DYNAMIC VOTING IN CLUBS

by

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Abstract

This paper examines the process and outcomes of democratic decision-making in clubs where a club is defined by its set of members whose preferences and decisions relate to the set of members in the club: the electorate to endogenous. Examples range from international organizations like the European Union and NATO to firms, workers' cooperatives and trade unions. Although the policy space is infinite, a majority voting equilibrium exists under plausible conditions and the equilibrium rule and the dynamics of clubs are characterized. Two types of club, one where a group funds some public good and the other where a given benefit is shared by the group, are analysed in detail.

Keywords: Cooperatives; local public goods; majority voting; median voter; organization size; partnerships; trade unions.

JEL Nos.: C72, D71, D92, H41, L20.

Introduction

The purpose of this paper is to investigate organizations where the decision makers are a group of members within the organization and decisions involve changing the group of members; more specifically, the paper examines the process and outcomes of democratic decision making in clubs where clubs are taken to be defined by a set of members whose preferences relate, directly or indirectly, to the set of members in the club. Democratic decision taking is interpreted to be majority voting by club members.

The literature on clubs initiated by Buchanan (1965) and further developed by Ng (1973) and Stiglitz (1977) views clubs as the providers of impure public goods: there is excludability so that provision may be restricted to members and there is partial rivalness through crowding and/or congestion. If the cost of provision is shared among members then individual preferences over club size will incorporate a trade-off between per capita cost reductions and increased congestion with increases in size. Whilst the literature has concentrated on concrete examples of public good provision, there are many other diverse examples. For instance, the costs and benefits of membership of international organizations like the European Union depends both directly on the set of States which form the Union and indirectly on the set of States through the decisions they take together relating to, for example, economic and legal matters. As can be evidenced by the recent Amsterdam Treaty, the size and composition of the Union is of dominant concern within the Union. At a different level, a trade union or a partnership may also be viewed as a club: it is interested in ensuring employment and high wages for its members; the larger the union membership, the more the goal of high wages may need to be compromised to ensure employment. The overall effect will be that union members or partners have preferences over the size of the union.

A club with democratic decision making is a paradigm for organizations where there is no single decision maker and decisions, particularly about the size of the organization, involve interests in the different parts of the organization. An example of this is where a firm expands by the construction of new plants and managers from these plants are involved in subsequent decision making. As multiple decision makers is a commonplace, this paper provides some insight into the growth and size of organizations

The early club literature focussed on (welfare) optimal public good provision and club size whereas it is clear that decisions are often taken through the operation of some voting procedure.[1] However, unlike voting over a conventional policy space, voting over club size gives rise to decisions over time that are time-inconsistent. A majority of members of a club may wish to change the club size and then fix it at the new level. But when the membership changes, a majority of the reformed club may wish to choose a new club size. Rational members will take account of future changes induced by their decisions when initially voting and this will affect the operation of the voting procedure. Previous analyses of voting with an endogenous electorate have ignored the dynamics induced by the voting mechanism. Stiglitz (1977) looked at a median voter choice where it is assumed that decisions will not induce further changes. Klevorick and Kramer (1973) adopt a similar approach and motivate a median voter rule by assuming one-period single-peaked preferences over the decision variable. Layard (1990) looks at a specific democratic trade union model and, assuming that voting is equivalent to a median voter choice, provides a restricted analysis of equilibrium under the assumption of zero discounting. It is also possible that present decisions affect future preferences and dynamics can be induced with a fixed electorate. An example of this which is fully consistent with forward looking voters is the interesting work of Krusell and Rios-Rull (1996).

It is well-known that majority voting can fail to produce a 'preferred' outcome and Arrow's impossibility theorem shows that this problem can be inherited by a very wide class of voting procedures. To overcome this, it is common to place restrictions on voters' preferences which use a dimensionality restriction on the policy space. Restrictions then take the form either of limiting individual preferences to be single-peaked (Black (1948)), or of placing a (single-crossing) restriction across preferences which allows individuals to be ordered by their marginal preference for the policy variable (Roberts (1977), Grandmont (1978), Rothstein (1990),Gans and Smart (1996)). However, with time-inconsistency, membership size may change many times and it will be impossible to restrict the

[1] Tiebout's (1956) classic analysis may be viewed as showing that competition between clubs may lead to optimality even though decisions are taken through a voting process. The literature on public good provision and voting can be traced back to Bowen (1943).

dimensionality of the policy space to apply a single-peakedness or single-crossing property.[2]

Despite this, a major purpose of this paper is to show that majority winners exist in dynamic voting problems if a plausible single-crossing condition is satisfied in a one-period version the problem. In addition, such a condition allows us to determine the nature of equilibrium - a median voter result applies even though the median voter is endogenous to choices that are being made - and the character of steady states and of the transition paths taken towards a steady state can be exposed.

The model is set up in the next section and section 3 investigates the characteristics of equilibrium in the model. The transition paths and steady states associated with equilibrium are examined in section 4. In the following two sections, two classes of example are developed and analysed. Section 5 looks at 'expansionist clubs' where, whatever the size of the club, a median voter would always prefer an increase in size. It is suggested that clubs providing public goods, and the European Union can be viewed as one such example, may possess this feature. In contrast, section 6 looks at 'contractionist clubs' where median voters always prefer a reduction in club size and a standard model of a democratic trade union is an example of this. Welfare implications of the club decisions are examined in section 7 and section 8 contains concluding remarks.

2. The Model

We consider a finite group X of infinitely lived individuals who are potential club members, $X = \{1,2...\bar{x}\}$. These individuals always wish to be members of the club though some may be excluded. It is assumed that there is a natural seniority system with regard to membership of the club such that when the club is of size x, its members are the set $\{1,2,...,x\}$. Thus, at any date t, the club is defined by its size x_t. If at some date the club size is x then the instantaneous utility of individual ξ is given by $u(x,\xi)$ and individual ξ wishes to maximise.

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[2] It is well-known that single-peakedness is not sufficient to ensure a majority winner when the policy space is not uni-dimensional and conditions for existence are restrictive e.g. Tullock (1967), Caplin and Nalebuff (1988). The single-crossing property is similar to that used a principal agent analysis. In that literature, the single-crossing conditions loses much of its usefulness when the dimensionality of the problem increases. One example of multi-dimensionality, where the dimensionality increases by allowing stochastic contracts, is Moore (1988). Variation over state of nature is similar to variation over time as studied here.

$$U = \sum_{0}^{\infty} \delta^t u(x_t,\xi) \qquad (1)$$

where δ, the discount factor satisfies $0 < \delta < 1$. As utilities are defined over a finite set, (1) is defined as long as $\delta < 1$.

Individual utility can be a direct function of club size through the sharing of the cost of provision of a public good or through congestion effects, and an indirect function of a club size through decisions taken by a club with a particular membership, e.g. the level of public good provision. An example of this will be considered in section 5. The function $u(x,\xi)$ is assumed to incorporate both elements and is assumed to satisfy the following restriction:

Strict Increasing Differences. For all $x > x'$, $\xi > \xi'$:

$$u(x,\xi) - u(x',\xi) > u(x,\xi') - u(x',\xi') \qquad (2)$$

This assumption is the critical assumption of the model. It is a discrete version of a single-crossing or Spence-Mirrlees condition and is sufficient to generate equilibria in static models of voting (Roberts (1977)), Rothstein (1990), Gans and Smart (1996)). For examples of dynamic games which utilize the weak and strong form of increasing differences, see Fudenberg and Tirole (1991).

