/2} h^2} \int_{x < |t| < \eta n^{1/2}} |dt| \\
= o \left( n^{1/2} + n^{-1} h^{-d-2} \right). \tag{3.45}
\]

(ii) \( N_1 \leq |t| \leq N_0 \)

For sufficiently large \( n \), there exists \( \xi > 0 \) such that \( |Y(t)| < 1 - \xi \) by assumption (x). We may take \( m = \left\lfloor -\frac{3 \log n}{\log (1 - \xi)} \right\rfloor \) to satisfy \( 1 \leq m \leq n - 1 \) for sufficiently large \( n \).

Since \( |Y(t)|^m \leq C n^{-3} \),

\[
\int_{N_1 \leq |t| \leq N_0} \frac{E e^{itB}}{t} |dt| \\
\leq \frac{C}{n^3 h^{3d+4}} \int_{N_1 \leq |t| \leq nh^d} |t| |dt| + C \left( \frac{\log n}{n^2 h^{d+2}} + \frac{\log n}{n^2 h^{d+4}} \right) \int_{N_1 \leq |t| \leq n^{1/2} \log n} |t| |dt| \\
+ C \left( \frac{\log n}{n^2 h^{d+2}} \right)^{3/2} \int_{N_1 \leq |t| \leq n^{1/2} \log n} |t|^2 |dt| \\
+ C \int_{N_1 \leq |t| \leq n^{1/2} \log n} \left\{ \frac{1}{|t|} + \frac{\log n}{n^{1/2} h} + \frac{(\log n)^2}{n h^2 |t|^3} \right\} |dt| \\
+ \frac{C \log n}{n^{7/2} h^3} \int_{N_1 \leq |t| \leq n^{1/2} \log n} |dt|
\]
\[ + \frac{C h(\log n)^{1/2}}{n^{1/2} (n h^{-d/2})^2} \int_{N_c \leq |t| \leq n h^{-d/2} \log n} |t| \, dt \]
\[ + \frac{C}{n^{5/2} h^2} \int_{N_c \leq |t| \leq n^{1/2} \log n} dt \]
\[ = o \left( n^{-1/2} + n^{-1} h^{-d/2} \right) \]
by (ix). Therefore, by (3.45) and (3.46),
\[(II) = o \left( n^{-1/2} + n^{-1} h^{-d/2} \right).\]

Estimation of (III).

\[(III) \leq C \left[ \int_p^\infty \frac{1}{t^2} e^{-t^2/2} \, dt + n^{1/2} h^2 \int_p^\infty e^{-t^2/2} \, dt + \frac{1}{n h^{-d/2}} \int_p^\infty e^{-t^2/2} \, dt \right. \]
\[+ \left. \frac{1}{n^{1/2}} \int_p^\infty (t + t^2) e^{-t^2/2} \, dt \right]. \quad (3.47)\]

The first integral in (3.47) is bounded by
\[\frac{1}{p^2} \int_p^\infty t e^{-t^2/2} \, dt = \frac{1}{p^2} e^{-p^2/2} = o(n^{-1}),\]
because \(p = \min(\log n, \varepsilon n^{1/2}).\) The remaining integrals are clearly \(o(1)\) as \(p \to \infty.\)

Therefore,
\[(III) = o \left( n^{-1/2} + n^{-1} h^{-d/2} + n^{1/2} L \right),\]
to complete the proof.

Proof of Theorem B.

In view of the proof of Theorem 2 of NR, \(\tilde{\kappa}_i \to \kappa_i, \ i = 1, 2, 3, 4,\) a.s. Combine this with Theorem A.
APPENDIX A

Lemma 1. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

\[ E|V_1W_{12}|^r = E|V_2W_{12}|^r = O(h^{-\frac{r(1-r)}{2}}) \quad \text{for} \quad 1 \leq r \leq 3. \]

Proof. Using Lemmas 1-(d) and 4 of NR,

\[ E|V_1W_{12}|^r \leq E\{ |V_1|^r E(|W_{12}|^r |1) \} \]
\[ \leq CE \{ (|Y_1|^r + 1)^2 \} h^{-\frac{r(1-r)}{2}} \]
\[ \leq Ch^{-\frac{r(1-r)}{2}} \quad \text{for} \quad 1 \leq r \leq 3 \quad \text{by (i)}. \]

\[ E|V_1W_{12}|^r = E|V_2W_{12}|^r \]

is obvious by the symmetry of \( W_{12} \) and (iii).

Lemma 2. Under assumptions (i), (iii), (iv), (v), (vi) and (viii),

\[ E|V_1|^r = O(1) \quad \text{for} \quad 1 \leq r \leq 6. \]

Proof. As in the proof of Lemma 3 of Robinson (1995),

\[ |\tilde{V}_1|^r = |E(V_2W_{12}|1)|^r \leq C(|Y_1|^r + 1) \quad \text{a.s.} \tag{A.1} \]

so (i) immediately produces the conclusion.

Lemma 3. Under assumptions (i), (iii), (iv), (v), (vi) and (viii),

(a) \( E|W_{11}|^r = O(h^{-\frac{r(d^2-2)}{2}}) \quad \text{for} \quad 1 \leq r \leq 3 \),

(b) \( E|W_{12}|^r = O(h^{-\frac{r(1-r)}{d-2}}) \quad \text{for} \quad 1 \leq r \leq 6 \).

Proof.

(a) \( \tilde{W}_{11} = E(W_{12}^2|1) \leq C(|Y_1|^2 + 1)h^{-d-2} \quad \text{a.s. by Lemma 4 of NR so again application of (i)} \).
completes the proof.

(b). Apply Lemma 6 of NR.

Lemma 4. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii), for \( d_1 \) given in (3.37),

(a) \( E|d_1 V_2|^r = O (1) \) for \( 1 \leq r \leq 3 \),
(b) \( E|d_1 V_1|^r = O (1) \) for \( 1 \leq r \leq 2 \).

Proof.

(a) By (iii), \( E|d_1 V_2|^r = E|d_1|^r E|V_2|^r \), where the second factor is bounded due to Lemma 1-(d) of NR. From Lemma 1-(d) of NR and (A.1),
\[
|d_1|^r \leq C \left( |V_1|^r + |V_1|^r + 1 \right) \leq C (|Y_1|^{2r} + 1) ,
\]
then apply (i).

(b) By an elementary inequality and (3.7), \( E|d_1 V_1|^r \leq C (E|V_1|^3|^r + E|V_1|^r + E|V_1|^r) \). By Lemma 1-(d) of NR and (A.1), \( E|V_1|^3|^r + E|V_1|^r = O (1) \) for \( 1 \leq r \leq 2 \), and
\[
E|V_1|^r \leq CE(|Y_1|^r + 1)^2 = O (1) \quad \text{for} \quad 1 \leq r \leq 3 \text{ by (i)}.
\]

Lemma 5. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

(a) \( E|\mathcal{W}_{12} V_1 V_3|^r = E|\mathcal{W}_{12} V_2 V_3|^r = O (h^{-r(r-1)} d^{-r}) \) for \( 1 \leq r \leq 3 \),
(b) \( E|\mathcal{W}_{12} V_1^2|^r = E|\mathcal{W}_{12} V_2^2|^r = O (h^{-r(r-1)} d^{-r}) \) for \( 1 \leq r \leq 2 \).

Proof.

(a). Using (iii), Lemma 1-(d) of NR and Lemma 1, for \( 1 \leq r \leq 3 \),
\[
E|\mathcal{W}_{12} V_1 V_3|^r = E|\mathcal{W}_{12} V_1|^r E|V_3|^r = O (h^{-r(r-1)} d^{-r}) .
\]
\( E|\mathcal{W}_{12} V_1 V_3|^r = E|\mathcal{W}_{12} V_2 V_3|^r \) is straightforward by (iii) and symmetry of \( \mathcal{W}_{12} \).
(b). By Lemmas 1-(d) and 4 of NR, the left side is
\[
E \left\{ |V_1|^2 E \left( |W_{12}|^r | 1 \right) \right\} \leq E \left\{ C \left( |Y_1|^3 + 1 \right) h^{-r \cdot (r-1) d-r} \right\} = O \left( h^{-r \cdot (r-1) d-r} \right).
\]
\[
E|W_{12}|^r = E|W_{12}|^r \quad \text{is straightforward by (iii) and symmetry of } W_{12}.
\]

Lemma 6. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii), with \( e_{12} \) given in (3.37),

(a) \[
E|e_{12}V_3|^r = O \left( h^{-(r-1) d-2r} \right) \quad \text{for } 1 \leq r \leq 3,
\]

(b) \[
E|e_{12}V_1|^r = E|e_{12}V_2|^r = O \left( h^{-(r-1) d-2r} \right) \quad \text{for } 1 \leq r \leq 2.
\]

Proof.

