Lucien Foldes and Pauline Watson

# Time series analysis of UK and US equity portfolios 1926-70

## Discussion paper

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by

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CONTENTS

# I INTRODUCTION

This paper deals with the time series properties of returns to equity portfolios in the UK and the US during the period 1926-70. The UK data used are two quarterly series of returns to a group of quoted companies for the period 1926-70 prepared by the authors, one for a portfolio weighted by the market value of equity, the other for an equally weighted portfolio; each series is available in money and in real terms. A description of the derivation of these series as well as the data themselves will be found in [1] and [1a]. For comparison with our value weighted portfolio we have used Ibbotson and Sinquefield's series for the US for the years 1926-75 [2]. No suitable quarterly series have been found to compare with our equally weighted series, but this is probably not of major importance since differences in behaviour between the two UK series as revealed by the techniques applied in this paper are usually small. Where only one set of figures is given below for the UK, these are value weighted unless the contrary is stated. Since the periods for which the two value weighted series are available do not correspond, we confine ourselves in this paper to the study of returns in the overlapping period 1926-70, which for purposes of analysis has been divided into the sub-periods 1926-39, 1939-51 and 1951-70 (sometimes referred to as sub-periods 1, 2 and 3 respectively). It will be seen that the behaviour in 1939-51 is consistently odd and perhaps not too much significance should be attached to the results for this period. In all, there are four series for the UK (value/equally weighted, real/money) and two for the US (real/money), each of which may be studied for the whole period and for the three sub-periods. In the preliminary discussion which follows we shall often refer simply to 'the series' without mentioning these distinctions.

Our statistical analysis takes as its starting point the strongest hypothesis about the cumulative total logarithmic returns process in the two countries which might reasonably be entertained, namely that it is a two-dimensional Brownian motion with drift; the process observed at quarterly intervals is then a normally distributed two-dimensional random walk with drift. This means that the quarterly total log-returns process is a sequence of independent, identically distributed, bivariate normal variables; then for each country separately the quarterly log-returns have independent, identical, normal distributions. We investigate the various ways in which this 'null' hypothesis might fail to hold, considering among other questions those of stationarity, normality and absence of auto-correlation. The general conclusion is that the hypothesis is an approximate description of the series and in most cases is not rejected by formal statistical tests; at the same time, close examination of the data suggests that there may well be certain systematic departures from the hypothesis. In the literature on efficient markets, weaker theories are also often considered, for example that the cumulative process is a martingale or sub-martingale. Some of the tests which we perform are also appropriate for testing these theories; thus, if the cumulative de-trended return is a martingale, it will be free from auto-correlation. In investigating the various properties we confine ourselves to the time series of returns themselves and make no attempt to describe their behaviour in terms of exogenous variables. The statistical discussion proceeds as follows. In Section II we consider briefly the question of lognormality, i.e. the normality of the logarithms of returns. There are some indications that the sample distributions are too highly concentrated about the mean for normality and that there are too many outlying values, but statistical tests are not sufficiently conclusive for the hypothesis to be rejected, particularly for the money terms series. In the further work we therefore do not consider the alternative hypotheses that the process is stable but of infinite variance or that the distribution is a mixture of a normal distribution and another which gives rise to the outlying values.

Stationarity of the series, at least over sub-periods, is desirable for statistical testing, and also highly relevant to the possibility of estimating the rates of return on equities to be expected in future. We have used two methods of testing for stationarity - comparing means between sub-periods and applying tests for trend to the series themselves. In addition, the sample variance has been examined for differences between periods. These tests and their results are described in Section III.

The above approach is not very sophisticated, but the tests have been enough to show that it is difficult to reject the hypothesis of stationarity of the mean in spite of large historical differences in returns. Our general conclusion is that the variability of the series is so great that we cannot make conclusive tests of differences between means. As regards variance, superficial examination suggests that in the UK this has been fairly constant over the period of our study. In the US, however, the variance does appear to be significantly greater in the first sub-period than in the later ones.

Other work which we have done - for example to test whether equities are on average more profitable than gilts - leads to similar difficulties. The sample variance is too great to allow means to be accurately estimated and comparisons between means are therefore inconclusive.

Strictly speaking, the tests for stationarity of means, variances and correlation coefficients used in Section III presuppose independence of the observations, which is tested only in later parts of the paper. If the hypothesis that quarterly log-returns for the UK and US are independent observations of the same bivariate normal distribution is accepted, either for the full period or for a sub-period, it remains to estimate the parameters of this distribution and in particular to test whether the means and standard deviations differ between the two countries and whether the two series are significantly correlated. Since these questions turn on the same statistics and methods as the discussion of stationarity, it is convenient to deal with them in the same Section. It is found that differences in mean returns between the two countries cannot be shown to be significant, but the US variances are significantly greater than those in the UK in 1926-39 and to a lesser extent in 1951-70. Point estimates for the parameters of real terms quarterly log-returns for 1926-70 as a whole are 0.014 and 0.018 for the UK and US means respectively (corresponding to about 1.4% and 1.8% rates of return per quarter), with 0.074 and 0.094 for the standard deviations and 0.357 for the correlation coefficient.

Despite the inconclusiveness of the tests of stationarity, we assume during the remaining investigations that the quarterly log-returns process is second order stationary. The next question to be considered is the existence of auto-correlation of any order. This is, of course, relevant to the possibility of making money by using knowledge of the past to improve prediction of future returns. Before applying more general techniques for identifying cycles, we carried out a separate investigation to see whether there are seasonal variations. This was prompted by the fact that we have quarterly data, and by the widely held belief that such cycles exist, which is expressed in such stock market lore as "Sell in May - go away". Such variations might be set up, for example, by the annual timetable of financial events and the publication of financial statistics. In Section IV we describe tests carried out on seasonal mean log-returns and on seasonal ranks. At first sight there do seem to be some signs of seasonal variation, at least in the frequency with which the four ranks occur in the quarters, but their significance is uncertain.

Section V describes further work in which the problem of auto-correlation has been approached in two ways: in the time domain by building auto-regressive models and in the frequency domain by spectral analysis. Two features are apparent, namely some positive first order auto-regressive behaviour in the US series and a cycle of approximate length 3 quarters in the UK. The latter feature can be linked with a certain pattern in the auto-covariance function of the series at low lags. As in other aspects of the work on these series, however, we cannot show that the features are statistically significant.

In Section VI we describe another way of approaching the question of variance and correlation which is more intuitively related to problems as seen by the investor, whose direct interest is in the movement of the return to his portfolio and the risk involved as he looks further into the future. Correlogram and spectrum are statistical representations of properties which are relevant to the investor's risk, but the relationship is not immediate. Even a perfectly estimated correlogram will not show at a glance how the risk, as measured by variance of log-return, moves as one looks into the future. We have therefore considered the matter in a third way, by examining the plot of average variance per quarter for different holding periods, which we have called the 'variance curve'. This shows how the risk of an investment behaves with increasing length of holding. The null hypothesis of a Brownian motion or random walk with drift implies a horizontal curve, i.e. constant average variance of log-return.

Despite its intuitive attraction, the variance curve is inconvenient as a basis for exact tests of significance, because successive values of the curve are not independent random variables. Such rough and ready tests as have been done lead to the usual conclusion - there do seem to be substantial deviations from the null hypothesis, but it cannot be formally rejected.

# II LOGNORMALITY

## (i) Discussion of results

Our investigations of lognormality, i.e. the normality of the variables $\log_e(1+r_t)$, where 1+r is the quarterly return, make use of the Kolmogorov-Smirnov maximum deviation test as well as a $\chi^2$ test. Histograms have been constructed for the UK and US series in money and real terms for the full period and the three sub-periods. In nearly all cases the values are found to cluster about the mean more than would be expected for a sample from a normal population. There are also in many cases several outlying values, particularly negative ones. All four series show variation in the shape of the histogram from one period to another and there is some indication from the full period that the distribution is negatively skewed.

The statistical tests have been carried out on all four series (UK/US, money/real) for the full period and the three sub-periods. The results are on the whole inconclusive. The only results of any significance are for the $\chi^2$ test for the UK real terms series in 1951-70 and the US real terms series in 1939-51, when the hypothesis of normality is rejected at the 1% level of significance. Thus we have no firm evidence for rejecting the hypothesis of normality for the money terms series in spite of the deviations mentioned above. For the real terms series there is more evidence for rejection, but it is inconclusive since results vary from period to period and according to the test performed.

The histograms for the equally weighted series appear to depart from the normal curve rather more than do those for the value weighted series, but the results of statistical tests are only slightly more significant and therefore are not presented here.

(ii) Cumulative relative frequency

Graph IIa shows the graphs of cumulative relative frequency derived from the histograms of the UK and US series in money terms, 1926-70, plotted against the corresponding normal curve. The step function derived from a normal distribution would lie wholly below the curve, touching it at the corners of the steps. In Graph IIa the actual step functions lie above the normal curve at around -3 standard deviations, indicating outlying negative values. Near the mean, the step functions lie below the normal curve, indicating that fewer than half the values are less than the mean, i.e. that the distribution is negatively skewed. At 1 standard deviation, however, the step functions have increased above the normal curve, showing a peak between the mean and 1 standard deviation which would be absent from a normal distribution.

(iii) Kolmogorov - Smirnov test

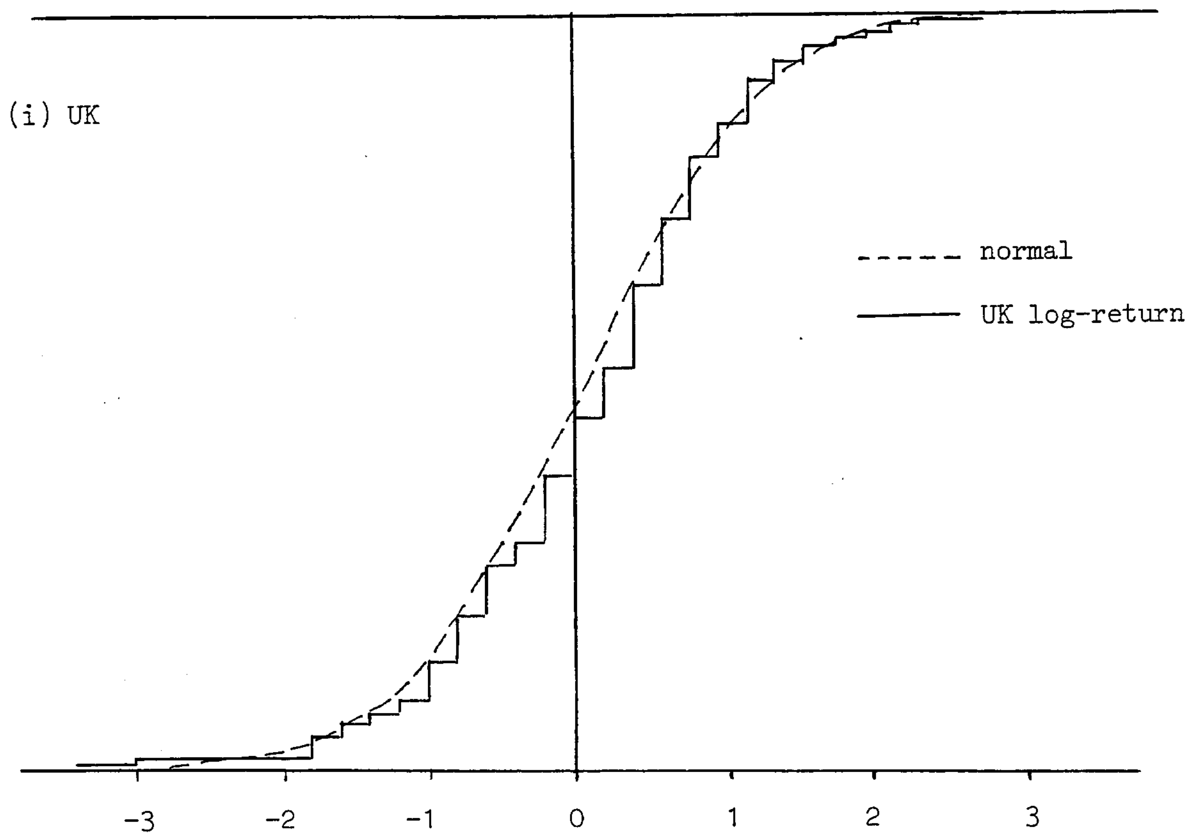
The departures from normality described above occur with considerable consistency from series to series and from period to period. We use the Kolmogorov - Smirnov maximum deviation test to determine whether these deviations warrant rejection of the hypothesis of normality. The test statistic is the maximum vertical distance between the curves in Graph IIa (measured from the corner of the step to the normal curve). Table IIa(i) gives this maximum deviation K for UK and US log-returns, in money and real terms, for all periods, and Table IIa(ii) gives critical values of K corresponding to different probability levels $\alpha$. In no case is K larger than the critical value even for $\alpha = 0.1$. Thus we cannot on this evidence rule out the possibility that the underlying distribution is normal despite the apparent deviations.

