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Majority voting on restricted domains

Working paper

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Abstract

In judgment aggregation, unlike preference aggregation, not much is known about domain restrictions that guarantee consistent majority outcomes. We introduce several conditions on individual judgments sufficient for consistent majority judgments. Some are based on global orders of propositions or individuals, others on local orders, still others not on orders at all. Some generalize classic social-choice-theoretic domain conditions, others have no counterpart. Our most general condition generalizes Sen's triplewise value-restriction, itself the most general classic condition. We also prove a new characterization theorem: for a large class of domains, if there exists any aggregation function satisfying some democratic conditions, then majority voting is the unique such function. Taken together, our results provide new support for the robustness of majority rule.

1 Introduction

In the theory of preference aggregation, it is well known that majority voting on pairs of alternatives may generate inconsistent (i.e., cyclical) majority preferences even when all individuals' preferences are consistent (i.e., acyclical). The most famous example is Condorcet's paradox. Here one individual prefers x to y to z, a second y to z to x, and a third z to x to y, and thus there are majorities for x against y, for y against z, and for z against x, a 'cycle'. But it is equally well known that if individual preferences fall into a suitably restricted domain, majority cycles can be avoided (for an excellent overview, see Gaertner [16]). The most famous domain restriction with this effect is Black's single-peakedness [1]. A profile of individual preferences is single-peaked if the alternatives can be ordered from 'left' to 'right' such that each individual has

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a most preferred alternative with decreasing preference for other alternatives as we move away from it in either direction. Inada [18] showed that another condition called single-cavedness and interpretable as the mirror image of single-peakedness also suffices for avoiding majority cycles: a profile is single-caved if, for some 'left'-'right' order of the alternatives, each individual has a least preferred alternative with increasing preference for other alternatives as we move away from it in either direction. Sen [40] introduced a very general domain restriction, called triplewise value-restriction, that garantees acyclical majority preferences and is implied by Black's, Inada's and other conditions; it therefore unifies several domain-restriction conditions, yet has a technical flavour without straightforward interpretation.

The wealth of domain-restriction conditions for avoiding majority cycles was supplemented by another family of conditions based not on 'left'-'right' orders of the alternatives, but on 'left'-'right' orders of the individuals. Important conditions in this family are Grandmont's intermediateness [17] and Rothstein's order restriction ([36], [37]) with its special case of single-crossingness (e.g., Roberts [34], Saporiti and Tohmé [38], Saporiti [39]). To illustrate, a profile of individual preferences is order-restricted if the individuals – rather than the alternatives – can be ordered from 'left' to 'right' such that, for each pair of alternatives x and y, the individuals preferring x to y are either all to the left, or all the right, of those preferring y to x.

Empirically, domain restrictions are important as many political and economic contexts induce a natural structure in preferences. For example, domain restrictions based on a 'left'-'right' order – whether of the alternatives or of the individuals – can capture situations in which preferences are structured by one normative or cognitive dimension, such as from socialist to libertarian, from urban to rural, or from secular to religious.

In the theory of judgment aggregation, by contrast, domain restrictions have received much less attention (the only exception is the work on unidimensional alignment, e.g., List [22]). This is an important gap in the literature since, here too, majority voting with unrestricted but consistent individual inputs may generate inconsistent collective outputs, while on a suitably restricted domain such inconsistencies can be avoided. As illustrated by the much-discussed discursive paradox (e.g., Pettit [32]), if one individual judges that a, $a \rightarrow b$ and b, a second that a, but not $a \rightarrow b$ and not b, and a third that $a \rightarrow b$, but not a and not b, there are majorities for a, for $a \rightarrow b$ and yet for not b, an inconsistency. But if no individual rejects $a \rightarrow b$, for example, this problem can never arise.

Surprisingly, however, despite the abundance of impossibility results generalizing the discursive paradox as reviewed below, very little is known about the domains of individual judgments on which discursive paradoxes can occur (as opposed to agendas of propositions susceptible to such problems, which have been extensively characterized in the literature). If we can find compelling domain restrictions to ensure majority consistency, this allows us to refine and possibly ameliorate the lessons of the discursive paradox. Going beyond the standard impossibility results, which all assume an unrestricted domain, we can then ask: in what political and economic contexts do the identified domain restrictions hold, so that majority voting becomes safe, and in what contexts are they violated, so that majority voting becomes problematic?

This paper has two goals. The first is to introduce several conditions on profiles of individual judgments that guarantee consistent majority judgments. As explained in a moment, these can be distinguished in at least two respects: first, in terms of whether they are based on orders of propositions, on orders of individuals, or not on orders at all; and second, if they are based on orders, in terms of whether these are 'global' or 'local'. We also discuss parallels and disanalogies with domain-restriction conditions on preferences and finally distinguish between product and non-product domains; this distinction is significant for game-theoretic applications.

The second goal of the paper is to present a new characterization result demonstrating the robustness of majority voting. In broad analogy with May's classic characterization of majority voting in binary choices [26] and Dasgupta and Maskin's theorem on the robustness of majority voting in preference aggregation [2], we show that, for a very large class of domains, if there exists any aggregation function at all that satisfies some minimal democratic conditions including consistency of its outcomes, then majority voting is the unique such aggregation function. In combination with our domain restriction conditions, this theorem provides a powerful argument in support of majority voting in appropriate circumstances.

Let us briefly comment on the two distinctions underlying our discussion. First, our conditions based on orders of the individuals are analogous to, and in fact generalize, some of the conditions on preferences reviewed above, particularly intermediateness and order restriction. By contrast, those conditions based on orders of the propositions are not obviously analogous to any conditions on preferences. While an order of individuals can be interpreted similarly in judgment and preference aggregation – namely in terms of the individuals' positions on a normative or cognitive dimension – an order of propositions in judgment aggregation is conceptually distinct from an order of alternatives in preference aggregation. Propositions, unlike alternatives, are not mutually exclusive. It is therefore surprising that sufficient conditions for consistent majority judgments can be given even based on orders of propositions. We also introduce a very general domain-restriction condition not based on orders at all: it generalizes Sen's condition of triplewise value-restriction. In concluding the paper, we characterize the maximal domain on which majority voting yields consistent collective judgments.

Secondly, our domain-restriction conditions based on orders admit global and local variants. In the global case, the individuals' judgments on all propositions on the agenda are constrained by the same 'left'-'right' order of propositions or individuals, whereas in the local case, that order may differ across subsets of the agenda. To give an illustration from the more familiar context of preference aggregation, single-peakedness and single-cavedness are global conditions, whereas the restriction of these conditions to triples of alternatives yields local ones. But while in preference aggregation local conditions result from the restriction of global conditions to triples of alternatives, the picture is more general in judgment aggregation. Here different 'left'-'right' orders may apply to different subagendas, which correspond to different semantic fields. We give precise criteria for selecting appropriate subagendas. An individual can be left-wing on a 'social' subagenda and right-wing on an 'economic' one, for example.

As already noted, some of our conditions generalize existing conditions in preference aggregation, notably Grandmont's intermediateness, Rothstein's order restriction and Sen's triplewise value-restriction, and reduce to them when the agenda of propositions under consideration contains binary ranking propositions suitable for representing preferences (such as xPy, yPz, xPz etc.).[1]

We pursue our two goals in reverse order, beginning with the characterization of majority voting, followed by the discussion of domain restrictions. We state our results for the general case in which individual and collective judgments are only required to be consistent; they need not be complete (i.e., they need not be opinionated on every proposition-negation pair). But whenever this is relevant, we also consider the important special case of full rationality, i.e., the conjunction of consistency and completeness.

A few remarks about the literature on judgment aggregation are due. The recent field of judgment aggregation emerged from the areas of law and political philosophy (e.g., Kornhauser and Sager [20] and Pettit [32]) and was formalized social-choice-theoretically by List and Pettit [24]. The literature contains several impossibility results generalizing the observation that on an unrestricted domain majority judgments can be logically inconsistent (e.g., List and Pettit [24] and [25], Pauly and van Hees [31], Dietrich [3], Gärdenfors [15], Nehring and Puppe [30], van Hees [41], Mongin [27], Dietrich and List [7], and Dokow and Holzman [13]). Other impossibility results follow from Nehring and Puppe's [28] strategy-proofness results on property spaces. Earlier precursors include works on abstract aggregation (Wilson [42], Rubinstein and Fishburn [35]). A liberal-paradox-type impossibility was derived in Dietrich and List [12]. Giving up propositionwise aggregation, possibility results were obtained, for example, by using sequential rules (List [23]) and fusion operators (Pigozzi [33]). Voter manipulation in the judgment-aggregation model was analysed in Dietrich and List [8]. But so far the only domain-restriction condition known to guarantee consistent majority judgments is List's unidimensional alignment ([21], [22]), a global non-product domain condition based on orders of individuals. Here we use Dietrich's generalized model [4], which allows propositions to be expressed in rich logical languages. We include some proofs in the main text, others in the appendix.

[1] The fact that these three existing conditions are already very general representatives of their respective families underlines the generality of our new conditions here.

2 The model

We consider a group of individuals $N = \{1, 2, \ldots, n\}$ ($n \geq 2$) making judgments on some propositions represented in logic (Dietrich [4], generalizing List and Pettit [24], [25]).

Logic. A *logic* is given by a *language* and a notion of *consistency*. The *language* is a non-empty set $\mathbf{L}$ of sentences (called *propositions*) closed under negation (i.e., $p \in \mathbf{L}$ implies $\neg p \in \mathbf{L}$, where $\neg$ is the negation symbol). For example, in standard propositional logic, $\mathbf{L}$ contains propositions such as a, b, $a \wedge b$, $a \vee b$, $\neg(a \to b)$ (where $\wedge$, $\vee$, $\to$ denote 'and', 'or', 'if-then', respectively). In other logics, the language may involve additional connectives, such as modal operators ('it is necessary/possible that'), deontic operators ('it is obligatory/permissible that'), subjunctive conditionals ('if p were the case, then q would be the case'), or quantifiers ('for all/some'). The notion of *consistency* captures the logical connections between propositions by stipulating that some sets of propositions $S \subseteq \mathbf{L}$ are *consistent* (and the others *inconsistent*), subject to some regularity axioms.[2] A proposition $p \in \mathbf{L}$ is a *contradiction* if $\{p\}$ is inconsistent and a *tautology* if $\{\neg p\}$ is inconsistent. For example, in standard logics, $\{a, a \to b, b\}$ and $\{a \wedge b\}$ are consistent and $\{a, \neg a\}$ and $\{a, a \to b, \neg b\}$ inconsistent; $a \wedge \neg a$ is a contradiction and $a \vee \neg a$ a tautology.

Agenda. The *agenda* is the set of propositions on which judgments are to be made. It is a non-empty set $X \subseteq \mathbf{L}$ expressible as $X = \{p, \neg p : p \in X_+\}$ for some set X_+ of unnegated propositions (this avoids double-negations in X). In our introductory example, the agenda is $X = \{a, \neg a, a \to b, \neg(a \to b), b, \neg b\}$. For convenience, we assume that X is finite.[3] As a notational convention, we cancel double-negations in front of propositions in X.[4] Further, for any $Y \subseteq X$, we write $Y^{\pm} = \{p, \neg p : p \in Y\}$ to denote the (single-)negation closure of Y.

Judgment sets. An individual's *judgment set* is the set $A \subseteq X$ of propositions in the agenda that he or she accepts (e.g., 'believes'). A *profile* is an n-tuple $(A_1, \ldots, A_n)$ of judgment sets across individuals. A judgment set is *consistent* if it is consistent in $\mathbf{L}$; it is *complete* if it contains at least one member of each proposition-negation pair $p, \neg p \in X$; it is *opinionated* if it contains precisely one such member. Our results mostly do not require completeness, in line with several works on the aggregation of incomplete judgments (Gärdenfors [15]; Dietrich and List [9], [10], [11]; Dokow and Holzman [14]; List and Pettit

[2] Self-entailment: Any pair $\{p, \neg p\} \subseteq \mathbf{L}$ is inconsistent. Monotonicity: Subsets of consistent sets $S \subseteq \mathbf{L}$ are consistent. Completability: $\emptyset$ is consistent, and each consistent set $S \subseteq \mathbf{L}$ has a consistent superset $T \subseteq \mathbf{L}$ containing a member of each pair $p, \neg p \in \mathbf{L}$. See Dietrich [4].

[3] For infinite X, our results hold either as stated or under compactness of the logic.

[4] More precisely, if $p \in X$ is already of the form $p = \neg q$, we write $\neg p$ to mean q rather than $\neg\neg q$. This ensures that, whenever $p \in X$, then $\neg p \in X$.

[24]). This strengthens our possibility results as the identified possibilities hold on larger domains of profiles. But we also consider the complete case.