(a). By (iii) and Lemma 1-(d) of NR, write
\[
E|e_{12}V_3|^r = E|e_{12}V_3|^r E|V_3|^r \leq C \left( E|V_1W_{12}|^r + E|V_1|^r E|W_{12}|^r \right).
\]

Then apply Lemmas 1, 2 and 3-(b).

(b). An elementary inequality gives
\[
E|e_{12}V_1|^r \leq C \left( E|V_1W_{12}|^r + E|V_1^2W_{12}|^r + E|V_1W_{12}|^r \right).
\]

Writing \( E|V_1W_{12}|^r = E \left\{ |V_1|^r \left( E\left( |W_{12}|^r \right) | 1 \right) \right\} \), the proof of Lemma 4 of NR applies to yield \( E \left( |V_2W_{12}|^r \right) \leq C \left( |Y_1|^r + 1 \right) h^{-r \cdot (r-1) d-r} \) a.s. Thus, for \( 1 \leq r \leq 3 \),
\[
E|V_1W_{12}|^r = O \left( h^{-r \cdot (r-1) d-r} \right).
\]

The second term in (A.4) has the same order bound as (A.5) by Lemma 5-(b) for \( 1 \leq r \leq 2 \). The third term in (A.4) is bounded due to (A.3), while the fourth term is bounded due to Lemma 1-(d) of NR and Lemma 2. We handle the last term in (A.4) similarly to Lemma 6 of NR:
\[
\left( E|e_{12}V_1|^r = E\left( E\left( e_{12}W_{23} \right)^r | 1, 2 \right) V_1|^r = O \left( h^{-r \cdot (r-1) d-2r} \right).
\]

(b) \[
E|e_{12}V_2|^r \quad \text{is straightforward by (iii) and symmetry of } e_{12}.
\]

Lemma 7. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

(a) \[
E|d_1W_{23}|^r = O \left( h^{-(r-1) d-r} \right) \quad \text{for } 1 \leq r \leq 3,
\]

(b) \[
E|e_{12}V_1|^r = E|e_{12}V_2|^r \quad \text{is straightforward by (iii) and symmetry of } e_{12}.
\]
(b) \( E|d_1W_{12}|^x = E|d_2W_{12}|^x = O(h^{-(r-1)d-x}) \) for \( 1 \leq r \leq 2 \).

**Proof.**

(a). Using (A.2) and Lemma 4 of Robinson (1995),

\[
E|d_1W_{23}|^x = E|d_1|^x E|W_{23}|^x = O(h^{-(r-1)d-x})
\]

for \( 1 \leq r \leq 3 \).

(b). Using (A.2) and Lemma 4 of NR the left side is

\[
E\left\{d_1^x E(|W_{12}|^x 1)\right\} \leq E\left\{(|Y_1|^{2x} + 1) C (|Y_1|^x + 1) h^{-(r-1)d-x}\right\}
\]

\[
\leq C E(|Y_1|^{3x} + 1) h^{-(r-1)d-x} = O(h^{-(r-1)d-x})
\]

for \( 1 \leq r \leq 2 \) under (i). \( E|d_1W_{12}|^x = E|d_2W_{12}|^x \) is straightforward by (iii) and symmetry of \( W_{12} \).

\( \square \)

**Lemma 8.** Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

(a) \( E|\hat{W}_{12}W_{12}|^x = O(h^{-(r-1)d-x}) \) for \( 1 \leq r \leq 3 \),

(b) \( E|\hat{W}_{12}W_{13}|^x = O(h^{-(r-1)d-x}) \) for \( 1 \leq r \leq 3 \),

(c) \( E|\hat{W}_{12}W_{23}|^x = O(h^{-(r-1)d-x}) \) for \( 1 \leq r \leq 3 \),

(d) \( E|\hat{W}_{12}W_{34}|^x = O(h^{-(r-1)d-x}) \) for \( 1 \leq r \leq 6 \).

**Proof.**

(a). In view of the proof of Lemma 6 of NR,

\[
E|\hat{W}_{12}W_{12}|^x = E|\hat{W}_{12}|^x |W_{12}|^x
\]

\[
\leq h^{-x(d-2)} CE(1 + |Y_1|^x + |Y_2|^x + |Y_1|^x |Y_2|^x) |W_{12}|^x
\]

\[
\leq C h^{-x(d-2)} \left( E|\hat{W}_{12}|^x + E|Y_1W_{12}|^x + E|Y_2W_{12}|^x + E|Y_1Y_2W_{12}|^x \right)
\].

The first term in parentheses is \( O(h^{-(r-1)d-x}) \) by Lemma 4 of Robinson (1995). From inspecting their proofs, Lemma 1 and (A.5) still hold with \( V_1 \) and \( V_2 \) replaced by \( Y_1 \) and \( Y_2 \) so that the other terms are \( O(h^{-(r-1)d-x}) \) for \( 1 \leq r \leq 3 \).
(b). Using Lemma 4 of NR, for $1 \leq r \leq 6$,
\[
E|\hat{W}_{12}\hat{W}_{13}|^{r} = E\left\{ |\hat{W}_{12}|^{r} E\left( |\hat{W}_{13}|^{r} \right) \right\} \\
\leq E\left\{ |\hat{W}_{12}|^{r} C( |\hat{Y}_{1}|^{r} + 1) h^{-(r-1)\alpha} \right\} \\
= C h^{-(r-1)\alpha} \left\{ E|\hat{W}_{12}\hat{Y}_{1}|^{r} + E|\hat{W}_{12}|^{r} \right\} .
\]
We may replace $V_{1}$ by $Y_{1}$ in (A.6), so that using also Lemma 3-(b),
\[
E|\hat{W}_{12}\hat{W}_{13}|^{r} = O\left( h^{-2(\varepsilon-1)\alpha} \right) \quad \text{for} \quad 1 \leq r \leq 3 .
\]

(c). The proof is as in (b).

(d). Writing $E|\hat{W}_{12}\hat{W}_{34}|^{r} = E|\hat{W}_{12}|^{r} E|\hat{W}_{34}|^{r}$ by (iii), the proof is straightforward by Lemma 4 of Robinson (1995) and Lemma 3-(b).

\[\square\]

Lemma 9. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),
\[
E|\hat{V}|^{r} = O\left( 1 \right) \quad \text{for} \quad 2 \leq r \leq 6 .
\]
Proof. Since $V_{i}, i=1,...,n$ is an iid sequence, the result follows straightforwardly by DFJ and Lemma 1-(d) of NR.

\[\square\]

Lemma 10. Under assumptions (i), (v), (vi), (vii) and (viii)
\[
|T_{1}|^{r} = O\left( n^{-\varepsilon} h^{-\alpha(d+2)} \right) \quad \text{for} \quad r > 0 .
\]
Proof. Using Lemma 4 of Robinson (1995) and $|\delta| < C$ due to (3.7),
\[
|T_{1}|^{r} \leq \frac{C}{n^{\varepsilon}} E(\hat{W}_{12}^{2}) |^{r} = O\left( n^{-\varepsilon} h^{-\alpha(d+2)} \right) .
\]

\[\square\]

Lemma 11. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),
\[ E|T_2|^r = O \left( n^{-\frac{r}{2}} \right) \quad \text{for} \quad 2 \leq r \leq 3. \]

**Proof.** Using (3.7), write

\[ E|T_2|^r \leq \frac{C}{n^r} E \sum_{i=1}^{n} (4V_i^2 - s^2)^{|r|} + \frac{C}{n^r} E \sum_{i=1}^{n} 8V_i^r. \]

Since \( E(4V_i^2) = s^2 \) and \( E(V_i) = 0 \), by (iii) both the \( 4V_i^2 - s^2 \) and \( V_i \) are martingale differences and thus the theorem of DFJ applies to yield

\[ E \left( \sum_{i=1}^{n} (4V_i^2 - s^2) \right)^{|r|} \leq C n^r E \left| 4V_1^r - s^2 \right|^r = O \left( n^r \right) \]

for \( 2 \leq r \leq 3 \) by (3.7) and Lemma 1-(d) of NR and

\[ E \left| \sum_{i=1}^{n} 8V_i^r \right|^r \leq C n^r E \left| V_1 \right|^r = O \left( n^r \right) \]

by Lemma 2.

\[ \square \]

**Lemma 12.** Under assumptions (i), (iii), (iv), (v), (vi) and (viii)

\[ E|T_3|^r = O \left( n^{-r} h^{-1(r-1)d-2r} \right) \quad \text{for} \quad 2 \leq r \leq 6. \]

