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SELF-SELECTION IN THE STATE SCHOOL SYSTEM

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ABSTRACT

With diminishing returns to the peer group, it is optimal social policy to mix children in schools. We consider what happens when, contrary to the outcome being determined by a social planner, schools and children are free to seek each other out: with some caveats, this leads to perfect segregation by child quality. It is shown that this is the worst possible outcome. We show also that a competitive system produces the optimal allocation of children to schools.

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SELF-SELECTION IN THE STATE SCHOOL SYSTEM

Donald Robertson and James Symons

1. Introduction

In this paper we consider the implications of the proposition that the level of achievement of children in schools is completely determined by their own quality and the quality of their peers in the school. We shall not specify what we mean by 'quality' here but we have in mind both intellectual ability and other characteristics such as ambition, docility, punctuality and so on, which we believe are derived in large part from the child's home environment. The view that performance in schools is largely unrelated to any characteristics of schools other than the quality of the students in the school was popularised by the Coleman Report (1966). For a comprehensive review of this literature, see Hanushek (1986). Recently Card and Krueger (1992) have presented evidence that conventional school inputs such as class size and teachers' wages are important in the attainment of children. This issue is largely irrelevant to the substance of this paper: all we need to take from the literature is that the peer group is important.[1]

Therefore let us assume that children arrive at school endowed with a single index of quality. The increase in quality produced by the school experience depends on the average quality of the individual's peer group. At once a welfare question emerges: is it better to mix children's qualities in schools, or should one educate them in groups segregated by quality?

In this study social welfare will be represented essentially by the sum over children of attainments on exit. This is not uncontentious. Most societies seek to create elites and if the peer group model is accepted, this will be done best by constructing at least some elite schools. However our concern is with the broad admissions policy of the state school system (not with the creation of research scientists or field marshals) and a utilitarian objective function seems appropriate here. If this is taken as given, optimal social policy depends on whether there are increasing or decreasing returns to peer groups in raising child quality. With decreasing returns it is fairly clear that one should mix children; with increasing returns one should segregate them into ability cohorts. Optimal policy thus turns on the empirical question. Henderson et al (1978) found that average class IQ showed diminishing returns in a study of the attainment of third grade children in the United States. Robertson and Symons (1995) show that average socioeconomic ratings of classmates showed diminishing returns on tests administered to British 11 year olds. Diminishing returns seem likely on a priori grounds, though

[1] Though see Evans et al.,1992 for arguments that empirical estimates of peer group effects may be overstated.

to make this case we would have to penetrate the peer group black box to establish behavioural causations.

In any case, we shall assume diminishing returns. We find below that mixing is then socially optimal, but there are some caveats. These arise because we consider dynamic aspects of a school i.e. that it is composed of a number of cohorts, youngest to oldest. Once one thinks of a school in this way, the appropriate peer group for an entrant child consists of all the children presently in the school plus all who will arrive subsequently while he or she is there.

We consider what happens in this framework when, contrary to the outcome being determined by a social planner, schools and children are free to seek each other out: we call this a free-matching equilibrium. We are able to show that, in what we consider is the most natural model, the free-matching equilibria give rise to perfect segregation i.e. children are sorted into schools by quality. However our analysis does throw up some rather odd sunspot-type equilibria wherein some mixing does occur. Finally we show that perfect segregation is the worst possible welfare outcome.

Thus, caveats aside, we show first that perfect mixing is the best in welfare terms; second that the free matching-produces perfect segregation; and third that perfect segregation is the worst possible welfare outcome. This occupies the next section. In Section 4 we show that the competitive solution in which students pay fees to schools (or receive inducements from them) achieves the optimal solution of perfect mixing. The final section of the paper concludes with a brief discussion of the implications of these results.