(iv) $\chi^2$ test

For each period and each series of log-returns, values of $\chi^2$ have been calculated and Tables used to obtain the associated values of Q, the probability that values of $\chi^2$ at least as large as those observed would have arisen by chance under the assumption of normality. These values of Q are set out in Table IIb. It is seen that the assumption of normality is rejected at the 1% level for the UK real terms series in 1951-70 and for the US real terms series in 1939-51. No other results are significant even at the 20% level. Thus we still have no convincing evidence that the money terms series are not normal, although two results for the real terms series indicate absence of normality.

Graph IIa

Cumulative relative frequency of log-returns in money terms compared with normal distribution, 1926-70



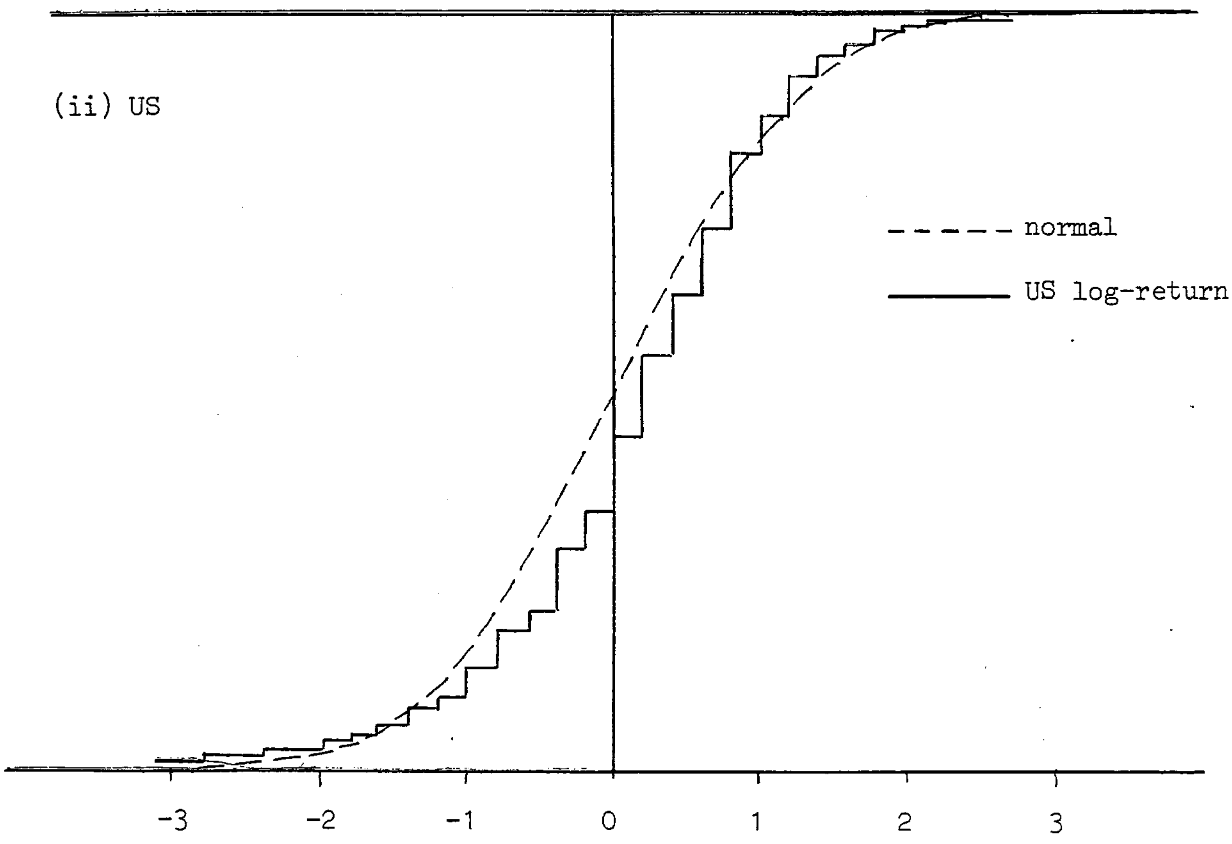



Table IIa

Kolmogorov - Smirnov test for normality of log-returns

(i) Values of maximum deviation K

| Period | UK | | US | |
|---|---|---|---|---|
| | Money Terms | Real Terms | Money Terms | Real Terms |
| 1926-70 | .049 | .054 | .074 | .053 |
| 1926-39 | .093 | .050 | .093 | .074 |
| 1939-51 | .116 | .100 | .090 | .090 |
| 1951-70 | .066 | .095 | .052 | .052 |

(ii) Critical values of K for various probability levels $\alpha$

| Period | Obs | $\alpha$ = .1 | $\alpha$ = .05 | $\alpha$ = .02 |
|---|---|---|---|---|
| 1926-70 | 179 | .121 | .134 | .150 |
| 1926-39 | 54 | .163 | .181 | .203 |
| 1939-51 | 46 | .177 | .196 | .219 |
| 1951-70 | 79 | .136 | .151 | .168 |

Table IIb

$\chi^2$ test for normality of log-returns

Values of Q

| Period | DF | UK | | US | |
|---|---|---|---|---|---|
| | | Money Terms | Real Terms | Money Terms | Real Terms |
| 1926-70 | 16 | .35 | .24 | .42 | .23 |
| 1926-39 | 6 | .54 | .996 | .27 | .72 |
| 1939-51 | 6 | .62 | .68 | .49 | .01 |
| 1951-70 | 8 | .71 | .01 | .48 | .29 |

## III STATIONARITY AND UK/US COMPARISONS

### (i) Discussion of results

Stationarity of the mean of the log-returns process has been investigated by two methods. The sample means of log-returns for the same country in different sub-periods have been tested for equality, and the log-return series themselves have been tested for trend. In addition, the mean log-returns in different countries have been compared period by period. Although some of the differences between periods and countries look large, there is very little evidence of significance. In some isolated cases, non-parametric tests for trend indicate significant changes from one sub-period to another. There is, however, no consistency in the results from different tests, and most of them are spectacularly inconclusive. In particular we cannot on this evidence reject the hypothesis of stationarity of the mean.

As regards variance, however, it does appear that the variance of the US series in 1926-39 is substantially and significantly greater than the UK variance in that sub-period, and also to a lesser extent during 1951-70. In addition, it seems that the US variance in 1926-39 was greater than that in the other two sub-periods. These results are obtained in both real and money terms. The UK series show no significant change in variance from period to period.

There is strong evidence of correlation between the UK and US series in all periods in both money and real terms, the coefficient being about .47 for the inter-war period and about .38 post-war.

All statistics in Table III below relate to value weighted portfolios. Differences in mean and variance from period to period for log-returns to the UK equally weighted portfolio are of the same order as those for the value weighted series and the results of relevant tests are also similar.

(ii) Comparisons of means

(a) Comparisons between periods

Estimates of means and standard deviations period by period for UK and US log-returns in money and real terms are shown in Table IIIa(i), and 95% confidence intervals for the money terms estimates in Table IIIa(ii). (Real terms confidence intervals are very similar.) Inspection shows that, except for the US in 1926-39, the standard deviation is fairly constant, so we first use a t-test for equality of means, see [3] p. 21. In each case, the probability Q that a difference in mean at least as great as that observed would occur even if the population means were the same is shown in Table IIIb under 'basic t-test'. It is seen that the lowest of these probabilities is 0.2, for the UK series in money terms, comparing sub-periods 1 and 3; thus, even in this case, a difference in means at least as great as that observed would occur in one sample out of five if the population means were the same.

We have in the above test assumed constant variance, but our investigations into the stationarity of variance described below indicate that this condition is not satisfied by our data, at least in the US. We have therefore carried out two further tests described in [4] Vol. II pp. 141-6 which are appropriate to this situation.

The first of these involves setting up a test statistic t which is a function of the ratio $\theta$ between the population variances; t can then be evaluated for various different values of $\theta$, whose true value is of course unknown. Table IIIb gives the probability Q described above associated with the maximum value of t as $\theta$ varies: lower values of t lead to less conclusive results. In all cases these values of Q are equal to or only a little smaller than those obtained for the basic test.

The second test which does not require the assumption of constant variance (Scheffé's test) consists of a t-test based on the smaller of the two samples under consideration and a random subset of equal size selected from the larger. The associated Q-values are also shown in Table IIIb. In most cases these values are a little larger than those obtained for the basic test.

Since the probabilities resulting from these last two tests are not greatly different from those of the basic test we may conclude that stationarity of variance is not fundamental to the results. In any case, we cannot conclude that the mean of any of the series is non-stationary.

(b) Comparison of UK and US means

To compare the means of the UK and US series we confine ourselves to comparisons within sub-periods and consider

$$y_i = x_{1,i} - x_{2,i}$$

where subscript 1 refers to the UK and 2 to the US. If the UK and US means are the same, $y_i$ has the theoretical mean zero and variance

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \ .$$

We may then use a t-test to discover whether $\bar{y}$ is significantly different from zero. The results of this test are given in Table IIIc and are seen to provide no significant evidence of difference between the means.

(c) Non-parametric tests for trend

Two non-parametric tests for trend described in [5] pp. 390-1 have been carried out on all four series for the full period and the three sub-periods. The first test involves computing the number of positive differences $x_{t+1} - x_t$ in the series and the second the calculation of Spearman's rank correlation coefficient between the $x_t$ arranged in increasing order and the same observations in temporal order. Table IIId gives the probabilities Q that values of the test statistics at least as large as those observed would occur even if there were no trend. The results of the first test are completely inconclusive, except for the UK series in real terms in 1926-39, where a value of .15 is obtained for Q. Smaller values of Q are obtained from Spearman's ρ, particularly for the US series (real and money) in 1951-70 where values of .02 to .03 are recorded. This means that in only one sample out of about forty would such a large value of ρ be obtained if there were no trend in the population. The corresponding results for the first test are, however, by no means so conclusive.

(iii) Comparisons of variances

(a) Comparisons between periods

We have tested for differences in variance between periods using an F-test based on the ratio of the two sample variances, see [3] pp. 38-9. These ratios for all four series in real and money terms are set out in Table IIIe(i). Table IIIe(ii) gives the critical values with which these ratios should be compared at different levels of significance.

It is seen that the F-values for the US series in the cases where sub-period 1 is compared with sub-periods 2 and 3 exceed the critical values at a significance level of 0.5% by a considerable amount. We must therefore conclude that the variance of the US series in sub-period 1 is significantly greater than that in the other two sub-periods, in both real and money terms.

(b) Comparison of UK and US variances

A different procedure is needed to compare variances between the two countries because returns are not independent. We use a method described by Kendall and Stuart [4] Vol. ii p.133 Exercise 20.19 to derive confidence intervals for the variance ratio $\lambda = \sigma_2^2/\sigma_1^2$ from a transformation of Student's t. The values of the observed ratio $L = s_2^2/s_1^2$ are shown in part (i) of Table IIIf and the confidence intervals for various probability levels in part (ii). It is seen that the null hypothesis $\lambda = 1$ is decisively rejected for the period 1926-39, when the observed variance for the US was more than 4 times that for the UK, both in money and in real terms. The rejection of the hypothesis for the full period 1926-70 appears to be due mainly to the figures for the inter-war period. The values of L for 1939-51 are entirely consistent with the null hypothesis, since the value of one lies well within all the confidence intervals considered. For 1951-70 this value lies outside the 99% interval but inside the 97.5% interval in the money case, providing fairly strong but not conclusive grounds for rejection; in the real case the evidence is somewhat weaker.

(iv) Correlation

We have tested for correlation between the UK and US series using the test statistic

$$t = \sqrt{\frac{(n-2)r^2}{1-r^2}}$$

where n is the number of observations and r is the correlation coefficient; this is distributed as Student's t with n-2 degrees of freedom, see [4] Vol. II p. 296. Values of Q, the probability that t-values at least as large as those obtained would occur if $\rho = 0$, are given in Table IIIg(i). This probability is less than 0.06% except in 1939-51 where it is approximately 6.5% in real terms and 2% in money terms. The same Table shows that the correlation coefficient is fairly stable from period to period, particularly in money terms. The confidence intervals shown in Table IIIg(ii) are quite wide, so that it is clear without further tests that the coefficients cannot be shown to differ significantly. On the other hand, we can confidently assert that there is significant positive correlation between the UK and US series.