**Proof.** Using (3.7), write

\[ E|T_3|^r \leq C n^{-2r} E \left| \sum_{k=1}^{n} Z_k \right|^r \tag{A.7} \]

where \( Z_k = \sum_{m=k+1}^{n} \tilde{W}_{km} \), for \( k = 1, \ldots , n-1 \). Since

\[ E(\tilde{W}_{12} \mid 2) = E(\tilde{W}_{12} \mid 1) = E\left\{ E(\tilde{W}_{13}\tilde{W}_{23} \mid 1, 2) \mid 1 \right\} = E(\tilde{W}_{13}\tilde{W}_{23} \mid 1) = 0 \quad \text{a.s.}, \]

\( Z_k \), \( k = n-1, \ldots , 1 \) is a martingale difference sequence. Thus we apply DFJ to bound (A.7) by \( C n^{-2r} (n-1) \begin{pmatrix} r-1 \\ r-1 \end{pmatrix} \sum_{k=1}^{n} E|Z_k|^r \). Since \( E(\tilde{W}_{km} \mid m) = 0 \) a.s. for \( m=k+1, \ldots , n \), the \( \tilde{W}_{km} \) are martingale differences. We use DFJ again and get, by Lemma 3-(b),

\[ E | Z_k |^r \leq C (n-k)^{r-1} \sum_{m=k+1}^{n} E |\tilde{W}_{km}|^r \leq C (n-k)^{r-1} (n-k) h^{-1(r-1)d-2r}. \]

\[ \square \]

**Lemma 13.** Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

\[ E|Q_1|^r = O \left( n^{-r} h^{-1(r-1)d-2r} \right) \quad \text{for} \quad 2 \leq r \leq 3. \]
Proof. Write \( P_{ij} = (V_i + V_j) W_{ij} - V_i - V_j \). Then \( \sum_{j=1}^{n} P_{ij} \) is a martingale difference sequence for \( i=n-1,\ldots,1 \). We can proceed by replacing \( W_{km} \) in Lemma 12 by \( P_{ij} \) due to the property \( E(P_{ij}|j) = 0 \) a.s. for \( i \neq j \). Applying DFJ and (3.7),

\[
E|Q_1|^{r} \leq C \left( \frac{n}{2} \right)^{r} E \left\{ \sum_{i=1}^{n-1} n \sum_{j=i+1}^{n} P_{ij} \right\}^{r} \leq C \left( \frac{n}{2} \right)^{r} n^{r-1} \sum_{i=1}^{n-1} E \left\{ \sum_{j=i+1}^{n} P_{ij} \right\}^{r} .
\]

Since \( P_{ij}, j=n, \ldots, i+1 \) is a martingale difference for fixed \( i \), we can apply the theorem of DJF again and obtain \( E \left| \sum_{j=1}^{n} P_{ij} \right|^{r} \leq C (n-i)^{r-1} \sum_{j=1}^{n-i} E|P_{ij}|^{r} \). By Lemmas 1 and 2,

\[ E|P_{ij}|^{r} \leq C [E|V_i|^{r} + E|V_i W_{ij}|^{r}] = O (h^{-r+1} \epsilon^{-d}) \text{ for } 1 \leq r \leq 3 . \]

Lemma 14. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

\[ E|Q_2|^{r} = O (n^{-r} h^{-r+1} \epsilon^{-d+r}) \text{ for } 2 \leq r \leq 6 . \]

Proof. By an elementary inequality and (3.7),

\[ E|Q_2|^{r} \leq \frac{C}{n^r} \left( \frac{n-1}{2} \right)^r E \left[ \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \sum_{m=k+1}^{n-i} V_i W_{km} \right]^{r} \]

\[ = \frac{C}{n^r} \left( \frac{n-1}{2} \right)^r n^{r-1} \sum_{i=1}^{n} E \left[ \sum_{k=1}^{n-i} \sum_{m=k+1}^{n-i} V_i W_{km} \right]^{r} . \]

\( V_i W_{km} \), \( m=k+1,\ldots,n \) is a martingale difference for fixed \( i, k, k \neq i \) and \( m \neq i \), and

\[ \sum_{m=k+1}^{n-i} V_i W_{km} \), \( k=n-1,\ldots,1 \) is also a martingale difference for fixed \( i \) and \( k \neq i \) so that we apply DFJ repeatedly as in the proof of the previous Lemma and get

\[
\sum_{i=1}^{n} E \left\{ \sum_{k=1}^{n-i} \sum_{m=k+1}^{n-i} V_i W_{km} \right\}^{r} \leq C \sum_{i=1}^{n} (n-2)^{r-1} \sum_{k=1}^{n-i} E \left\{ \sum_{m=k+1}^{n-i} V_i W_{ij} \right\}^{r}
\]

\[
\leq C (n-1)^{r-1} \sum_{i=1}^{n} \sum_{k=1}^{n-i} E (n-k)^{r-1} \sum_{m=k+1}^{n-i} E \left| V_i W_{km} \right|^{r}
\]

\[
\leq C n^{r-1} h^{-r+1} \epsilon^{-d+r}
\]

for \( 2 \leq r \leq 6 \) by (iii), Lemma 1-(d) of NR and Lemma 4 of Robinson (1995). \qed

Lemma 15. Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

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Proof. Writing $E|R_i|^r \leq C\left(\frac{n}{2}\right)^{-r} E|\sum_{i=1}^{n} \sum_{j=i+1}^{n} V_i V_j|^r$ due to (3.7), as in Lemma 12 or 13, $V_i V_j$, $i=1,...,j-1$ is a martingale difference sequence for fixed $j$ as well as $\sum_{j=i+1}^{n} V_i V_j$, $i=1,...,n$. We use DFJ repeatedly again and (i), (iii) and Lemma 1-(d) of NR to obtain

$$E|\sum_{i=1}^{n} \sum_{j=i+1}^{n} V_i V_j|^r \leq C (n-1)^{\frac{r-1}{2}} \sum_{i=1}^{n} E|V_i V_j|^r \leq C (n-1)^{\frac{r-1}{2}} \sum_{i=1}^{n} (n-i)^{\frac{r-1}{2}} \sum_{j=i+1}^{n} E|V_i V_j|^r = O(n^r).$$

Lemma 16. Under assumptions (i), (iii), (iv), (v), (vi) and (viii),

$$E|R_2|^r = O(n^{-3(r-1)}h^{-2(r-1)d-2r}) \quad \text{for} \quad 2 \leq r \leq 3.$$

Proof. Using (3.7), write

$$E|R_2|^r = \frac{C}{n^r} \left(\frac{n}{2}\right)^{-r} E|\sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} (W_{ik} W_{im} - \bar{W}_{ik} \bar{W}_{im})|^r.$$

Since $R_2$ has the same martingale structure as $Q_2$, the same method of proof as in Lemma 14 applies. The difference is in the moment bounds of the two summands, i.e.

$$E|V_i W_{km}|^r = O(h^{-(r-1)d-r}) \quad \text{for} \quad 1 \leq r \leq 6$$

and

$$E|\bar{W}_{ik} \bar{W}_{im} - \bar{W}_{km}|^r = O(h^{-2(r-1)d-2r}), \quad i \neq k \neq m$$

by Lemmas 1-(d), 4 of NR and Lemma 3-(b).

Lemma 17. Under assumptions (i), (iii), (iv), (v), (vi) and (viii)

$$E|R_3|^r = O(n^{-2r}h^{-2(r-1)d-2r}) \quad \text{for} \quad 2 \leq r \leq 3.$$

Proof. Write $E|R_3|^r \leq \frac{C}{n^r} \left(\frac{n}{2}\right)^{-r} E|\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=1}^{n} \sum_{m=1}^{n} \{W_{ij}^2 - \bar{W}_{ij} - \bar{W}_{jj} + E(W_{ij}^2)\}|^r$ using (3.7). Since
\[
E(\tilde{W}_{ij} - \tilde{W}_{ii} - \tilde{W}_{jj} + E(\tilde{W}_{12}) \mid j) = E(\tilde{W}_{ij} - \tilde{W}_{ii} - \tilde{W}_{jj} + E(\tilde{W}_{12}) \mid i) = 0
\]

for \( j > i \), \( R_3 \) has the same martingale structure as \( T_3 \). Therefore, we apply DFJ to obtain

\[
E \left| \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \{ \tilde{W}_{ij} - \tilde{W}_{ii} - \tilde{W}_{jj} + E(\tilde{W}_{12}) \} \right|^r 
\]

\[
\leq C (n-1)^{r-1} \sum_{i=1}^{n-1} E \left| \sum_{j=i+1}^{n} \{ \tilde{W}_{ij} - \tilde{W}_{ii} - \tilde{W}_{jj} + E(\tilde{W}_{12}) \} \right|^r 
\]

\[
\leq C (n-1)^{r-1} \sum_{i=1}^{n-1} (n-i)^{r-1} \sum_{j=i+1}^{n} \left| \tilde{W}_{ij} - \tilde{W}_{ii} - \tilde{W}_{jj} + E(\tilde{W}_{12}) \right|^r 
\]

\[
= O \left( n^r h^{-(2r-1)d-2r} \right)
\]

by Lemma 4 of Robinson (1995) and Lemma 3-(a).