The Strict Increasing Differences condition embodies two different aspects. First, and most directly, it says that individuals may be ordered by their preference over club size. This is a mild assumption. Second, it says that an individual who prefers a smaller club size will be admitted to the club before somebody who prefers a larger size. This feature is often implied by more primitive considerations of preferences (see section 6 below) or follows naturally from the problem under investigation. For instance, in the case of the potential expansion eastwards of the European Union, it is natural to believe that countries have a preference to be towards the "centre" of the Union; in this case, more easterly countries have a greater preference for an eastwards expansion and geographical location induces both a ranking for admittance and a preference over size.

The club size can vary over time. We restrict attention to situations that give rise to

Markov transition rules y that map from X to X: a transition rule defines the path of x recursively such that for all t: $x_{t+1} = y(x_t)$. If there is an x such that x = y(x) then x is a steady state. The rule y(.) can be a deterministic rule but it is also permitted to be stochastic. If individual ξ's discounted utility under y(.), starting at x, is given by V(x,ξ,y) then V is defined recursively by[3]

$$V(x,\xi,y) = u(x,\xi) + \delta V(y(x),\xi,y) \qquad (3)$$

If y(.) is stochastic then (1) is assumed to be the von Neumann-Morgenstern utility function and the second term in the right-hand side of (4) is replaced by an expectation over V.

An individual's preferences over club size will be given by V and will be conditional on the transition rule y(.). We first define a

Markov Voting Equilibrium MVE. Given any transition rule y*(.), for each x ε X, let Y*(x) be the set of y such that for all z ε X:

$$\begin{aligned} &\# \{\xi:\xi \le x \;\&\; V(y,\xi,y^*) > V(z,\xi,y^*)\} \\ \ge \quad &\# \{\xi,\xi \le x \;\&\; V(y,\xi,y^*) < V(z,\xi,y^*)\} \end{aligned} \qquad (4)$$

If y*(x)ε Y*(x) for all x then y*(.) is an MVE.

This condition says that, given that y*(.) is followed in the future, no club size defeats y*(x) in pairwise majority voting as the choice for the next period - y*(x) is the Condorcet winner for the club of size x. (If y*(.) is stochastic then (4) must hold for all realisations of y*(.) that occur with non-zero probability). Voting for the state that maximises V is a weakly dominant strategy for each member - there are degenerate Nash equilibria where, for instance, everybody votes for the *status quo* (because no single member can disrupt such an outcome) but such possibilities are ruled out by assumption. Note that as the electorate at t+1 will be $\{1,...y^*(x_t)\}$, the majority winner chosen by this group will in general be different to that chosen by $\{1,....x_t\}$ and so $y^*(y^*(x_t))$ may differ from $y^*(x_t)$.

In the next section we will wish to compare an MVE with a median-voter rule where,

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[3] The third argument of V is the function y rather than the value of y evaluated at some x.

for every club size, an individual with median seniority chooses the club size for the next period. A median voter is a function m(.) such that, at club size x, m(x) is (x+1)/2 when x is odd and either x/2 or (x/2)+1 when x is even. The group of individuals both below and above the median voter, including the median voter, constitute a weak majority.

We can now define a

Median Voter Rule. Given any transition rule $\tilde{y}(.)$, for each $x \varepsilon X$, let $\tilde{Y}(x)$ be the set of y such that for all $z \varepsilon X$:

x odd:		$V(y,m(x),\tilde{y}) \geq V(z,m(x),\tilde{y})$	
x even:	either	$V(x,m(x),\tilde{y}) \geq V(z,m(x),\tilde{y})$ for each m(x)	(5)
	or	$V(x,m(x),\tilde{y}) > V(z,m(x),\tilde{y})$ for some m(x)	

If $\tilde{y}(x) \varepsilon \tilde{Y}(x)$ for all x then $\tilde{y}(.)$ is a median voter rule.

The connection between an MVE and a median voter rule will be investigated in the next section.

3. Equilibrium

The difficulty in the analysis of voting equilibria comes from the dynamics induced by a changing electorate. Consider, first, what would happen if a club of size x could choose a new size which would be implemented forever. In this case, preferences would be based upon instantaneous utility and the median voter choice $\tilde{y}$ would satisfy (applying (5)):

x odd:		$u(\tilde{y},m(x)) \geq u(z,m(x))$	
x even:	either	$u(\tilde{y},m(x)) \geq u(z,m(x))$ for each m	(6)
	or	$u(\tilde{y},m(x)) > u(z,m(x))$ for some m.	

Such a $\tilde{y}$ exists as X is finite: trivially when x is odd, and when x is even, let $\tilde{y}$ be a best choice for m(x) = (x/2)+1 from among the best outcomes for m(x) = x/2. Assume that $\tilde{y} > z$ for some z (the alternative case $z > \tilde{y}$ is treated similarly). Applying the strict increasing differences condition, we have

$$u(\tilde{y},\xi) > u(z,\xi) \qquad (7)$$

for all $\xi > \frac{x+1}{2}$, x odd, and $\xi > \frac{x}{2}$, x even - both median voters cannot be indifferent when x is even. Thus

$$\#\{\xi:\xi \leq x \;\&\; u(\tilde{y},\xi) > u(z,\xi)\} \geq \#\{\xi,\xi \leq x \& u(\tilde{y},\xi) < u(z,\xi)\} \qquad (8)$$

and ỹ will be a (majority) voting equilibrium: a voting equilibrium exists which is a median voter rule. This is, in essence, the approach taken in Roberts (1977).[4]

When future club sizes are not fixed, the situation is less straightforward. Assume that if a club size of y_1 is chosen next period then there will be very little future variability whereas the opposite is true if y_2 is chosen. There will be a tendency for individuals with more concave utility functions to prefer y_1 and those with less concave functions to prefer y_2. But as concavity bears no necessary relationship to increasing differences, there will be no natural order of preference across individuals and y_2 may be chosen over y_1 because it is preferred by a coalition of low ξ and high ξ individuals whereas a median voter prefers y_1 to y_2.

To proceed, we look at the implications of the increasing differences condition with regard to variable club sizes and then consider whether the dynamic path of club sizes that result from an MVE can be restricted. A straightforward implication of increasing differences is:

<u>Lemma 1</u>. Given any x ε X, let $(y_0,....,y_t,...)$ be a sequence such that y_t ε X and $y_t \geq x$ ($y_t < x$, resp.) for all t, $y_t > x$ ($y_t < x$, resp.) for some t. If an individual ξ weakly prefers a constant x to the stream (y_t) then there is strict preference for all $\xi',\xi' < \xi$ ($\xi' > \xi$, resp.):

$$\sum_0^\infty \delta^t u(x,\xi) \geq \sum_0^\infty \delta^t u(y_t,\xi) \qquad \textbf{(9)}$$

[4] A straightforward presentation of this approach and a formal statement of results is to be found in Grans and Smart (1996).

$$=>\sum_{0}^{\infty} \delta^t u(x,\xi') > \sum_{0}^{\infty} \delta^t u(y_t,\xi')$$

Proof. We consider the case where $y_t \geq x$. Using increasing differences, we have

$$u(y_t\ \xi) - u(x,\xi) \geq u(y_{t,}\xi') - u(x,\xi')$$

with the inequality strict from some t. Weighting by δ^t and summing over all t gives (9) ■

An immediate strengthening of Lemma 1 is

Lemma 1'. Let $(y_0,y_1,....),(y'_0,y'_1,....)$ be two sequences such that $y'_t \geq y_t$ ($y'_t \leq y_t$ resp.) for all t with the inequality strict from some t. If an individual weakly prefers the stream (y_t) to the stream (y'_t) then there is strict preference of (y_t) over (y'_t) for all $\xi',\xi' < \xi$,($\xi' > \xi$, resp.).

We now turn our attention to the characteristics of an MVE, y*, and a median voter rule, ỹ. The following result is central to our characterization (proofs not provided in the text are given in the Appendix).