Table IIIa

(i) Means and standard deviations of UK and US log-returns

| Period | | UK | | US | |
|---|---|---|---|---|---|
| | | Money Terms | Real Terms | Money Terms | Real Terms |
| 1926-70 | Mean | .02144 | .01398 | .02249 | .01808 |
| | S.D. | .07327 | .07433 | .09437 | .09435 |
| 1926-39 | Mean | .01328 | .01502 | .01177 | .01656 |
| | S.D. | .06270 | .06303 | .13651 | .13312 |
| 1939-51 | Mean | .01602 | .00204 | .02775 | .01453 |
| | S.D. | .07068 | .07434 | .07934 | .08445 |
| 1951-70 | Mean | .03016 | .02023 | .02677 | .02119 |
| | S.D. | .08097 | .08122 | .06259 | .06436 |

(ii) 95% Confidence Intervals for means and standard deviations of UK and US log-returns in money terms

| Period | Mean | | Standard Deviation | |
|---|---|---|---|---|
| | UK | US | UK | US |
| 1926-70 | (.0107, .0322) | (.0087, .0363) | (.0644, .0850) | (.0830, .1095) |
| 1926-39 | (-.0053, .0319) | (-.0289, .0523) | (.0521, .0788) | (.1134, .1715) |
| 1939-51 | (-.0033, .0353) | (.0061, .0495) | (.0595, .0871) | (.0668, .0977) |
| 1951-70 | (.0121, .0483) | (.0128, .0408) | (.0700, .0959) | (.0541, .0741) |

Table IIIb

Comparisons of means between periods

Values of Q for various t-statistics

| Periods compared | Test | UK | | US | |
|---|---|---|---|---|---|
| | | Money Terms | Real Terms | Money Terms | Real Terms |
| 1 and 2 | Basic t-test | .84 | .35 | .49 | .93 |
| | Max t = t(θ) | .84 | .35 | .47 | .93 |
| | Random subset | .85 | .41 | .51 | .93 |
| 2 and 3 | Basic t-test | .33 | .22 | .94 | .62 |
| | Max t = t(θ) | .30 | .20 | .94 | .62 |
| | Random subset | .40 | .29 | .94 | .65 |
| 3 and 1 | Basic t-test | .20 | .69 | .40 | .79 |
| | Max t = t(θ) | .18 | .68 | .39 | .79 |
| | Random subset | .20 | .70 | .46 | .81 |

Table IIIc

Comparison of means between UK and US

Values of Q for t-test

| Period | Money Terms | Real Terms |
|---|---|---|
| 1926-70 | .88 | .57 |
| 1926-39 | .93 | .92 |
| 1939-51 | .36 | .38 |
| 1951-70 | .70 | .92 |

Table IIId

Non-parametric tests for trend

Values of Q

| Period | Test | UK | | US | |
|---|---|---|---|---|---|
| | | Money Terms | Real Terms | Money Terms | Real Terms |
| 1926-70 | +ve diffs | .38 | .30 | .46 | .46 |
| | Spearman's ρ | .08 | .24 | .30 | .16 |
| 1926-39 | +ve diffs | .32 | .15 | .46 | .46 |
| | Spearman's ρ | .25 | .13 | .41 | .33 |
| 1939-51 | +ve diffs | .38 | .46 | .46 | .46 |
| | Spearman's ρ | .24 | .47 | .17 | .16 |
| 1951-70 | +ve diffs | .38 | .50 | .38 | .50 |
| | Spearman's ρ | .37 | .41 | .03 | .02 |

Table IIIe

Comparisons of variances between periods

(i) Values of $F = s_a^2/s_b^2$

| Periods compared | UK | | US | |
|---|---|---|---|---|
| | Money Terms | Real Terms | Money Terms | Real Terms |
| 1 and 2 | 1.27 | 1.39 | 2.97 | 2.48 |
| 2 and 3 | 1.31 | 1.19 | 1.61 | 1.72 |
| 3 and 1 | 1.67 | 1.66 | 4.28 | 4.76 |

(ii) Percentage points of the F-distribution

| Periods compared | Degrees of Freedom | Probability Level | | | |
|---|---|---|---|---|---|
| | | 5% | 2.5% | 1% | 0.5% |
| 1 and 2 | $\nu_1 = 53$<br>$\nu_2 = 45$ | 1.63 | 1.80 | 2.00 | 2.17 |
| 2 and 3 | $\nu_1 = 45$<br>$\nu_2 = 78$ | 1.55 | 1.68 | 1.86 | 1.99 |
| 3 and 1 | $\nu_1 = 78$<br>$\nu_2 = 53$ | 1.46 | 1.65 | 1.82 | 1.94 |

## Table IIIf

### Comparison of variances between UK(=1) and US(=2)

#### (i) Values of $L = s_2^2/s_1^2$

| Period | Money Terms | Real Terms |
|---|---|---|
| 1926-70 | 1.66 | 1.61 |
| 1926-39 | 4.74 | 4.46 |
| 1939-51 | 1.26 | 1.29 |
| 1951-70 | 1.66 | 1.59 |

#### (ii) Confidence intervals for $\lambda = \sigma_2^2/\sigma_1^2$

| Prob. level | 95% | 97.5% | 99% | 99.5% |
|---|---|---|---|---|
| 1926-70 | | | | |
| Money | (1.48, 1.86) | (1.46, 1.89) | (1.43, 1.93) | (1.41, 1.95) |
| Real | (1.45, 1.79) | (1.43, 1.82) | (1.40, 1.85) | (1.38, 1.87) |
| 1926-39 | | | | |
| Money | (2.80, 8.04) | (2.56, 8.78) | (2.36, 9.52) | (2.20,10.23) |
| Real | (2.63, 7.56) | (2.41, 8.25) | (2.22, 8.95) | (2.07, 9.61) |
| 1939-51 | | | | |
| Money | (0.72, 2.21) | (0.66, 2.41) | (0.59, 2.68) | (0.56, 2.82) |
| Real | (0.76, 2.19) | (0.70, 2.38) | (0.63, 2.63) | (0.60, 2.77) |
| 1951-70 | | | | |
| Money | (1.11, 2.49) | (1.02, 2.69) | (0.96, 2.88) | (0.91, 3.03) |
| Real | (1.06, 2.38) | (0.98, 2.58) | (0.92, 2.76) | (0.87, 2.90) |

# Table IIIg

## (i) Correlation between UK and US series

| Period | Correlation Coefficient | | Value of Q | |
|---|---|---|---|---|
| | Money Terms | Real Terms | Money Terms | Real Terms |
| 1926-70 | .371 | .357 | <.0002 | <.0002 |
| 1926-39 | .469 | .471 | <.0002 | .0004 |
| 1939-51 | .344 | .274 | .0180 | .0656 |
| 1951-70 | .381 | .382 | .0006 | .0006 |

## (ii) 95% Confidence intervals for UK/US correlation coefficients in money terms

| Period | 95% Confidence interval |
|---|---|
| 1926-70 | (.10, .58) |
| 1926-39 | (.22, .67) |
| 1939-51 | (.15, .56) |
| 1951-70 | (.25, .49) |

# IV SEASONAL VARIATION OF LOG-RETURNS

## (i) Discussion of results

In this section we consider in a simple way the significance of differences in mean quarterly log-returns at different times of the year. We consider the series in money terms only, since there are seasonal fluctuations in the price index which will show up in the real terms series. Table IVa gives means and standard deviations of UK and US quarterly log-returns for the full period 1926-70 and for all three sub-periods. The row headed 'Feb' gives figures for the series of log-returns arising in quarters beginning on 1st February each year and so on.

Table IVa shows that there was considerable variation of seasonal mean log-returns in both countries. For the UK value weighted series the overall mean log-return for all quarters is .0214, composed of .0398 in the August quarter, .0269 in November, .0165 in February and .0027 in May. The same pattern is apparent in the seasonal means of the equally weighted series, the overall mean of .0270 comprising .0487 in August, .0326 in November, .0199 in February and .0068 in May. At first sight these figures do lend some support to the Stock Exchange saying "Sell in May - go away". There is variation of a similar order in the US mean log-returns, but in this case the largest value occurs in May and the smallest in August. The total mean is .0225, comprising .0061 in August, .0345 in November, .0116 in February and .0380 in May.

Tests for determining the significance of these differences in mean log-returns are discussed below. Briefly, if we could take several sample sets of quarterly mean log-returns for the full period we would expect differences as great as those shown by our data in one sample out of ten for the UK value weighted series, one out of twenty for the UK equally weighted series or one sample out of three or four for the US, even though there was no underlying reason for it. For most sub-periods the probability of the observed differences occurring by chance is greater. If ranks are assigned to the quarters in each year (1 for the lowest log-return and 4 for the highest) and tests of significance for quarterly mean ranks are performed, the differences are such as would occur in less than one sample out of 40 in the UK for the full period (for both value weighted and equally weighted series) and about one in 33 for the US in 1926-39. The results for sub-periods in the UK are less striking, and for the US there is little evidence of a significant variation in the full period or the remaining sub-periods. Of course, even if the seasonal differences were significant, it would be necessary to postulate a mechanism for producing them which would work differently in the two countries.

(ii) Tests for differences among mean log-returns

We test the null hypothesis that the population means for each set of quarterly log-returns are the same. This may be tested separately in each sub-period. For present purposes we may assume normality, since for the money series, which interests us here, this hypothesis was not rejected by the tests carried out in Section II. Inspection of Table IVa suggests that it is reasonable to assume that all observations have a common standard deviation $\sigma$, at least within the period or sub-period considered. Simple standard tests for differences among means are therefore available.

An appropriate procedure for testing the hypothesis is an analysis of variance test, which assesses the significance of differences among the sample means for the four quarters by comparing them with the differences among the observations within samples. The resulting test statistic $F^0$ has the distribution of Fisher's F with 4-1=3 and N-4 degrees of freedom, where N is the total sample size, see [4] Vol. II pp. 503-4 and [3] p. 169. For large samples there is an alternative test statistic S which is distributed as $\chi^2$ with 3 degrees of freedom.

A more refined test first eliminates the variance arising from the fact that in each quarter we have observations from the same n years (4n=N assuming a full set of observations in each year). The resulting test statistic $F^1$ is also distributed as F but with 3 and 3(n-1) degrees of freedom.

Values of $F^0$ and $F^1$ for all periods and of S for the full period are given in Table IVc below; their significance will be assessed after a brief discussion of rank tests.

(iii) Tests for differences among ranks

It is of some interest to consider whether high or low mean log-returns in particular quarters are due to the frequency with which these quarters have higher or lower log-returns than other quarters in the same years, rather than exceptionally large or small values on a few occasions. Some insight into this question can be obtained by considering statistics of the quarterly ranks of log-returns rather than the log-returns themselves. In each year the rank 1 has been assigned to the quarter having the lowest log-return, the rank 4 to that having the highest, and the mean quarterly ranks have been tabulated in Table IVb. Note that the random variable 'rank' has theoretical and sample mean $(1+2+3+4)/4 = 2.5$ and standard deviation $\sqrt{\{(1^2+2^2+3^2+4^2-2.5^2 x4)/4\}} = 1.118$ in each year.

The tests for differences among mean ranks used here are the same as those used for mean log-returns. The variable 'rank' is not normally distributed, but since the sample sizes are quite large it seems reasonable to use the F-test described above in this case also: the test is described by Kendall and Stuart as "remarkably robust to departures from normality" [4] Vol. II p. 504. The values of F obtained for ranks appear on the right hand side of Table IVc(i). As a check on the result for the full period, we also give the value of S for ranks; in large samples this statistic, like its counterpart for log-returns, has the distribution of $\chi^2$ with 3 degrees of freedom.

(iv) Significance of results

The significance of the results given in Table IVc(i) can be assessed by comparing the values of F and S with the percentage points of the corresponding distributions, some of which are set out in Tables IVc(ii) and (iii). Briefly, the values of the test statistics for log-returns seem too low to be interesting individually. The values of $F^0$ for the UK value weighted series for the full period and 1939-51 are at about the 10% level, corresponding to chances of 9:1 against the occurrence of differences among quarters at least as great as those observed if the null hypothesis is true. Values of $F^1$ for these two periods and of S for the full period are around the same level. For the equally weighted series values of all statistics for the full period are around the 5% level, but sub-period values are much less significant. The most significant US result is around the 15% level.

The statistics for ranks are more significant, particularly in the UK where in the full period both $F^0$ and S are significant at the 2.5% level for both series. This is equivalent to chances of more than 40:1 against the occurrence of differences in mean as large as those observed. The values of $F^1$ for this period are both about the 5% level, as are the values of $F^0$ for both UK series in 1951-70. The equally weighted series also has a value of $F^0$ at about the 3% level in 1926-39. For the US in 1926-39, $F^0$ is also around this level, but other values of the US statistics are not at all significant.

## (v) Outlying observations

The improved results obtained for the UK series when ranks are used instead of log-returns indicate that the differences among mean log-returns are not on the whole caused by a few very large or small values. To demonstrate this we show the effect on mean log-returns of values lying more than two standard deviations from the mean. Table IVd shows, for the full period, the incidence among quarters of such observations and the effect on mean log-returns of omitting them. It is seen that for the UK value weighted series, values lying more than two standard deviations from the mean have little effect. The UK equally weighted series behaves somewhat differently since the two large negative values in May are seen to play a considerable part in depressing the return in that quarter. In the US, the low return in August is almost entirely due to the occurrence of five large negative values in that quarter.