\[\square\]

**Lemma 18.** Under assumptions (i), (iv), (v), (vi) and (viii)

\[
E|R_4|^r = O \left( n^{-\frac{3r}{2}} h^{-r(d+2)} \right) \quad \text{for} \quad 1 \leq r \leq 3.
\]

**Proof.** Write \( E|R_4|^r \leq C \frac{1}{n^{2r}} E \left| \sum_{i=1}^{n} \{ \tilde{W}_{ii} - E(\tilde{W}_{12}) \} \right|^r \) using (3.7). Since \( \tilde{W}_{ii} - E(\tilde{W}_{12}) \) is a martingale difference, by (iii), DFJ and Lemma 3-(a),

\[
E \left| \sum_{i=1}^{n} \{ \tilde{W}_{ii} - E(\tilde{W}_{12}) \} \right|^r \leq C n^{\frac{r}{2} - 1} \sum_{i=1}^{n} E \left| \tilde{W}_{ii} - E(\tilde{W}_{12}) \right|^r = O \left( n^r h^{-r(d+2)} \right).
\]

\[\square\]

**Lemma 19.** Under assumptions (i), (iii), (iv), (v), (vi), (vii) and (viii),

\[
E|R_3|^r = O \left( n^{-2r} h^{-r(d+2)} \right) \quad \text{for} \quad 1 \leq r \leq 3.
\]

**Proof.** Using (3.7), DFJ and Lemma 6 of Robinson(1995),

\[
E|R_3|^r \leq C \frac{1}{n^{4r}} \left| \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \tilde{W}_{ij} \right|^{2r} \leq C \frac{1}{n^{4r}} \left( n-1 \right)^{r-1} \sum_{i=1}^{n-1} E \left| \sum_{j=i+1}^{n} \tilde{W}_{ij} \right|^{2r} 
\]

\[
= O \left( n^{-2r} h^{-r(d+2)} \right).
\]

\[\square\]

**APPENDIX B**
Here, we present some of the derivations used in the proof of Theorem A, namely:

(a) \( E(\tilde{\beta}_2 e^{it\tilde{b}_2}) \)
\[
= -\{\gamma(t)\}^{-n-1}\left\{ \frac{i t}{n} E(W_{12}^2) + O\left( \frac{|t|}{n^2 h^{d+2}} + \frac{t^2}{n^3 h^{d+2}} + \frac{|t| h^2}{n h^{d+2}} \right) \right\}
- \{\gamma(t)\}^{-n-1}\left\{ 4E(V_1^3) + 8E(W_{12}V_1 V_2) \right\} \frac{\gamma(t)}{n^1/2} + O\left( \frac{|t|}{n} \right)
- \{\gamma(t)\}^{-n-2}\left\{ \frac{(it)^2}{n^{1/2}} \left( 4E(V_1^3) + 8E(W_{12}V_1 V_2) \right) \right\} \frac{\gamma(t)}{n^2} + O\left( \frac{|t|^2 + |t|^3}{n^{1/2}} + \frac{t^4}{n^{3/2}} \right)
+ \{\gamma(t)\}^{-n-3}\left\{ 0 \left( \frac{|t|^3 + |t|}{n} + \frac{t^2}{n^{3/2} h^{d+2}} + \frac{t^6}{n^{3} h^{d+2}} + \frac{|t|^5}{n^{5/2} h^{d+2}} + \frac{t^4 + |t|^3}{n^2 h^{d+2}} \right) \right\},
\]

(b) \(|E(\tilde{\beta}_{2m} e^{it\tilde{b}_n})| \leq \frac{Cm}{n^{1/2} h^2} |\gamma(t)|^{-m-4},
\]

(c) \(|E(\tilde{\beta}_{3m} e^{it\tilde{b}_n})| \leq \frac{Cm}{n^{1/2} h^3} |\gamma(t)|^{-m-4},
\]

(d) \(E|\tilde{\beta}_{2m}|^2 \leq Cm \left( \frac{1}{n^2 h^{2d+4}} + \frac{1}{n^2} \right),
\]

(e) \(E|\tilde{\beta}_{3m}|^2 \leq \frac{Cm}{n^4 h^{3d+6}},
\]

(f) \(|E\tilde{\beta}_{2m} e^{it\tilde{b}_n}| \leq \frac{Cn^{1/2}}{h^2} |\gamma(t)|^{-m-5}.
\]

for \(1 \leq m \leq n-1\).

Proof.

(a) Write
\[
E(\tilde{\beta}_2 e^{it\tilde{b}_2}) = E(T\tilde{\beta}_2 e^{it\tilde{b}_2})
\]
\[
= E(T_1 \frac{2}{\sqrt{n}} \sum_{j=1}^{n} V_j e^{itb_2}) + E(T_2 \frac{2}{\sqrt{n}} \sum_{j=1}^{n} V_j e^{itb_2}) + E(T_3 \frac{2}{\sqrt{n}} \sum_{j=1}^{n} V_j e^{itb_2})
\]
\[
= (A) + (B) + (C) \quad \text{. (B.1)}
\]

Thus
\[
(A) = -\frac{4n^{1/2}}{(n-2)^2 s^3} E(W_{12}^2) \sum_{j=1}^{n} E(V_j e^{itb_2}) \quad \text{. (B.2)}
\]
Due to (iii), (3.17), \( \gamma(t) = E(e^{i t \frac{2}{\sqrt{s}} V_j}) \), \( 4E(V_j^2) = s^2 \), (3.7) and Lemma 1-(d) of NR,
\[
E(V_j e^{i t b_j}) = E(V_j e^{i t \frac{2}{\sqrt{s}} V_j}) E(e^{i t \frac{2}{\sqrt{s}} \sum_{i=1}^n V_i})
\]
\[
= \left[ E\left(V_j \left(e^{i t \frac{2}{\sqrt{s}} V_j} - 1 - i t \frac{2 V_j}{n^{1/2} s}\right)\right) + (i t) E\left(\frac{2 V_j^2}{n^{1/2} s}\right) \right] (\gamma(t))^{n-1}
\]
\[
= \left\{ \gamma(t) \right\}^{n-1} \frac{\frac{i t s}{2 n^{1/2}} + O\left(\frac{t^2}{n}\right)}{n^{1/2}}
\]
\[
= \left\{ \gamma(t) \right\}^{n-1} \frac{i t}{2 n^{1/2}} + O\left(\frac{t^2}{n} + \frac{|t|h^L}{n^{1/2}}\right) \quad (B.3)
\]
Substituting (B.3) into (B.2),
\[
(A) = -\left\{ \gamma(t) \right\}^{n-1} \frac{4n^{3/2}}{(n-2)^2 s^3} E(W_{12}^2) \left\{ \frac{i t}{2 n^{1/2}} + O\left(\frac{t^2}{n} + \frac{|t|h^L}{n^{1/2}}\right) \right\}
\]
\[
= -\frac{\gamma(t)^{n-1}}{s^2} \left\{ \frac{2 i t}{n} E(W_{12}^2) + O\left(\frac{|t|}{n^{1/2}} + \frac{t^2}{n^{3/2} h^{d+2}} + \frac{|t|h^L}{n h^{d+2}}\right) \right\} . \quad (B.4)
\]
Now write \( (B) = (B)' + (B)'' \), where
\[
(B)' = -\frac{1}{n^{3/2} s^3} \sum_{j=1}^n E(4V_j^2 - s^2 + 8V_j) V_j e^{i t b_j} , \quad (B.5)
\]
\[
(B)'' = -\frac{1}{n^{3/2} s^3} \sum_{j=1}^n \sum_{k=1}^n E(4V_j^2 - s^2 + 8V_j) V_k e^{i t b_k} . \quad (B.6)
\]
The summand of \( (B)' \) is, using (3.17) and Lemma 1-(d) of NR,
\[
E\left(4V_1^2 - s^2 + 8V_1\right) V_1 e^{i t \frac{2}{\sqrt{s}} V_1} E(e^{i t \frac{2}{\sqrt{s}} \sum_{i=1}^n V_i})
\]
\[
= \left\{ \gamma(t) \right\}^{n-1} E\left(4V_1^3 + 8E(W_{12} V_1 V_2) + O\left(\frac{|t|}{n^{1/2}}\right) \right) . \quad (B.7)
\]
Substituting (B.7) into (B.5),
\[
(B)' = -\frac{1}{n^{1/2} s^3} \left\{ \gamma(t) \right\}^{n-1} \left\{ 4E(V_1^3) + 8E(W_{12} V_1 V_2) + O\left(\frac{|t|}{n^{1/2}}\right) \right\} . \quad (B.8)
\]
For \( j \neq k \), the summand of \( (B)'' \) is, due to (iii),
\[
E\left(4V_1^2 - s^2 + 8V_1\right) V_2 e^{i t \frac{2}{\sqrt{s}} V_1} E(e^{i t \frac{2}{\sqrt{s}} \sum_{i=1,2} V_i})
\]
\[
= \left\{ \gamma(t) \right\}^{n-2} E\left(4V_1^2 - s^2 + 8V_1\right) V_2 e^{i t \frac{2}{\sqrt{s}} V_1} \}
\]
\[
= \left\{ \gamma(t) \right\}^{n-2} E\left(4V_1^2 - s^2 + 8V_1\right) e^{i t \frac{2}{\sqrt{s}} V_1} E(V_2 e^{i t \frac{2}{\sqrt{s}} V_1})
\]
\[
= \left\{ \gamma(t) \right\}^{n-2} E\left(4V_1^2 - s^2 + 8V_1\right) e^{i t \frac{2}{\sqrt{s}} V_1} E(V_2 e^{i t \frac{2}{\sqrt{s}} V_1})
\]
\[\{\mathcal{Y}(t)\}^{-2} \left[ \frac{4V_1^2 - s^2 + 8V_1}{n^2 - 1} e^{\frac{itV_1}{\sqrt{n}s}} + \frac{it}{\sqrt{n}s} \left( 4V_1^2 - s^2 + 8V_1 \right) \right] \]