Aggregation functions. A *domain* is a set D of profiles, interpreted as admissible inputs to the aggregation. An *aggregation function* is a function F that maps each profile $(A_1, \ldots, A_n)$ in a given domain D to a collective judgment set $F(A_1, \ldots, A_n) = A \subseteq X$. While the literature focuses on the *universal domain* (which consists of all profiles of consistent and complete judgment sets), we here focus mainly on domains that are less restrictive in that they allow for incomplete judgments, but more restrictive in that we impose some structural conditions. We call an aggregation function *consistent* or *complete*, respectively, if it generates a consistent or complete judgment set for each profile in its domain. The *majority outcome on a profile* $(A_1, ..., A_n)$ is the judgment set

$$\{p \in X : \text{there are more individuals } i \in N \text{ with } p \in A_i \text{ than with } p \notin A_i\}.$$

The aggregation function that generates the majority outcome on each profile in its domain D is called *majority voting on D*.[5]

Preference aggregation as a special case. To relate our results to existing results on preference aggregation, we must explain how preference aggregation can be represented in our model.[6] Since preference relations are binary relations on some set, they allow a logical representation. Take a simple predicate logic $\mathbf{L}$ with a set of two or more constants $K = \{x, y, ...\}$ representing alternatives and a two-place predicate P representing (strict) preference. For any $x, y \in K$, xPy means 'x is preferable to y'. Define any set $S \subseteq \mathbf{L}$ to be *consistent* if $S \cup Z$ is consistent in the standard sense of predicate logic, where Z is the set of rationality conditions on strict preferences.[7] Now the *preference agenda* is $X_K = \{xPy \in \mathbf{L} : x, y \in K\}^{\pm}$. Preference relations and opinionated judgment sets stand in a bijective correspondence:

- to any preference relation (arbitrary binary relation) $\succ$ on K corresponds the opinionated judgment set $A_{\succ} \subseteq X_K$ with

$$A_{\succ} = \{xPy : x, y \in K \& x \succ y\} \cup \{\neg xPy : x, y \in K \& x \not\succ y\};$$

- conversely, to any opinionated judgment set $A \subseteq X_K$ corresponds the preference relation $\succ_A$ on K with

$$x \succ_A y \Leftrightarrow xPy \in A \ \forall x, y \in K.$$

[5] Other widely discussed aggregation functions include dictatorships, supermajority functions, and premise-based or conclusion-based functions.

[6] For details of the construction, see Dietrich and List [7], extending List and Pettit [25].

[7] Z consists of $(\forall v_1)(\forall v_2)(v_1 P v_2 \rightarrow \neg v_2 P v_1)$ (asymmetry), $(\forall v_1)(\forall v_2)(\forall v_3)((v_1 P v_2 \wedge v_2 P v_3) \rightarrow v_1 P v_3)$ (transitivity), $(\forall v_1)(\forall v_2)(\neg v_1 = v_2 \rightarrow (v_1 P v_2 \vee v_2 P v_1))$ (connectedness) and, for each pair of distinct constants $x, y \in K$, $\neg x = y$ (exclusiveness of alternatives).

A preference relation $\succ$ is fully rational (i.e., asymmetric, transitive and connected) if and only if $A_{\succ}$ is consistent, because we have built the rationality conditions on preferences into the logic. Under this construction, a judgment aggregation function (for opinionated judgment sets) represents a preference aggregation function, and majority voting as defined above corresponds to pairwise majority voting in the standard Condorcetian sense.

3 Why majority voting?

To motivate our focus on majority voting, we begin by presenting a new characterization of it on a large class of domains. We use two democratic conditions in addition to the requirement of consistent collective judgment sets.

Anonymity. For any profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*)$ in the domain of F that are permutations of each other, $F(A_1, \ldots, A_n) = F(A_1^*, \ldots, A_n^*)$.

Acceptance/rejection neutrality. For any profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*)$ in the domain of F and any proposition $p \in X$,

$$[\text{for all } i \in N,\, p \in A_i \Leftrightarrow p \notin A_i^*] \;\Rightarrow\; [p \in F(A_1, \ldots, A_n) \Leftrightarrow p \notin F(A_1^*, \ldots, A_n^*)].$$

Both conditions are familiar from May's classic characterization of majority voting in a single binary choice [26]. Anonymity requires equal treatment of all individuals, and acceptance/rejection neutrality prevents the aggregation function from favouring the acceptance of a proposition over its rejection or vice versa; that is, if the individuals accepting a given proposition in one profile are the same as those rejecting it in another, then the proposition must be collectively accepted in the first profile if and only if it is collectively rejected in the second.[8]

Let D_2 be the set of all profiles $(A_1, ..., A_n)$ of consistent individual judgment sets where at most two of the A_i's are distinct.

Theorem 1 *If an aggregation function on a domain $D \supseteq D_2$ is consistent, anonymous and acceptance/rejection neutral, then it is majority voting on D.*[9]

[8] If we require consistency *and* completeness of individual and collective judgment sets, acceptance/rejection neutrality becomes equivalent to 'unbiasedness' (Dietrich and List [6]) and, suitably translated, 'neutrality-within-issues' (Nehring and Puppe [29]).

[9] Theorem 1 requires X to contain no tautologies (unlike all other results in this paper). As a counterexample when X contains a tautology t, let n be odd and let the aggregation rule F on $D = D_2$ be given by: (i) $t \in F(A_1, ..., A_n) \Leftrightarrow |\{i : t \in A_i\}| < n/2$; (ii) $\neg t \notin F(A_1, ..., A_n)$; (iii) for all $p \in X \backslash \{t, \neg t\}$, $p \in F(A_1, ..., A_n) \Leftrightarrow |\{i : p \in A_i\}| > n/2$. By (i), F is not majority voting on D. But F has all required properties: anonymity is obvious; acceptance/rejection neutrality can be shown by observing that n is odd and (with regard to $\neg t$) that in $D = D_2$ no individual ever accepts a contradiction; consistency can be shown by observing that the domain is D_2 and that consistent sets remain consistent by adding a tautology.

This result is surprising in at least two respects. First, unlike May's theorem, it requires no monotonicity condition on the aggregation function; monotonicity follows from the other conditions. Second, unlike almost all results in the field of judgment aggregation, it requires no assumptions about the agenda. Existing theorems usually need some agenda complexity assumptions, for example to derive monotonicity if it is not explicitly imposed; so the validity of a theorem for *all* agendas is rather atypical.

How can we interpret theorem 1? As noted in the introduction, its lesson is somewhat similar to that of Dasgupta and Maskin's much-discussed theorem on the robustness of majority voting in preference aggregation [2]. Theorem 1 shows that, for a very large class of domains (namely all those including D_2), if there is any consistent aggregation function at all that satisfies anonymity and acceptance/rejection neutrality, then majority voting is the unique such function. Non-degenerate domains, such as those introduced below, clearly fall into this class of domains.[10]

To prove theorem 1, we first state a lemma, proved in the appendix. Using standard terminology, call aggregation function F *independent* if, for any profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*)$ in the domain of F and any proposition $p \in X$,

$$[\text{for all } i \in N, \, p \in A_i \Leftrightarrow p \in A_i^*] \;\Rightarrow\; [p \in F(A_1, \ldots, A_n) \Leftrightarrow p \in F(A_1^*, \ldots, A_n^*)].$$

Lemma 2 *Every consistent and acceptance/rejection neutral aggregation function on a domain $D \supseteq D_2$ is independent.*

Proof of theorem 1. Consider any agenda X without tautologies, and let F and D be as specified. By lemma 2, F is independent. For every $p \in X$, let $\mathcal{K}_p$ be the set of numbers $k \in \{0, ..., n\}$ such that $p \in F(A_1, ..., A_n)$ for some (and hence, by independence and anonymity, every) profile $(A_1, ..., A_n) \in D$ with $|\{i : p \in A_i\}| = k$. We prove three claims, the second one being the key step.

Claim 1: For all $p \in X$ and all $k \in \{0, ..., n\}$, $k \in \mathcal{K}_p \Leftrightarrow n - k \notin \mathcal{K}_p$.

Consider any $p \in X$ and any $k \in \{0, .., n\}$. Let $C \subseteq N$ be a coalition of size k. As X contains no tautologies and thus no contradictions, $\{p\}$ is consistent. So there exists a profile $(A_1, ..., A_n) \in D_2$ ($\subseteq D$) such that $\{i \in N : p \in A_i\} = C$ (e.g., $A_i = \{p\}$ for $i \in C$, and $A_i = \emptyset$ for $i \notin C$). Analogously, there exists a profile $(A_1^*, ..., A_n^*) \in D_2$ ($\subseteq D$) such that $\{i \in N : p \in A_i^*\} = N \backslash C$. By acceptance/rejection neutrality, $p \in F(A_1, ..., A_n) \Leftrightarrow p \notin F(A_1^*, ..., A_n^*)$. In this equivalence, the left-hand-side is equivalent to $k \in \mathcal{K}_p$, and the right-hand-side to $n - k \notin \mathcal{K}_p$. So $k \in \mathcal{K}_p \Leftrightarrow n - k \notin \mathcal{K}_p$, as desired.

Claim 2: For all $p \in X$ and all $k \in \{0, ..., n\}$, $k \in \mathcal{K}_p \Rightarrow k > n/2$.

Let $p \in X$, and assume for a contradiction that $\mathcal{K}_p$ contains $k \leq n/2$. By claim 1, $\mathcal{K}_{\neg p}$ contains exactly one of $k, n - k$. Define k^* as k if $k \in \mathcal{K}_{\neg p}$ and as $n - k$ if $n - k \in \mathcal{K}_{\neg p}$. As $k \leq n/2$, we have $k + k^* \leq n$. So (also using the fact that $\{p\}$ and $\{\neg p\}$ are consistent) there exists a profile $(A_1, ..., A_n) \in D_2$

[10] Our theorem also supports the robustness of majority voting on the preference agenda.

($\subseteq D$) in which exactly k of the sets A_i contain p (e.g., equal $\{p\}$) and exactly k^* of them contain $\neg p$ (e.g., equal $\{\neg p\}$). As $k \in \mathcal{K}_p$ and $k^* \in \mathcal{K}_{\neg p}$, we have $p, \neg p \in F(A_1, ..., A_n)$, contradicting consistency.

Claim 3: For all $p \in X$ and all $k \in \{0, ..., n\}$, $k \in \mathcal{K}_p \Leftrightarrow k > n/2$ (which completes the proof that F is majority voting on D).

Let $p \in X$ and $k \in \{0, ..., n\}$. By claim 2, $k \in \mathcal{K}_p \Rightarrow k > n/2$. Conversely, let $k \notin \mathcal{K}_p$. Then $n - k \in \mathcal{K}_p$ by claim 1. So, by claim 2, $n - k > n/2$, i.e. $k < n/2$. Hence $k \not> n/2$, as desired. ■

4 Conditions for majority consistency based on global orders

We have seen that, on every sufficiently large domain, if there is any consistent aggregation function at all that satisfies anonymity and acceptance/rejection neutrality, then majority voting is the unique such function. But is majority voting consistent on any such domains? We already know from the discursive paradox that without any domain restriction it is not (unless the agenda is trivial).[11] However, we now show that there exist many compelling domains on which majority voting is consistent. On these domains, then, majority voting not only follows from the conditions of theorem 1 but also satisfies them.[12]

4.1 Conditions based on orders of propositions

We begin with two conditions based on 'global' orders of the propositions. An *order of the propositions (in X)* is a linear order $\leq$ on X.[13]

Single-plateauedness. A judgment set A is *single-plateaued relative to* $\leq$ if

$$A = \{p \in X : p_{\text{left}} \leq p \leq p_{\text{right}}\} \text{ for some } p_{\text{left}}, p_{\text{right}} \in X,$$

and a profile is $(A_1, ..., A_n)$ is *single-plateaued relative to* $\leq$ if every A_i is *single-plateaued* relative to $\leq$.

Single-canyonedness. A judgment set A is *single-canyoned relative to* $\leq$ if

$$A = X \backslash \{p \in X : p_{\text{left}} \leq p \leq p_{\text{right}}\} \text{ for some } p_{\text{left}}, p_{\text{right}} \in X,$$

[11] Majority inconsistencies can arise whenever the agenda has a minimal inconsistent subset of three or more propositions. For a proof of this fact under consistency alone, see Dietrich and List [9]; under full rationality, see Nehring and Puppe [29].

[12] If n is odd. For even n, majority voting is not acceptance/rejection neutral. (Rejection by exactly $n/2$ individuals leads to rejection but acceptance by the same $n/2$ individuals does not lead to acceptance.) A characterization of majority voting for arbitrary n is possible by subtly weakening acceptance/rejection neutrality, but we leave these technicalities aside.