## Table IVa

### Means and standard deviations of quarterly log-returns

| Quarter Beginning | 1926-70 | | 1926-39 | | 1939-51 | | 1951-70 | |
|---|---|---|---|---|---|---|---|---|
| | Mean | S.D. | Mean | S.D. | Mean | S.D. | Mean | S.D. |
| (a) UK VW | | | | | | | | |
| Feb | .0165 | .0623 | -.0013 | .0523 | .0033 | .0439 | .0362 | .0733 |
| May | .0027 | .0831 | .0135 | .0660 | -.0235 | .1086 | .0096 | .0792 |
| Aug | .0398 | .0827 | .0422 | .0623 | .0361 | .0609 | .0404 | .1061 |
| Nov | .0269 | .0583 | -.0002 | .0668 | .0438 | .0353 | .0348 | .0600 |
| Total | .0214 | .0733 | .0133 | .0627 | .0160 | .0707 | .0302 | .0810 |
| (b) UK EW | | | | | | | | |
| Feb | .0199 | .0586 | .0046 | .0479 | .0071 | .0395 | .0377 | .0704 |
| May | .0068 | .0847 | .0134 | .0672 | -.0072 | .1113 | .0100 | .0830 |
| Aug | .0487 | .0823 | .0551 | .0832 | .0357 | .0646 | .0522 | .0936 |
| Nov | .0326 | .0588 | -.0001 | .0721 | .0537 | .0330 | .0417 | .0546 |
| Total | .0270 | .0733 | .0179 | .0699 | .0233 | .0705 | .0353 | .0772 |
| (c) US VW | | | | | | | | |
| Feb | .0116 | .0832 | -.0167 | .1202 | .0140 | .0750 | .0300 | .0472 |
| May | .0380 | .0941 | .0687 | .1290 | .0326 | .0804 | .0195 | .0681 |
| Aug | .0061 | .1047 | -.0296 | .1567 | .0370 | .0901 | .0108 | .0605 |
| Nov | .0345 | .0934 | .0224 | .1318 | .0267 | .0796 | .0477 | .0710 |
| Total | .0225 | .0944 | .0118 | .1365 | .0278 | .0793 | .0268 | .0626 |

VW: Value weighted
EW: Equally weighted

Table IVb

Mean ranks of quarterly log-returns

| Quarter Beginning | 1926-70 | 1926-39 | 1939-51 | 1951-70 |
|---|---|---|---|---|
| (a) UK VW | | | | |
| Feb | 2.311 | 2.286 | 2.091 | 2.450 |
| May | 2.200 | 2.571 | 2.182 | 1.950 |
| Aug | 2.911 | 3.154 | 2.677 | 2.900 |
| Nov | 2.546 | 2.077 | 2.917 | 2.632 |
| (b) UK EW | | | | |
| Feb | 2.333 | 2.286 | 2.091 | 2.500 |
| May | 2.178 | 2.571 | 2.182 | 1.900 |
| Aug | 2.867 | 3.231 | 2.667 | 2.750 |
| Nov | 2.591 | 2.000 | 2.917 | 2.790 |
| (c) US VW | | | | |
| Feb | 2.378 | 2.143 | 2.182 | 2.650 |
| May | 2.711 | 3.143 | 2.455 | 2.550 |
| Aug | 2.245 | 2.000 | 2.667 | 2.150 |
| Nov | 2.636 | 2.692 | 2.667 | 2.579 |

## Table IVc

### Tests for differences among quarterly mean log-returns and ranks

### (i) Values of test statistics

| Period | N-4 | UK | | | | | | US | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | VW | | | EW | | | VW | | |
| | | $F^0$ | $F^1$ | S | $F^0$ | $F^1$ | S | $F^0$ | $F^1$ | S |
| (a) Log-returns | | | | | | | | | | |
| 1926-70 | 175 | 2.11 | 2.15 | 6.21 | 2.75 | 2.82 | 8.02 | 1.30 | 1.48 | 3.88 |
| 1926-39 | 50 | 1.40 | 1.89 | | 1.78 | 2.02 | | 1.48 | 1.84 | |
| 1939-51 | 42 | 2.42 | 1.95 | | 1.85 | 1.54 | | 0.17 | 0.16 | |
| 1951-70 | 75 | 0.59 | 0.56 | | 1.10 | 1.10 | | 1.26 | 1.19 | |
| (b) Ranks | | | | | | | | | | |
| 1926-70 | 175 | 3.72 | 2.69 | 10.70 | 3.42 | 2.47 | 9.85 | 1.73 | 1.26 | 5.13 |
| 1926-39 | 50 | 2.43 | 1.74 | | 3.21 | 2.30 | | 3.24 | 2.22 | |
| 1939-51 | 42 | 1.44 | 0.98 | | 1.44 | 0.98 | | 0.47 | 0.33 | |
| 1951-70 | 75 | 2.72 | 1.88 | | 2.85 | 1.98 | | 0.80 | 0.55 | |

Table IVc (contd.)

Tests for differences among quarterly mean log-returns and ranks

(ii) Percentage points of F with 3 and M degrees of freedom

| M | Probability level | | | |
|---|---|---|---|---|
| | 25% | 10% | 5% | 2.5% |
| 40 | 1.42 | 2.23 | 2.84 | 3.46 |
| 60 | 1.41 | 2.18 | 2.76 | 3.34 |
| 120 | 1.39 | 2.13 | 2.68 | 3.23 |
| $\infty$ | 1.37 | 2.08 | 2.60 | 3.12 |

(iii) Percentage points of $\chi^2$ with 3 degrees of freedom

| Probability level | | | |
|---|---|---|---|
| 25% | 10% | 5% | 2.5% |
| 4.11 | 6.25 | 7.81 | 9.35 |

Table IVd

Effect of outlying values on mean log-returns, 1926-70

| Quarter beginning | Number of values outside 2 standard deviations | | Mean excluding these values | Overall mean |
|---|---|---|---|---|
| | +ve | -ve | | |
| (a) UK VW | | | | |
| Feb | - | - | .0165 | .0165 |
| May | 1 | 2 | .0087 | .0027 |
| Aug | 3 | 1 | .0350 | .0398 |
| Nov | - | - | .0269 | .0269 |
| (b) UK EW | | | | |
| Feb | - | - | .0199 | .0199 |
| May | - | 2 | .0168 | .0068 |
| Aug | 2 | 1 | .0446 | .0487 |
| Nov | - | 1 | .0369 | .0326 |
| (c) US VW | | | | |
| Feb | - | 1 | .0185 | .0116 |
| May | 1 | 1 | .0377 | .0380 |
| Aug | - | 5 | .0348 | .0061 |
| Nov | 1 | 1 | .0368 | .0345 |

# V AUTO-CORRELATION

## (i) Discussion of results

In this section we investigate more generally the existence of auto-correlation in the series; in particular, we wish to test the hypothesis that the series are purely random or 'white noise'. First we use spectral analysis to indicate cycles in the data and show up auto-regressive patterns. Where appropriate, auto-regressive models may then be fitted to the data and their goodness of fit analysed. Money terms series have again been used to avoid picking up cycles which originate in the price index, but in fact it is shown below that the spectra of the real and money terms series are very similar.

The analysis yields no evidence which is significant at more than the 10% level that the series are not random. The most significant feature is a slight auto-regressive pattern in the US series over the full period 1926-70. Further, the UK spectra have a peak at a frequency of approximately .3 cycles per quarter (representing a wavelength of about three quarters) which, although not particularly significant, does occur consistently in the full period and in all three sub-periods. It is shown that this peak is related to a particular pattern in the auto-covariance function at low lags, i.e. the auto-covariances at lags 1 and 2 are both less than those at lags 3 and 4. The first pair of auto-covariances may be positive or negative, but typically the second pair are positive. This auto-covariance pattern is absent from the US series.

(ii) Methods

Spectral analysis depends on the fact that a time series may be considered as a weighted sum of cyclical components of different frequencies. The spectrum (spectral density function) of the series shows what proportion of the total variance of the series is associated with each component - see [6] for a full discussion. A random or 'white noise' series has equal amounts of variance arising at all frequencies, so that its spectrum is a horizontal line. Thus any departure of the spectrum of a series from the horizontal indicates some cycle or trend, and if the deviation is greater than would be expected to arise from sampling error, we may conclude that the series is not random.

The significance of deviations from the horizontal may be assessed in two ways. First it is possible to compute confidence intervals for the logarithm of spectral density at different levels of significance. (The logarithm is used since the width of each confidence interval will then be the same at all frequencies.) If the log spectral density is plotted, any feature in the curve whose amplitude exceeds the confidence interval at a particular level of significance may be considered significant at that level. Secondly, the integrated spectrum may be used for a significance test. The integrated spectrum at a given frequency is the sum of the variances due to components at all frequencies less than or equal to the given one. For a random series, since the spectrum is a horizontal line, this sum of variances will be proportional to frequency, and the integrated spectrum will be a line passing through the origin at an angle of $45^{\circ}$ to the axes. The significance of any departure of the integrated spectrum from this line may be determined by means of the Kolmogorov - Smirnov test [6] pp. 234-7.

(iii) Spectra of log-return series in money terms

Graphs Va(i)-(iv) show the spectra of UK and US log-returns in money terms for the full period 1926-70 and the three sub-periods. Except for the UK in 1939-51, all the spectra have a peak at small values of f, indicating long term components, and the UK spectra also have a peak at approximately f = .3 cycles per quarter, although this is not very marked in 1951-70. This peak is totally absent from the US spectra, which are all rather dissimilar, although the full period and 1926-39 show a slight auto-regressive pattern (spectrum decreasing from f = 0 to f = 0.5). Confidence intervals show that none of these features can be considered significant at the 5% level.

Graph Vb shows the integrated spectra for the UK and the US series in the full period. It is seen that for the US there is a departure from the $45^{\circ}$ line which is significant at about the 10% level.

That this is due to the slight auto-regressive nature of the series can be shown by fitting a first order auto-regressive equation to the data (see (v) below).

(iv) Auto-covariances

Although the peak at $f \simeq .3$ in the UK spectra is not a feature of any great statistical significance, it does occur consistently and reasons for it and for the absence of a similar peak from the US spectra have been sought. Examination of the first few auto-covariances of the UK series shows that there is a consistent pattern which is absent in the case of the US. For the UK, auto-covariances at lags 1 and 2 are relatively low compared with those at lags 3 and 4, this pattern being more marked in some periods than in others. The US auto-covariances do not conform to this pattern in any period. Since the auto-covariances at low lags have more influence on the shape of the spectrum than those at higher lags, it seems likely that the pattern of two low values followed by two higher ones is linked with the appearance of the peak at $f \simeq .3$. This can be shown by computing the spectra of several simple auto-covariance functions in which the first four values have this pattern and the values at higher lags are zero. All such spectra have a peak at $f \simeq .3$, whereas there is no such peak in the spectra of other simple auto-covariance functions which approximate those of the US series. It may further be shown that the basic shape of the spectra is not unduly influenced by the 'tail' of the auto-covariance function, i.e. the values for lags greater than 4.

We may thus conclude that the peak at $f \approx .3$ in the UK spectra is associated with relatively low auto-covariance at lags 1 and 2 (i.e. between values arising three or six months apart) combined with relatively high auto-covariance at lags 3 and 4 (i.e. between values arising nine months or one year apart).

(v) Auto-regression

The spectrum of the US series for the full period shows a slight auto-regressive pattern, in that there is a tendency for the spectral density to decrease from $f = 0$ to $f = 0.5$. The integrated spectrum test shows that the same series displays a departure from white noise which is significant at the 10% level and which may be due to this auto-regressive pattern. To investigate this possibility we examine the residuals of a first order auto-regressive equation fitted by ordinary least squares. The results of the regression are given in Table Va; it is apparent that the fit is not particularly good, since the value of $R^2$ is small and the standard error of residuals $\sigma_\varepsilon^*$ is little less than the original standard deviation of the series (notation as in Malinvaud [5]).

Graph Vc shows the spectrum and integrated spectrum of the residuals of this auto-regression compared with those of the original series. The spectrum of the residuals is much closer to a horizontal line and the integrated spectrum departs very little from a line at $45^\circ$ to the axes, indicating that despite the poor fit the auto-regressive equation has accounted for the bulk of the observed departure from white noise.

(vi) Log-returns in real terms and UK equally weighted portfolio

Graphs Vd and Ve compare the spectra of money terms and real terms log-returns for the UK and the US in the full period. Graph Vd also shows the spectrum of the UK equally weighted series in money terms. The real terms series have been derived using price indices smoothed to eliminate seasonal fluctuations. The graphs show differences between the real and money spectra and between the equally weighted and money weighted spectra to be very small. No further analysis of this type has therefore been considered necessary for real terms or equally weighted series.