2. The Framework

Each period a cohort of children is to be admitted to a system comprising n schools of identical size. Children remain at school for m years. We assume the incoming cohort at date t is represented by $\Omega_t = (\Omega; t)$ where Ω is the unit interval $[0; 1]$. Equip Ω with the Lebesgue measure μ. Each child $\omega \in \Omega$ has an associated quality $q(\omega)$ where $q : \Omega \to \Re^+$ is an increasing measurable function with finite integral. Thus it is implicit that the distribution of qualities does not change over time. An assignment of children to schools is a partition $\{\Omega_{it}\}$ of Ω_t into disjoint measurable subsets of equal measure ($= 1/n$). We think of Ω_{it} as the cohort entering school i at time t. Define for $Z \subseteq \Omega$

$$I(Z) = \int_{\omega \in Z} q(\omega) d\mu(\omega) \qquad (2.1)$$

Define also $q_{it} = nI(\Omega_{it})$ and $\bar{q} = I(\Omega)$. Then

$$n\bar{q} = \sum_{i=1} q_{it} \tag{2.2}$$

is independent of t. The number q_{it} is the average quality of children admitted to school i at time t, while $\bar{q}$ is the average quality of children in the population.

Let A be the set of vectors in $\Re^n$ with coordinates $q_i = nI(\Omega_i); \quad i = 1;\ldots;n;$ where $\{\Omega_i\}$ ranges over all possible decompositions of Ω into measurable subsets of equal measure. The set A thus represents all average quality vectors achievable by assignments of children to schools. Let $m_i = nI(J_i); \quad i = 1;\ldots;n$ where J_i is the interval $(i - 1/n; i/n]$. The following characterisation of the set A will be useful below. The proof is given in an Appendix.

Proposition The set A of average quality vectors achievable by assignments of children to schools forms a convex polyhedron in $\Re^n$ with vertices given by all permutations of the coordinates of the vector $(m_1;\ldots;m_n)$.

A school system may be represented by a tableau in which rows corresponds to schools and columns give the average qualities of admitted children. In the example in Figure 1 we assume that each of two schools contains, at any one time, two cohorts of children, one just admitted and one admitted in the previous period.

Figure 1
A School System

time	...	$t-1$	t	$t+1$	...
school 1	...	1	2	1	...
school 2	...	2	1	2	...

At t, school 1 has one older cohort of average quality 1 and a newly admitted cohort of average quality 2. vice versa for school 2. Next period the box moves one unit to the right and the enrolment changes. Note that the sum of the columns is constant, reflecting our assumption that the quality distribution does not change over time.

Students are assumed to rank schools by the quality improvements they offer, determined by the peer group. We shall assume quality improvements are a function of the weighted average

$$V_{it} = \sum_{r=-m} w_r q_{it+r} \tag{2.3}$$

where $w_r \geq 0$ for all r. The set $\{q_{it-1}, \ldots, q_{it-m}\}$ is the average qualities of children in school i when a new cohort is admitted, while $\{q_{it+1}, \ldots, q_{it+m}\}$ refers to those who will come subsequently while the cohort is in the school. In the main we shall assume perfect foresight so there are no expectational considerations. Usually we shall assume the weights are tent-shaped i.e. the function $r \to w_r$ is non- decreasing for negative r and non-increasing for positive r. This means that, in forming a valuation of a school, a student values less the qualities of students with whom he or she will spend less time. In the event that $w_r = 0$ for $r > 0$ we shall say that the weights (or the system in general) are backward-looking. In the system exhibited in Figure 1, if students entering at t evaluate the quality of those there when they arrive (including their own cohort) and the quality of those who will arrive at $t + 1$, with equal weights on the cohorts (normalised to sum to unity), school 1 then has value $V_{it} = 4/3$ while school 2 has value $5/3$.

Gains in quality on exit of a student entering school i at t are given by $f(V_{it})$ where f is some concave function. Thus there are diminishing returns to school quality. System-wide gain in aggregate quality is given by

$$\sum_{i=1} f(V_{it}) \tag{2.4}$$

We assume social welfare at t is given by

$$W_t = \sum_{s=t}^{\infty} (\Sigma_{i=1}^{n} f(V_{is}))\beta^{s-t+m} \tag{2.5}$$

where $0 < \beta < 1$ is a discount factor. At each date t a social planner should seek an assignment of children to schools to maximise (2.5).