Proposition 1. Consider a dynamic path (y_t) generated by an MVE, y*: $y_{t+1} = y^*(y_t)$, $t \geq 0$ or by a median voter rule ỹ: $y_{t+1} = \tilde{y}(y_t)$, $t \geq 0$. It is impossible that either $y_0 > y_1 \leq y_t$, $t \geq 2$ with $y_1 < y_\tau$ for some $\tau \geq 2$ or $y_0 < y_1 \geq y_t$, $t \geq 2$ with $y_1 > y_t$ for some $\tau \geq 2$ (if y* is stochastic then there are no strictly positive probability realisations of this form).

Proposition 1 rules out extreme turning points in the size of club membership. We use it first to investigate the occurrence of cycles. A transition rule y(.) generates a cycle if there is a dynamic path (y_t), $y_{t+1} = y(y_t)$ for all $t \geq 0$, such that $y_s=y_t$, $s < t$ and $y_\tau \neq y_s,y_t$ for some τ, $s < \tau < t$. A stochastic rule generates cycles if a cycle occurs with strictly positive probability.

<u>Proposition 2</u>. An MVE transition rule y* generates no cycles and a median voter rule ỹ generates no cycles.

This result follows from Proposition 1. If there is a cycle then let $\underline{x}$ be the <u>lowest</u> membership size belonging to the cycle. If y(.) is the transition rule then $\underline{x} = y(x')$ for some $x' > \underline{x}$ (equality is ruled out because $\underline{x}$ would then be a steady state ($\underline{x} = y(\underline{x})$) which rules out a cycle). If y(.) is stochastic then there is a strictly positive realisation with these properties. From the definition of $\underline{x}$, the transition path (y_t) starting at $x'(y_0 = x')$ satisfies the property that $y_0 > y_1 \le y_t$, $t \ge 2$ with $y_1 < y_t$ for some t, e.g. t = 2. Using Proposition 1, cycles are not possible if the transition rule is an MVE or a median voter rule.

Proposition 2 shows that the transition rules under consideration do not give rise to a perpetual cycle and, as x is finite, the dynamic paths generated by y* and ỹ must, in a finite time, reach a steady state $\underline{x}$, $\underline{x} = y(\underline{x})$ where y(.) is y*(.) or ỹ(.). An induction argument, moving backwards from a steady state, allows us to apply Proposition 1 to show that transitions to a steady state must involve a degree of monotonicity.

<u>Proposition 3</u>. Let (y_t) be a dynamic path generated by an MVE or a median voter rule. Then the path is monotonic:

$$y_{t+1} \underset{<}{>} y_t \Rightarrow y_\tau \underset{\le}{\ge} y_{t+1} \qquad (10)$$

for all t, τ, τ > t.

Proposition 3 does not show that all dynamic paths are monotonic in the same direction and we will see in Section 4 that commonly there will be a mixture of monotonically increasing and decreasing paths. What can be shown is that paths do not cross and this is shown by a similar induction argument to that used to prove Proposition 3:

<u>Proposition 4</u>. If (y_t) and (y'_t) are dynamic paths generated by an MVE or a median voter rule then $y_0 \ge y'_0 \Rightarrow y_t \ge y'_t$ for all $t \ge 0$.

These results show that majority voting and median voter transition rules are not ‘too exotic’ and possess similar properties. They also allow us to tie together the two rules.

Proposition 5. An MVE transition rule y* is a median voter rule ỹ and vice versa.

This result has three purposes. First, it allows a simple characterization of an MVE. Second, the computation of an MVE in any example is greatly eased by the ability to concentrate on median voter rules. Third, the result allows us to tackle the issue of the existence of appropriate transition rules; thus far, our results have been predicated on the assumption that rules exist. Our route to the existence of Markov Voting Equilibria will be based upon showing first that median voting rules exist by constructing a normal form game where Nash equilibria correspond to median voter rules.

Proposition 6. A (possibly stochastic) median voter rule ỹ (.) exists.

A combination of Propositions 5 and 6 give us a voting equilibrium existence result.

Proposition 7. A Markov Voting Equilibrium MVE exists.

The equilibrium may involve mixed strategies. In a similar though simpler problem of bequest games, Leininger (1986) has shown the existence of pure strategy equilibria under plausible conditions. Under a weak further restriction on preferences, it is possible to show that transition rules have a recursive structure which permits us to show that pure strategy equilibria exist. We return to this issue when we have better understood the dynamics paths generated by transition rules (see Proposition 15 below).

4. Steady States and Transition Paths

We have seen that, starting from any level of club size, a steady state is attained in finite time and the transition is monotonic. In this section we investigate and characterize steady states and transition paths in greater detail.

It is easiest to uncover the structure of MVEs by looking at median voter rules. Given an x let $\mu(x)$ be the club size that would be optimal for a median voter who could commit the club not to change its size in the future. Then we have:

$$\text{x odd:} \qquad \mu(x) = \text{argmax } u(.,\frac{x+1}{2}) \qquad (11)$$

$$\text{x even:} \qquad \mu(x)\, \varepsilon\, [\mu^*,\mu^{**}] \qquad (12)$$

where

$$\mu^* = \text{argmax } u(.,x/2) \qquad \mu^{**} = \text{argmax } u(.,(x/2)+1)$$

For the purpose of illustration, Figure 1 plots an example of $\mu(x)$ with x treated as a continuous variable. The monotonicity of $\mu(x)$ follows from the condition of strict increasing differences.

4.1 Steady States

Consider a club size like x* in Figure 1 where μ is unique. We have

$$u(x^*,m(x^*)) > u(x,m(x^*)) \qquad \text{for all } x,\ x \neq x^* \qquad (13)$$

Let $\tilde{y}(.)$ be the median voter transition rule. Now, if $\tilde{y}(x^*) \neq x^*$ then

$$V(x^*,m(x^*),\tilde{y}) = u(x^*,m(x^*)) + \delta V(\tilde{y}(x^*),m(x^*),\tilde{y}) > V(\tilde{y}(x^*),m(x^*),\tilde{y}) \qquad (14)$$

as from (13)

$$V(\tilde{y}(x*),m(x*),\tilde{y}) = \sum_{t=0}^{\infty} \delta^t u(y_t,m(x*)) < (\sum_{t=0}^{\infty} \delta^t)u(x*,m(x*)). \qquad (15)$$

As (14) violates (5), we have:

<u>Proposition 8</u>. If there is a club size x* where the unique value of μ(x*) is x* then x* is a

steady state of an MVE.

The intuition is straightforward - if a club size is reached which the median voter views as optimal then he will not wish to vote for a change in its size and, as he is a median voter, he can always enlist a majority in ensuring no change. We call such a position an extrinsic steady state because it is a steady state irrespective of the transition rule adopted away from this state. In Figure 1 there are three extrinsic steady states.

Are there steady states other than extrinsic steady states? Consider a situation where

$$\begin{aligned} &\mu(x^*) = \mu(x^*+1) = \mu(x^*+2) = x^* \\ &\mu(x^*+3) = x^*+2 \\ &\mu(x^*+k) = x^*+5,\ k \geq 4 \end{aligned} \qquad (16)$$

Here, x* is an extrinsic steady state. When the club size is x*+1 or x*+2, the median voter will choose a club size of x* in the knowledge that x* will be chosen forever and this must dominate any other choice. At a club size of x*+3, a decision to increase the size to x*+4 will lead the then median voter to raise it to x*+5 in the knowledge that it will remain at that level forever; on the other hand, a decision to reduce the club size to x*+2 say, will then lead to it falling to x*. Let the median voter at x*+3 have preferences of the form.

$$u(x,m(x^*+3)) = -(x-(x^*+2))^2 \qquad (17)$$

The future discounted utility of m(x*+3) from changing the club size is:

$$\begin{aligned} &\text{increase:} \quad -2^2 - \frac{\delta}{1-\delta^2}\,3 = -\left(\frac{4+5\delta}{1-\delta}\right) \\ &\text{no change:} \quad -\frac{1}{1-\delta} \qquad (18) \\ &\text{reduce:} \quad -\frac{\delta}{1-\delta}\,2^2 = -\frac{4\delta}{1-\delta} \end{aligned}$$

The optimal rule is

1) No change if $\delta \geq$ ¼
2) Reduce to x*+2 if $\delta <$ ¼

If discounting is not too high, the club level of x*+3 is a steady state even though the median voter would prefer a smaller club and this proves

<u>Proposition 9</u>. There can be steady states of an MVE which involve club sizes that are sub-optimal for the median voter at that club size.