\[\times \left\{ \frac{V_2}{n^2} \left( e^{\frac{itV_2}{\sqrt{n}s}} - 1 - it \frac{V_2}{\sqrt{n}s} \right) + \frac{it}{\sqrt{n}s} \frac{V_2^2}{\sqrt{n}s} \right\} \]

\[= \{\mathcal{Y}(t)\}^{-2} \left[ \frac{it}{\sqrt{n}s} \left( 4V_1^2 - s^2 + 8V_1 \right) + O \left( \frac{|t|^2}{n} \right) \right] \]

\[= \{\mathcal{Y}(t)\}^{-2} \left[ \frac{it}{\sqrt{n}s} (E(4V_1^3) + 8E(W_{12}V_1V_2)) + O \left( \frac{|t|^2}{n} \right) \right] \]

\[= \{\mathcal{Y}(t)\}^{-2} \left[ \frac{(it)^2}{n} (E(4V_1^3) + 8E(W_{12}V_1V_2)) + O \left( \frac{|t|^3}{n^{3/2}} + \frac{|t|^4}{n^2} \right) \right] \]

(B.9)

by (3.17). Therefore, substituting (B.9) into (B.6) yields

\[(B)^{II} = -\frac{n(n-1)}{n^{3/2}s^3} \left[ \frac{(it)^2}{n} (4E(V_1^3) + 8E(W_{12}V_1V_2)) + O \left( \frac{|t|^3}{n^{3/2}} + \frac{|t|^4}{n^2} \right) \right] \{\mathcal{Y}(t)\}^{-2}.

(B.10)

By (B.8) and (B.10),

\[(B) = -\frac{\{\mathcal{Y}(t)\}^{-n-1}}{s^3} \left\{ \frac{4E(V_1^3) + 8E(W_{12}V_1V_2)}{n^{1/2}} + O \left( \frac{|t|}{n} \right) \right\} \]

\[= \frac{\{\mathcal{Y}(t)\}^{-n-2}}{s^3} \left[ \frac{(it)^2}{n^{1/2}} (4E(V_1^3) + 8E(W_{12}V_1V_2)) \right] \]

\[+ O \left( \frac{t^2}{n} + \frac{|t|^3}{n^{3/2}} + \frac{|t|^4}{n^2} \right) \]

(B.11)

Now, write

\[(C) = -\frac{4}{n^{1/2}s^3} \left\{ \frac{n-1}{2} \right\} \sum_{1 \leq j < k \leq n} \frac{1}{\sqrt{n}1^2} \sum_{1 \leq j < k \leq n} V_1 e^{itb_2} \]

\[= -\frac{4}{n^{1/2}s^3} \left( \frac{n-1}{2} \right)^2 \sum_{i=1}^{n} \sum_{j<i}^{n} \left( 1^i \right) E(W_{jk}V_1 e^{itb_2}) \]

\[= -\frac{4}{n^{1/2}s^3} \left( \frac{n-1}{2} \right)^2 \sum_{j=1}^{n} \sum_{k<j}^{n} E(W_{jk}V_1 e^{itb_2}) \]

\[= -\frac{4}{n^{1/2}s^3} \left( \frac{n-1}{2} \right)^2 \sum_{j=1}^{n} \sum_{k<j}^{n} E(W_{jk}V_1 e^{itb_2}) \]

\[= (C) + (C)^{II} + (C)^{III}. \]
Using (iii), Lemma 1-(d) of NR, \( E(V_j) = 0 \) and \( E(W_{km}) = E(W_{km}|k) = E(W_{km}|m) = 0 \), the summand of (C)' is
\[
E \left\{ \sum_{i=1}^{n-3} \frac{\nabla^2}{2} (V_{i+1}V_{i+2}) \right\} E \left( e^{-it\sqrt{n}} (W_{12}) \right) E \left( e^{-it\sqrt{m}} (V_{1}) \right) = E \left\{ \sum_{i=1}^{n-3} \frac{\nabla^2}{2} (V_{i+1}V_{i+2}) \right\} \{ \gamma(t) \}^{n-3}
\]
\[
= \left\{ E \left( e^{-it\sqrt{n}} (W_{12}) V_1 \right) \right\} + \frac{4}{n^2} E \left( e^{-it\sqrt{n}} (W_{12}) \right) \{ \gamma(t) \}^{n-3}
\]
\[
= \left\{ E \left( e^{-it\sqrt{n}} (W_{12}) V_1 \right) \right\} \{ \gamma(t) \}^{n-3}.
\]
The last equality uses (3.17) and
\[
E \left| W_{12} V_1 V_2 \right|^2 \leq \{ E \left| W_{12} \right|^3 \}^{1/2} \{ E \left| V_1 V_2 \right|^3 \}^{2/3} \leq C h^{-2} = O(h^{-d-2})
\]
due to Hölder's inequality, Lemma 3-(b), (i), (iii) and Lemma 1-(d) of NR.

(C)' = \[
\frac{\gamma(t)^{n-3} 8 n (n-1) (n-2)}{2^{3/2}} \left( \frac{t^6}{n^2 h^{d+2} n^{5/2}} + \frac{t^6}{n^3 h^{d+2}} \right) \{ \gamma(t) \}^{n-3}.
\]
\[
= \left( \frac{\gamma(t)}{n^2 h^{d+2}} \right)^{n-3} \left( \frac{t^6}{n^2} + \frac{t^6}{n^5 h^{d+2}} \right) \{ \gamma(t) \}^{n-3}.
\]
\[
= \left( \frac{\gamma(t)}{n^2 h^{d+2}} \right)^{n-3} \left( \frac{t^6}{n^2} + \frac{t^6}{n^5 h^{d+2}} \right) \{ \gamma(t) \}^{n-3}.
\]
Here we use, due to (iii) and Lemma 2,
\[
E \left( W_{12} V_1 V_2 \right) = E \left( W_{13} W_{23} V_1 V_2 \right) = \left[ E \left( W_{13} V_1 | 3 \right) E \left( W_{23} V_2 | 3 \right) \right] = E(V_3) = O(1) \quad (B.13)
\]
The summand of (C)'' can be expressed as follows using (iii), \( E(W_{12} V_1 e^{-it\sqrt{n}}) = 0 \), Lemma 3-(b) and (3.17).
Similarly, by (3.7), (B.1), (B.4), (B.11) and (B.16),

\[
\begin{align*}
E(\tilde{B}_2 e^{i\bar{\theta}_n}) &= (A) + (B) + (C) \\
&= -\{\gamma(t)\}^{n-1} \left\{ \frac{it}{n} E(W_2) + O \left( \frac{|t|}{n^2 h^{d+2}} + \frac{t^2}{n^{3/2} h^{d+2}} + \frac{|t| h^2}{n h^{d+2}} \right) \right\} \\
&- \{\gamma(t)\}^{n-1} \left\{ \frac{4E(V_1^3)}{n^{1/2}} + \frac{8E(W_1 V_2)}{n^{1/2}} + O \left( \frac{|t|}{n} \right) \right\} \\
&- \{\gamma(t)\}^{n-2} \left\{ \frac{(it)^2}{n^{1/2}} \left( 4E(V_1^3) + 8E(W_1 V_2) \right) + O \left( \frac{t^2 + |t|^3}{n^{3/2}} + \frac{t^4}{n^{3/2}} \right) \right\} \\
&+ \{\gamma(t)\}^{n-3} \left\{ O \left( \frac{|t|^3}{n^2} + \frac{t^2}{n^{3/2} h^{d+2}} + \frac{t^6}{n^3 h^{d+2}} + \frac{|t|^5}{n^{5/2} h^{d+2}} + \frac{t^4 + |t|^3}{n^2 h^{d+2}} \right) \right\}.
\end{align*}
\]