The welfare index W is a function of the matrix $Q_t = (q_{is})$; $i = 1, \ldots, n$; $s = -m, -m+1, \ldots$ i.e. $W_t = W(Q_t)$. Exploiting the concavity of f in (2.5) one can easily show that W is a concave function of its matrix argument. Define a collection U_i; $i = 1, \ldots, n$; of assignment matrices as follows: $U_1 = Q_t$; & U_{r+1} is obtained from U_r by replacing the first row by the second, the second by the third, etc. and finally the n^{th} row by the first. The U_i are thus obtained merely by permuting the rows of Q_t; so by symmetry of the welfare function $W(U_i) = W(Q_t)$ for all i. Since the set A in the above Proposition is convex, $\tilde{Q}_t = \sum_{i=1}^{n} U_i/n$ is also the matrix of an assignment. Moreover

$$\begin{aligned} W(\tilde{Q}_t) &\geq [W(U_1) + \ldots + W(U_n)]/n \\ &= W(Q_t) \end{aligned}$$

Thus an assignment at all dates which equalises across schools the average quality of the incoming cohort is both feasible and optimal. We summarise the above discussion in

Theorem 1 There is a socially optimal assignment of children to schools in which each school receives an incoming cohort of equal average quality. Thus the valuations of all schools V_{it} are equal at each date.

If the function f is quadratic, $f(V) = V - \alpha V^2$; $\alpha > 0$; then, summing over i; it is easy to see that

$$\sum_{i=1}^{n} f(V_{it}) = n(\bar{q} - \alpha \bar{q}^2 - \alpha V(V_{it}))$$

where $V(V_{it})$ is the variance of the numbers V_{it}; $i = 1, \ldots, n$. Since $\bar{q}$ is constant by assumption, $\sum_{i=1}^{n} f(V_{it})$ is maximised by minimising the variance of the school valuations.

This optimal assignment need not be unique. As an example, consider again the tableau in Figure 1 and extend it into the past and future by alternating 1s and 2s in the rows. For the valuation function $V_{it} = \frac{1}{2}q_{it} + \frac{1}{2}q_{it-1}$ (backward-looking equal weights) the assignment produces equal valuations at all dates so the variance is zero and this outcome is thus optimal for quadratic $f(.)$.

A perhaps more realistic problem is to take the assignment of cohorts to schools at dates earlier than t as given and to look for assignments at t and later to maximise (2.5). For example in Figure 1 we could imagine the planner formulating his policy at t, taking the assignment at $t - 1$ as given. In this case perfect mixing ($q_{is} = 3/2$ for $i = 1, 2$; $s = t, t+1, \ldots$) does not produce an optimum for the above valuation function: alternating 1s and 2s sets the variance equal to zero at all dates, while mixing produces non-zero variance at date t.

We do not have general results in this case. For $m = 1$ (i.e. moving averages of length 3) one can show, fairly trivially, that if the characteristic polynomial of the moving average has a root inside the unit circle the optimal assignment will converge to perfect mixing.

3. Free Matching

So much for the optimal assignment. We now consider what might happen if schools and children are free to seek each other out. We must be quite careful exactly what we mean by this. We assume that each child ranks schools on the basis of the valuation V_{it}. In the event of draws on this basis, we imagine they are resolved according to some arbitrary ranking of the schools such as the order in the phone book. We shall refer to this as the prior ranking of schools and denote it by $\succ_p$. Thus we have defined a ranking of schools at t by: $i \succ_v j$ if and only if $V_{it} > V_{ij}$ or $V_{it} = V_{ij}$ and $i \succ_p j$. Clearly $\succ_v$ well-orders the set of schools.

We assume further that schools prefer students of higher quality, which might follow because they are easier to teach, and thus are preferred by teachers, or because school policy is determined by a board of parents who seek to increase V_{it} for the benefit of their own children.

Thus schools have preferences over children and children have preferences over schools. This is a matching problem of the sort considered by Roth and Marilda Sotomayor (1990). This theory typically considers two groups of agents (call them schools and students). Each school has a preference ordering over students, vice versa for students. In contrast to our case, different students (and schools) may have different preferences. The key concept is the **stable** matching i.e. a pairing of students and schools wherein the following never occurs:

(i) schools i and i' are paired to students ω and ω', respectively;

(ii) i' prefers ω to ω' and ω prefers i' to i.