Such steady states will be referred to as <u>intrinsic steady states</u> because they are sustained as steady states by the transition rule operated away from the steady state rather than by the preferences of the median voter for that steady state.[5] The example developed to show Proposition 9 also makes clear the role of discounting:

<u>Proposition 10</u>. If δ is sufficiently small then there are no intrinsic steady states.

To see this, note that, with δ small, the choice by m(x) of a utility maximising club size followed by any dynamic path dominates any other possibility.

As steady states are reached in finite time, low discounting implies that the value of a dynamic path is dominated by the value of the steady state that will be reached. We thus have:

<u>Proposition 11</u>. If S is the set of steady states for all values of the discount factor close to unity, then[6]

(i) $x \varepsilon S \Rightarrow u(x,m(x)) \geq u(z,m(x)) \; \forall z \varepsilon S$ **(19)**

[5] It is possible to have steady states as corner solutions, e.g. at $\bar{x}$ with $\mu(\bar{x}) > \bar{x}$. For interpretation purposes, such a steady state can be viewed as being extrinsic.

[6] Of course, there is no guarantee that S exists as there may be a discontinuity at δ=1.

(ii) $x \varepsilon X/S$ => $u(x,m(x)) \leq u(z,m(x))$ for some $z \varepsilon S$.

Proof. To show (i), assume that $u(x,m(x)) < u(z,m(x))$ for some $x \varepsilon S$. Then at a club size of x, the median voter would gain from changing the club size to z rather than remaining at x. This is true whatever the discount factor. To show (ii), note that if $u(z,m(x)) < u(x,m(x))$ for all $z \varepsilon S$ then with δ close to unity, all dynamic paths will be inferior to remaining at x ■

For an open and dense set of utility functions (in the Euclidean metric) the inequalities in Proposition 11 will be strict and the set S will be defined by the given inequalities. Proposition 9 will then become an 'if and only if' statement and the inequalities will provide a method for computing sets of steady states of voting equilibria.

4.2 Transition Paths

We now investigate transition paths and, in particular, transitions when close to a steady state. It is convenient to assume that $\mu(x)$ captures one period preferences - the median voter prefers sizes closer to $\mu(x)$. We thus adopt the following assumption:

Strict Quasi-Concavity (SQC):[7] Each individual has a strictly quasi-concave utility function $u(.,\xi)$.

Consider, first, an extrinsic steady state x* and consider some x' above x*. Assume that for all x, $x^* < x \leq x'$:

$$\mu(x) < x \tag{20}$$

(recall that $\mu(x^*) = x^*$). Given (20), the median voter at (x^*+1) most prefers a club size of x* and this choice, as it involves x* then being chosen forever must dominate any other dynamic path. At (x^*+2), the median voter most prefers a club size of x* or x*+1. Given SQC, any dynamic path starting above (x^*+2) is dominated by choosing (x^*+2) until the path drops below (x^*+2) and then replicating it. The same applies to paths starting below x*. Thus, the

[7] This is the same as single peakedness but its role here is different because it is still the case that V may not be single-peaked in its first argument.

median voter at (x*+2) always chooses a club size between x* and (x*+2). The same argument applies inductively for all x up to x' and a similar argument then applies below x*.

<u>Proposition 12.</u> **Assume SQC and let x* be an extrinsic steady state. If $\mu(x) < x$ for all x, $x^* < x \leq x'$ then $y^*(x')\varepsilon[x^*,x']$. Similarly, if $\mu(x) > x$ for all x, $x'' \leq x < x^*$ then $y^*(x'')\varepsilon[x'',x^*]$. Under these conditions, the steady state is an attractor.**

Returning to Figure 1, the two extrinsic steady states other than x* will be attractors.

The alternative situation is when $\mu(x) > x$ for all x, $x^* < x \leq x'$ where x* is an extrinsic steady state. If it is optimal to reduce the club size from x' then, as the MVE is monotonic (in the sense of Proposition 3), it will continue to fall and this path will be inferior for the median voter to remaining at x' or, perhaps, to allowing the club size to increase.

<u>Proposition 13</u>. **Assume SQC and let x* be an extrinsic steady state. If $\mu(x) > x$ for all x, $x^* < x \leq x'$ then $y^*(x')\ \varepsilon\ [x',\bar{x}]$. Similarly, if $\mu(x) < x$ for all x, $x'' \leq x < x^*$ then $y^*(x')\varepsilon[0,x'']$. Under these conditions, the steady state is a repeller.**

In Figure 1, x* is a repeller steady state. In Proposition 12 and 13, it is not guaranteed that movement will be towards or away from the steady state: club sizes close to the steady state may be intrinsic steady states. However, let us now examine <u>gradualism</u> where voting involves changes over a number of periods before a steady state is reached. In particular, let us look at the conditions under which the club size changes by the minimal amount (unity) each period until the steady state is attained. Let utilities be of the form

$$u(x',m(x)) = (x'-(x^*+\nu(x-x^*)))^2 \qquad (21)$$

so that

$$\mu(x) = x^* + \nu(x-x^*) \qquad (22)$$

The preferences in (21) may be viewed as an approximation to more general preferences around the extrinsic steady state x*. It is assumed that $\nu < 1$ so that x* is an attractor.

To see when gradualism is likely, assume that $\delta \rightarrow 0$. A median voter will always wish to choose the club size given according to instantaneous utility and the median voter at x*+k will most prefer x*+k-1 if

$$1 - \frac{1}{2k} > \nu > 1 - \frac{3}{2k}. \tag{23}$$

For instance, with k=1 it will be necessary that $\nu < ½$ (a result that is independent of δ) whereas with k = 4 it will be necessary that $\nu \in [5/8, 7/8]$ which is incompatible with $\nu < ½$. Thus, gradualism with minimal changes at each period is likely only over small ranges.

Convergence to a steady state is possible even though jumps may be by discrete amounts exceeding unity. With preferences given by (21), a median voter at x*+k will always prefer x* to (x*+k) if $\nu < ½$: in this case, a jump first to somewhere above x* will be optimal when k is large enough, e.g. when $\nu = 1/5$, the median voter at x*+3 will most prefer the club size to first drop to x*+1 and then it will fall to x* in the subsequent period. When $\nu > ½$, x*+1 will be an intrinsic steady state and then convergence to x* will not occur.

We can summarise the foregoing analysis: if preferences of different median voters are similar ($\nu < 1$) then an extrinsic steady state is a weak attractor; if they are very similar ($\nu < ½$) then convergence always occurs though the transition is in discrete jumps; if they are less similar ($½ < \nu < 1$) then convergence does not occur from some club sizes - there will be intrinsic steady states. When preferences are dissimilar ($\nu > 1$), transitions are away from the steady state.