(b) Writing, using (3.7) and Lemma 4 of Robinson (1995),

\[
|E(\tilde{B}_{2m} e^{i\bar{\theta}_n})| \leq \frac{C}{n^{3/2} h^{d+2}} \sum_{j=1}^{m} |E(V_j e^{i\bar{\theta}_n})|
\]

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\[
+ \frac{C}{n^{3/2}} \left\{ \sum_{j=1}^{n} \sum_{k=1}^{m} |E(d_j V_k e^{it \hat{\theta}_n})| + \sum_{j=1}^{n} \sum_{k=1}^{m} |E(d_j V_k e^{it \hat{\theta}_n})| \right\}
+ \frac{C}{n^{5/2}} \left\{ \sum_{j=1}^{n} \sum_{k=1}^{m} |E(e_j V_s e^{it \hat{\theta}_n})| + \sum_{j=1}^{n} \sum_{k=1}^{m} |E(e_j V_s e^{it \hat{\theta}_n})| \right\}
+ \frac{C}{n^{3/2}} \left\{ \sum_{j=1}^{n} \sum_{k=1}^{m} |E(V_j W_{kl} V_s e^{it \hat{\theta}_n})| + \sum_{j=1}^{n} \sum_{k=1}^{m} |E(V_j W_{kl} V_s e^{it \hat{\theta}_n})| \right\},
\]

\[
|E(V_j e^{it \hat{\theta}_n})| = |E(V_j e^{it \frac{2}{\sqrt{m}} V_j}) E\left(e^{it \frac{2}{\sqrt{m}} \sum_{j=1}^{n} V_j} \right)| |E(e^{it \frac{1}{\sqrt{m}} \sum_{j=1}^{n} V_j})| \leq E|V_j| |\gamma(t)| \leq 1. \tag{B.17}
\]

For \( j=1, \ldots, m \), since \( b_3 - b_{3m} + \hat{E}_3' - \hat{E}_3'' + \hat{E}_{3m}' - \hat{E}_{3m}' \) is independent of \( V_1, \ldots, V_m \).

For \( j \leq m \), \( k \leq m \) and \( j \neq k \),
\[
|E(d_j V_k e^{it \hat{\theta}_n})| = |E[d_j V_k e^{it \frac{2}{\sqrt{m}} V_j} + \sum_{k=1}^{m} V_k] E\left(e^{it \frac{1}{\sqrt{m}} \sum_{j=1}^{n} V_j} \right)| \leq E|d_j V_k| |\gamma(t)| \leq E|d_j V_k| |\gamma(t)|. \tag{B.19}
\]

For \( j \leq k \leq m \),
\[
|E(d_j V_k e^{it \hat{\theta}_n})| = |E[d_j V_k e^{it \frac{2}{\sqrt{m}} V_j} + \sum_{k=1}^{m} V_k] E\left(e^{it \frac{1}{\sqrt{m}} \sum_{j=1}^{n} V_j} \right)| \leq E|d_j V_k| |\gamma(t)| \leq E|d_j V_k| |\gamma(t)|. \tag{B.20}
\]

For \( j \leq m \) and \( k \geq m+1 \),
\[
|E(d_j V_k e^{it \hat{\theta}_n})| = |E[d_j V_k e^{it \frac{2}{\sqrt{m}} V_j} + \sum_{k=1}^{m} V_k] E\left(e^{it \frac{1}{\sqrt{m}} \sum_{j=1}^{n} V_j} \right)| \leq E|d_j V_k| |\gamma(t)| \leq E|d_j V_k| |\gamma(t)|. \tag{B.21}
\]

For \( j \geq m+1 \) and \( k \leq m \), similarly to (B.21),
\[
|E(d_j V_k e^{it \hat{\theta}_n})| \leq E|d_j V_k| |\gamma(t)|. \tag{B.22}
\]
Therefore, by (B.19)-(B.22) and Lemma 4, for all $j, k$,
\[
|E(d_j V_k e^{it \hat{\theta}})| \leq E|d_j V_k| |\gamma(t)|^{m-2}
\]  
(B.23)

Similarly to the derivation of (B.23), for any $j, k, l, s$,
\[
|E(e_{jk} V_s e^{it \hat{\theta}})| \leq E|e_{jk} V_s| |\gamma(t)|^{m-3},
\]  
(B.24)

\[
|E(V_j \hat{w}_k V_s e^{it \hat{\theta}})| \leq E|V_j \hat{w}_k V_s| |\gamma(t)|^{m-4}.
\]  
(B.25)

Substituting (B.18), (B.23)-(B.25) into (B.17), using $|\gamma(t)| \leq 1$,
\[
|E(B_2m e^{it \hat{\theta}})| \leq C |\gamma(t)|^{m-4} 
\]

\[
\left[ \frac{1}{n^{3/2} h^{d+2}} \sum_{j=1}^{m} E|V_j| + \frac{1}{n^{3/2}} \left( \sum_{j=1}^{m} \sum_{k=1}^{m} E|d_j V_k| + \sum_{j=1}^{m} \sum_{k=m+1}^{m} E|d_j V_k| \right) \right]
\]
\[
+ \frac{1}{n^{5/2}} \left( \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{s=1}^{m} E|e_{jk} V_s| + \sum_{j=1}^{m} \sum_{k=1}^{m} E|e_{jk} V_s| \right)
\]
\[
+ \frac{1}{n^{7/2}} \left( \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{s=1}^{m} E|V_j \hat{w}_k V_s| + \sum_{j=1}^{m} \sum_{k=1}^{m} E|V_j \hat{w}_k V_s| \right)
\]
\[
+ \sum_{j=1}^{m} \sum_{k=m+1}^{m} \sum_{s=m+1}^{m} E|V_j \hat{w}_k V_s| \right].
\]  
(B.26)

The summations in the square brackets have the following bounds.
\[
\sum_{j=1}^{m} E|V_j| \leq C m \quad \text{by Lemma 1-(d) of NR.} 
\]  
(B.27)

\[
\sum_{j=1}^{m} \sum_{k=1}^{m} E|d_j V_k| = \sum_{j=1}^{m} \sum_{k=1}^{m} E|d_j V_k| + \sum_{j=1}^{m} \sum_{k=1}^{m} E|d_j V_k| 
\]  
(B.28)

\[
\leq C (m + mn) \quad \text{by Lemma 4.} 
\]  
(B.29)

\[
\sum_{j=1}^{m} \sum_{s=m+1}^{m} E|d_j V_s| \leq C mn \quad \text{by Lemma 4-(a).} 
\]  
(B.30)
by Lemma 6, \( \sum_{s}^{(i_1, i_2, \ldots, i_r)} \) denoting summations excluding \( s = i_1, i_2, \ldots, i_r \).

\[
\sum_{j=1}^{m} \sum_{k=j+1}^{m} \sum_{s=m+1}^{n} |e_{jk} V_s| = \sum_{j=1}^{m} \sum_{k=j+1}^{m} \sum_{s=m+1}^{n} (j, k, 1) E|e_{jk} V_s| + \sum_{j=1}^{m} \sum_{k=j+1}^{m} |e_{jk} V_k| \\
\leq C (mn^2 + mn) h^{-2} \quad \text{by Lemma 6.} \quad (B.31)
\]

\[
\sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{s=1}^{n} E|J_j W_{k,1} V_s| \\
= \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{s=1}^{n} (j, k, 1) E|J_j W_{k,1} E|V_s| + \sum_{j=1}^{m} \sum_{k=1}^{n} E|J_j W_{k,1}^2| \\
\leq C (mn^3 + mn^2 + m^2 n) h^{-1} \quad (B.32)
\]

by (iii), Lemma 1-(d) of NR, Lemma 4 of Robinson (1995) and Lemma 5.