If (i) and (ii) were to hold, one says the pairing (i', ω) **blocks** the matching (i, ω), (i', ω'). It is argued that in considering possible outcomes of a matching, attention should be restricted to stable matchings. Gale and Shapley (1962) have given an algorithm for computing a stable matching for finite sets of agents. In general, stable matchings are not unique. In the event of common preferences between schools and students, however, the stable matching is unique: the best school gets the best student etc.

We wish to generalise the notion of stable matches to our context. Define an assignment Ω_{it}; $i = 1, \ldots, n$ of children to schools at t as stable if $i \succ_v j$ implies there exist no $E_i \subseteq \Omega_{it}$, $E_j \subseteq \Omega_{jt}$ where E_i and E_j have positive measure and $q(\omega_j) > q(\omega_i)$ for all $\omega_i \in E_i$; $\omega_j \in E_j$. The following is easy to prove:

Theorem 2 *If Ω_{it}; $i = 1, \ldots, n$ is a stable assignment of children to schools at t, the sets Ω_{it} coincide, almost everywhere, with the subsets of the decomposition of $\Omega = [0, 1]$ into the subintervals $[0, 1/n]$, $(1/n, 2/n] \ldots (1 - 1/n, 1]$.*

Thus if we assume all assignments are stable, the best school automatically receives the best tranche of students, the second-best the second-best tranche, and so on. In this case we may treat each ability tranche as if it were a single unit and, in the valuation of a school by formula (1), the numbers q_{it} are the time invariant means of the ability tranches of Ω.

We wish to restrict somewhat further our family of admissible assignments. What we wish to exclude is the possibility that the current cohort arriving at a school could do better by going to another school who would be prepared to have them in place of their existing arriving cohort. For example consider a finite cohort of students and assume the quality of the top cohort is so high that any school they choose is the best by the $\succ_v$ criterion. Assume the cohort is large and the $\succ_v$ ordering is unchanged by a single defection. Then any choice by the top cohort can be part of a stable matching. In these circumstances it is plausible that the top tranche will spontaneously choose the top school since this will lead to a Pareto improvement for all their members over any other choice (c.f. Harsanyi and Selton (1988)).

In our context we define as assignment as **Pareto-consistent** if whenever $q_{it} > q_{jt}$ then

$$V_{it} \geq V_{jt} - w_0 q_{jt} + w_0 q_{it}$$

with equality only if $i \succ_p j$. The term on the right in the above inequality is the valuation of school j if its current cohort is replaced by Ω_{it}. Define

$$V^*_{it} = V_{it} - w_0 q_{it} \tag{3.1}$$

i.e. the value of school i excluding the arriving cohort; and define $i \succ_* j$ to mean $V^*_{it} > V^*_{jt}$ or $V^*_{it} = V^*_{jt}$ and $i \succ_p j$. Then our definition of Pareto-consistency is equivalent to the statement that $q_{it} > q_{jt}$ implies $i \succ_* j$. Among the set of stable assignments $q_{it} > q_{jt}$ is equivalent to $i \succ_v j$; so Pareto-consistency asserts precisely that $i \succ_v j$ implies $i \succ_* j$. In these circumstances the two orderings $\succ_v$ and $\succ_*$ are the same.

Call an assignment **free matching** if it is both stable and Pareto-consistent. We summarise the foregoing discussion as:

Theorem 3 A free matching assignment matches the j^{th} quality tranche with the j^{th} school according to the $\succ_*$ ordering.

A free matching equilibrium is an assignment of children to schools Ω_{it}; $i = 1, \ldots, n$ which is free matching at all dates t. We wish to characterise the set of free matching equilibria.

The problem is complicated by the fact that the ranking of schools depends on the future as well as the past. If valuations are backward-looking, however, we have a simple result. Define **perfect segregation** as an assignment of children to schools in which schools are composed, at all times, of the same quality tranches (up to sets of measure zero).

Theorem 4 If valuations are backward looking, the free matching equilibria are perfectly segregated.

Proof If V^*_{it} is the valuation of school i according to (3.1), we have

$$V^*_{it} = w_1 q_{i\,t-1} + \ldots + w_m q_{i\,t-m}$$

from which it follows that

$$\begin{aligned} V^*_{it+1} - V^*_{it} &= w_1 q_{it} + q_{it-1}(w_2 - w_1) + \ldots + q_{it-m+1}(w_m - w_{m-1}) - w_m q_{it-m} \\ &\geq w_1(q_{it} - q_m) \end{aligned}$$

where q_m is the maximum of the q_{it} (i.e. the average quality of the top tranche).