Finally, we look at transition close to an intrinsic steady state where $\mu(x^*) > x^*$. Consider any club size x between x* and $\mu(x^*)$, $x \in [x^*, \mu(x^*)]$. As preferences exhibit strict increasing differences, $\mu(x) \geq \mu(x^*)$, so the median voter at x will, under SQC, prefer to keep the club at size x rather than let it fall (it may be desirable to increase its size). Median voters above x also prefer a size of x to anything smaller so that no transitions to x* will occur as part of voting equilibria. Now consider an initial club size of x below x*. If $\mu(x) \geq x^*$ then a median voter will prefer a club size of x* to any size below x*. Thus, if some dynamic path starting below x* is preferred to x* then it must be because it involves a movement above x*; however, this would be dominated by moving above x* directly. Finally, if it would be optimal to choose a dynamic path starting above x* then, as dynamic paths are monotonic, this path would also dominate x* for the median voter at x* (Lemma 1), contradicting x* as a steady state. We have thus shown:

<u>Proposition 14</u>. Assume SQC and let x* be an intrinsic steady state with μ(x*) > x* (the case μ(x*) < x* is symmetrically treated). If the initial club size is above x* then it never falls to x*. If the initial club size x is below x* but μ(x) ≥ x* then members will vote for a club size of x*.

Proposition 14 shows that intrinsic steady states have a 'knife-edge' property - they are stable in one direction but unstable in the other. If the parameters of the model were to be subject to unexpected shocks then, with these dynamics, intrinsic steady states could be expected to be broken down over time: consider an unexpected shock which leads x* to lose its status as an intrinsic steady state but μ(x*) > x* both before and after the shock; adjustment in club size will be upwards irrespective of the form of the shock! If, instead, the shocks were expected then the status of intrinsic steady states would be unclear (one specific model with uncertainty is analysed in Roberts (1989.

The foregoing analysis shows that equilibrium transitions can be solved recursively, moving away from steady states. As the choice space, in terms of club sizes that can be chosen, is finite, the equilibrium transition can always be a pure selection from some 'best' set. This allows us to obtain.

<u>Proposition 15</u>. Assume SQC. A Markov Voting Equilibrium exists in pure strategies.

In this result, strict quasi-concavity could be replaced by quasi-concavity. In normal form games, a quasi-concavity assumption on payoffs can be used to show the existence of pure strategy equilibria but the role played by quasi-concavity is wholly different in the present context. To see this, it can be noted that the choice space is discrete so the usual role of quasi-concavity to obtain continuous optimal choices is inapplicable; in addition, the relevant payoff functions in our analysis are the functions V and they do not inherit quasi-concavity from the underlying one-period utility functions.

5. Expansionist Clubs: Non-Rival Public Good Provision

The classic example of a club arises with the provision of excludable and non-rival public goods. Many organisations where membership confers benefits approximate this example irrespective of whether the organization is small-scale like a local sports club or large scale like the European Union and other international organizations.

We will consider a simple model of public good provision and suggest that particular preferences with regard to membership size are implied. Given these implied preferences, we will then investigate equilibrium voting behaviour.

Assume that the extent of provision of some public good is given by q where q, a scalar, denotes quantity, quality, location or other attribute. With the cost of provision C(q) being shared equally among x members, assume that the utility of some member ξ is given by the quasi-linear function

$$U = v(q,\xi) - C(q)/x \qquad (24)$$

Members determine changes in the club size and the variable q. Assume that q is chosen each period by majority vote after the club size for the next period has been determined. Then with a Markov transition rule, y*, the choice of q at time t will be a straightforward static choice by the x_t members. Assume that

$$v_{q\xi} > 0, \qquad (25)$$

$$C_{qq} \geq 0. \qquad (26)$$

Condition (25) is a single-crossing condition which says that individuals with a higher ξ prefer a high level of q. Using the approach outline in section 3, a majority voting equilibrium will exist with q chosen to be the level most preferred by the median voter (or a level between the two most preferred levels of the two median voters when x is even). In the case of x odd, the choice q(x) will satisfy

$$v_q(q(x),m(x)) = \frac{C'}{x} \qquad (27)$$

and utility as a function of membership size will be given by

$$u(x,\xi) = v(q(x),\xi) - C(q(x))/x \qquad (28)$$

Consider u(.,m(x)), the median voter's utility over membership size. Close to x, this may be approximated by a Taylor expansion:

$$u(x+d,m(x)) \approx u(x,m(x)) + [v_q(q(x),m(x)) - C'/x](q(x+d)-q(x)) + \frac{C}{x^2} d \qquad (29)$$

When d is suitably small, and on the assumption that integer values of d are 'suitably small', (29) implies (using (27)) that:

$$u(x+d,m(x)) - u(x,m(x)) \approx \frac{C}{x^2} . d \qquad (30)$$

which tells us that, in terms of single period utility, the median voter gains from a higher membership level and loses from a lower level. Thus

$$\mu(x) > x \qquad (31)$$

and, whatever the club size, the median voter at that club size would most prefer the size to increase. Notice that the argument leading to (31) demonstrates the dominance of the cost sharing aspects of public good provision - there may be other decisions being made by the club but because these are chosen through majority voting, the median voter can ignore the effect of club size on these decisions; in essence we are invoking an envelope condition. Clubs satisfying (31) will be termed expansionist clubs.

We now explore the dynamics of club size for an expansionist club and note first that if (54) holds for all x then all steady states must be intrinsic.[8] Let x* be an intrinsic steady state. We know from Proposition 14 that x* will be approached from below and diverged from above. To be concrete, let

$$\mu(x) = x + k$$
$$u(x,m(\xi)) = -(x-\mu(\xi))^2 \qquad (32)$$

These may be viewed as local approximations around x* to more general functions. The equilibrium voting rule y*(x) may be determined by working recursively backwards from x*.

Let $z_i = (1+\delta+\delta^2+\ldots\ldots+\delta^i)^{-1} = (1-\delta)/(1-\delta^{i+1})$. Then the voting equilibrium is:

I.	$x \in (x^*-kz_0, x^*)$	$y^*(x) = x^*$
II.	$x \in (x^*-k(z_0+z_1), x^*-kz_0)$	$y^*(x) = x + kz_0$
III.	$x \in (x^*-k(z_0+z_1+z_2), x^*-k(z_0+z_1))$	$y^*(x) = x + kz_1$

$$x\varepsilon\ (x*-k\sum_{0}^{j+1} z_i, x*-k\sum_{0}^{j} y_i) \qquad y*(x)=x+kz_j$$

and this holds for all x below x* but above the adjacent intrinsic steady state x** where the process restarts. Figure 2 shows this voting rule.

To understand why this rule will be chosen in equilibrium, consider any x in region I. The median voter does not wish for the club size to go above x* (we know this because the median voter at x* prefers to stay at x*), and the best he can do to raise the club size as close to x+k as possible which involves choosing x*, the steady state. In region II, the median voter knows that he can either keep the club size constant or allow it to rise. The best he can achieve is to choose his most preferred size which will remain for one period before further members are admitted to take the club to x*. Region III is more interesting: here the median voter at x chooses a new club size knowing that this gives direct benefits and then affects the club size that will be chosen in subsequent periods. The optimal

[8] As noted in the last section, a corner steady state at $\bar{x}$ is best interpreted as extrinsic so (31) implies that all interior steady states that are intrinsic.

admission is kz_1 which optimises the trade-off of inter-temporal gains and losses.

Assume that the initial club size is x' in Figure 2. Then the size will pass through four levels before eventually reaching x* in five periods time. When δ is close to unity, there exists an x in region IV which is close to x*-2k where the median voter prefers to keep the club size constant rather than allowing it to rise - there is another intrinsic steady state: there can be four changes in club size but no more. As $\delta \to 0$, the number of regions tends to infinity with no intrinsic steady states existing below x*. In this case, median voters always choose to increase the club size by k and there can be multiple changes. Thus, whatever the level of δ, there are starting positions that give rise to least four changes in club size before a steady state is reached. Intriguingly, apart from the last movement, the magnitude of the change in club size increases with each subsequent movement towards the steady state.