[13] Thus $\leq$ is reflexive ($x \leq x\ \forall x$), transitive ($[x \leq y$ and $y \leq z] \Rightarrow x \leq z\ \forall x, y, z$), connected ($x \neq y \Rightarrow [x \leq y$ or $y \leq x]\ \forall x, y$) and antisymmetric ($[x \leq y$ and $y \leq x] \Rightarrow x = y\ \forall x, y$).

and a profile is $(A_1, ..., A_n)$ is *single-canyoned relative to* $\leq$ if every A_i is *single-canyoned* relative to $\leq$.[14]

An order $\leq$ that renders a profile single-plateaued or single-canyoned is called a *structuring order*; it need not be unique. If a profile is single-plateaued or single-canyoned relative to some $\leq$, we also call it *single-plateaued* or *single-canyoned* simpliciter. Both conditions are illustrated in Figure 1.

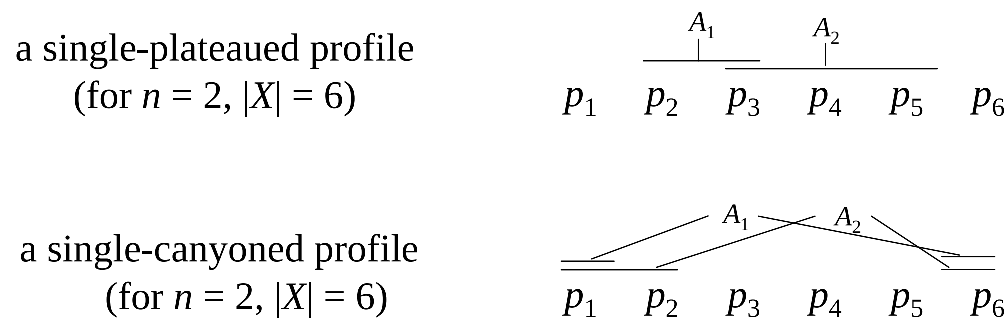

Figure 1: Single-plateauedness and single-canyonedness

The order $\leq$ may represent a normative or cognitive dimension on which propositions are located. If the agenda contains scientific propositions about global warming, for example, individuals may hold single-plateaued judgment sets relative to an order of the propositions from 'most pessimistic' to 'most optimistic', and the location of each individual's plateau may reflect his or her scientific position. If the agenda contains propositions about the effects of various tax or budget policies, the propositions may be ordered from 'socialist' to 'libertarian'. If the agenda contains propositions concerning biological issues, the order may range from 'closest to theory X' (e.g., evolutionary theory) to 'closest to theory Y' (e.g., creationism).

We first observe that every single-canyoned profile is single-plateaued, as proved in the appendix. Our proof reorders the propositions so as to 'glue together' any individual's two extreme sets of propositions into a single plateau.

Proposition 3 *Every single-canyoned profile* $(A_1, ..., A_n)$ *of consistent judgment sets is single-plateaued.*

As anticipated, majority voting preserves consistency on single-plateaued profiles. On single-canyoned profiles, it does even more: it also preserves single-canyonedness.

Proposition 4 *For any profile* $(A_1, ..., A_n)$ *of consistent judgment sets,*

[14]In the definitions of single-plateauedness and single-canyonedness, we do not require $p_{\text{left}} \leq p_{\text{right}}$, i.e., $\{p : p_{\text{left}} \leq p \leq p_{\text{right}}\}$ may be empty.

(a) if $(A_1, ..., A_n)$ is single-plateaued, the majority outcome is consistent;

(b) if $(A_1, ..., A_n)$ is single-canyoned, the majority outcome is consistent and single-canyoned (relative to the same structuring order).

Proof. Consider a profile $(A_1, ..., A_n)$. The following notation is used in this and other proofs. Let A be the majority outcome. For each $p \in X$, define $N_p = \{i \in N : p \in A_i\}$. Whenever we consider an order $\leq$ of X, let $[p, q] = \{r \in X : p \leq r \leq q\}$, for each $p, q \in X$. An order $\leq$ is sometimes identified with the corresponding ascending list of propositions $p_1 ... p_{2k}$ (from 'left' to 'right'), where $2k$ is the size of X (which is even as X is a union of pairs $\{p, \neg p\}$). Now let each A_i be consistent.

(a) Assume single-plateauedness, say relative to $\leq$. Among all propositions in A, let p and q be, respectively, the smallest and largest proposition with respect to $\leq$. So $A \subseteq [p, q]$. As N_p and N_q each contain a majority of the individuals, we have $N_p \cap N_q \neq \emptyset$, and so there is an $i \in N_p \cap N_q$. As A_i is single-plateaued and $p, q \in A_i$, we have $[p, q] \subseteq A_i$ and thus $A \subseteq A_i$. Therefore A is consistent.

(b) Let $(A_1, ..., A_n)$ be single-canyoned, say relative to $\leq$. By part (a) and proposition 3, A is consistent. As one easily checks, A is single-canyoned relative to $\leq$ *if and only if*, for all $p \in A$, we have $\{q \in X : q \leq p\} \subseteq A$ or $\{q \in X : q \geq p\} \subseteq A$. So it suffices to establish the right-hand side of this equivalence. Consider any $p \in A$. Check that either (i) $|\{q \in X : q \leq p\}| \leq k < |\{q \in X : p \leq q\}|$ or (ii) $|\{q \in X : p \leq q\}| \leq k < |\{q \in X : q \leq p\}|$. We assume (i) and show that $\{q \in X : q \leq p\} \subseteq A$ (analogously, (ii) implies $\{q \in X : p \leq q\} \subseteq A$). For each $i \in N_p$, single-canyonedness implies that $\{q \in X : q \leq p\} \subseteq A_i$ or $\{q \in X : p \leq q\} \subseteq A_i$. But the latter is impossible: otherwise $|A_i| > k$ by (i), so that A_i would contain a pair $p, \neg p$, contradicting consistency. So we have $\{q \in X : q \leq p\} \subseteq A_i$ for all $i \in N_p$ and thus for a majority of the individuals. It follows that $\{q \in X : q \leq p\} \subseteq A$, as desired. ■

4.2 Conditions based on orders of individuals

Let us now turn to two conditions based on 'global' orders of the individuals. An *order of the individuals (in N)* is linear order Ω on N. For any sets of individuals $N_1, N_2 \subseteq N$, we write $N_1 \Omega N_2$ if $i \Omega j$ for all $i \in N_1$ and $j \in N_2$.

Unidimensional orderedness.[15] A profile $(A_1, ..., A_n)$ is *unidimensionally ordered relative to* Ω if, for all $p \in X$,

$$\{i \in N : p \in A_i\} = \{i \in N : i_{\text{left}} \Omega i \Omega i_{\text{right}}\} \text{ for some } i_{\text{left}}, i_{\text{right}} \in N.$$

[15] In this definition, we do not require $i_{\text{left}} \Omega i_{\text{right}}$, i.e., $\{i : i_{\text{left}} \Omega i \Omega i_{\text{right}}\}$ may be empty.

Unidimensional alignment. (List [22]) A profile $(A_1, ..., A_n)$ is *unidimensionally aligned relative to* Ω if, for all $p \in X$,

$$\{i \in N : p \in A_i\}\Omega\{i \in N : p \notin A_i\} \text{ or } \{i \in N : p \notin A_i\}\Omega\{i \in N : p \in A_i\}.$$

In analogy to the earlier definition, an order Ω that renders a profile unidimensionally ordered or unidimensionally aligned is called a *structuring order*; again, it need not be unique. If a profile is unidimensionally ordered or unidimensionally aligned relative to some Ω, we also call it *unidimensionally ordered* or *unidimensionally aligned* simpliciter. Both conditions are illustrated in Figure 2.

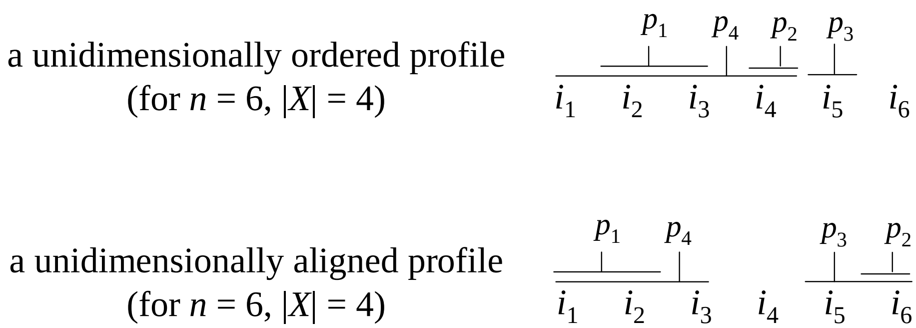

Figure 2: Unidimensional orderedness and unidimensional alignment

Unidimensional alignment is a special case of unidimensional orderedness: it is the case in which, for every $p \in X$, at least one of $i_{\text{left}}, i_{\text{right}}$ is 'extreme', i.e., the left-most or right-most individual in the structuring order Ω.

Proposition 5 *Every unidimensionally aligned profile* $(A_1, ..., A_n)$ *is unidimensionally ordered.*

How can we interpret the two conditions? A profile is unidimensionally ordered if the individuals can be ordered from 'left' to 'right' such that, for each proposition, the individuals accepting it are all adjacent to each other; a profile is unidimensionally aligned if, in addition, the individuals accepting each proposition are either all to the left or all to right of those rejecting it. The order of the individuals can be interpreted as reflecting their location on some underlying normative or cognitive dimension. The idea underlying unidimensional orderedness is that each proposition, like each individual, is located somewhere on the dimension and is accepted by those individuals whose location is 'close' to it, hence by some interval of individuals 'around' it. In a decision problem about climate policies, for example, the proposition 'taxation on emissions should be moderately increased' might have a central location and might therefore be accepted by a 'central' interval of individuals. In the case of unidimensional alignment, the extreme positions on the given dimension correspond to either clear acceptance or clear rejection of each proposition, and, for each proposition, there is a threshold between these extremes (which may vary across propositions) that divides the 'acceptance-region' from the 'rejection-region'.[16]

On unidimensionally ordered profiles, majority voting preserves consistency, and we can say something about the nature of its outcome: it is a subset of the middle individual's judgment set (or, for even n, a subset of the intersection of the two middle individuals' judgment sets). If the profile is unidimensionally aligned, the majority outcome is not just included in that set but coincides with it.

Proposition 6 *For any profile $(A_1, ..., A_n)$ of consistent judgment sets,*

(a) if $(A_1, ..., A_n)$ is unidimensionally ordered, the majority outcome A is consistent and

$$A \subseteq \begin{cases} A_m & \text{if } n \text{ is odd,} \\ A_{m_1} \cap A_{m_2} & \text{if } n \text{ is even,} \end{cases}$$

where m is the middle individual (if n is odd) and m_1, m_2 the middle pair of individuals (if n is even) in any structuring order Ω;

(b) (List [22]) if $(A_1, ..., A_n)$ is unidimensionally aligned, the majority outcome is as stated in part (a) with $\subseteq$ replaced by $=$.

Proof. Let each A_i be consistent. We use earlier proof notation.

(a) Suppose unidimensional orderedness, say relative to Ω. For all $p \in A$, N_p is some interval $[i_{\text{left}}, i_{\text{right}}]$. By $|N_p| > n/2$, $[i_{\text{left}}, i_{\text{right}}]$ is long enough to contain the middle individual m (if n is odd) or the middle pair of individuals m_1, m_2 (if n is even); so $p \in A_m$ (if n is odd) or $p \in A_{m_1} \cap A_{m_2}$ (if n is even). Therefore $A \subseteq A_m$ (if n is odd) or $A \subseteq A_{m_1} \cap A_{m_2}$ (if n is even), as desired. By implication, A is consistent.

(b) See List [22], or check that, under unidimensional alignment, the converse inclusions $A_m \subseteq A$ (if n is odd) or $A_{m_1} \cap A_{m_2} \subseteq A$ (if n is even) also hold in the proof of (a). ■

4.3 The logical relationships between the four conditions

We have already seen that single-canyonedness implies single-plateauedness, and that unidimensional alignment implies unidimensional orderedness. A natural question is how the first two conditions, which are based on orders of the propositions, are related to the second two, which are based on orders of the individuals. The following result answers this question.[17]

Proposition 7 *(a) Restricted to profiles of consistent judgment sets,*

[16] In List [21], unidimensional alignment is interpreted in terms of 'meta-agreement'.

[17] The non-implication claims in (a) do not refer to a *fixed* agenda X and group size n. Rather, for *some* (in fact, most) X and n, there are profiles satisfying one condition but not the other. For special X or n, e.g., for $X = \{p, \neg p\}$ or $n = 2$, all conditions hold trivially.