Note that the above inequality makes use of the monotonicity of the weights. It thus follows that if i is the best school at t, so that $q_{it} = q_m$, its *-valuation increases or stays the same. Now the set of all possible *-valuations is …nite, so the sequence of *-valuations of the best school must attain its upper bound and stay constant henceforth. Consider the best school at s after the upper bound is attained. At s + 1 it is possible that another school could have equal *-valuation, in which case we would have to appeal to the $\hat{A}_p$ ordering of schools to resolve who is now the best. If no such school emerges within m periods of s then the original school becomes the unambiguous best, for then it would be populated entirely by q_m cohorts and its *-valuation would strictly dominate all others. If such a school does emerge and stands higher on the $\hat{A}_p$ ranking, either it remains the best for m periods or is supplanted by another still higher on the $\hat{A}_p$ ranking. But the number of schools is …nite so this procedure must stop in a …nite time. Thus, eventually, a school emerges which remains the best forever. Once this happens the best school and the best students e¤ectively disappear from the analysis. We may now consider the second-to-top school and show that it will eventually be populated exclusively by the second top tranches. Thus after a …nite time, the schools are perfectly segregated. Clearly it is possible to produce a bound on the number of periods it takes for this to happen which is independent of the initial date t. To complete the proof we need to show that perfect segregation always obtains, rather than eventually. But this is obvious since the time to perfect segregation is bounded: if some part of the assignment were not perfect segregation, it would be enough to consider a su¢ciently early t to obtain a contradiction.

Corollary to Proof If a backward-looking system becomes free matching at some date, it evolves towards perfect segregation in …nite time.

Theorem 4 is a little more subtle than it looks and is overturned if we remove some seemingly innocuous assumptions. Consider …rst the assumption that the distribution of qualities of the incoming cohort of students is constant over time i.e. the quality function q(:) is time invariant. Consider two schools, comprising three cohorts (m = 3) with the tableau

School 1	:::	10	10	:1	:1	:001	:001	:::
School 2	:::	1	1	:01	:01	:0001	:0001	:::

Assume the *-valuation gives equal weights to the two senior levels. Then the pattern in the above tableau is consistent with the best children being allocated to the best school: but the best school cycles over time.

Consider next our assumption that the weight function is monotonic. Consider as before a two- school, three-level system with tableau

School 1	:::	1	0	1	0	1	0	:::
School 2	:::	0	1	0	1	0	1	:::

If the *-valuation attaches a weight of unity to the top level and zero to the lower level (the little boys love only the big boys), then the tableau is consistent with the best school receiving the best tranche. Once again, the top school cycles over time.

We now turn attention to the case of two-sided weights i.e. students take into account future schoolmates as well as those currently observable.

Theorem 5 If there are two schools, the free matching-equilibria are perfectly segregated.

Proof The problem is obviously invariant to an increasing affine transformation of qualities so we assume that q_{it} is 0 or 1. It then follows that

$$V^*_{1t} + V^*_{2t} = 1 - w_0$$

for all t. By an argument given in Theorem 4, monotonicity of weights gives an inequality:

$$(*) \qquad V^*_{it+1} - V^*_{it} \geq w_{-1}(q_{it} - 1) - w_1 q_{it+1}$$

The proof is completed by showing the following tableau is unobtainable:

	...	t	t + 1	...
School 1	...	1	0	...
School 2	...	0	1	...

Assume it is. Then the following chain of inequalities holds

$$(**) \qquad V^*_{1t+1} \geq V^*_{1t} \geq V^*_{2t} \geq V^*_{2t+1} \geq V^*_{1t+1}$$

The first follows by setting $q_{1t} = 1$, $q_{1t+1} = 0$ in (*). The second follows because school 1 is preferred at t. The third follows because V^*_{1t} and V^*_{2t} sum to a constant for all t and V^*_{1t} is larger at t + 1 than at t. The fourth follows since school 2 is preferred at t + 1. The upshot is that the inequalities in (**) are all equalities. But since $V^*_{1t} = V^*_{2t}$ and school 1 was chosen, it follows that school 1 is higher on the prior ranking. This is at variance with $V^*_{1t+1} = V^*_{2t+1}$ and school 2 being chosen at t + 1. The contradiction delivers the result.