Table Va

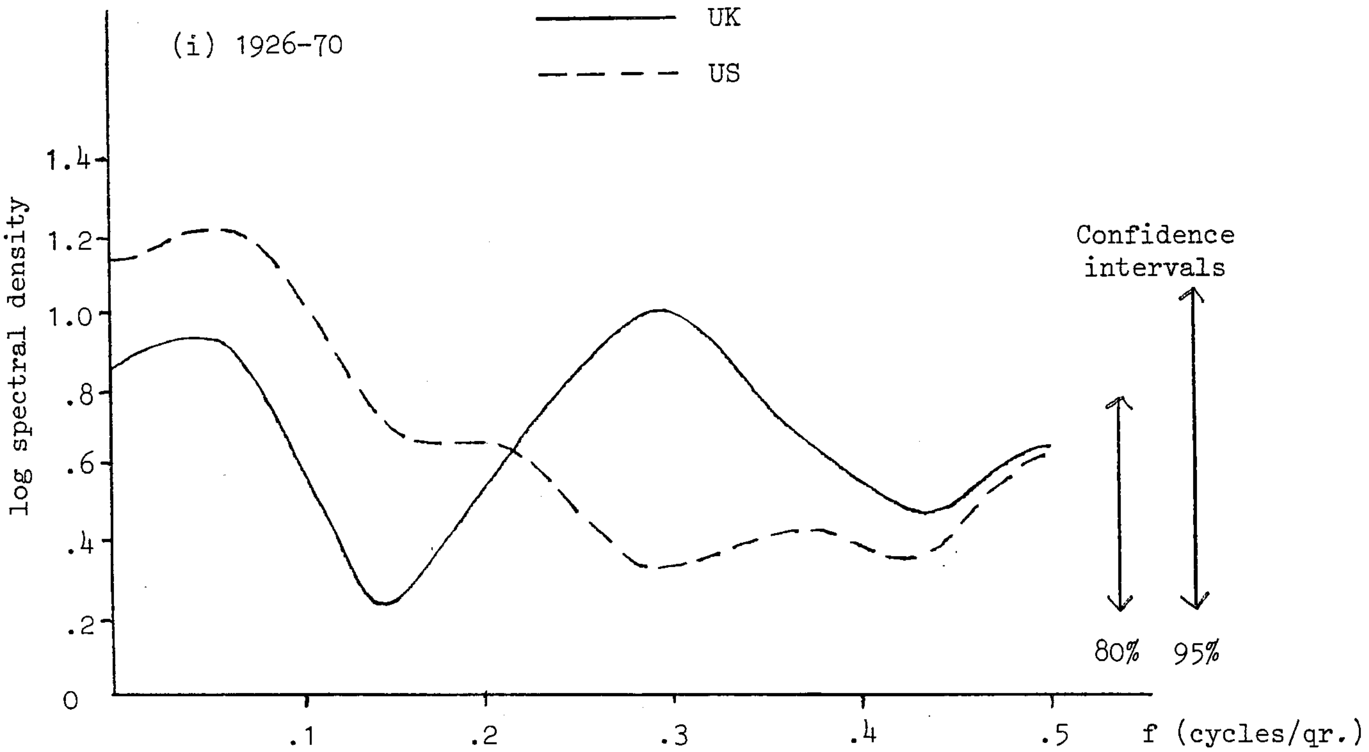
First order auto-regression of US log-return series in money terms

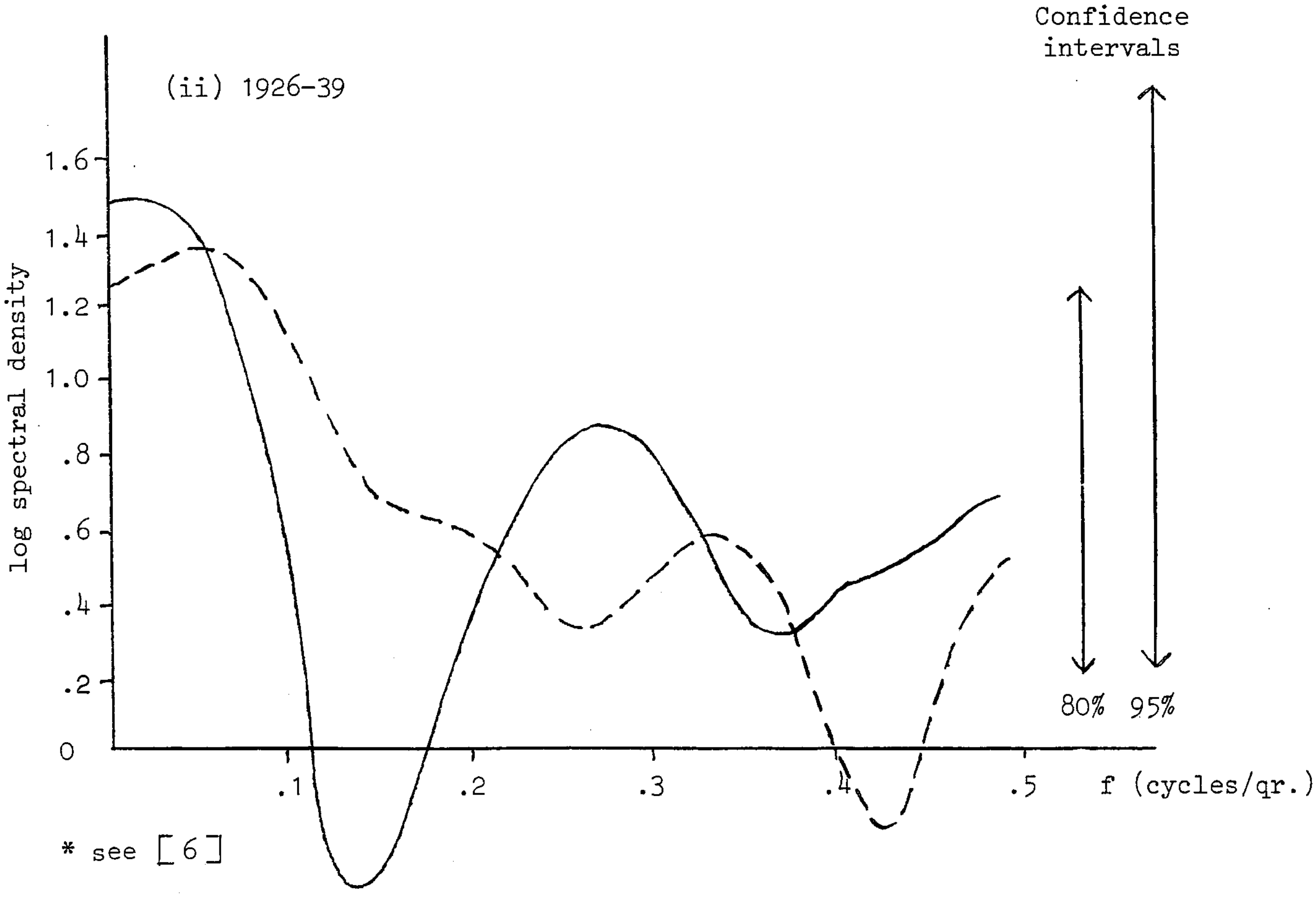
Auto-regression equation: $x_t = ax_{t-1} + b + \varepsilon_t$

| | |
|---|---|
| $\mu_{x_t} = .0225$<br>$\sigma_{x_t} = .0947$ | $R^2 = .0443$<br>$\sigma^*_\varepsilon = .0927$ |
| $a = .2101$<br>$\sigma_a = .0738$<br>$T_a = 2.85$ | $b = .0180$<br>$\sigma_b = .0072$<br>$T_b = 2.52$ |
| Durbin-Watson statistic = 2.02 | |

Graph Va

Spectra of UK and US log-returns to value-weighted portfolio in money terms, gross of tax. Parzen window, truncation point 15*.

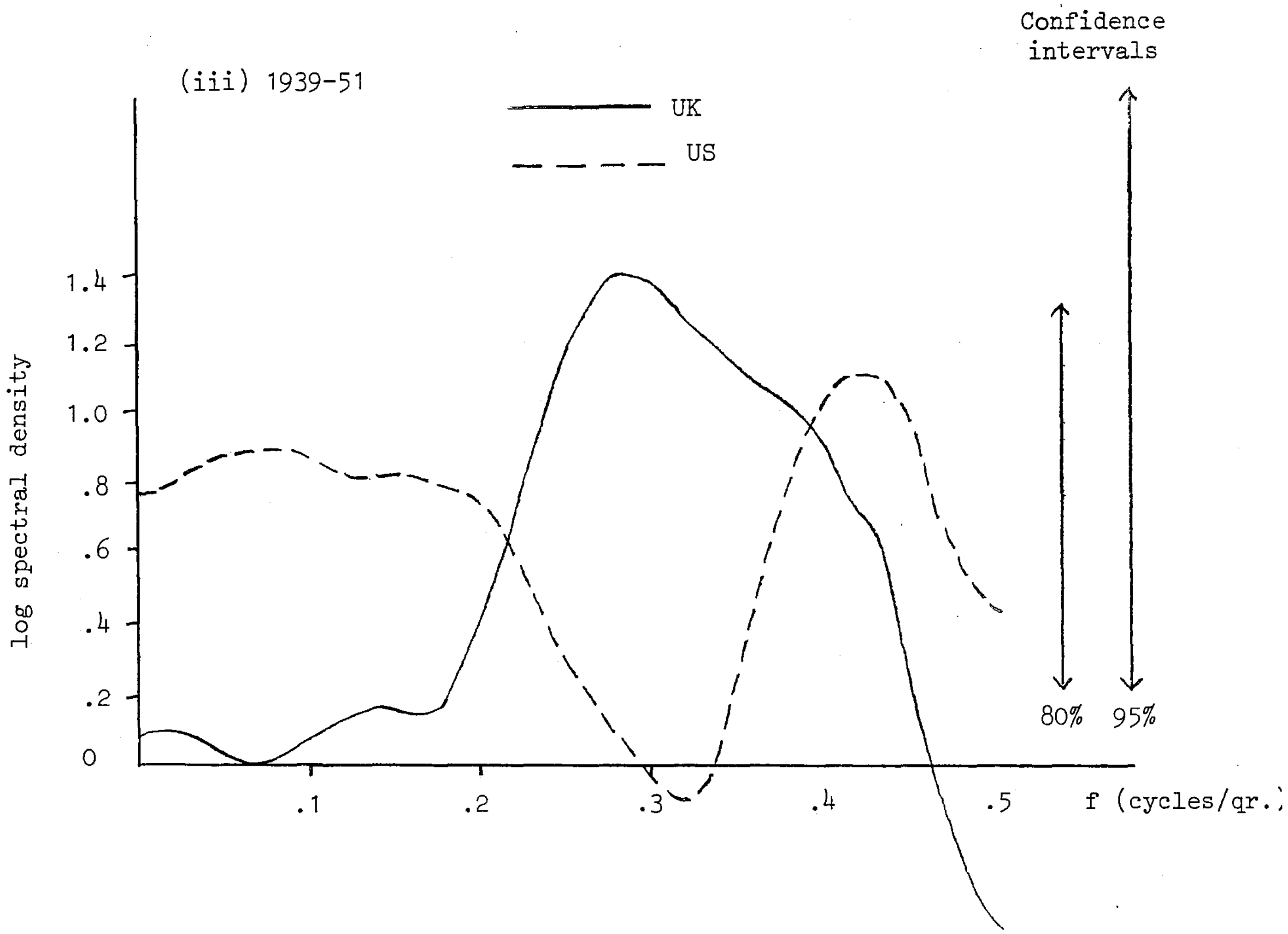


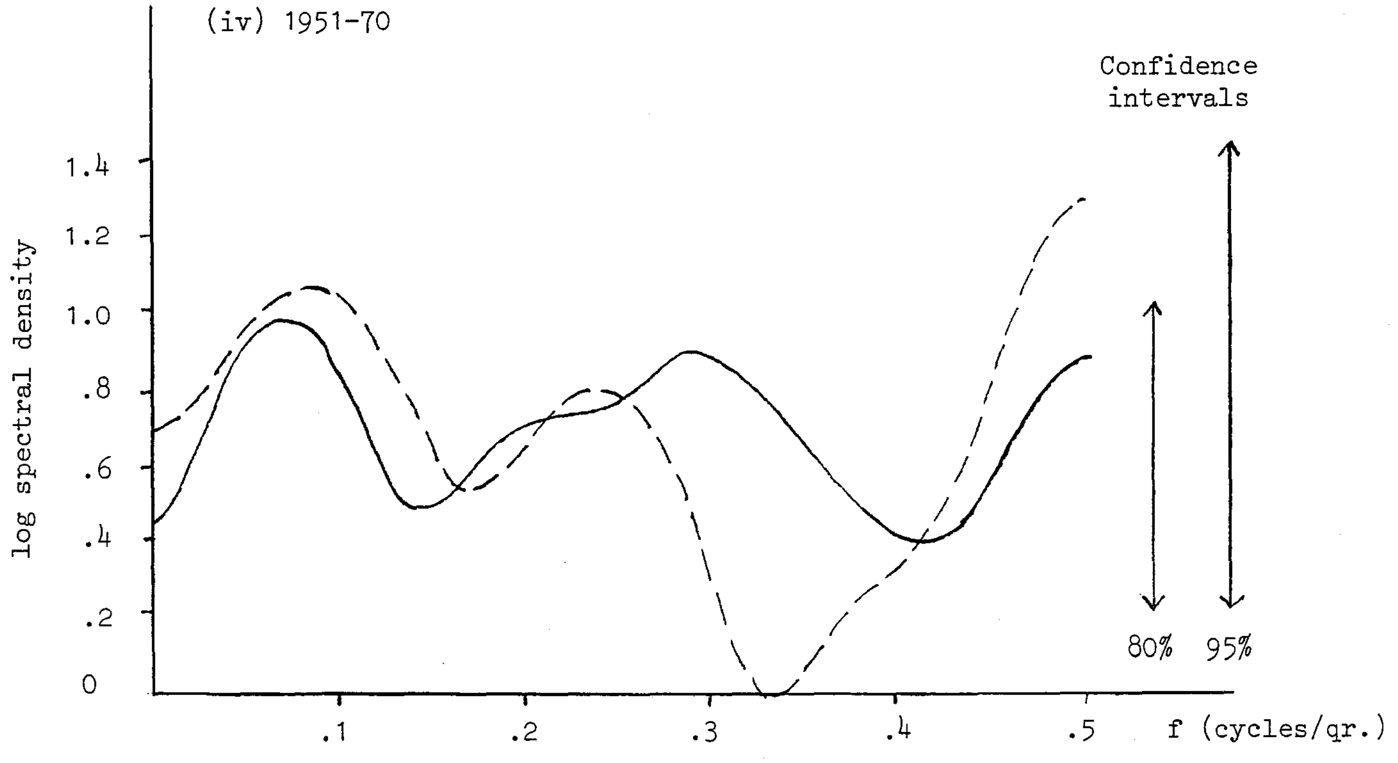




* see [6]

Graph Va (contd.)