- *unidimensional alignment implies any of the other three conditions;*
- *single-canyonedness implies single-plateauedness;*
- *there are no other pairwise implications between the four conditions.*

(b) Restricted to profiles of consistent and complete (or just of opinionated) judgment sets, the four conditions are equivalent.

Proof. (a) We already know that single-canyonedness implies single-plateauedness, and that unidimensional alignment implies unidimensional orderedness. To show that unidimensional alignment implies the other conditions too, it suffices to establish that it implies single-canyonedness. We do this in the appendix, where we also show by counterexamples that there are no other implications.

(b) Let $(A_1, ..., A_n)$ be a profile of consistent and complete (or just opinionated) judgment sets. Then each A_i contains exactly $k = |X|/2$ propositions. Since, by part (a), unidimensional alignment implies single-canyonedness, and single-canyonedness implies single-plateauedness, the equivalence of all four conditions follows from the following additional implications, which we now prove using the fact that $|A_i| = k$ for all i. We use the notation from an earlier proof.

Single-plateauedness $\Rightarrow$ unidimensional orderedness. Suppose single-plateauedness, say relative to the order $p_1...p_{2k}$. Then, for all i, there is (using $|A_i| = k$) an index $j(i) \in \{1, ..., 2k\}$ such that $A_i = [p_{j(i)}, p_{j(i)+k-1}]$. Consider an order of the individuals $i_1...i_n$ such that $j(i_1) \leq j(i_2) \leq ... \leq j(i_n)$. To check unidimensional orderedness relative to $i_1...i_n$, note that, for all $p = p_l \in X$, we have

$$\begin{aligned}\{i : p_l \in A_i\} &= \{i : p_l \in [p_{j(i)}, p_{j(i)+k-1}]\} = \{i : j(i) \leq l < j(i) + k\} \\ &= \{i : -l \leq -j(i) < k - l\} = \{i : l - k < j(i) \leq l\},\end{aligned}$$

which is an interval of the order $i_1...i_n$, as desired.

Unidimensional orderedness $\Rightarrow$ unidimensional alignment. Let $(A_1, ..., A_n)$ be unidimensionally ordered, say relative to the order Ω. To see that it is also unidimensionally aligned relative to the same order Ω, consider any $p \in X$. As each A_i contains exactly one member of each pair $p, \neg p \in X$, $N_{\neg p} = N \backslash N_p$. Further, by unidimensional orderedness, N_p and $N_{\neg p}$ are (Ω-)intervals. So N_p and $N \backslash N_p$ are intervals. Hence $N_p \Omega N \backslash N_p$ or $N \backslash N_p \Omega N_p$, as desired. ■

4.4 Applications to preference aggregation: order restriction and intermediateness

What do our present domain-restriction conditions amount to when translated into the classical framework of preference aggregation? As we have already noted, the conditions based on orders of the propositions, although applicable to the preference agenda, have no obvious counterparts in preference aggregation. But those based on orders of individuals do. We now relate unidimensional orderedness to Grandmont's intermediateness [17] and unidimensional alignment to Rothstein's order restriction ([36], [37]).

To introduce intermediateness and order restriction, define a *(strict) preference relation* be a binary relation $\succ$ on K (so far, we do not impose any rationality conditions on preferences), and define a *preference profile* to be an n-tuple $(\succ_1, ..., \succ_n)$ of such relations.[18]

Intermediateness. (Grandmont [17]) A preference profile $(\succ_1, ..., \succ_n)$ is *intermediate relative to* Ω if, for all $x, y \in K$ for all $i, j, k \in N$ with $i\Omega j\Omega k$,

$$[x \succ_i y \text{ and } x \succ_k y] \Rightarrow x \succ_j y.$$

Order restriction. (Rothstein [36], [37]) A preference profile $(\succ_1, ..., \succ_n)$ is *order restricted relative to* Ω if, for all $x, y \in X$,

$$\{i \in N : x \succ_i y\}\Omega\{i \in N : y \succ_i x\} \text{ or } \{i \in N : y \succ_i x\}\Omega\{i \in N : x \succ_i y\}.$$

The following is easy to check:

Remark 8 (a) *A preference profile $(\succ_1, ..., \succ_n)$ is order restricted (relative to some Ω) if and only if the corresponding judgment profile $(A_{\succ_1}, ..., A_{\succ_n})$ is unidimensionally aligned (relative to the same Ω).*

(b) *An opinionated preference profile $(\succ_1, ..., \succ_n)$ is intermediate (relative to some Ω) if and only if the corresponding judgment profile $(A_{\succ_1}, ..., A_{\succ_n})$ is unidimensionally ordered (relative to the same Ω), where opinionation means that, for each $i \in N$ and all distinct $x, y \in K$, precisely one of $x \succ_i y$ or $y \succ_i x$ holds.*

The restriction to opinionated preference profiles in part (b) can be dropped under an alternative correspondence between preference relations and judgment sets.[19]

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[18] Rothstein and Grandmont formulate their definitions more generally for weak preference relations $\succeq_i$.

[19] Without opinionation of each $\succ_i$, intermediateness of $(\succ_1, ..., \succ_n)$ is not equivalent to unidimensional orderedness of $(A_{\succ_1}, ..., A_{\succ_n})$. For all $x, y \in K$, the former requires that $\{i \in N : xPy \in A_i\}$ be an interval, the latter that $\{i \in N : \neg xPy \in A_i\}$ be an interval too. But under another correspondence between preference relations $\succ \in K \times K$ and judgment sets $A \subseteq X_K$, intermediateness becomes equivalent to unidimensional orderedness (and order restriction remains equivalent to unidimensional alignment). On our earlier definition, the judgment set $A_\succ$ corresponding to a preference relation $\succ$ is always opinionated. But a judgment set $A \subseteq X_K$ need not be opinionated. In particular, if $x \not\succ y$, this can have two distinct interpretations: either 'not considering x preferable to y' or 'considering x not preferable to y', corresponding to not accepting p and accepting $\neg p$, where p is 'x is preferable to y'. Our earlier definition of $A_\succ$ assumes the second (stronger) interpretation of $x \not\succ y$, because $A_\succ$ contains $\neg xPy$ if $x \not\succ y$. While a preference relation $\succ \subseteq K \times K$ is ambiguous

5 Conditions for majority consistency based on local orders

For many agendas, the four domain-restriction conditions discussed so far are stronger than necessary for achieving majority consistency. Our goal in this section is to weaken them by applying them not to judgments on all propositions in X, but rather to judgments on various subagendas of X, thereby allowing the relevant structuring order of individuals or propositions to vary across different subagendas. This move parallels the move in preference aggregation from single-peakedness to single-peakedness restricted to triples of alternatives. We begin by introducing the general form of our 'local' domain restriction conditions; then we discuss two approaches to specifying the relevant subagendas.

5.1 The general form of the local conditions

A *subagenda (of X)* is a subset $Y \subseteq X$ that is itself an agenda (i.e., non-empty and closed under single negation). For each of our four global domain-restriction conditions, we say that a profile $(A_1, ..., A_n)$ satisfies the given condition *on a subagenda* $Y \subseteq X$ if the restricted profile $(A_1 \cap Y, ..., A_n \cap Y)$, viewed as a profile of judgment sets on the agenda Y, satisfies it. The relevant structuring order is then called a *structuring order on* Y and denoted $\leq_Y$ (if it is an order of propositions) or Ω_Y (if it is an order of individuals). Whenever one of the conditions is satisfied globally, then it is also satisfied on every $Y \subseteq X$. But we now define a local counterpart of each global condition. Let $\mathcal{Y}$ be some set of subagendas.

Local single-plateauedness / single-canyonedness / unidimensional orderedness / unidimensional alignment. A profile $(A_1, ..., A_n)$ satisfies *the local counterpart of each global condition* (with respect to a given set of subagendas $\mathcal{Y}$) if it satisfies the global condition on every $Y \in \mathcal{Y}$.

This allows different structuring orders $\leq_Y$ or Ω_Y for different subprofiles $(A_1 \cap Y, ..., A_n \cap Y)$ (with $Y \in \mathcal{Y}$). Any implications and equivalences between our four global conditions, as stated in proposition 7, carry over to their local counterparts (each defined with respect to the same $\mathcal{Y}$).[20]

between the two interpretations, a judgment set $A \subseteq X_K$ is not. For any distinct $x, y \in K$, a preference relation $\succ$ can display four different patterns: $x \succ y \& y \not\succ x$, $x \not\succ y \& y \succ x$, $x \not\succ y \& y \not\succ x$, or $x \succ y \& y \succ x$; a judgment set $A \subseteq X_K$ can display $2^4 = 16$ different patterns, depending on which of $xPy, \neg xPy, yPx, \neg yPx$ are contained in A. Under the weaker interpretation of $x \not\succ y$, we define $A_\succ = \{xPy : x, y \in K \& x \succ y\}$ (an incomplete judgment set, unless $\succ$ is the total relation). Now a preference relation $\succ$ is fully rational (i.e., asymmetric, transitive and connected) if and only if $A_\succ$ is consistent *and* contains a member of each pair $xPy, yPx \in X$ with $x \neq y$. Intermediateness of $(\succ_1, ..., \succ_n)$ then translates into unidimensional orderedness of $(A_{\succ_1}, ..., A_{\succ_n})$.

[20] Analogously to proposition 7, the non-implication claims in (a) do not refer to a *fixed*

Corollary 9 *(a) Restricted to profiles of consistent judgment sets,*

- *local unidimensional alignment implies any of the other three local conditions;*
- *local single-canyonedness implies local single-plateauedness;*
- *there are no other pairwise implications between the four local conditions.*

(b) Restricted to profiles of consistent and complete (or just of opinionated) judgment sets, the four local conditions are equivalent.

The subagendas with respect to which our local conditions are defined – i.e., those in $\mathcal{Y}$ – must be carefully chosen. Choosing them according to their size (e.g., by including in $\mathcal{Y}$ all subagendas of size less than some k) or according to the syntactic form of propositions in them (e.g., by including in $\mathcal{Y}$ all subagendas whose propositions contain only a certain type or number of logical connectives) does not generally work. Our choice of subagendas is guided by two goals. The first is to ensure that a consistent majority outcome for every subagenda implies a consistent majority outcome overall (just as acyclicity on triples of alternatives in preference aggregation implies acyclicity overall). The second is to minimize the total number and size of subagendas, so as to make our local domain-restriction conditions as unrestrictive as possible.

5.2 Selecting subagendas I: minimal inconsistent sets

What set of subagendas $\mathcal{Y}$ should be chosen? In this subsection, we take the following approach. Note that a judgment set $A \subseteq X$ is inconsistent if and only if it has a *minimal inconsistent* subset $Y \subseteq X$, i.e., a subset that is inconsistent but all of whose proper subsets are consistent. So a consistent majority outcome can be achieved by each of our local domain-restriction conditions where $\mathcal{Y}$ is defined as

$$\mathcal{Y} = \{Y^{\pm} : Y \text{ is a minimal inconsistent subset of } X\}. \tag{1}$$

Proposition 10 *For any profile $(A_1, ..., A_n)$ of consistent judgment sets,*

(a) if $(A_1, ..., A_n)$ satisfies any of the four local conditions with respect to $\mathcal{Y}$ as defined in (1), the majority outcome A is consistent;

(b) in the case of local unidimensional orderedness,

$$A \subseteq \begin{cases} \cup_{Y \in \mathcal{Y}}(A_{m_Y} \cap Y) & \text{if } n \text{ is odd,} \\ \cup_{Y \in \mathcal{Y}}(A_{m_{Y,1}} \cap A_{m_{Y,2}} \cap Y) & \text{if } n \text{ is even,} \end{cases}$$

where, for each $Y \in \mathcal{Y}$, m_Y is the middle individual (if n is odd) and $m_{Y,1}, m_{Y,2}$ the middle pair of individuals (if n is even) in any structuring

agenda X, set of subagendas $\mathcal{Y}$, and group size n. Rather, for *some* (in fact, most) X, $\mathcal{Y}$ and n, there are profiles satisfying one condition but not the other. In special cases, e.g., for $\mathcal{Y} = \emptyset$, all conditions hold trivially.

order Ω_Y on Y;[21]

(c) in the case of local unidimensional alignment, A is as stated in part (b) with $\subseteq$ replaced by $=$.

Proof. Let $\mathcal{Y}$ and $(A_1, ..., A_n)$ be as specified, with majority outcome A.