For more than two schools, however, there is bad news.

Theorem 6 Assume equal weights. For more than two schools there are non-segregated free matching equilibria.

Proof We exhibit such an equilibrium for the case n = 3.

School 1	...	1	2	2	1	0	0	1	2	2	1	0	0	1	2	2	...
School 2	...	2	1	0	0	1	2	2	1	0	0	1	2	2	1	0	...
School 3	...	0	0	1	2	2	1	0	0	1	2	2	1	0	0	1	...

It can be easily checked that the numbers on either side of a 0 sum to 1, those on either side of a 1 sum to 2 and those either side of a 2 sum to 3. Thus the tableau is a free matching equilibrium of a system comprising two levels (lower and upper) with equal weights assigned to current and future schoolmates. Higher order examples ($n > 3$) may be obtained trivially by appending rows of constant q_{it} to this example. For example we could obtain a non-segregated tableau for $n = 4$ by writing a row of 3s across the top of the existing tableau.

The tableau in Theorem 4 is not the only non-segregated free matching equilibrium. In fact one can judiciously insert sections of the perfectly segregated tableau. Consider:

School 1	...	1	2	2	2	...	2	2	2	1	...
School 2	...	2	1	1	1	...	1	1	0	0	...
School 3	...	0	0	0	0	...	0	0	1	2	...

This tableau consists of three parts. The first and third are sections of the tableau from Theorem 6. The middle is the segregated case. To resolve draws we assume the prior ordering of schools is $1 \succ_p 2 \succ_p 3$. It will be seen that this has produced another free matching equilibrium.

These equilibria strike us as extremely artificial and suggest that, in a sense, the model is under-determined. Clearly the area where the model is most deficient is in expectation formation. One escape is to assume that expectations of the future depend on the past. If we assume that the expectation of future quality in school i is a weighted sum of past qualities where the weights are positive (past quality predicts future quality) and declining (proximate qualities are better predictors) then we may substitute out the future values in (3.1) to obtain a V^* function with one-sided weights. An application of Theorem 4 then delivers perfect segregation. Moreover expectations would be trivially correct. Thus we have:

Theorem 7 If expectations of future quality are formed as weighted averages of existing qualities with positive declining weights then the free matching-equilibria are perfectly segregated and expectations are correct, ex-post.

Our final result shows that perfect segregation produces, in welfare terms, the worst possible outcome.

Theorem 8 Perfect segregation minimises the welfare function W_t at each date t:

Proof Consider the welfare index W as a function of an assignment vector at t; $W = W(q_{1t}; \ldots; q_{nt})$, holding fixed arguments at other dates. The domain of the restricted function is the convex polyhedron characterised in the Proposition above. But a concave function defined on a convex polyhedron achieves a minimum at some vertex, and, by the Proposition, these are segregated assignments with quality vectors $(m_1; m_2; \ldots; m_n)$ (and those obtained by permuting these co-ordinates). It follows that a minimum of $W = W(Q_{it})$ is obtained by segregated assignments at each date. We complete the proof by showing that, among the class of outcomes in which schools receive segregated assignments, the welfare index is minimised when each school receives the same quality tranche at each date.

Consider the term in (2.4) , $\sum_{i=1}^{n} f(V_{is})$. We have

$$\begin{aligned}
\sum_{i=1} f(V_{is}) &= \sum_{i=1} f\left(\sum_{r=-m} w_r q_{i\,t+r}\right) \\
&\geq \sum_{i=1}\sum_{r=-m} w_r f(q_{i\,t+r}) \qquad \text{(concavity of } f\text{)} \\
&= \sum_{r=-m} w_r \sum_{i=1} f(q_{i\,t+r}) \\
&= \sum_{r=-m} w_r \sum_{i=1} f(m_i) \qquad \text{(segregated assignments)} \\
&= \sum_{i=1} f(m_i) \qquad (\Sigma w_r = 1)
\end{aligned}$$

It is now enough to note that this final term corresponds to an assignment where each school i receives the m_i quality tranche each period. This completes the proof.