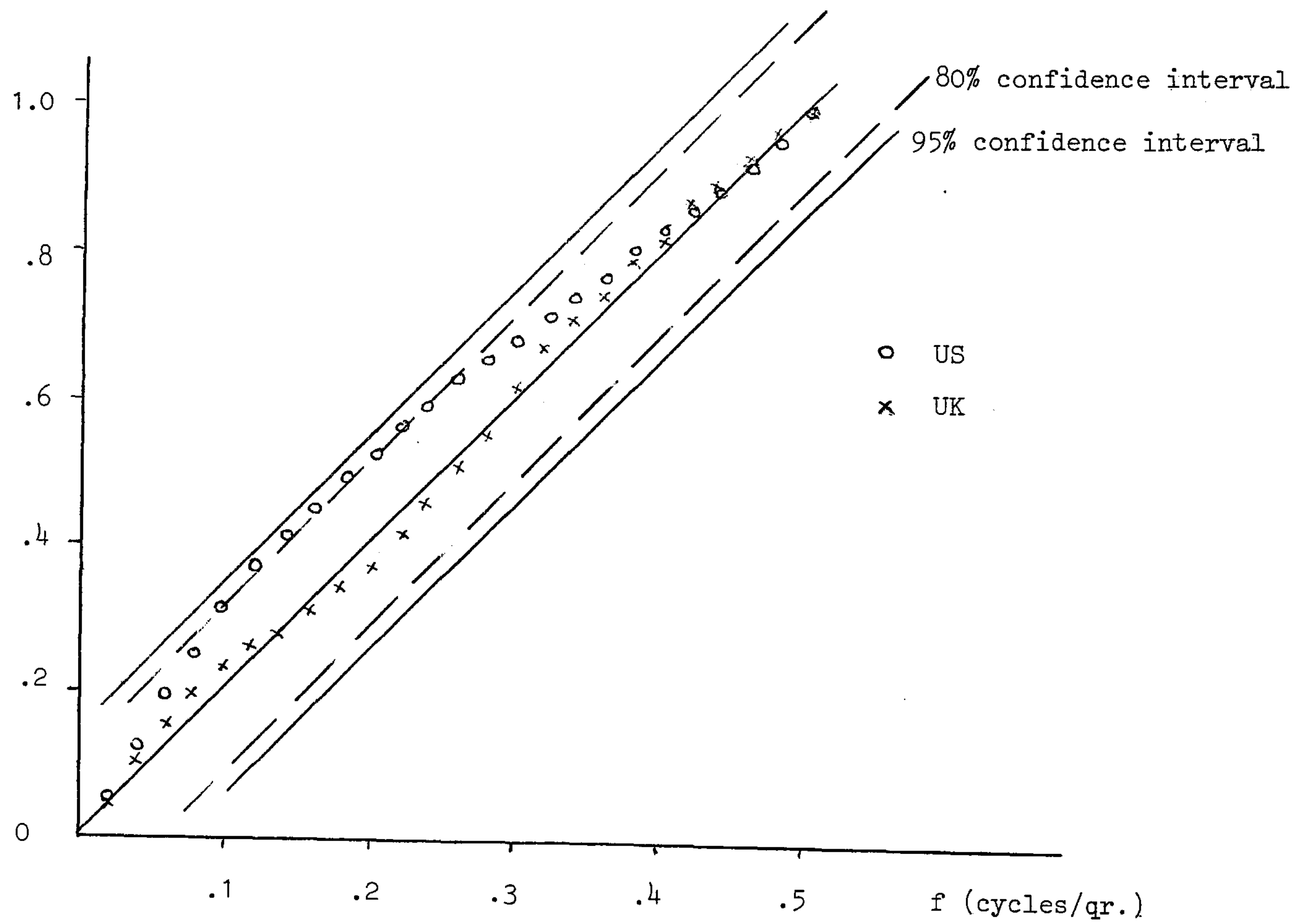
Spectra of UK and US log-returns to value weighted portfolio in money terms, gross of tax. Parzen window, truncation point 15.



Graph Vb

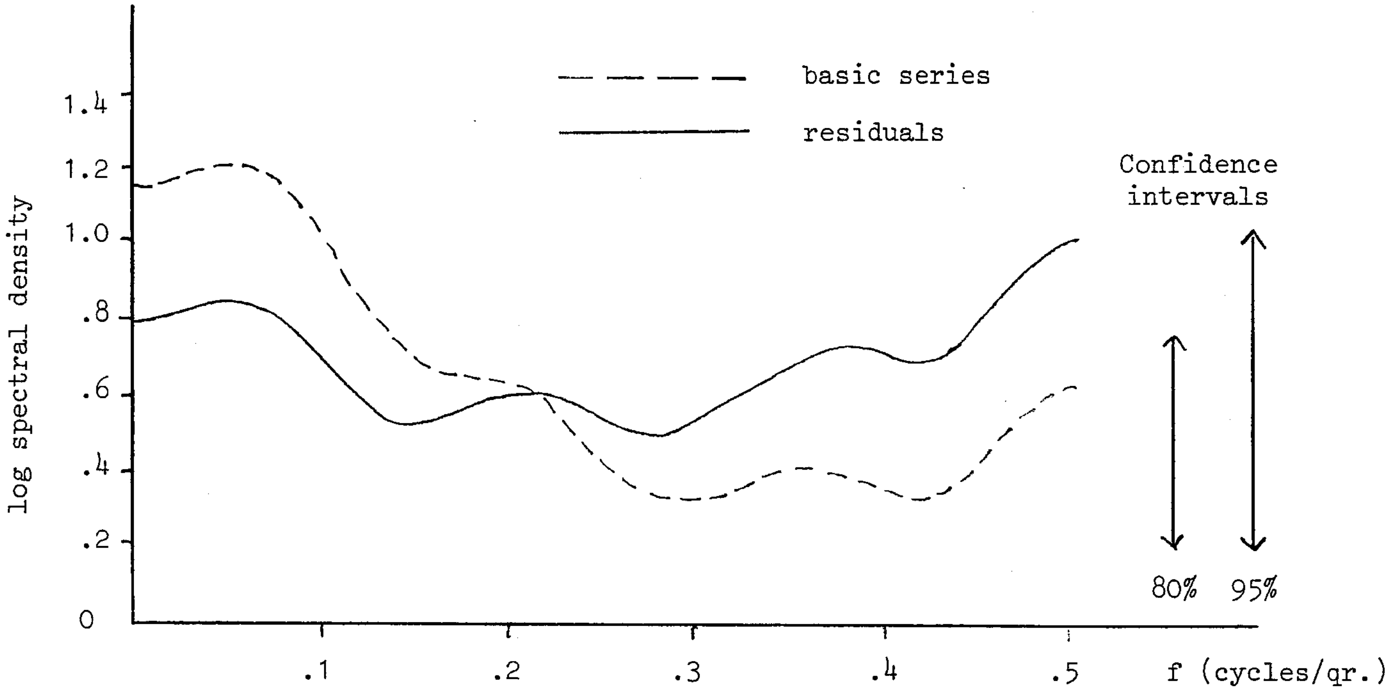
Integrated spectra of UK and US log-returns to value weighted portfolio in money terms, gross of tax, 1926-70.



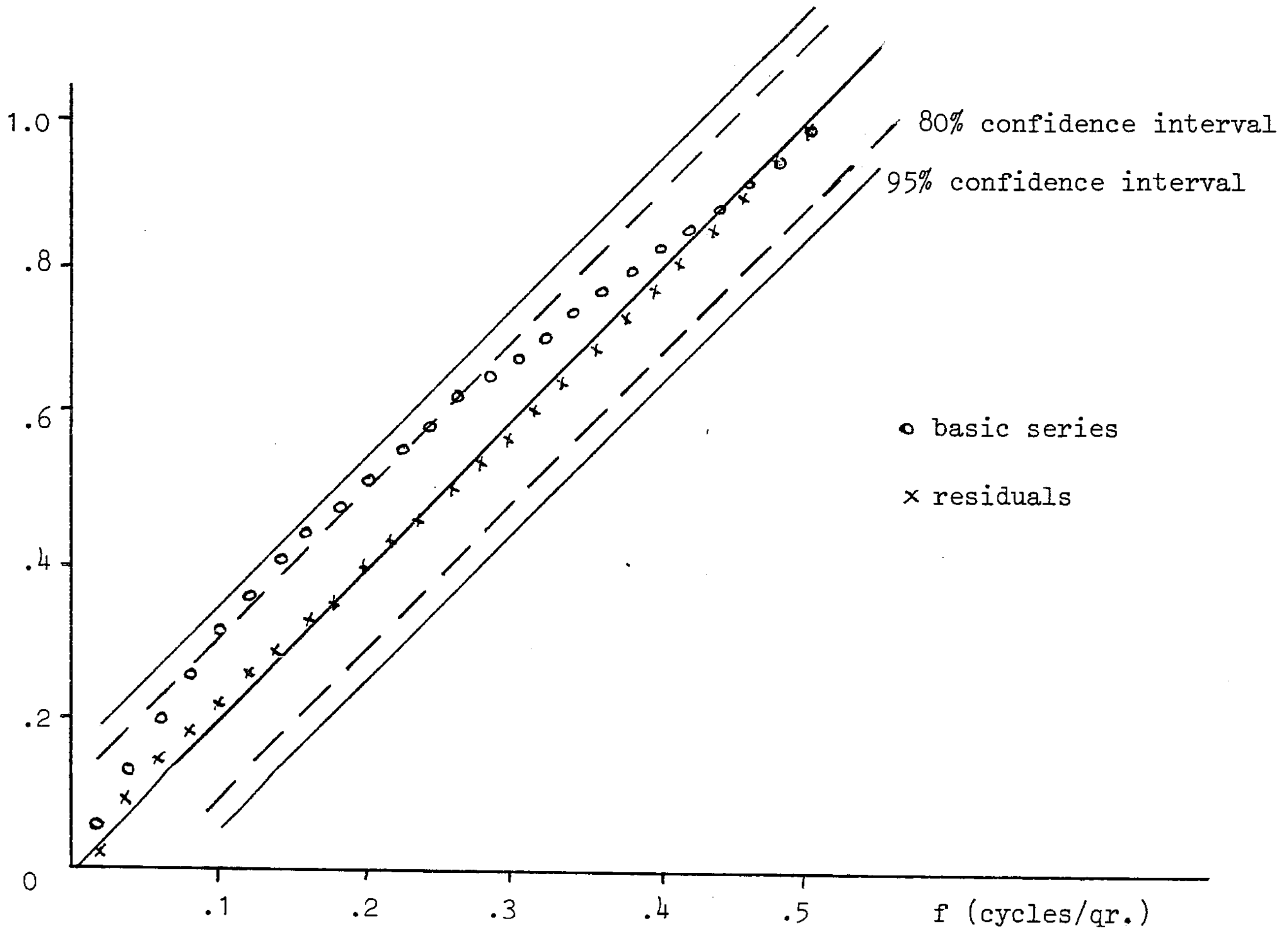
Graph Vc

Effect of first order auto-regression on spectrum of US log-returns, 1926-70.