(a) To prove A's consistency, it suffices to prove that A has no minimal inconsistent subset, hence to prove that $A \cap Y$ is consistent for all $Y \in \mathcal{Y}$. So consider any subagenda $Y \in \mathcal{Y}$. As $(A_1, ..., A_n)$ is, for example, single-plateaued on Y (the proof is similar for single-canyonedness or unidimensinoal orderedness/alignment), $(A_1 \cap Y, ..., A_n \cap Y)$ is single-plateaued for the agenda Y and hence has a consistent majority outcome by proposition 4. But this majority outcome is $A \cap Y$. So $A \cap Y$ is consistent, as desired.

(b) Assume unidimensional orderedness and let the individuals $(m_Y)_{Y \in \mathcal{Y}}$ (if n is odd) or $(m_{Y,1}, m_{Y,2})_{Y \in \mathcal{Y}}$ (if n is even) be as specified. To show that $A \subseteq \cup_{Y \in \mathcal{Y}}(A_{m_Y} \cap Y)$ (if n is even) or $A \subseteq \cup_{Y \in \mathcal{Y}}(A_{m_{Y,1}} \cap A_{m_{Y,2}} \cap Y)$ (if n is odd), it is by $A = \cup_{Y \in \mathcal{Y}}(A \cap Y)$ sufficient to show that, for all $Y \in \mathcal{Y}$, $A \cap Y \subseteq A_{m_Y} \cap Y$ (if n is even) or $A \cap Y \subseteq A_{m_{Y,1}} \cap A_{m_{Y,2}} \cap Y$ (if n is odd). This follows from part (a) of proposition 6 because $A \cap Y$ is the majority outcome on the unidimensionally ordered profile $(A_1 \cap Y, ..., A_n \cap Y)$.

(c) The proof is analogous to that of part (b), with each "$\subseteq$" replaced by "$=$" and where we now make use of part (b) (not (a)) of proposition 6. ■

5.3 Selecting subagendas II: irreducible sets

The set of subagendas generated from all minimal inconsistent subsets of the agenda can be large, but using this rich set has been necessary in order to guarantee majority consistency on domains that allow even for incomplete individual judgment sets. However, in the important special case of individual completeness, it is enough for majority consistency to impose any of our four local domain-restriction conditions with a much slimmer definition of the relevant set of subagendas. We generate these subagendas not from all minimal inconsistent subsets of the agenda, but only from those that are *irreducible* in the following sense.[22] For any inconsistent set $Y \subseteq X$, we call another inconsistent set $Z \subseteq X$ a *reduction* of Y if

$$|Z| < |Y| \text{ and each } p \in Z \backslash Y \text{ is entailed by some } V \subseteq Y \text{ with } |Y \backslash V| > 1,$$

[21] The result continues to hold if every occurrence of the quantification $Y \in \mathcal{Y}$ in part (b) is weakened to the quantification $Y \in \mathcal{Y}^*$, where $\mathcal{Y}^* \subseteq \mathcal{Y}$ is any subset of subagendas covering X, i.e., with $\cup_{Y \in \mathcal{Y}^*} Y = X$. There are many ways to cover X; trivial ones are $\mathcal{Y}^* = \{\{p, \neg p\} : p \in X_+\}$ and $\mathcal{Y}^* = \mathcal{Y}$. The representation of A becomes slim if $\mathcal{Y}^*$ *minimally covers* X, i.e., covers X but no $\mathcal{Z} \subsetneq \mathcal{Y}^*$ does so too.

[22] Dietrich [5] has subsequently generalized this concept.

and we call Y *irreducible* if it has no reduction.[23] For instance, the inconsistent set $\{a, a \to b, b \to c, \neg c\}$ (where a, b, c are distinct atomic propositions) is reducible to $Z = \{b, b \to c, \neg c\}$, since b is entailed by $\{a, a \to b\}$, whereas Z is irreducible. Now define

$$\mathcal{Y} = \{Y^{\pm} : Y \text{ is an irreducible subset of } X\}. \tag{2}$$

The set of subagendas defined in (2) is a subset (usually a proper subset[24]) of the one defined in (1) above, since every irreducible set is minimal inconsistent (a non-minimal inconsistent set is reducible to any of its inconsistent proper subsets). The local domain-restriction conditions resulting from (2) are therefore less restrictive than those resulting from (1) above.

The following lemma is crucial; a proof is given in the appendix.

Lemma 11 *Every complete and inconsistent judgment set $A \subseteq X$ has an irreducible subset.*

Using lemma 11, we can prove our central claim: if individuals hold not only consistent but also complete judgment sets, our local domain-restriction conditions defined in terms of irreducible sets are enough to guarantee majority consistency. The assumption of individual completeness ensures an (apart from ties) complete majority outcome, so as to make lemma 11 applicable in the proof.

Proposition 12 *For any profile $(A_1, ..., A_n)$ of consistent and complete judgment sets, if $(A_1, ..., A_n)$ satisfies any (hence by corollary 9 all) of the four local conditions with respect to $\mathcal{Y}$ as defined in (2), the majority outcome is consistent.*

Proof. We consider a profile $(A_1, ..., A_n)$ of the specified kind and use the earlier notation.

Case 1: n is odd. Then A is complete. So, by proposition 11, to prove A's consistency, it suffices to prove that A has no irreducible subset, hence to prove that $A \cap Y$ is consistent for all $Y \in \mathcal{Y}$. The latter follows by an argument analogous to the one in the proof of part (a) of proposition 10.

Case 2: n is even. Let A_{n+1} be any complete and consistent judgment set such that $(A_1, ..., A_{n+1})$ still satisfies the local condition, e.g. single-plateauedness on $\mathcal{Y}$, now for group size $n + 1$ (one might take $A_{n+1} = A_1$). By case 1 the majority outcome on $(A_1, ..., A_{n+1})$ is a consistent judgment set $\widetilde{A}$. Check that $A \subseteq \widetilde{A}$. So A is consistent, as desired. ■

[23] In the definition of reduction, the clause $|Y \backslash V| > 1$ is essential. Dropping it would render all inconsistent sets $Y \subseteq X$ of size three or more reducible, namely to any pair $\{p, \neg p\}$ with $p \in Y$; $\neg p$ is entailed by $Y \backslash \{p\}$.

[24] It is usually a proper subset since many minimal inconsistent subsets of the agenda, such as $\{a, a \to b, b \to c, \neg c\}$, are reducible.

5.4 Applications to preference aggregation: order restriction and intermediateness on k-tuples of alternatives

What do our local conditions look like when applied to the preference agenda? To answer this question, we must identify the set of subagendas $\mathcal{Y}$ under each of our two criteria for selecting subagendas. A few definitions are needed. By our definition of the logic of preferences, for any distinct $x, y \in K$, $\neg xPy$ and yPx are equivalent. Call two judgment sets *essentially identical* if one arises from the other by (zero, one or more) replacements of propositions by equivalent propositions. For any distinct $x_1, ..., x_k \in K$ ($k \geq 1$), the cyclical preferences $x_1 \succ x_2 \succ ... \succ x_k \succ x_1$ can be represented by the set $\{x_1Px_2, x_2Px_3, ..., x_{k-1}Px_k, x_kPx_1\}$. We call such a set, and any set essentially identical to it, a *cycle* (of length k).

We are now in a position to identify the minimal inconsistent subsets of the preference agenda.

Remark 13 *The minimal inconsistent sets $Y \subseteq X_K$ are the cycles.*

Proof. This follows from the definition of the logic **L** for representing preferences. First, any cycle is obviously minimal inconsistent in **L**. Second, suppose $Y \subseteq X_K$ is minimal inconsistent. One can check that, by Y's inconsistency, some subset $Y^* \subseteq Y$ is a cycle. By minimal inconsistency, then, $Y = Y^*$. ■

Next let us identify the irreducible subsets of the preference agenda. Not all cycles fall into this category. To illustrate, observe that any cycle of length k

$$Y = \{x_1Px_2, x_2Px_3, ..., x_{k-1}Px_k, x_kPx_1\}$$

with $k \geq 4$ is reducible, e.g., to the 3-cycle $\{x_1Px_2, x_2Px_3, x_3Px_1\}$, as x_3Px_1 is entailed by $\{x_3Px_4, x_4Px_5, ..., x_kPx_1\}$.

Remark 14 *The irreducible sets $Y \subseteq X_K$ are the cycles of length 1, 2 or 3.*

Proof. First, consider any cycle Y of length at most three. If Y is a 1-cycle, i.e., $Y = \{xPx\}$ for some $x \in K$, or a 2-cycle, i.e., $Y = \{xPy, yPx\}$ with distinct $x, y \in K$, then Y is obviously irreducible. Now let Y be a 3-cycle, i.e., $Y = \{xPy, yPz, zPx\}$ for distinct $x, y, z \in K$. Suppose, for a contradiction, that Y is reducible, say to $Z \subseteq X$. Then $|Z| \leq 2$. Moreover each $p \in Z$ is entailed by a single member of Y, i.e. by one of xPy, yPz, zPx. But the only proposition in X entailed by xPy is xPy (and the logically equivalent $\neg yPx$), and similarly for yPz and zPx. So each $p \in Z$ is one of xPy, yPz, zPx (or one of $\neg yPx, \neg zPy, \neg xPz$). Hence Z is (essentially identical to) a proper subset of $Y = \{xPy, yPz, xPz\}$. So Z is consistent, a contradiction.

Second, suppose $Y \subseteq X_K$ is irreducible. Hence Y is minimal inconsistent. So, by part (a) of Remark 13, Y is a cycle, hence (essentially identical to) a set of type $\{x_1Px_2, x_2Px_3, .., x_{k-1}Px_k, x_kPx_1\}$ ($k \geq 1$). Now $k \leq 3$, as otherwise Y would be reducible to $Z := \{x_1Px_2, x_2Px_3, x_3Px_1\}$. So Y is a 1- or 2- or 3-cycle. ■

By remark 13, the set of subagendas generated from minimal inconsistent sets is

$$\mathcal{Y} = \{Y^{\pm} : Y \subseteq X_K \text{ is a cycle}\},$$

and by remark 14, the set of subagendas generated from irreducible sets is the smaller set

$$\mathcal{Y} = \{Y^{\pm} : Y \subseteq X_K \text{ is a cycle of length 1, 2 or 3}\}.$$

Just as in the global case, we are thus able to relate local unidimensional orderedness and local unidimensional alignment to local versions of intermediateness and order restriction. Consider the following two local conditions on preference profiles:

Intermediateness on triples. (Grandmont [17]) A preference profile $(\succ_1, ..., \succ_n)$ is *intermediate on triples* if, for every subset $K' \subseteq K$ with $|K'| = 3$, the preference profile restricted to K', i.e., $(\succ_1 |_{K'}, ..., \succ_n |_{K'})$, is intermediate (as defined above).

Order restriction on triples. (Rothstein [36], [37]) A preference profile $(\succ_1, ..., \succ_n)$ is *order restricted on triples* if, for every subset $K' \subseteq K$ with $|K'| = 3$, the preference profile restricted to K', i.e., $(\succ_1 |_{K'}, ..., \succ_n |_{K'})$, is order restricted (as defined above).

It is easy to see that, when $\mathcal{Y}$ is defined as the set of subagendas of X_K generated from all cycles, unidimensional orderedness and unidimensional alignment with respect to $\mathcal{Y}$ are more demanding than intermediateness and order restriction on triples, respectively. Unlike the two triplewise conditions on preference profiles, our conditions require a structuring order of the individuals for every k-tuple of alternatives, not just for every triple. As already noted, our stronger requirement is warranted when we want to guarantee majority consistency even in the absence of individual completeness; order restriction or intermediateness on triples do not guarantee acyclic majority preferences when individual incompleteness is allowed.

But in the case of individual completeness, it suffices for majority consistency to define our local conditions in terms of irreducible sets, i.e., by defining $\mathcal{Y}$ as the set of subagendas of X_K generated from all cycles of length up to three. Local unidimensional orderedness and alignment then become equivalent to the triplewise variants of Grandmont's and Rothstein's conditions, as shown in the appendix:

Proposition 15 *A profile* $(\succ_1, ..., \succ_n)$ *of strict linear orders*[25] *on* K *is intermediate (equivalently, order restricted) on triples if and only if the associated judgment profile* $(A_{\succ_1}, ..., A_{\succ_n})$ *is locally unidimensionally ordered (equivalently, aligned) with respect to* $\mathcal{Y}$ *as defined by (2).*

6 Conditions for majority consistency not based on orders

Although our domain-restriction conditions based on local orders are already much less restrictive than those based on global orders, it is possible to weaken them further. Just as the various conditions based on orders in preference aggregation – single-peakedness, single-cavedness etc. – can be generalized to a weaker, but less easily interpretable, condition – namely Sen's triplewise value-restriction [40] – so in judgment aggregation the conditions based on orders can be weakened to a more abstract condition, to be called *value-restriction.* When applied to the preference agenda, this condition becomes non-trivially equivalent to Sen's condition. But despite generalizing Sen's condition, our condition is simpler to state; we thus also hope to offer a new perspective on Sen's condition.