Theorems 1 and 8 show that perfect segregation is a global welfare minimum. Theorem 7 and 5 give conditions under which free matching leads to perfect segregation. If these conditions hold, then our results imply that letting children and schools 'seek each other out' leads to the worst possible result.

4. Competitive Allocation

We consider now the effects of allowing prices to enter the model i.e allowing students to bid for places and schools to bid for students. Recently Rothschild and White (1995) have shown that competitive prices can support efficient outcomes, even when peer-group effects are present, and we shall demonstrate this in our context.

The school experience is in fact a conflation of two economic functions. Firstly the student supplies factor services to his peer-group. Secondly the student is the recipient of a portion of the factor services supplied by his peer-group. The key point is that both of these activities can clearly be supported by competition when considered individually. It is thus natural to hypothesise that prices charging for the net gain (services received less those delivered) will support the desired outcome.

We abstract from dynamic considerations and consider a school i with the aggregate production function $y = af(\bar{q}_i)$ where a is the number on the school roll and q_i is the average quality of children at i. For this production function, the marginal product of the quality q of a given student at i is $f'(q_i)$., which would thus be the price of the factor input under competition. It would follow that such an individual would be paid $qf'(\bar{q})$ in a regime of perfect mixing where here $\bar{q}$ is the system-wide average quality. However all individuals in this school also absorb benefits each of the value $f(\bar{q})$. Under competition each will pay the same for these benefits: and this must be $\bar{q}f(\bar{q})$ if the total payments to the factors are to equal the revenue raised. Consider therefore price given by the net payment of an individual of quality q:

$$p(q) = (\bar{q} - q)f'(\bar{q}) \tag{4.1}$$

Theorem 9 We assume the quality function f is properly concave, $f'' < 0$. With quality payments given by (4.1), under perfect mixing there are no incentives for students to change schools, nor for other schools to be set up.

Proof A student of quality q gains

$$f(\bar{q}_s) - (\bar{q} - q)f'(\bar{q})$$

from attending school s of average quality $\bar{q}_s$ if fees are determined by (4.1): Under perfect mixing $\bar{q}_s = \bar{q}$ for all schools so gains are equalised and students are indifferent between schools.

With regard to setting up new schools, note that it is clear from the form of (4.1) that all schools make zero profits. This observation can be sharpened to show that no new school, charging whatever it pleases, can make non-negative profits by charging net fees other than those given by (4.1). To see this, note that a new school S must satisfy

$$f(\bar{q}_S) - p_\omega \geq f(\bar{q}) - p(q(\omega)) \tag{4.2}$$

for all pupils $\omega \in S$, where p_ω is the fee paid by each ω. This condition ensures that S's enrolment do at least as well as they might in the perfectly mixed schools. A profit maximising new school will raise p_ω so that (4.2) binds. We thus have

$$\begin{aligned} p_\omega &= f(\bar{q}_S) - f(\bar{q}) + p(q(\omega)) \\ &= f(\bar{q}_S) - f(\bar{q}) - (q(\omega) - \bar{q})f'(\bar{q}) \end{aligned}$$

If S is to make profits, the average of this over students at S must be non-negative:

$$f(\bar{q}_S) - f(\bar{q}) - (\bar{q}_S - \bar{q})f'(\bar{q}) \geq 0$$

But if f is properly concave the left-hand-side has a maximum of zero for $\bar{q}_S = \bar{q}$, so that S is perfectly mixed, as required.

5. Concluding Discussion

A combination of parental choice and free selection of pupils by schools in the state system will most likely lead to bad results: bad schools populated by the worst students together with elite schools populated by the educational elite. In our set-up transfer of children from the elite schools to the bad schools would effect a welfare gain, the losses of the elite being more than compensated by gains to those in the worse schools. It is interesting to note that if parents and schools are free to choose in the manner outlined above, the less information parents have about schools the better (from society's point of view). Information will facilitate segregation. This is in contrast to the usual argument for the laissez-faire, wherein choice leads to the good driving out the bad. In our simplified framework, these Darwinist possibilities are absent: the only difference between schools is their enrolment. Freedom to choose here leads to the worst result of perfect segregation. Remarkably, however, allowing a little more choice - the ability for schools to charge fees and maximise profits - leads to the best result of perfect mixing.