(i) Spectra of basic series and residuals of auto-regression

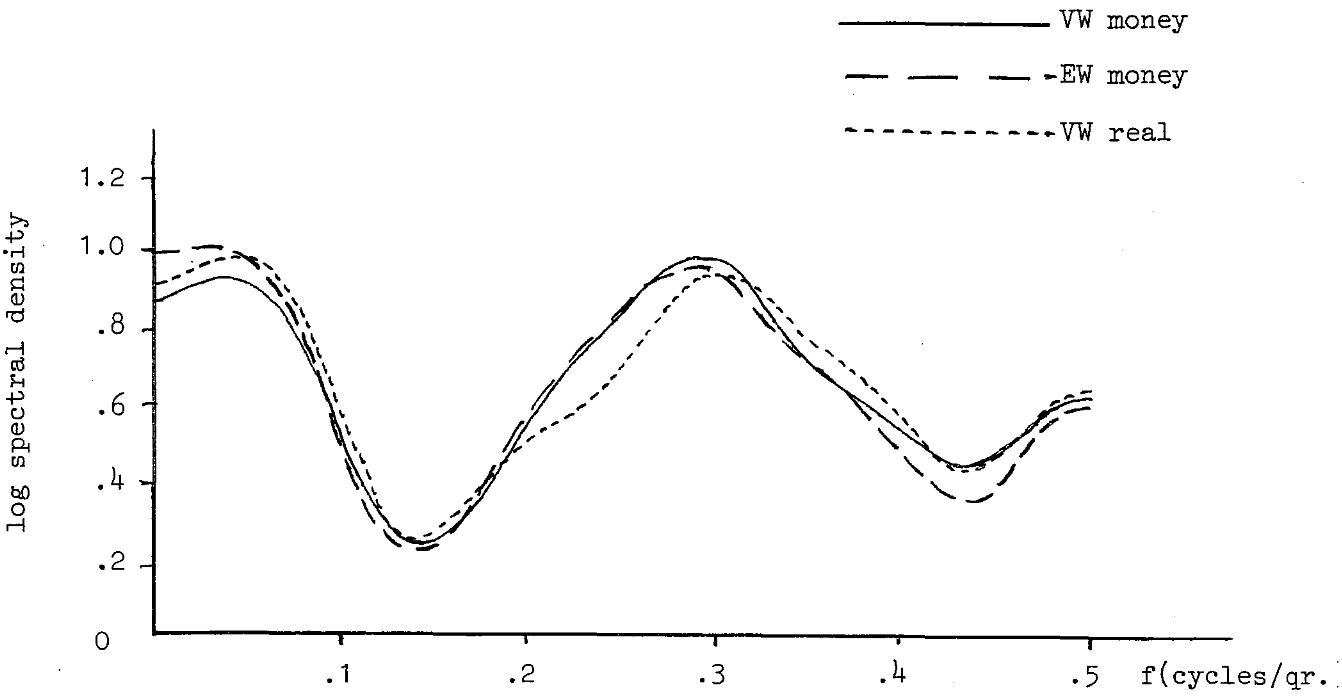



(ii) Integrated spectra of basic series and residuals



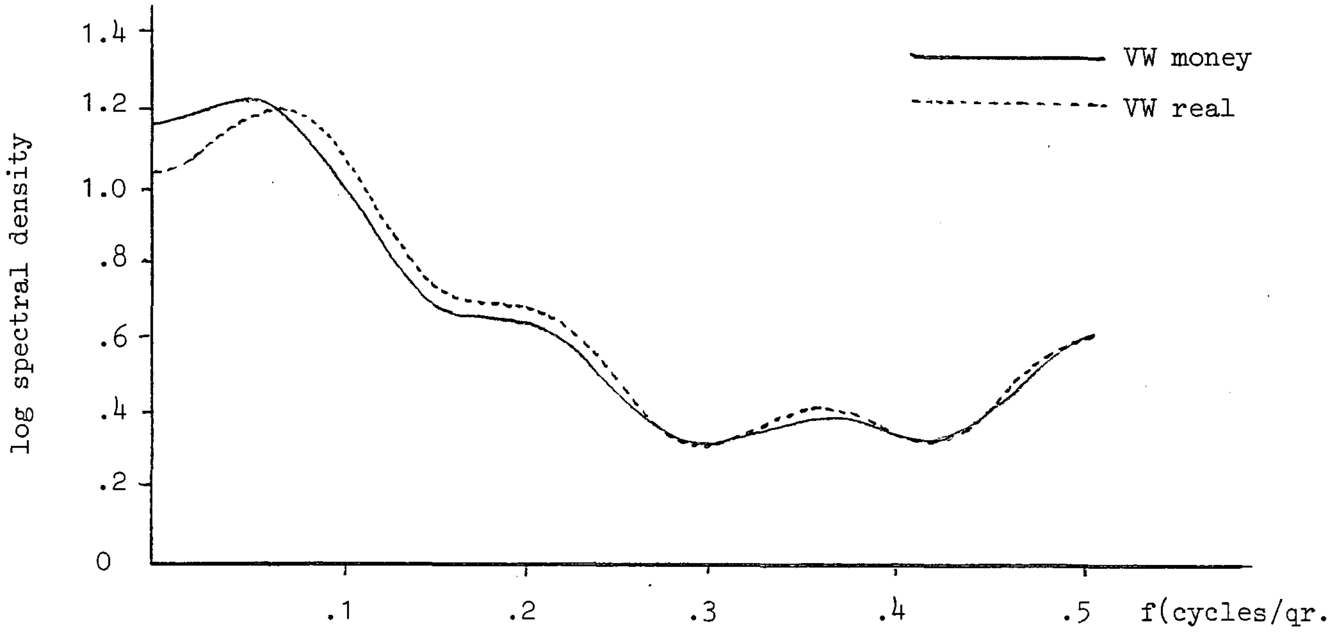
Graph Vd

Spectra of log-returns to UK value weighted portfolio in money and real terms and equally weighted portfolio in money terms only, 1926-70




Graph Ve

Spectra of log-returns to US value weighted portfolio in money and real terms, 1926-70.



# VI VARIANCE CURVE

## (i) Definition

Part of the purpose of the present work is to shed some light on the risk properties of the log-return series. Although we have gained some information from the preceding analysis, it does not give straightforward answers to the questions an investor might ask about risk. In search of a more intuitive approach we examine the average variance per quarter for different holding periods. This is defined as follows: let $\sigma_\tau^2$ be the theoretical variance of log-return for investments with holding period $\tau$ quarters; then the average variance is $\sigma_\tau^2/\tau$ and the 'variance curve' is obtained by plotting $\sigma_\tau^2/\tau$ against $\tau$. In order to estimate this curve we plot $v_\tau$, an unbiased estimator of $\sigma_\tau^2/\tau$ whose derivation is given in the Appendix. Graph VIa shows the estimated curves for all four value weighted series (UK/US, money/real).

It should be noted that these curves are closely related to the auto-correlation coefficients discussed above. A positive/negative sum of correlation coefficients for holding periods not exceeding $\tau$ gives rise to an increase/decrease in average variance between $\tau$ and $\tau + 1$. Precise formulae will be found in the Appendix.

These formulae also demonstrate the major difficulty encountered in attempting to derive statistical tests based on the variance curve.

Although in large samples the auto-correlation coefficients of different orders are approximately independent, $v_\tau$ involves a cumulative sum of such coefficients and successive values are therefore related. A second difficulty arises, particularly in the case of sub-period variance curves (see below), when investments with long holding period are considered. Comparatively few disjoint observations are available for calculating average variance, and although it is possible to overcome this problem to some extent by taking overlapping periods of constant length, this makes the derivation of tests more complex. For this latter reason also, not too much significance should be attached to values of average variance for long holding periods.

(ii) Behaviour of risk with holding period

The 'geometric' Brownian motion theory of stock market returns suggests that $\sigma_\tau^2/\tau$ should be constant. Indeed, if log-returns in different quarters are independent and have the same variance $\sigma^2 = \sigma_1^2$, then $\sigma_\tau^2 = \tau\sigma_1^2$ and risk as measured by the variance of log-return grows linearly with the length of period. An alternative theory - that all risks cancel out in the long run - can be translated into the assertion that $\sigma_\tau^2/\tau \to 0$ as $\tau \to \infty$. Graphs of UK and US variance in money and real terms (Graph VIa) do not bear out either of these simple hypotheses. It seems that risk per quarter increases sharply for holding periods of up to 6 or 7 quarters, then falls consistently until around 25-30 quarters.

Thereafter there are further fluctuations, which should not be regarded as very significant, but the graphs do suggest that the risk settles down to a 'permanent' level which cannot be eliminated by further prolonging the investment.

(iii) Sub-period variance curves

Variance curves in money terms for the three sub-periods are plotted on Graph VIb. Because of the smaller number of values available for the shorter series the graphs are plotted for holding periods of less than 25 quarters only, but even then not too much attention should be paid to values at the higher end of this range. It is seen that the peak in the curve at holding periods of 6 or 7 quarters is not a completely consistent feature, varying considerably in size from period to period and being totally absent for the sub-period 1939-51 in the UK. The size of the peak is determined by the first few auto-correlation coefficients, large peaks occurring when several of these are large and positive (i.e. around .2 to .3) as in the UK in 1926-39. Smaller peaks occur when some of these coefficients are small or negative. Individual quarterly log-returns can have considerable influence on the sub-period variance curves. In particular, three large values in the UK series occurring in 1932 and 1933 seem to be mainly responsible for the very large peaks in the variance curves of the sub-period 1926-39.

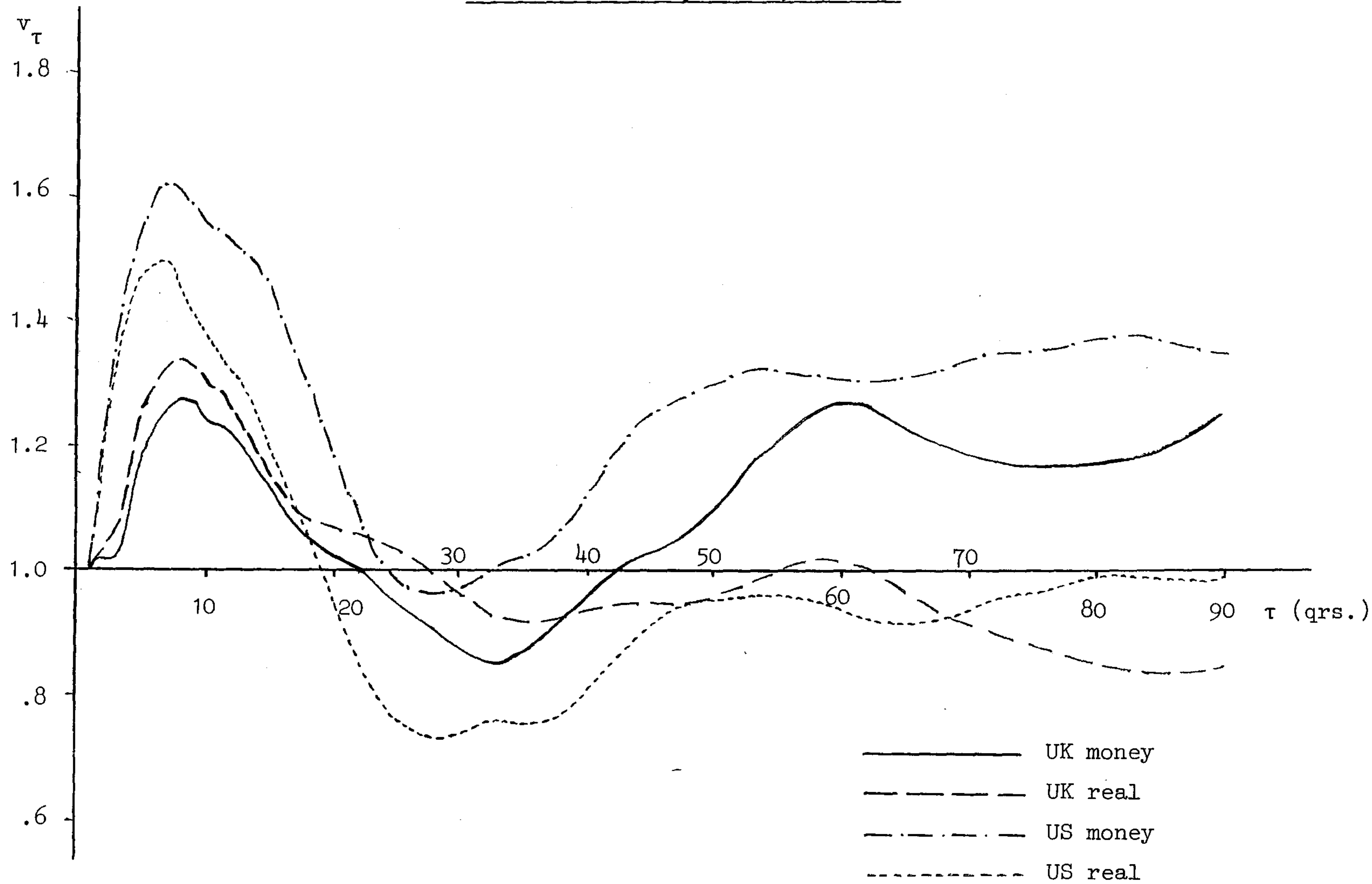
(iv) Equally weighted series

Graph VIc shows the variance curves for the UK equally weighted series in money and real terms, 1926-70, compared with the curves for the corresponding value weighted series. The curves are seen to be similar in shape although the equally weighted curve lies above its value weighted counterpart in both money and real terms, the two pairs of curves being almost parallel for holding periods of more than 25 quarters. The difference is mainly due to higher auto-correlation at low lags in the equally weighted series, particularly in real terms, since at lag 10 the distance between the curves is half its maximum value in money terms and two thirds in real.