6.1 Value-restriction

We state two variants of our condition, one based on minimal inconsistent sets, the other based on irreducible sets.

Value-restriction. A profile $(A_1, ..., A_n)$ is *value-restricted* if every (non-singleton[26]) minimal inconsistent set $Y \subseteq X$ has a two-element subset $Z \subseteq Y$ that is not a subset of any A_i.

Weak value-restriction. A profile $(A_1, ..., A_n)$ is *weakly value-restricted* if every (non-singleton) irreducible set $Y \subseteq X$ has a two-element subset $Z \subseteq Y$ that is not a subset of any A_i.

Informally, value-restriction reflects a particular kind of agreement: for every minimal inconsistent (or irreducible in the weak case) subset of the agenda, there exists a particular conjunction of two propositions in this subset that no individual endorses. Like our previous domain-restriction conditions, the two new conditions are each sufficient for consistent majority outcomes (the weaker condition in the important special case of individual completeness).

[25] A *strict linear order* is an irreflexive, antisymmetric, transitive and connected binary relation.

[26] The qualification 'non-singleton' in this definition and the next is unnecessary if X contains only contingent propositions, since this rules out singleton inconsistent sets.

Proposition 16 *For any profile $(A_1, ..., A_n)$ of consistent judgment sets,*

(a) if $(A_1, ..., A_n)$ is value-restricted, the majority outcome is consistent;

(b) if $(A_1, ..., A_n)$ is weakly value-restricted and each A_i is complete, the majority outcome is consistent.

Proof. Let $(A_1, ..., A_n)$ consist of consistent judgment sets.

(a) Suppose $(A_1, ..., A_n)$ is value-restricted, but the majority outcome, A, is inconsistent. Then A has a minimal inconsistent subset Y. Obviously, Y is non-singleton (otherwise a majority would support a contradiction). So, by value-restriction, Y has a two-element subset $Z \subseteq Y$ that is not a subset of any A_i. However, since $Z \subseteq A$, there is a majority for each of the two elements of Z. Since two majorities must overlap, some A_i contains both of these elements, whence $Z \subseteq A_i$ for some $i \in N$, a contradiction.

(b) Suppose $(A_1, ..., A_n)$ is weakly value-restricted and each A_i is complete. There are two cases.

Case 1: n is odd. Then A is also complete (because there cannot be majority ties). Suppose for a contradiction that the majority outcome, A, is inconsistent. Then A has an irreducible subset Y by proposition 11, and one can derive a contradiction analogously to part (a).

Case 2: n is even. Let A_{n+1} be any complete and consistent judgment set such that $(A_1, ..., A_{n+1})$ is still weakly value-restricted, now for group size $n+1$ (of course, there is such an A_{n+1}: e.g., take $A_{n+1} = A_1$). Let A' be the majority outcome on $(A_1, ..., A_{n+1})$. By case 1, A' is consistent. Check that the majority outcome on $(A_1, ..., A_n)$ is a subset of A'; hence it is consistent too, as desired. ■

How general are our two value-restriction conditions? The following proposition, proved in the appendix, answers this question.

Proposition 17 *(a) Each of our four conditions based on global orders implies value-restriction.*

(b) Each of our four conditions based on local orders, with respect to $\mathcal{Y}$ defined in terms of minimal inconsistent sets, implies value-restriction.

(c) Each of our four conditions based on local orders, with respect to $\mathcal{Y}$ defined in terms of irreducible sets, implies weak value-restriction.

6.2 Applications to preference aggregation: triplewise value-restriction

We now show that, when applied to the preference agenda, our two value-restriction conditions surprisingly both collapse into Sen's triplewise value-restriction. Let us recapitulate Sen's condition:

Triplewise value-restriction. (Sen [40]) A preference profile $(\succ_1, ..., \succ_n)$ is *triplewise value-restricted* if, for every triple of distinct alternatives $x, y, z \in K$, there is one alternative, say x, that is either not ranked top by any individual (no i has $x \succ_i y$ and $x \succ_i z$), or not ranked middle by any individual (no i has $y \succ_i x \succ_i z$ or $z \succ_i x \succ_i y$) or not ranked bottom by any individual (no i has $y \succ_i x$ and $z \succ_i x$).

An alternative, but equivalent definition of triplewise value-restriction requires that, for each triple of alternatives, the individuals' preferences are either single-peaked or single-caved or separable in a sense defined by Inada [18]. (See also Elsholtz and List [19].) The following is the central result of this subsection, proved in the appendix.

Proposition 18 *For any profile $(A_1, ..., A_n)$ of consistent and complete judgment sets on the preference agenda, the following are equivalent:*
(a) $(A_1, ..., A_n)$ is value-restricted,
(b) $(A_1, ..., A_n)$ is weakly value-restricted,
(c) the associated preference profile $(\succ_{A_1}, ..., \succ_{A_n})$ is triplewise value-restricted.

7 Conclusion

We have introduced several domain-restriction conditions on profiles of individual judgment sets that are sufficient for consistent majority outcomes. Some of our conditions are based on global orders of either the propositions or the individuals, others on local orders of them, and yet others not on orders at all. We have justified our focus on majority voting by providing a new characterization result showing that, for a large class of domains, if there is any consistent aggregation function at all that satisfies certain democratic conditions, then majority voting is the unique such function.

While all domain-restriction conditions discussed in this paper are sufficient for consistent majority outcomes, it is useful to compare them with a necessary and sufficient condition.

Majority-consistency. A profile $(A_1, ..., A_n)$ is *majority-consistent* if every minimal inconsistent set $Y \subseteq X$ contains a proposition not contained in a majority of the A_is.[27]

If (and only if) this condition is met, no minimal inconsistent set of propositions can be accepted under majority voting, and thus the majority outcome is consistent. But there are some important differences between majority-consistency and the various conditions introduced earlier. First, unlike majority-consistency, the various earlier conditions are easily interpretable: they embody particular types of agreement within the group, for instance agreements on normative or cognitive dimensions underlying individual judgments. Secondly, the

[27] It is easy to see that, when the majority outcome is complete, it is enough to quantify over all irreducible (as opposed to all minimal inconsistent) sets $Y \subseteq X$.

earlier conditions are *structural* (as opposed to *numerical*) in the sense of depending only on *whether or not* certain patterns occur in each judgment set in a given profile, but not on *how often* those patterns occur (Elsholtz and List [19]). By contrast, majority-consistency is a numerical condition. Thirdly, as we show in a moment, each of our earlier conditions can be used to define product domains, whereas majority-consistency cannot.

A domain D of admissible profiles of an aggregation function (say, majority voting) is called a *product domain* if it can be expressed as

$$D = D_1 \times D_2 \times ... \times D_n,$$

where, for each $i \in N$, D_i is the set of admissible judgment sets of individual i (typically, D_i is the same for all i). A domain is called a *non-product domain* if it does not admit such an expression, i.e., if the judgment set an individual can submit may depend on the judgment sets submitted by others. For example, in preference aggregation, single-peakedness and single-cavedness relativized to an antecedently fixed order of alternatives specify product domains, while single-peakedness and single-cavedness simpliciter do not.

The distinction between product and non-product domains is important both theoretically and practically. It is theoretically important in game-theoretic analyses of aggregation problems. If we want to interpret an aggregation problem as a game, where the individuals' possible inputs – i.e., their preferences or judgments – are their strategies, then the domain of admissible profiles must be a Cartesian product of the strategy sets across individuals. Standard definitions of strategy-proofness following Gibbard and Satterthwaite employ precisely this representation, although they can be modified so as to accommodate non-product domains (Saporiti and Tohmé [38]; see also Dietrich and List [8]). Practically, product domains matter when an aggregation function represents a voting procedure in the ordinary sense. Here each voter must be given a list of admissible choices – i.e., a set D_i of admissible judgment sets (typically the same across voters) – and cannot be told that certain choices are inadmissible depending on the choices made by others.

The product domains induced by our various conditions are as follows:

- The product domain of single-plateaued/canyoned profiles relative $\leq$ (a fixed order on X): each D_i is the set of consistent judgment sets $A \subseteq X$ that are single-plateaued/canyoned relative to $\leq$. In the case of unidimensional orderedness/alignment, the construction is slightly more elaborate.[28]
- The product domain of locally single-plateaued/canyoned profiles relative to $(\leq_Y)_{Y \in \mathcal{Y}}$ (a family of fixed orders $\leq_Y$ on the subagendas $Y \in$

[28]Let $(A_1, ..., A_{n^*})$ be any profile of consistent judgment sets satisfying unidimensional orderedness or alignment (relative to some Ω), where $n^* \geq 1$ is any arbitrary group size (not necessarily identical to n). If we define each D_i to be the set of all A_js occurring in $(A_1, ..., A_{n^*})$, then $D = D_1 \times D_2 \times ... \times D_n$ is a product domain of unidimensionally ordered or aligned profiles of consistent judgment sets.

$\mathcal{Y}$): each D_i is the set of consistent judgment sets $A \subseteq X$ that are single-plateaued/canyoned on each $Y \in \mathcal{Y}$ relative to $\leq_Y$. Again, a more elaborate construction is possible for local unidimensional orderedness/alignment.

- The product domain of value-restricted profiles relative to $(Z_Y)_{Y \in \mathcal{MI}}$ (a family of fixed two-element subsets $Z_Y \subseteq Y$, with Y ranging over the set $\mathcal{MI}$ of minimal inconsistent subsets of X): each D_i is the set of consistent judgment sets $A \subseteq X$ that are not supersets of any Z_Y.[29]

By contrast, the condition of majority-consistency does not induce a product domain in this way because majority-consistency is a numerical condition, not a structural one.

In conclusion, figure 3 summarizes the logical relationship between all the domain-restriction conditions discussed in this paper, in each case applied to profiles of consistent individual judgment sets.

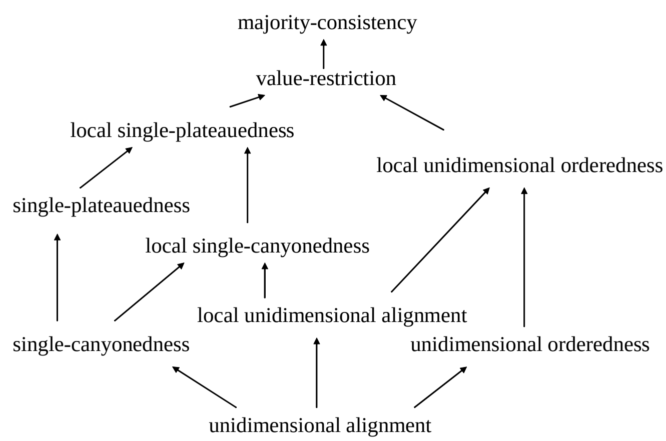

Figure 3: The logical relationship between the domain-restriction conditions

References

[1] Black, D. (1948) On the Rationale of Group Decision-Making. *Journal of Political Economy* 56: 23-34

[2] Dasgupta, P., Maskin, E. (1998/2007) On the Robustness of Majority Rule. Working paper, Institute of Advanced Study, Princeton

[3] Dietrich, F. (2006) Judgment Aggregation: (Im)Possibility Theorems. *Journal of Economic Theory* 126(1): 286-298

[29] In the case of weak value restriction, D_i can be defined analogously, with $\mathcal{MI}$ replaced by the (smaller) set of irreducible subsets of X.