Whereas this may be construed as an argument for the provision of education by unregulated competitive markets, it should be noted that the fee structure is rather unappealing: the worst students would pay the highest fees. Thus in a system of compulsory education we might well be seeking to extract large fees from those least able to pay. In these circumstances the competitive solution would clearly need to be supplemented by a scheme to enable the poor to pay market fees. If the state were able to identify the least able and provide them with some sort of dowry to offer prospective schools, the welfare optimum could be achieved. In practice, means tested vouchers may approximate.

APPENDIX

Proof of the Proposition:

Let $B \subseteq \Re^n$ be the set of vectors q such that

$$\text{(A.1)} \qquad \sum_{i=1} q_i \geq \sum_{i=1} m_i \qquad (s = 1, \ldots, n-1)$$

and

$$\text{(A.2)} \qquad \sum_{i=1} q_i = \sum_{i=1} m_i$$

where $(q_1, \ldots, q_n)$ is any permutation of the coordinates of q.

First, we assert $A = B$. If $q \in \Re^n$ is a vector of school means, then the inequalities in (A.1) follow because $q(\omega)$ is an increasing function while the equality (A.2) is trivial. Conversely let $q \in B$; $q = (q_1, \ldots, q_n)$. We wish to show that q can arise from an assignment of children to schools. We prove this result by induction. The result is trivial for $n = 1$; assume it is true up to and including $n-1$. Define

$$G(x) = nI([x, x + 1/n])$$

for $x \in [0, 1]$. Then $G(0) = m_1$ and $G(1 - 1/n) = m_n$. Thus since $m_1 \leq q_1 \leq m_n$ the Intermediate Value Theorem shows that $nI(Z) = q_1$ where $Z = [z, z + 1/n]$ for some z with $0 \leq z \leq 1 - 1/n$.

We now only outline the proof. The interval S will meet at most two of the sub-intervals $[(j-1)/n, j/n]$; $j = 1, \ldots, n$; see the Figure. As drawn S meets $[1/n, 2/n]$ and $[2/n, 3/n]$ and $I(S) = q_1$ is the hatched area. The key idea is that if the hatched area were to be removed, the function $q(\omega)$ would remain increasing and the problem would involve $n-1$ subintervals, $[1/n, 3/n] \setminus S$ replacing $[1/n, 2/n]$ and $[2/n, 3/n]$. All that needs to be done is to show that a version of (A.1) and (A.2) will hold for the collapsed interval. This is obvious from the Figure.

We have shown that $A = B$ i.e. the system (A.1) and (A.2) characterises the set of vectors that can represent assignments. We shall have occasion to use a slight generalisation of this result wherein the domain E of the function q is any bounded measurable set in $\Re$ and $m_i = I(E_i)$ for all i where $\{E_i\}$ is an ordered partition of E (i.e. a partition in which $e_1 < e_2 < \ldots < e_n$ for each $e_i \in E_i$) into n subsets of equal measure.

To complete the proof let q be a vertex of A and assume $q_i = I(F_i)$ $i = 1, 2$ are two coordinates of q, the first say. Let E_1, E_2 be an ordered partition of $F_1 \cup F_2$ with $p_i = I(E_i)$; $i = 1, 2$ and let $C \subseteq \Re^n$ be the set of vectors obtained by swapping measure between F_1 and F_2, keeping constant the remaining ordinates of q. By the ‘slight generalisation’ just discussed, this is the line segment joining $(p_1; p_2; q_3; ::::q_n)$ and $(p_2; p_1; q_3; ::::q_n)$ in $<^n$. Since q is a vertex it is now immediate that p_1 and p_2 coincide with q_1 and q_2 in some order, so that F_1 and F_2 coincide with E_1 and E_2, up to measure zero. Thus F_1 and F_2 form an ordered partition of F_1 [F_2: An extension of this argument shows that $F_1; F_2::::F_n$ forms an ordered partition of - = [0; 1]. Clearly the F_i must coincide, up to sets of measure zero, with J_i.

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