6. Contractionist Clubs: Pure Congestion

As opposed to the clubs examined in the last section, there are clubs where membership confers benefits but exclusivity is preferred. For instance a club may own costlessly a public good which is subject to rivalness - a member desires access to the good but prefers the smallest size club that is compatible with retaining membership. One class of economic examples relates to the ability of a group of workers to extract surplus, e.g. a workers' cooperative, a partnership or a trade union. Workers gain from being a member of the club but the return per worker may decline with of the number of workers. If individual preferences are of this form, what can be said about the dynamic of club size?

Any individual club member has a utility function that declines with membership but is subject to the proviso that the member is not excluded from the club. Thus we have for $x \geq \xi$:

1) $u(x,\xi)$ declines in x

(33)

2) $u(x,\xi) > \underline{u}$

where $\underline{u}$ is the utility of not being a club member.[9] The function u reaches a maximum at x = ξ and the preferred club size of the median voter is m(x):

$$\mu(x) = m(x) < x \tag{34}$$

Clubs where there is a preference for reduction may be termed contractionist clubs. As in the case of expansionist clubs, there are no interior extrinsic steady states - any steady states are intrinsic with transition paths characterized by reductions in club size.

The voting equilibrium can be solved recursively as for the expansionist clubs. The principal difference comes from the possibility of exclusion as club size falls. The equilibrium depends upon the costs of excludability. We look at two cases:

1. <u>Low Exclusion Cost</u>

Let $(1-\delta)u(m(x),m(x))+\delta\underline{u} \geq u(x,m(x))$ for all x.

If x* is a steady state within an MVE, y*, then:

$$y^*(x) = x^*, \quad x \in [x^*,2x^*]$$

$$y^*(x) = \frac{x}{2}, \quad x \geq 2x^*$$

II. <u>High Exclusion Cost</u>

Let $(1-\delta)u(m(x),m(x)) + \delta\underline{u} \leq u(x,m(x))$ for all x.

If x* is a steady state within an MVE, y*, then:

$$y^*(x) = x^*, \quad x \in [x^*,2x^*]$$

$y^*(2x^*) = 2x^*$, i.e. 2x* is a steady state.

To understand these, consider any x between x* and 2x*. The median voter will not wish the club size to fall below x* - the median voter at x* does not want this, by assumption, and, subject to this, x* offers the highest utility and dominates any dynamic path. Thus, there is a single adjustment to a steady state at x*. At 2x* + 1, the median voter can vote for a reduction in the club size, optimally to m(2x*+1), but this will be

[9] When agents are not members, the increasing differences will not be <u>strict</u>. The results of previous sections are not sensitive to this relaxation of strict increasing differences.

followed one period later by a further reduction in the club size when the voter will be excluded. The payoff is:

$$V = u(m(2x^*+1),m(2x^*+1)) + \frac{\delta u}{1-\delta} \quad (35)$$

whereas the best payoff from non-reduction is to retain the club size of 2x*+1 and obtain

$$V = u(2x^*+1,m(2x^*+1))/(1-\delta) \quad (36)$$

If case I applies, (35) exceeds (36) and the reduction strategy is optimal. Now consider 2x*+2. The median voter is excluded within two periods if reduction occurs so the same rules apply. In fact, if the rule applies up to any x,x > 2x*, then the same considerations apply to x - voting for a reduction involves exclusion within two periods. On the other hand, when the conditions for case II apply, the median voter at 2x* prefers to keep the club size at 2x* and this becomes another steady state.[10]

7. Welfare Considerations

The analysis that has been conducted has been concerned entirely with a positive analysis of club decisions. The purpose of this section is to compare these decisions with what would be optimal in a welfare maximising context.

Consider the choice of a club size in the pursuit of maximizing the welfare of current members. With a utilitarian objective and present club size of x*, the objective is to maximize

$$U = \sum_{1}^{x^*} V(x,\xi,y(.)) \quad (37)$$

where y(.) is the utilitarian planner's rule that will be followed in the future. Notice that if the club size changes from x* then the objective will change - the planner's rule will be time-inconsistent (Phelps and Pollak (1968)). Assume that the distribution of discounted utility differences in the club is non-skewed in the sense that average discounted utility

[10] Specific models of democratic trade unions are examined in Roberts (1989)and Layard (1990).

differences is the same as discounted utility differences for some median individual, i.e. for all x,

$$V(x*,m(x*),y) - V(x,m(x*),y) = \frac{1}{x*}\sum_{1}^{x*}[V(x*,\xi,y) - V(x,\xi,y]. \qquad (38)$$

Consider the median voter rule ỹ (.). If (38) is satisfied then, for all x,

$$\sum_{1}^{x*} V(\tilde{y}(x*),\xi,\tilde{y}) \geq \sum_{1}^{x*} V(x,\xi,\tilde{y}) \qquad (39)$$

so the median voter rule and, by implication, the markov voting equilibrium are (time-inconsistent) utilitarian optimal transition rules under non-skewness: the non-optimality of markov voting equilibria can be traced directly to the skewness in the distribution of preferences.

The welfare function (37) fails to be consistent with regard to future choices[11]. A more plausible welfare objective could include both current and potential members of the club and seek to maximize

$$U = \sum_{1}^{\bar{x}} V(x,\xi,y(.)). \qquad (40)$$

A rule y(.) is optimal if

$$\sum_{1}^{x*} V(y(x*),\xi,y) + [\sum_{x*+1}^{\bar{x}} [V(y(x*),\xi,y) - V(x,\xi,y)]] \geq \sum_{1}^{x*} V(x,\xi,y). \qquad (41)$$

The term in brackets is the gain accruing to non-members from the optimal decision at x* as opposed to choosing some other x. Assume that utility is linear in incomes, so that utility is transferable, and that individuals are unaffected by club size if they are not members. Discounted utilities will then only be non-zero for individuals who will become members in the future and, for a perfectly discriminating club, the bracketed term is the difference in

[11] However, as a society may respect the welfare only of the living when decisions about population policy are taken, (37) may be reasonable in such contexts.

fees that could be extracted by choosing y(x*) instead of x with the threat that membership will be denied. Assume that fees are shared between current members. If $\tilde{y}$ (.) is the median voter rule that results from this sharing then, for all x,

$$V(\tilde{y}(x*),m(x*),\tilde{y})+\frac{1}{x*}[\sum_{x*+1}^{\bar{x}}[V(y(x*),\xi,\tilde{y})-V(x,\xi,\tilde{y})]] \geq V(x,m(x*),\tilde{y}) \qquad (42)$$

and if utility differences are non-skewed then (38) in (42) tells us that (41) is satisfied by $\tilde{y}$: the median voter rule and markov voting equilibrium are rules that maximize (40) if there is 1) non-skewness and 2) linear utility in income and perfect extraction of surplus from joining members. Under these restrictive but interpretable conditions, welfare and voting equilibrium transition rules will coincide.

Let us note one implication of this equivalence. Equation (40) is the discounted sum of everybody's utility and the optimal policy will be to choose, immediately, a membership size that maximizes

$$U = \sum_{1}^{\bar{x}} u(x,\xi). \qquad (43)$$

Thus, if the non-skewness assumption is satisfied, the complex dynamic structure of clubs induced by voting can be traced to the imperfect inability to extract fees from new members. Under such conditions, equilibrium is equivalent to the maximisation of a time-inconsistent welfare functions whereas perfect extraction would lead to a simple equilibrium where the steady state is achieved in one step and accords with the maximization of a time consistent welfare function.