[4] Dietrich, F. (2007) A generalised model of judgment aggregation. *Social Choice and Welfare* 28(4): 529-565

[5] Dietrich, F. (forthcoming) The possibility of judgment aggregation on agendas with subjunctive implications. *Journal of Economic Theory*

[6] Dietrich, F., List, C. (2005) The impossibility of unbiased judgment aggregation. Working paper, London School of Economics

[7] Dietrich, F., List, C. (2007) Arrow's theorem in judgment aggregation. *Social Choice and Welfare* 29(1): 19-33

[8] Dietrich, F., List, C. (2007) Strategy-proof judgment aggregation. *Economics and Philosophy* 23(3)

[9] Dietrich, F., List, C. (2007) Judgment aggregation by quota rules: majority voting generalized. *Journal of Theoretical Politics* 19(4): 391-424

[10] Dietrich, F., List, C. (forthcoming) Judgment aggregation without full rationality. *Social Choice and Welfare*

[11] Dietrich, F., List, C. (2007) Judgment aggregation with consistency alone. Working paper, London School of Economics

[12] Dietrich, F., List, C. (forthcoming) A liberal paradox for judgment aggregation. *Social Choice and Welfare*

[13] Dokow, E., Holzman, R. (forthcoming) Aggregation of binary evaluations. *Journal of Economic Theory*

[14] Dokow, E., Holzman, R. (2006) Aggregation of binary evaluations with abstentions. Working paper, Technion Israel Institute of Technology

[15] Gärdenfors, P. (2006) An Arrow-like theorem for voting with logical consequences. *Economics and Philosophy* 22(2): 181-190

[16] Gaertner, W. (2001) *Domain Conditions in Social Choice Theory*. Cambridge (Cambridge University Press)

[17] Grandmont, J.-M. (1978) Intermediate Preferences and the Majority Rule. *Econometrica* 46(2): 317-330

[18] Inada, K.-I. (1964) A Note on the Simple Majority Decision Rule. *Econometrica* 32(4): 525-531

[19] Elsholtz, C., List, C. (2005) A Simple Proof of Sen's Possibility Theorem on Majority Decisions. *Elemente der Mathematik* 60: 45-56

[20] Kornhauser, L. A., Sager, L. G. (1986) Unpacking the Court. *Yale Law Journal* 96(1): 82-117

[21] List, C. (2002) Two Concepts of Agreement. *The Good Society* 11(1): 72-79

[22] List, C. (2003) A Possibility Theorem on Aggregation over Multiple Interconnected Propositions. Mathematical Social Sciences 45(1): 1-13 (Corrigendum in *Mathematical Social Sciences* 52:109-110)

[23] List, C. (2004) A Model of Path-Dependence in Decisions over Multiple Propositions. *American Political Science Review* 98(3): 495-513

[24] List, C., Pettit, P. (2002) Aggregating Sets of Judgments: An Impossibility Result. *Economics and Philosophy* 18: 89-110

[25] List, C., Pettit, P. (2004) Aggregating Sets of Judgments: Two Impossibility Results Compared. *Synthese* 140(1-2): 207-235

[26] May, K. O. (1952) A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision. *Econometrica* 20(4): 680-684

[27] Mongin, P. (forthcoming) Factoring out the impossibility of logical aggregation. *Journal of Economic Theory*

[28] Nehring, K., Puppe, C. (2002) Strategy-Proof Social Choice on Single-Peaked Domains: Possibility, Impossibility and the Space Between. Working paper, University of California at Davies

[29] Nehring, K., Puppe, C. (2007) Abstract Arrovian Aggregation. Working paper, University of Karlsruhe

[30] Nehring, K., Puppe, C. (forthcoming) Consistent judgement aggregation: the truth-functional case. *Social Choice and Welfare*

[31] Pauly, M., van Hees, M. (2006) Logical Constraints on Judgment Aggregation. *Journal of Philosophical Logic* 35: 569-585

[32] Pettit, P. (2001) Deliberative Democracy and the Discursive Dilemma. *Philosophical Issues* 11: 268-299

[33] Pigozzi, G. Belief merging and the discursive dilemma: an argument-based account to paradoxes of judgment aggregation. *Synthese* 152(2): 285-298

[34] Roberts, K. W. S. (1977) Voting over Income Tax Schedules. *Journal of Public Economics* 8(3): 329-340

[35] Rubinstein, A., Fishburn, P. (1986) Algebraic Aggregation Theory. *Journal of Economic Theory* 38: 63-77

[36] Rothstein, P. (1990) Order Restricted Preferences and Majority Rule. *Social Choice and Welfare* 7(4): 331-342

[37] Rothstein, P. (1991) Representative Voter Theorems. *Public Choice* 72(2-3): 193-212

[38] Saporiti, A., Tohmé, F. (2006) Single-crossing, strategic voting and the median choice rule. *Social Choice and Welfare* 26(2): 363-383

[39] Saporiti, A. (forthcoming) On the existence of Nash equilibrium in electoral competition. *Game Theory and Information*

[40] Sen, A. K. (1966) A Possibility Theorem on Majority Decisions. *Econometrica* 34(2): 491-499

[41] van Hees, M. (2007) The limits of epistemic democracy. *Social Choice and Welfare* 28(4): 649-666

[42] Wilson, R. (1975) On the Theory of Aggregation. *Journal of Economic Theory* 10: 89-99

A Appendix: Additional proofs

Proof of lemma 2. Consider any agenda (possibly containing tautologies), and let F and D be as specified. Consider any $p \in X$ and any $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in D$ in which the same set of individuals $C \subseteq N$ accepts p. We must show that $p \in F(A_1, ..., A_n) \Leftrightarrow p \in F(A_1^*, ..., A_n^*)$. By consistency of F, if p is a contradiction, it belongs to neither $F(A_1, ..., A_n)$ nor $F(A_1^*, ..., A_n^*)$, hence $p \in F(A_1, ..., A_n) \Leftrightarrow p \in F(A_1^*, ..., A_n^*)$. Now suppose p is not a contradiction (but perhaps a tautology). Then the profile $(A_1', ..., A_n')$ given by $A_i' = \emptyset$ for all $i \in C$ and $A_i' = \{p\}$ for all $i \notin C$ is in D_2, hence in D. By acceptance/rejection neutrality, $p \in F(A_1, ..., A_n) \Leftrightarrow p \notin F(A_1', ..., A_n')$. Further, by acceptance/rejection neutrality, $p \in F(A_1^*, ..., A_n^*) \Leftrightarrow p \notin F(A_1', ..., A_n')$. So $p \in F(A_1, ..., A_n) \Leftrightarrow p \in F(A_1^*, ..., A_n^*)$, as desired. ■

Proof of proposition 3. We use the notation introduced in the proof of proposition *4*. Consider a profile $(A_1, ..., A_n)$ of consistent individual judgment sets, and let $(A_1, ..., A_n)$ be single-canyoned, say relative to the order $p_1...p_{2k}$. We consider any A_i and show that A_i is single-plateaued relative to the new order $p_{k+1}...p_{2k}p_1...p_k$. By assumption, (i) $A_i = \{p_1, ...p_j\} \cup \{p_{j'}, ..., p_{2k}\}$ for some $0 \leq j \leq j' \leq 2k + 1$. As A_i is consistent, A_i contains no pair $p, \neg p \in X$; so $|A_i| \leq |X|/2 = k$, whence (ii) $j \leq k$ and $j' \geq k + 1$. Using both (i) and (ii), one can check that A_i is an interval relative to the new order $p_{k+1}...p_{2k}p_1...p_k$, as desired. More precisely,

$$A_i = \begin{cases} [p_{j'}, p_j] & \text{if } j \neq 0 \text{ and } j' \neq 2k+1, \\ [p_1, p_j] & \text{if } j \neq 0 \text{ and } j' = 2k+1, \\ [p_{j'}, p_{2k}] & \text{if } j = 0 \text{ and } j' \neq 2k+1, \\ \emptyset & \text{if } j = 0 \text{ and } j' = 2k+1. \end{cases} \blacksquare$$

Supplementary parts of the proof of proposition 7. We use the notation from the earlier proofs as well as the abbrevations SP (single-plateauedness), SC (single-canyonedness), UO (unidimensional orderedness) and UA (unidimensional alignment).

UA $\Longrightarrow$ SC. Let $(A_1, ..., A_n)$ be a profile of consistent judgment sets, and suppose UA, for simplicity relative to the order (Ω) $1, 2, ..., n$ We show SC relative to the order ($\leq$) $p_1 p_2 ... p_{2k}$ that

- begins with the propositions $p \in X$ with $N_p = \{1, ..., n\}$,
- followed by the propositions $p \in X$ with $N_p = \{1, ..., n-1\}$,

...

- followed by the propositions $p \in X$ with $N_p = \{1\}$,
- followed by the propositions $p \in X$ with $N_p = \emptyset$,
- followed by the propositions $p \in X$ with $N_p = \{n\}$,
- followed by the propositions $p \in X$ with $N_p = \{n-1, n\}$,

...

- ending with the propositions $p \in X$ with $N_p = \{2, ..., n\}$.

	1	2	3	4	5
p_1	Y	Y	Y	Y	Y
p_2	Y	Y	Y	Y	
p_3	Y				
p_4					
p_5					Y
p_6		Y	Y	Y	Y

Table 1: Example of the order $p_1, ..., p_{2k}$ for $n = 5$ individuals and $2k = 6$ propositions; a 'Y' indicates acceptance of the row proposition by the column individual

This procedure to construct $p_1 ... p_{2k}$ is well-defined, since, by UA, each $p \in X$ is of one of the forms considered in the procedure. In the example profile of Table 1, it is obvious that $(A_1, ..., A_n)$ is SC relative to $p_1 ... p_{2k}$: $A_1 = X \backslash [p_4, p_6]$, $A_2 = A_3 = A_4 = X \backslash [p_3, p_5]$ and $A_4 = A_5 = X \backslash [p_2, p_4]$.

For the general proof of SC, consider any A_h ($1 \leq h \leq n$) and let us show that A_h is SC relative to $\leq$. It suffices to prove that, for all $p \in X$, either $[p_1, p] \subseteq A_h$ or $[p, p_{2k}] \subseteq A_h$. Consider any $p \in X$. By UA, either $N_p = \{1, ..., k\}$ for some k, or $N_p = \{k, ..., n\}$ for some $k \geq 2$. By construction of the order $p_1 ... p_{2k}$, in the first case $[p_1, p] \subseteq A_h$ and in the second case $[p, p_{2k}] \subseteq A_h$, as desired.

SP $\not\Rightarrow$ SC. Consider an agenda X and a profile $(A_1, ..., A_n)$ consisting of pairwise disjoint consistent judgment sets, at least three of which are non-empty. The profile is SP, namely relative to an order starting with the propositions in A_1, followed by those in A_2, ..., and ends with those in A_n. But the profile is not SC: if it were SC, say relative to an order $\leq$, then each non-empty A_i would contain an extreme (i.e., left- or right-most) proposition; so that, as at least three A_is are non-empty but there are only two extreme propositions, the A_is would not be pairwise disjoint, a contradiction.

SC $\not\Rightarrow$ UO. Consider an agenda X, group N and profile $(A_1, ..., A_n)$ such that $n = 4$, $A_1 = \{p, p', q, q'\}$, $A_2 = \{p, p'\}$, $A_3 = \{q, q'\}$, $A_4 = \{p, q\}$, where $p, p', q, q' \in X$ are pairwise distinct. This profile is SC: consider an order $\leq$ such that $p < p' < ... < q' < q$ (where '...' contains all remaining propositions). Suppose for a contradiction UO holds, say relative to an order $i_1...i_n$. As $N_{p'} = \{1, 2\}$, individuals 1 and 2 are neighbours (in $i_1...i_n$). As $N_{q'} = \{1, 3\}$, 1 and 3 are neighbours. So 1 is 'surrounded' by 2 and 3, i.e., $i_1...i_n$ contains the sublist 213 or 312; suppose it contains the sublist 213 (the proof continues analogously for the sublist 312). Also, as $N_p = \{1, 2, 4\}$, 4 is a neighbour of 1 or of 2; since 4 cannot be a neighbour of 1 (which is surrounded by 2 and 3), it is a neighbour of 2. So $i_1...i_n$ contains the sublist 4213. Finally, as $N_q = \{1, 3, 4\}$, 4 is a neighbour of 1 or 3, which is not the case since $i_1...i_n$ contains the sublist 4213.

SC $\not\Rightarrow$ UA. This follows from SC $\not\Rightarrow$ UO by UO $\Rightarrow$ UA.

SP $\not\Rightarrow$ UO. This follows from SC $\not\Rightarrow$ UO by SC $\Rightarrow$ SP.

SP $\not\Rightarrow$ UA. This follows from SC $\not\Rightarrow$ UA by SC $\Rightarrow$ SP.

UO $\not\Rightarrow$ UA. Consider an agenda X, group N and profile $(A_1, ..., A_n)$ such that $n \geq 3$ and the A_is are pairwise disjoint and singleton. As every N_p is empty or singleton, the profile is UO (relative to any order of N). It is not UA: if it were, say relative to the order Ω of N, then each $i \in N$ would have to be extreme, i.e., smallest or largest in Ω (as i is the only individual accepting the proposition in A_i), which is not possible as there are $n \geq 3$ individuals but only two extreme positions.

UO $\not\Rightarrow$ SP. Consider a group, agenda X and profile $(A_1, ..., A_n)$ with $n = 3$ and $A_1 = \{p, p_1\}$, $A_2 = \{p, p_2\}$ and $A_3 = \{p, p_3\}$, where $p, p_1, p_2, p_3 \in X$ are pairwise distinct. This profile is UO, relative to any order of N. But it is not SP: if it were SP, say relative to an order $p_1...p_{2k}$ of X, then in this order p would have to be a neighbour of p_1 (by $A_1 = \{p, p_1\}$), and one of p_2 (by $A_2 = \{p, p_2\}$), and also one of p_3 (by $A_3 = \{p, p_3\}$), a contradiction.