\[
\sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{s=1}^{n} E|J_j W_{k,1} V_s| \\
= \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{s=1}^{n} (j, 1) E|J_j W_{k,1} E|V_s| + \sum_{j=1}^{m} \sum_{k=1}^{n} E|J_j W_{k,1}^2| \\
\leq C (mn^3 + mn^2) h^{-1} \quad (B.33)
\]

by (iii), Lemma 1-(d) of NR, Lemma 4 of Robinson (1995) and Lemma 5.

\[
\sum_{j=1}^{m} \sum_{m<k<1}^{n} \sum_{s=m+1}^{n} E|J_j W_{k,1} V_s| \\
= \sum_{j=1}^{m} \sum_{m<k<1}^{n} (k, 1) E|J_j W_{k,1} E|V_s| + \sum_{j=1}^{m} \sum_{m<k<1}^{n} (E|J_j W_{k,1} V_k| + E|J_j W_{k,1} V_1|) \\
\leq C (mn^3 + mn^2) h^{-1} \quad (B.34)
\]

by (iii), Lemma 1-(d) of NR, Lemma 4 of Robinson (1995) and Lemma 5. Therefore, substituting

(B.27)-(B.34) into (B.26), using \( 1 \leq m \leq n-1 \),
Similarly to (B.23)-(B.25), for all \( \ell \)，(B.36) gives

\[
|E(\mathcal{L}_m' e^{it\tilde{\theta}_n})| \leq C |\gamma(t)|^{m-\delta} \left( \frac{m}{n^{3/2} h^{d+2}} + \frac{mn}{n^{3/2} h^{d}} + \frac{mn^2}{n^{5/2} h^2} + \frac{mn^3}{n^{7/2} h^3} \right)
\]

\[
\leq \frac{C_1 m}{n^{1/2} h^2} |\gamma(t)|^{m-4},
\]

the third term in parentheses dominating for sufficiently large \( n \) by assumption (ix).

(c) Using (3.7) and Lemma 4 of Robinson (1995), we write

\[
|E(\mathcal{L}_m'e^{it\tilde{\theta}_n})| \leq C \left[ \frac{1}{n^{5/2} h^{d+2}} \sum_{j=1}^{m} \sum_{j=1}^{n} |E(\mathcal{W}_{1s} e^{it\tilde{\theta}_n})| + \frac{1}{n^{5/2}} \left( \sum_{j=1}^{m} \sum_{k=1}^{n} |E(\mathcal{W}_{1s} e^{it\tilde{\theta}_n})| + \sum_{j=1}^{m} \sum_{k=1}^{n} |E(\mathcal{W}_{1s} e^{it\tilde{\theta}_n})| \right) + \frac{1}{n^{7/2}} \left( \sum_{j=1}^{m} \sum_{k=1}^{n} |E(\mathcal{W}_{1s} e^{it\tilde{\theta}_n})| + \sum_{m<k} \sum_{j=1}^{m} \sum_{s=1}^{n} |E(\mathcal{W}_{1s} e^{it\tilde{\theta}_n})| \right) \right].
\]

Similarly to (B.23)-(B.25), for all \( j, k, l, s \),

\[
|E(\mathcal{W}_{1s} e^{it\tilde{\theta}_n})| \leq |E|\mathcal{W}_{1s}| |\gamma(t)|^{m-2},
\]

\[
|E(\mathcal{W}_{1s} e^{it\tilde{\theta}_n})| \leq |E|\mathcal{W}_{1s}| |\gamma(t)|^{m-3}.
\]

Substituting (B.37)-(B.39) into (B.36), we have, due to \( |\gamma(t)| \leq 1 \),

\[
|E(\mathcal{L}_m'e^{it\tilde{\theta}_n})| \leq C |\gamma(t)|^{m-4} \left[ \frac{1}{n^{5/2} h^{d+2}} \sum_{j=1}^{m} \sum_{j=1}^{n} |E|\mathcal{W}_{1s}| + \frac{1}{n^{5/2}} \left( \sum_{j=1}^{m} \sum_{k=1}^{n} |E|\mathcal{W}_{1s}| + \sum_{j=1}^{m} \sum_{k=1}^{n} |E|\mathcal{W}_{1s}| \right) + \frac{1}{n^{7/2}} \left( \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{s=1}^{n} |E|\mathcal{W}_{1s}| + \sum_{m<k} \sum_{j=1}^{m} \sum_{s=1}^{n} |E|\mathcal{W}_{1s}| \right) \right].
\]

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Applying Lemma 4 of Robinson (1995), Lemmas 7 and 8, and (ix),
\[
|E(\tilde{b}_3 e^{it\tilde{\theta}})| \leq C |\gamma(t)|^{m-4} \left( \frac{mn}{n^{5/2}h^{d+3}} + \frac{mn^2}{n^{5/2}h} + \frac{1}{n^{7/2}} \left( \frac{mn}{h^{d+3}} + \frac{mn^3}{h^3} \right) \right)
\]
\[
= C m |\gamma(t)|^{m-4} \left( \frac{1}{n^{3/2}h^{d+3}} + \frac{1}{n^{1/2}h} + \frac{1}{n^{5/2}h^{d+3}} + \frac{1}{n^{1/2}h^3} \right)
\]
\[
\leq \frac{C m}{n^{1/2}h^3} |\gamma(t)|^{m-4} .
\]

(d) Write, using (3.7) and Lemma 4 of Robinson (1995),
\[
|E|\tilde{b}_2|^2 \leq C \left[ \frac{1}{n^{3}h^{2d+4}} E \left| \sum_{i=1}^{m} V_i \right|^2 + \frac{1}{n^3} \left( E \left| \sum_{i=1}^{n} \sum_{s=1}^{m} d_i V_s \right|^2 + E \left| \sum_{i=1}^{m} \sum_{s=1}^{m} d_i V_s \right|^2 \right) \right.
\]
\[
+ \frac{1}{n^3} \left( E \left| \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{s=1}^{m} e_{ij} V_s \right|^2 + E \left| \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{s=1}^{m} e_{ij} V_s \right|^2 \right)
\]
\[
+ \frac{1}{n^3} \left( E \left| \sum_{i=1}^{n} \sum_{k=1}^{n} \left( \sum_{s=1}^{m} V_i W_{k1} V_s \right)^2 + E \left| \sum_{i=1}^{m} \sum_{k=1}^{m} \left( \sum_{s=1}^{m} V_i W_{k1} V_s \right)^2 \right. \right)
\]
\[
+ E \sum_{i=1}^{m} \sum_{k=m+1}^{n} \left( \sum_{s=1}^{m} V_i W_{k1} V_s \right)^2 \right) \right] . \quad (B.40)
\]

We show bounds only of some typical terms. Since \( V_i \) is an iid sequence with zero mean, due to Lemma 1-(d) of NR, \( E \left| \sum_{i=1}^{m} V_i \right|^2 = m E |V_1|^2 \leq C m . \)

Writing
\[
E \left| \sum_{i=1}^{m} \sum_{s=1}^{m} d_i V_s \right|^2 \leq C \left( E \left| \sum_{i=1}^{m} d_i V_i \right|^2 + E \left| \sum_{i=1}^{m} \sum_{s=1}^{m} d_i V_s \right|^2 + E \left| \sum_{i=1}^{m} \sum_{s=1}^{m} d_i V_s \right|^2 \right) , \quad (B.41)
\]
the first term in parentheses is bounded by
\[
m E |d_1 V_1|^2 + m (m-1) E |d_1 V_1| |d_2 V_2| \leq C m^2 \quad (B.42)
\]
due to (iii) and Lemma 4-(b). Since \( d_i \) and \( V_s \) are iid with zero mean,
\[
E \left| \sum_{i=1}^{m} \sum_{s=1}^{m} d_i V_s \right|^2 = \sum_{i=1}^{m-1} E (d_i^2) \sum_{s=i+1}^{m} (V_s^2) \leq C m^2 \quad (B.43)
\]
by Lemma 1-(d) of NR and (A.2) under (i). Similarly, using Lemma 4-(a),
\[
E \left| \sum_{s=1}^{m} \sum_{i=s}^{m} d_i V_s \right|^2 \leq \sum_{s=1}^{m} \sum_{i=s}^{m} E (d_i^2) E (V_s^2) \leq C m n . \quad (B.44)
\]

From (B.41)-(B.44),
\[
E \left| \sum_{i=1}^{n} \sum_{s=1}^{m} d_i V_s \right|^2 \leq C (m^2 + mn) .
\]

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Similarly,
\[ E \left| \sum_{i=1}^{m} \sum_{j=m+1}^{n} d_{ij} V_{s} \right|^2 = \sum_{i=1}^{m} E \left( d_{i}^2 \right) \sum_{s=m+1}^{n} E \left( V_{s}^2 \right) \leq Cmn. \]