8. Concluding Remarks

This paper has examined the structure of voting problems when the electorate is endogenous. This necessitates the examination of voting in a dynamic context which introduces complexities not present in static problems; despite this, it has been shown that dynamic majority voting equilibria exist under conditions no stronger than those that have been used commonly in static problems. As well as addressing the existence issue, it has been shown that majority voting is equivalent to a median voter rule with the relevant median voter endogenous to the process because the electorate itself is endogenous. As the median voter changes through time, voting outcomes embody a 'time-inconsistency' with steady states being reached after passing through several intermediate steps, a dynamic process which is neither forced upon the problem by exogenous constraints like costs of adjustment, nor in the interests of any single voter.

We have investigated and provided a characterization of both the transition paths and steady states that result under majority voting. Of particular importance has been the characterization of intrinsic steady states which are determined by the dynamics of voting rather than by the extrinsic properties of the steady state itself. As the classes of models examined in sections 5 and 6 show, intrinsic steady states are likely to be pervasive. These steady states exhibit a 'knife-edge' stability and would be broken down over time through unexpected shocks to the system. A complete analysis of dynamic voting with (expected) shocks would be valuable.

There are two sets of problems which have been assumed away by assumption in our analysis but warrant further consideration. First, it has been assumed that the ordering by which individuals join or leave clubs is pre-prescribed through a system like seniority. It would be of interest to relax this assumption and the possibility of membership fees provides one mechanism to determine growth of club membership with the individuals who constitute new members being determined by their willingness to pay. Whether or not the condition of increasing differences continues to hold would be a relevant question. Second, the analysis of this paper has shown that existing members of a club may face a dilemma because of their wish to enjoy the benefits of the larger club without suffering from the decisions that will then be taken by the larger club. The two effects could be divorced by considering the possibility of both voting and non-voting members. Without the existence of membership fees, the scales are tipped in favour of existing club members voting to admit only non-voting members. But as majority voting does not necessarily produce 'good' outcomes, the possibility of introducing voting members could, through the decisions in which they are involved, improve the outcome of the voting process to existing members. This topic deserves to be analyzed in greater detail.

APPENDIX

Proposition 1. Consider a dynamic path (y_t) generated by an MVE, y^*: $y_{t+1}=y^*(y_t)$, $t \geq 0$ or by a median voter rule $\tilde{y}$:$y_{t+1} = \tilde{y}(y_t)$. It is impossible that either $y_0 > y_1 \leq y_t$, $t \geq 2$ with $y_1 < y_\tau$ for some $\tau \geq 2$ or $y_0 < y_1 \geq y_t$, $t \geq 2$ with $y_1 > y_\tau$ for some $\tau \geq 2$ (if the rule is stochastic then there are no strictly positive probability realisations of this form).

Proof. Assume that under some MVE, y^*, there is a dynamic path with $y_0 > y_1 \leq y_t$, $t \geq 2$ and $y_1 < y_\tau$ for some $\tau \geq 2$. It will be shown that a contradiction is obtained (the or case is treated similarly). Now,

$$V(y_1,\xi,y^*) = u(y_1,\xi) + \delta V(y^*(y_1),\xi,y^*) \qquad \text{(A.1)}$$

so that

$$V(y_1,\xi,y^*) \lesseqgtr V(y_2,\xi,y^*) \qquad \text{(A.2)}$$

$$\Rightarrow (\sum_2^\infty \delta^t) u(y_1,\xi) \lesseqgtr \sum_2^\infty \delta^t u(y_t,\xi). \qquad \text{(A.3)}$$

Now $y_t \geq y_1$ for all $t \geq 2$, with the inequality being strict for some t, so applying Lemma 1, there must exist ξ^*, ξ^{**}, $\xi^* \leq \xi^{**}$ such that

$$\begin{aligned} \xi \leq \xi^* &\Leftrightarrow V(y_1,\xi,y^*) > V(y_2,\xi,y^*) \\ \xi^* < \xi \leq \xi^{**} &\Leftrightarrow V(y_1,\xi,y^*) = V(y_2,\xi,y^*) \\ \xi^{**} < \xi &\Leftrightarrow V(y_1,\xi,y^*) < V(y_2,\xi,y^*) \end{aligned} \qquad \text{(A.4)}$$

(As the increasing difference condition is strict, it must in fact be the case that either $\xi^{**} = \xi^*$ or $\xi^{**} = \xi^* + 1$). Applying (4), y_1 is chosen at y_0 so that:

$$\xi^* \geq y_0 - \xi^{**} \tag{A.5}$$

and y_2 is chosen at y_1 so:

$$y_1 - \xi^{**} \geq \xi^* \tag{A.6}$$

which combine to give

$$y_1 \geq \xi^* + \xi^{**} \geq y_0 \tag{A.7}$$

which is the contradiction.

To prove the same result for median voter rules, the proof can be replicated with y* replaced by $\tilde{y}$. Instead of (A.5) we have, applying (5), that as y_1 is a median voter choice at y_0:

$$\begin{aligned} &y_0 \text{ odd:} \qquad m(y_0) = (y_0+1)/2 \leq \xi^{**} \\ &y_0 \text{ even:} \qquad m(y_0) = y_0/2 \leq \xi^* \end{aligned} \tag{A.8}$$

(Both $y_0/2$ and $(y_0/2)+1$ cannot be indifferent). Also as y_2 is the median voter choice at y_1:

$$\begin{aligned} &y_1 \text{ odd:} \qquad m(y_1) = (y_1+1)/2 > \xi^* \\ &y_1 \text{ odd:} \qquad m(y_1) = (y_1+1)/2 > \xi^{**} \end{aligned} \tag{A.9}$$

which gives a contradiction for all four permutations as $\xi^{**}=\xi^*$ or $\xi^* +1$. ■

<u>Proposition 3</u>. Let (y_t) be a dynamic path generated by an MVE or a median voter rule. Then the path is monotonic:

$$y_{t+1} \underset{<}{>} y_t \Rightarrow y_\tau \underset{\leq}{\geq} y_{t+1}, \quad \text{for all } t,\tau,\tau>t.$$

<u>Proof</u>. By induction. From Proposition 2, a steady state is reached after a finite number of transitions - the maximum number of changes of state before a steady state is reached. The result is vacuously true for dynamic paths where at t the path is within one transition of a steady state. Assume that it is true for dynamic paths within r transitions of a steady state (with probability unity for stochastic rules) but fails for a path (y_t) which reaches a steady state after (r+1) periods. Then such a path must take the form $y_0>y_1<y_t \geq 2$, or $y_0<y_1>y_t$ $t \geq 2$. Each case is similarly treated and we examine the first. Let $x=y_0$ and $\underline{x}=y_1$ which, by appeal to Proposition 1, provides a contradiction. Thus, all paths within (r+1) transitions of reaching a steady state are monotonic; by induction, the general result

is proven. ■

Proposition 4. If (y_t) and (y'_t) are dynamic paths generated by an MVE or a median voter rule then $y_0 \geq y'_0 \Rightarrow y_t \geq y'_t$ for all $t > 0$.

Proof. By induction on the number of transitions before a steady state is reached, as in Proposition 3. Take dynamic paths generated by MVE transition rules first. If y_0 and y'_0 are steady states then the result is true. Now assume that it is true if pairs of membership levels are within r transitions of a steady state. Let y_0 and y'_0 be within (r+1) transitions of a steady state and assume that $y_0 > y'_0$ (if $y_0 = y'_0$ then the result is obvious) but $y_\tau < y'_\tau$ for some $\tau > 0$. Then $y_t \leq y'_t$ for all $t \geq 1$ by the induction assumption and Lemma 1' implies that there exist $\xi^*, \xi^{**}, \xi^* \leq \xi^{**}$ (and in fact at most unity apart) such that

$$\begin{aligned} \xi \leq \xi^* \quad &\Rightarrow \quad V(y_1,\xi,y^*) > V(y'_1,\xi,y^*) \\ \xi^* < \xi \leq \xi^{**} \quad &\Rightarrow \quad V(y_1,\xi,y^*) = V(y'_1,\xi,y^*) \\ \xi^{**} < \xi \quad &\Rightarrow \quad V(y_1,\xi,y^*) < V(y'_1,\xi,y^*) \end{aligned} \qquad (A.10)$$

From (4) we have

$$\xi^* \geq (y_0 - \xi^{**})$$

and (A.11)

$$\xi^* \leq (y'_0 - \xi^{**})$$

which contradicts $y_0 > y'_0$.