UO $\not\Rightarrow$ SC. This follows from UO $\not\Rightarrow$ SP by SC $\Rightarrow$ SP. ■

Proof of lemma 11. Let $A \subseteq X$ be complete and inconsistent. Among all inconsistent subsets of A, let B be one of smallest size $|B|$. We show that B is irreducible. Suppose for a contradiction that B is reducible to $C \subseteq X$. We will define an inconsistent subset of A smaller than B, in contradiction to the choice of B. By $|C| < |B|$ and the choice of B, we have $C \nsubseteq A$. So there is a $p \in C \backslash A$. Since A is complete, we have $\neg p \in A$. As C is a reduction of B, there is a subset $B^* \subseteq B$ with $|B \backslash B^*| \geq 2$ and $B^* \vdash p$. Now $B^* \cup \{\neg p\}$ is an inconsistent subset of A smaller than B:

- $B^* \cup \{\neg p\}$ is a subset of A by $B^* \subseteq B \subseteq A$ and $\neg p \in A$;
- $B^* \cup \{\neg p\}$ is inconsistent by $B^* \vdash p$;
- $|B^* \cup \{\neg p\}| \leq |B^*| + 1 = |B| - |B \backslash B^*| + 1 \leq |B| - 2 + 1 < |B|$. ■

The proof of proposition 17 requires a lemma:

Lemma 19 *Let $S \neq \emptyset$ be a set of subsets $I \subseteq N$ that are each intervals relative to some fixed linear order on N. If the elements of S are pairwise non-disjoint (i.e., $I \cap J \neq \emptyset$ for all $I, J \in S$), they are all non-disjoint (i.e., $\cap_{I \in S} I \neq \varnothing$).*

Proof. Let S be as defined in the lemma. Note that S must be finite. So a proof by induction on the size of S is possible. More precisely, we prove by induction that $\cap_{i \in S} I = [\max_{i \in I} \min I, \min_{I \in S} \max I] \neq \emptyset$.

First let S have size 1, say $S = \{I\}$. The claim then holds, since $\cap_{J \in S} J = I = [\min I, \max I]$, which is non-empty because it can be written as $I \cap I$, a non-empty set by pairwise non-disjointness.

Now suppose the claim holds for sets of a size k (≥ 1), and consider a set S of size $k+1$, say $S = S' \cup \{J\}$ where S' has size k. We have $\cap_{I \in S} I = J \cap (\cap_{I \in S'} I)$, where by induction hypothessis, $\cap_{I \in S'} I = [\max_{I \in S'} \min I, \min_{I \in S'} \max I] \neq \emptyset$. So

$$\cap_{I \in S} I = J \cap [\max_{I \in S'} \min I, \min_{I \in S'} \max I].$$

This set obviously equals $[\max_{I \in S} \min I, \min_{I \in S} \max I]$. To complete the proof, suppose for a contradiction that this interval is empty. The intersection of two intervals (here, of J and $[\max_{I \in S'} \min I, \min_{I \in S'} \max I]$) can only be empty if the largest element of one of the intervals is smaller than the smallest element of the other interval. So either $\min_{I \in S'} \max I < \min J$ or $\max J < \max_{I \in S'} \min I$. In the first case, there is an $I \in S'$ such that $\max I < \min J$, so that $I \cap J = \emptyset$. In the second case, there is an $I \in S'$ with $\max J < \min I$, so that again $I \cap J = \emptyset$. So in any case pairwise non-disjointness is violated, a contradiction. ■

Proof of proposition 15. Let $(\succ_1, ..., \succ_n)$ be as specified, and denote by $(A_1, ..., A_n)$ the corresponding judgment profile, whose judgment sets A_i ($= A_{\succ_i}$) are complete and consistent as each $\succ_i$ is also fully rational. For all i and all distinct $x, y \in K$, $x \succ_i y \Leftrightarrow y \not\succ_i x$; so that for $(\succ_1, ..., \succ_n)$ intermediateness on triples is indeed equivalent to order restriction on triples. Moreover, as each A_i is complete and consistent, for $(A_1, ..., A_n)$ local unidimensional orderedness with respect to $\mathcal{Y}$ is indeed equivalent to local unidimensional alignment with respect to $\mathcal{Y}$ (see corollary 9). So it remains to show that $(\succ_1, ..., \succ_n)$ is intermediate on triples if and only if $(A_1, ..., A_n)$ is locally unidimensionally ordered (with respect to $\mathcal{Y}$).

To prove the latter, recall that the irreducible sets are, by Remark 14, the cycles of length 1 or 2 or 3, i.e. the subagendas essentially identical to a subagenda of type

$$\{xPx\}^{\pm} \text{ or } \{xPy, yPx\}^{\pm}\ (x \neq y) \text{ or } \{xPy, yPz, zPx\}^{\pm}\ (x, y, z \text{ distinct}). \quad (3)$$

So, using that unidimensional orderedness on a subagenda is equivalent to unidimensional orderedness on any essentially identical subagenda, $(A_1, ..., A_n)$ is locally unidimensionally ordered if and only if it is unidimensionally ordered on any subagenda of one of the three types in (3). Unidimensional orderedness holds trivially on subagendas of the first type $\{xPx\}^{\pm}$; and similarly for subagendas of type $\{xPy, yPx\}^{\pm}$ $(x \neq y)$: consider an order of N beginning with the individuals i with $x \succ_i y$, and followed by the individuals i with $y \succ_i x$. So local unidimensional orderedness is equivalent to unidimensional orderedness on each of the subagendas

$$\{xPy, yPz, zPx\}^{\pm} \ (x, y, z \in K \text{ distinct}).$$

But this is equivalent to intermediateness of $(\succ_1, ..., \succ_n)$, as one easily checks (using that, for distinct $x, y \in K$, $\neg xPy \in A_i \Leftrightarrow yPx \in A_i$ for all A_i). ■

Proof of proposition 17. We prove part (a). Parts (b) and (c) follow analogously. Consider a profile $(A_1, ..., A_n)$ of consistent judgment sets. By propositions 3 and 5, it suffices to show that (i) single-plateauedness implies value-restriction and that (ii) unidimensional orderedness implies value-restriction.

(i) Suppose $(A_1, ..., A_n)$ is single-plateaued, say relative to the order $\leq$. To show value-restriction, consider any non-singleton minimal inconsistent set Y. We must specify a two-element subset of Y not contained in any A_i. Define it as consisting of the smallest element p and the largest element q of Y (relative to the order $\leq$). As desired, no A_i can contain both p and q: otherwise it would (by single-plateauedness) include the entire interval from p to q, hence include the inconsistent set Y, a contradiction.

(ii) Suppose, for a contradiction, that $(A_1, ..., A_n)$ is unidimensionally ordered but not value-restricted. Let Y be a minimal inconsistent set for which value-restriction is violated. Let S be the set $\{\{i \in N : p \in A_i\} : p \in Y\}$. By unidimensional orderedness, S consists of intervals (relative to a structuring order Ω). Further, these intervals are pairwise non-disjoint: otherwise there would be $p, q \in Y$ such that $\{i \in N : p \in A_i\} \cap \{i \in N : q \in A_i\} = \varnothing$, so that no A_i contains both p and q, whence value-restriction would not be violated for Y. So, by lemma 19, S has a non-empty intersection. In other words, some A_i contains all $p \in Y$. But then A_i is inconsistent, a contradiction. ■

Proof of proposition 18. Let $(A_1, ..., A_n)$ be as specified, and denote by $(\succ_1, ..., \succ_n)$ the corresponding preference profile. We first show that (b) is equivalent to (c), and then that (a) is equivalent to (b).

(c) $\Longrightarrow$ (b). First suppose $(\succ_1, ..., \succ_n)$ is triplewise value-restricted. Consider any non-singleton irreducible $Y \subseteq X_K$. By remark 14, Y is a cycle of length 2 or 3. If Y has length 2, hence is a binary inconsistent set, we can take $Z = Y$, and by individual consistency no A_i includes Z. Now let Y be a 3-cycle, hence essentially identical to aset of the form $\{xPy, yPz, zPx\}$ for distinct $x, y, y \in K$. By triplewise value-restriction, some of x, y, z is in $(\succ_1, ..., \succ_n)$ either never ranked between, or never above, or never below, the two other alternatives. We go through all nine cases:

- if x is never ranked between y and z, no A_i is a superset of $Z = \{zPx, xPy\}$;
- if y is never ranked between x and z, no A_i is a superset of $Z = \{xPy, yPz\}$;

- if z is never ranked between x and y, no A_i is a superset of $Z = \{yPz, zPx\}$;
- if x is never ranked above y and z, no A_i is a superset of $Z = \{xPy, yPz\}$;
- if y is never ranked above x and z, no A_i is a superset of $Z = \{yPz, zPx\}$;
- if z is never ranked above x and y, no A_i is a superset of $Z = \{zPx, xPy\}$;
- if x is never ranked below y and z, no A_i is a superset of $Z = \{yPz, zPx\}$;
- if y is never ranked below x and z, no A_i is a superset of $Z = \{zPx, xPy\}$;
- if z is never ranked below x and y, no A_i is a superset of $Z = \{xPy, yPz\}$.

(b) $\Longrightarrow$ *(c)*. Now let $(A_1, ..., A_n)$ be weakly value-restricted. To show that $(\succ_1, ..., \succ_n)$ is triplewise value-restricted, consider any distinct alternatives $x, y, z \in K$. By remark 14, the sets $Y = \{xPy, yPz, zPx\}$ and $Y' = \{zPy, yPx, xPz\}$ are irreducible and non-singleton. So, by weak value-restriction, Y has a two-element subset Z not included in any A_i, and similarly Y' has a two-element subset Z' not included in any A_i. Assume $Z = \{xPy, yPz\}$ (the proof is analogous for other binary subsets of Y). Since each A_i is neither a superset of Z nor one of Z', we can conclude the following:

- if $Z' = \{zPy, yPx\}$, then no A_i ranks y between x and z;
- if $Z' = \{yPx, xPz\}$, then no A_i ranks z below x and y;
- if $Z' = \{xPz, zPy\}$, then no A_i ranks x above y and z.

So, whatever Z' is, we have triplewise value-restriction.

(a) $\Longrightarrow$ *(b)*. Trivial, since irreducible sets are minimal inconsistent.

(b) $\Longrightarrow$ *(a)*. Suppose $(A_1, ..., A_n)$ is weakly value-restricted. To show value-restriction, consider any non-singleton minimal inconsistent set $Y \subseteq X_K$. By remark 13, Y is a cycle of some length k, hence is essentially identical to – we may assume identical to – a set of type

$$Y = \{x_1Px_2, x_2Px_3, ..., x_{k-1}Px_k, x_kPx_1\}, \text{ with distinct } x_1, ..., x_k \in K$$

for some $k \geq 2$. We show by induction on the size k of Y that Y has a two-element subset Z that is not included in any A_i.

First let $k = 2$ or $k = 3$. Then Y is by remark 14 irreducible, hence has by weak value restriction a two-element subset Z not included in any A_i.

Now suppose $k \geq 4$, and let the claim hold for sets of size less than k. Consider the non-singleton irreducible sets $Y' = \{x_1Px_2, x_2Px_3, x_3Px_1\}$ and $Y'' = \{x_1Px_3, x_3Px_4, ..., x_{k-1}Px_k, x_kPx_1\}$. By induction hypothesis,

(*) Y' has a binary subset Z' not included in any A_i; and

(**) Y'' has a binary subset Z'' not included in any A_i.

We distinguish three cases.

Case 1: $x_3Px_1 \notin Z'$. Then $Z' \subseteq Y$, and we may put $Z = Z'$.

Case 2: $x_1Px_3 \notin Z''$. Then $Z'' \subseteq Y$, and we may put $Z = Z''$.

Case 3: $x_3Px_1 \in Z'$ and $x_1Px_3 \in Z''$. Then $Z' = \{p, x_3Px_1\}$ for some $p \in \{x_1Px_2, x_2Px_3\}$, and $Z'' = \{q, x_1Px_3\}$ for some $q \in \{x_3Px_4, ..., x_{k-1}Px_k, x_kPx_1\}$. Define $Z = \{p, q\}$. Obviously, Z is a two-element subset of Y. Further, no A_i includes Z:

- if $p \in A_i$, then $x_3Px_1 \notin A_i$ (as A_i does not include Z'), so $x_1Px_3 \in A_i$ (as A_i is complete and consistent), and so $q \notin A_i$ (as A_i does not include Z'');

- if $q \in A_i$, then $x_1 P x_3 \notin A_i$ (as A_i does not include Z''), so $x_3 P x_1 \in A_i$ (as A_i is complete and consistent), and so $p \notin A_i$ (as A_i does not include Z'). ■