The sub-period variance curves for the equally weighted series are very similar to those of Graph VIb.

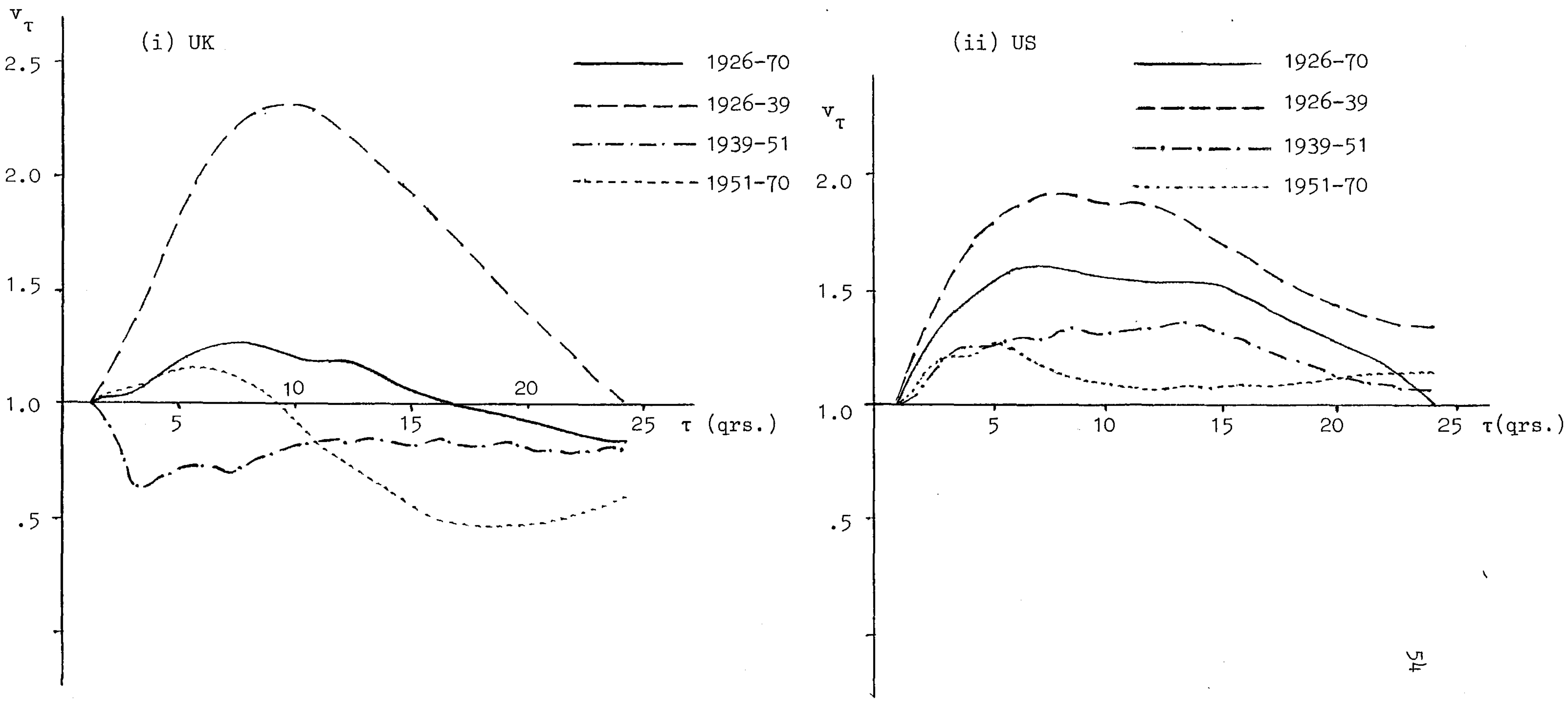
Graph VIa

Variance curve of log-returns to UK and US value weighted portfolios in real and money terms, 1926-70.



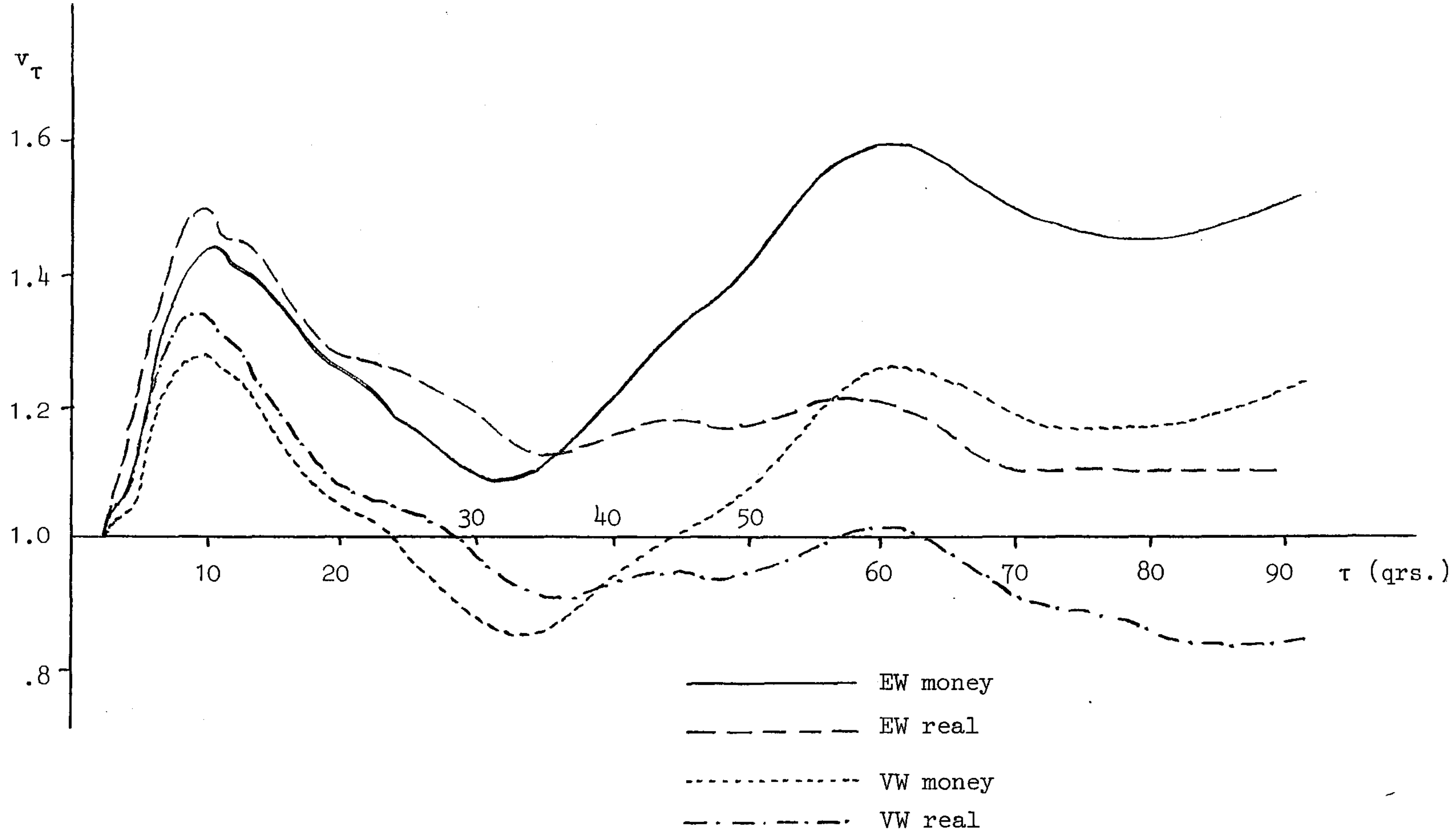
Graph VIb

Variance curve of log-returns to value weighted portfolio in money terms only by sub-periods



Graph VIc

Variance curve for UK log-returns to value weighted and equally weighted portfolios, 1926-70.



# APPENDIX

## Variance curve formulae

### (i) Unbiased estimator of average variance

Let $Y_t$, $t = 1,\ldots,N$ be random variables of the form $\log_e(1+r_t)$ with common mean $\mu$ and variance $\sigma^2$. Let $y_t$ be observations on $Y_t$.

Define $z_{t,\tau} = \sum_{j=0}^{\tau-1} y_{t+j}$, where $y_{t+j} = y_{t+j-N}$ if $t+j > N$.

Then $m_\tau = \frac{1}{N}\sum_1^N z_{t,\tau} = m\tau$ and $s_\tau^2 = \frac{1}{N}\sum_1^N (z_{t,\tau} - m_\tau)^2$.

We seek an estimator $v_\tau$ of $\sigma_\tau^2 = \mathrm{Var}(z_{t,\tau})$ such that $E(v_\tau) = \sigma_\tau^2/\tau$.

$$\text{Set} \quad s_\tau^2 = \frac{1}{N}\sum_1^N \left[z_{t,\tau} - \mu_\tau - (m_\tau - \mu_\tau)\right]^2$$

$$= \frac{1}{N}\sum_1^N (z_{t,\tau} - \mu_\tau)^2 - \tau^2(m - \mu)^2 \quad \text{since } \mu_\tau = \tau\mu \ .$$

$$E(s_\tau^2) = E\left[\frac{1}{N}\sum_1^N (z_{t,\tau} - \mu_\tau)^2\right] - \tau^2 E(m - \mu)^2$$

$$= \sigma_\tau^2 - \frac{\tau^2\sigma^2}{N} \quad \text{since } E(m - \mu)^2 = \mathrm{Var}(m) = \sigma^2/N$$

$$= \sigma_\tau^2 - \frac{\tau^2 E(s_1^2)}{N} \cdot \frac{N}{N-1} \quad \text{since } E(s_1^2) = \sigma_1^2 - \frac{\sigma_1^2}{N} = \frac{N-1}{N}\sigma_1^2 \ .$$

$$\therefore \quad E\left(s_\tau^2 - \frac{\tau^2 s_1^2}{N-1}\right) = \sigma_\tau^2$$

Thus $\quad E\left(\frac{s_\tau^2}{\tau} - \frac{\tau s_1^2}{N-1}\right) = \frac{\sigma_\tau^2}{\tau}$, so

$v_\tau = \frac{s_\tau^2}{\tau} + \frac{\tau s_1^2}{N-1}$ is the required estimator.

(ii) Serial correlation and average variance (theoretical parameters)

From above

$$\sigma_\tau^2 = \mathrm{var}(z_{t,\tau}) = \mathrm{var}(Y_1+Y_2+\ldots\ldots+Y_{t+\tau-1})$$

Thus

$$\sigma_2^2 = \mathrm{var}(Y_1+Y_2) = \mathrm{var}(Y_1) + \mathrm{var}(Y_2) + 2\ \mathrm{cov}(Y_1,Y_2)$$

$$= 2\sigma^2 + 2\sigma^2\rho_1 = 2\sigma^2(1+\rho_1)$$

where $\rho_\tau$ is the auto-correlation coefficient at lag $\tau$.

$$\sigma_3^2 = \mathrm{var}(Y_1+Y_2+Y_3) = \mathrm{var}((Y_1+Y_2) + Y_3)$$

$$= \sigma_2^2 + \sigma^2 + 2\ \mathrm{cov}(Y_1+Y_2,Y_3)$$

$$= \sigma_2^2 + \sigma^2 + 2\sigma^2(\rho_1+\rho_2)$$

$$= \sigma_2^2 + \sigma^2(1+2(\rho_1+\rho_2)) \qquad (1)$$

$$= \sigma^2(3+4\rho_1+2\rho_2) \qquad (2)$$

Generalising formula (1) we have

$$\sigma^2_{\tau+1} = \sigma^2_\tau + \sigma^2(1+2(\rho_1+\rho_2+....+\rho_\tau))$$

or
$$\frac{\sigma^2_{\tau+1} - \sigma^2_\tau - \sigma^2}{2\sigma^2} = \rho_1+\rho_2+....+\rho_\tau$$

This can be used to calculate the $\rho_\tau$ iteratively given values of the $\sigma^2_\tau$.

Generalising formula (2) we have

$$\sigma^2_\tau = \sigma^2 \left[\tau + 2\sum_1^{\tau-1} \rho_i(\tau-i)\right] , \quad \text{or}$$

$$\frac{\sigma^2_\tau}{\tau} = \sigma^2 \left[1 + 2\sum_1^{\tau-1} \rho_i(1 - \frac{i}{\tau})\right]$$

Hence by differencing and rearranging

$$\frac{\sigma^2_{\tau+1}}{\tau+1} - \frac{\sigma^2_\tau}{\tau} = \frac{2\sigma^2}{\tau(\tau+1)} \sum_{i=1}^{\tau} i\rho_i$$

This can be used to calculate values of the $\sigma^2_\tau/\tau$ given values of the $\rho_\tau$ and $\sigma^2$.

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