We next consider
\begin{align*}
E \left| \sum_{i=1}^{m} \sum_{j=i+1}^{n} \sum_{s=1}^{n} e_{ij} V_{s} \right|^2 & \leq C \left\{ E \left[ \sum_{i=1}^{m} \sum_{j=i+1}^{n} \sum_{s=1}^{n} e_{ij} V_{s} \right]^2 + E \left[ \sum_{i=1}^{m} \sum_{j=i+1}^{n} e_{ij} V_{j} \right]^2 \\ & + E \left[ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{s=j+1}^{n} e_{ij} V_{s} \right]^2 + E \left[ \sum_{i=1}^{m} \sum_{j=1}^{n} e_{ij} V_{j} \right]^2 \right\}. \end{align*}

Due to (iii), \( E(e_{ij} | i) = E(e_{ij} | j) = 0 \), \( E(V_{s}) = 0 \) and Lemma 6, the triple summation terms on the right of (B.45) is \( O \left( (m^3 + m^2n + mn^2) h^{-\delta - 4} \right) \). Using Lemma 6 and Hölder's inequality, the second term in (B.45) is
\[ \sum_{i=1}^{m} \sum_{j=i+1}^{n} E(e_{ij} V_{i})^2 + 2 \sum_{i=1}^{m} \sum_{k=i+1}^{n} \sum_{j=k+1}^{n} E(e_{ij} V_{i} e_{kj} V_{k}) \leq C \left[ mn E(e_{12} V_{1})^2 + m^2 n \{ E(e_{13} V_{1})^2 E(e_{23} V_{2})^2 \}^{1/2} \right] \leq C m^2 n h^{-\delta - 4}. \]

Similarly, the fourth term of (B.45) is \( O \left( m^3 h^{-\delta - 4} \right) \). Using Lemma 5, as above, the terms involving \( V_{i} V_{k} V_{s} \) in (B.40) are \( O \left( (m^4 + m^3n + m^2n^2 + mn^3) h^{-\delta - 2} \right) \), so by (ix)
\[ E|\hat{\beta}_{2m}|^2 \leq C \left( \frac{m}{n h^{d+2}} \left( \frac{m}{n^2 h^{d+2}} \right) + \frac{C}{n^3} (m^2 + mn) \right) \]
\[ + \frac{C}{n^5} (m^3 + m^2n + mn^2) h^{-\delta - 4} \]
\[ + \frac{C}{n^7} (m^4 + m^3n + m^2n^2 + mn^3) h^{-\delta - 2} \]
\[ \leq C \left( \frac{1}{n^3 h^{2d+4}} + \frac{1}{n^2} \right). \]

(e) The derivation is similar using Lemma 4 of Robinson (1995), Lemmas 7 and 8. As in (d), we can show
\[ E|\hat{\beta}_{3m}|^2 \leq \frac{C}{n^5 h^{2d+4}} mnh^{-\delta - 2} + \frac{C}{n^5} (m^3 + m^2n + mn^2) h^{-\delta - 2} \]
\[ + \frac{C}{n^7} (m^4 + m^3n + m^2n^2 + mn^3) h^{-3d+6} \]
\[ \leq \frac{C m}{n^4 h^{3d+6}}. \]
(f) Write

\[ |E \beta_j'' e^{it \beta_n}| = |E Q \tilde{w} e^{it \beta_n}| \leq |E Q \tilde{w} e^{it \beta_n}| + |E Q \tilde{w} e^{it \beta_n}|. \]  

By (3.7),

\[ |E (Q \tilde{w} e^{it \beta_n})| \]

\[ \leq \frac{C}{n^{7/2}} \sum_{j=1}^{n-1} \sum_{l=1}^{n-1} \sum_{s=1}^{n-1} E \left\{ |(V_j + V_k) W_{jk} - V_j - V_k| W_{ls} e^{it \beta_n} \right\} \]

\[ \leq \frac{6C}{n^{7/2}} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \sum_{s=1}^{n-1} E \left\{ |(V_j + V_k) W_{jk} - V_j - V_k| W_{ls} \right\} |Y(t)|^{m-4} \]

\[ + \frac{6C}{n^{7/2}} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \sum_{s=1}^{n-1} E \left\{ |(V_j + V_k) W_{jk} - V_j - V_k| W_{ks} \right\} |Y(t)|^{m-3} \]

\[ + \frac{C}{n^{7/2}} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} E \left\{ |(V_j + V_k) W_{jk} - V_j - V_k| W_{jk} \right\} |Y(t)|^{m-2} \]

\[ \leq \frac{C n^{1/2}}{n^{3/2}} E \left\{ |(Y_1 + Y_2) W_{12} - V_1 - V_2| W_{34} \right\} |Y(t)|^{m-4} \]

\[ + \frac{C}{n^{1/2}} E \left\{ |(Y_1 + Y_2) W_{12} - V_1 - V_2| W_{13} \right\} |Y(t)|^{m-3} \]

\[ + \frac{C}{n^{1/2}} E \left\{ |(Y_1 + Y_2) W_{12} - V_1 - V_2| W_{12} \right\} |Y(t)|^{m-3}. \]  

(B.48)

Using (i), (iii), Lemmas 1-(d), 4 of NR, (A.1) and Lemma 4 of Robinson (1995), the first expectation of (B.48) is bounded by

\[ C E\left\{ \left( |Y_1| + |Y_2| + 1 \right) |W_{12}| \right\} E|W_{34}| \leq C h^{-2}. \]

Using (i), Lemmas 1-(d), 4 of NR, (A.1) and Lemma 4 of Robinson (1995), the second expectation of (B.48) is bounded by

\[ C E\left\{ \left( |Y_1| + |Y_2| + 1 \right) |W_{12}| \right\} \leq C h^{-1} E\left\{ \left( |Y_1| + |Y_2| + 1 \right) \right\} \]

\[ \leq C h^{-1} E\left\{ \left( |Y_1| + |Y_2| + 1 \right) \right\} . \]

Similarly to Lemma 1 and (A.5), \( E|Y_1 W_{12}| + E|Y_1^2 W_{12}| + E|Y_1 Y_2 W_{12}| = O(h^{-1}) \) so that the above quantity is \( O(h^{-2}) \). The third expectation of (B.48) is bounded by

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due to Lemmas 1-(d), 4 of NR, Lemma 4 of Robinson (1995) and Lemma 2. Therefore,

$$|E(\Omega_{1} W e^{it\theta_{1}})| \leq C \left( \frac{n^{1/2}}{h^{2}} + \frac{1}{n^{3/2} h^{d+2}} \right) |\gamma(t)|^{m-4} \leq C \frac{n^{1/2}}{h^{2}} |\gamma(t)|^{m-4} .$$

The second term of (B.47) is bounded by, using (3.7),

$$\frac{C}{n^{9/2}} \left| \sum_{r=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n-1} \sum_{l=1}^{n} \sum_{s=1}^{n} E(V_{1} W_{12} W_{13}^{t} e^{it\theta_{r}}) \right|$$

$$\leq \frac{C}{n^{9/2}} \left( \sum_{r=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n-1} \sum_{l=1}^{n} \sum_{s=1}^{n} E(V_{1} W_{12} W_{13}^{t} e^{it\theta_{r}}) \right) |\gamma(t)|^{-m+5}$$

$$+ \frac{C}{n^{9/2}} \left( \sum_{r=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n-1} \sum_{l=1}^{n} \sum_{s=1}^{n} E(V_{1} W_{12} W_{13}^{t} e^{it\theta_{r}}) \right) |\gamma(t)|^{-m+4}$$

$$+ \frac{C}{n^{9/2}} \left( \sum_{r=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n-1} \sum_{l=1}^{n} \sum_{s=1}^{n} E(V_{1} W_{12} W_{13}^{t} e^{it\theta_{r}}) \right) |\gamma(t)|^{-m+3}$$

$$\leq C \frac{n^{1/2}}{n^{9/2}} |V_{1}| |E|W_{23}| |W_{45}| |\gamma(t)|^{-m+5}$$

$$+ \frac{C}{n^{1/2}} \left( |V_{1}| |E|W_{23}|W_{24}| + |V_{1}| W_{14} |E|W_{23}| \right) |\gamma(t)|^{-m+4}$$

$$+ \frac{C}{n^{3/2}} \left( |V_{1}| |E|W_{23}|^{2} + |V_{1}| W_{13} |E|W_{23}| \right) |\gamma(t)|^{-m+3}$$

$$\leq C \left( \frac{n^{1/2}}{h^{2}} + \frac{1}{n^{1/2} h^{2}} + \frac{1}{n^{3/2} h^{d+2}} \right) |\gamma(t)|^{-m+5}$$

by (i), (iii), Lemmas 1-(d), 4 of NR, Lemma 4 of Robinson (1995) and Lemma 1. Then apply (ix) □
References

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