If the dynamic path is generated by a median voter rule then, with ξ^* and ξ^{**} defined by (A.10) under the rule $\tilde{y}$, we have:

$$\begin{aligned} &y_0 \text{ odd:} \quad m(y_0) = (y_0+1)/2 \leq \xi^{**} \\ &y_0 \text{ even:} \quad m(y_0) = y_0/2 \leq \xi^* \\ &y'_0 \text{ odd:} \quad m(y'_0) = (y'_0+1)/2 > \xi^* \\ &y'_0 \text{ even:} \quad m(y'_0) = (y'_0/2)+1 > \xi^{**} \end{aligned} \qquad (A.12)$$

which contradicts $y_0 > y'_0$ for all possible permutations as $\xi^{**} = \xi^*$ or $\xi^* + 1$. ■

Proposition 5. An MVE transition rule y^* is a median voter rule $\tilde{y}$ and vice versa.

Proof. Let $\bar{y}(.)$ be any transition rule satisfying the ordering condition of Proposition 4. The rules y^* and $\tilde{y}$ are two examples. It is sufficient to show that $Y^*(x) = \tilde{Y}(x)$ for all x, where the sets are defined by (4) and (5) under the rule $\tilde{y}$. Take $y \in Y^*(x)$. Then, for any z:

$$\begin{aligned} &\# \{\xi : \xi \leq x \ \&\ V(y,\xi,\bar{y}) > V\{z,\xi,\bar{y})\} \\ \geq &\# \{\xi : \xi \leq x \ \&\ V(y,\xi,\bar{y}) < V\{z,\xi,\bar{y})\} \end{aligned} \tag{A.13}$$

As $\bar{y}$ satisfies the ordering condition, Lemma 1' implies that the two sets in (A.13) take the form $(1,\ldots.,\xi^*)$ and $(\xi^{**}+1,\ldots.x),(\xi^{**} = \xi^*$ or $\xi^*+1)$.
If $y > z$ then (A.13) gives

$$x - \xi^{**} \geq \xi^* \tag{A.14}$$

If x is odd then $m(x) > \xi^*$; if x is even then $x - \xi^{**} \geq x/2$ so $m(x) = x/2 + 1 \geq \xi^{**} + 1$.

In both cases (5) is implied. Repeating the exercise for $z > y$ gives $y \in \tilde{Y}(x)$. Now take any $y \in \tilde{Y}(x)$. Assume that x is odd (the case of x even is dealt with similarly): then, using (5), we have for all z

$$V(y,\frac{x+1}{2},\bar{y}) \geq V(z,\frac{x+1}{2},\bar{y}) \tag{A.15}$$

If $y > z$ then Lemma 1' implies (given that $\bar{y}$ satisfies the ordering condition):

$$\begin{aligned} \xi \leq \xi^* \quad &\Rightarrow V(y,\xi,\bar{y}) < V(z,\xi,\bar{y}) \\ \xi^* < \xi \leq \xi^{**} \quad &\Rightarrow V(y,\xi,\bar{y}) = V(z,\xi,\bar{y}) \end{aligned} \tag{A.16}$$

$\xi > \xi^{**} \qquad\qquad \Rightarrow V(y,\xi,\bar{y}) > V(z,\xi,\bar{y})$

where $\xi^{**}=\xi^*$ or ξ^*+1. (A.15) implies that $\frac{x+1}{2} > \xi^*$ which gives $x-\xi^{**} \geq \xi^*$, or

$$\begin{aligned} &\#\{\xi:\xi \leq x \,\&\, V(y,\xi,\bar{y}) > V(z,\xi,\bar{y})\} \\ &\geq \#\{\xi:\xi \leq x \,\&\, V(y,\xi,\bar{y}) < V(z,\xi,\bar{y})\} \end{aligned} \qquad (A.17)$$

Repeating the exercise for $z>y$ gives the same conclusion so that $y \in Y^*(x)$. We have thus shown that $Y^*(x)=\bar{Y}(x)$ which implies that y^* is a median voter rule under transition rule y^* and $\tilde{y}$ is an MVE under the transition rule $\tilde{y}$, i.e. y^* is a median voter rule and $\tilde{y}$ is an MVE. ■

Proposition 6. A (possibly stochastic) median voting rule $\tilde{y}$ (.) exists.

Proof. We construct a normal form game G with a finite number of players and (pure) strategies. A Nash equilibrium in mixed strategies always exists in such games (Fudenberg and Tirole (1991),Theorem 1.1)). The equilibrium will be shown to be a median voting rule.

The Game G comprises:

1) A set of players $X = \{1,\ldots,\bar{x}\}$
2) A strategy space X for each player.
3) A payoff for each player v_i: if player j chooses strategy s(j) then

$$v_i = \begin{cases} V(s(i),(i+1)/2,s(.)) & \text{i odd} \\ \tfrac{1}{2}\, V(s(i),i/2,s(.)) + V(s(i),(i/2)+1,s(.)) & \text{i even} \end{cases} \qquad (A.18)$$

For this game, let s*(.) Be a Nash equilibrium. Then

$$v(s^*(i),s^*(-i)) \geq v_i(s(i),s^*(-i)) \qquad (A.19)$$

where s(i) is any other feasible strategy and $s^*(-i)=(s^*(1),\ldots s^*(i-1),s^*(i+1),\ldots.s^*(\bar{x}))$.

To complete the argument, it is necessary to examine the difference between a median voter choosing a state conditional on the transition rule which includes that median voter's behaviour in the future, and a median voter choosing a state recognising that he will also make choices in the future (as is implicit in this constructed game).

Fix y(-i), the transition rule chosen at all club sizes other than i. Given a starting state x, let p(t,x) be the probability of a first return to membership level i after t periods and let W(x,ξ,t) be the expected discounted utility of individual ξ up to and including t when first return is at t (p(∞,x) and W(x,ξ,∞) refer to situations where return does not occur). The functions p and W are independent of y(i).

Then,

$$V(x,\xi,(y(i),y(-i))) = \sum_{t=1}^{\infty} p(t,x)[W(x,\xi,t)+\delta^{t+1}V(y(i),\xi,(s(i),s(-i)))]+p(\infty,x)W(x,\xi,\infty)). \qquad \text{(A.20)}$$

To make the arguments of functions clear, this may be written as:

$$V(x,\xi,(y(i),y(-i))) = A(x,\xi) + B(x,\xi)V(y(i),\xi,(y(i),y(-i))). \qquad \text{(A.21)}$$

If s*(i) is the Nash equilibrium strategy then it maximises V for m(i) given that s* (i) is the decision at i and it determines the future transition rule. Thus:

$$V(s^*(i),m(i),(s^*(i),s^*(-1))) \geq V(s(i),\xi,(s(i),s^*(-i))) \qquad \text{(A.22)}$$

Using (29) this becomes:

$$\frac{A(s*(i),m(i))}{1-B(s*(i),m(i))} \geq \frac{A(s(i),m(i))}{1-B(s(i),m(i))} \qquad \text{(A.23)}$$

Now

$$V(s(i),m(i),(s*(i),s*(-i))) = A(s(i),m(i)) + \frac{B(s(i),m(i))A(s*(i),m(i))}{1-B(s*(i),m(i))} \quad (A.24)$$

$$\leq V(s*(i),m(i),s*(i),s*(-i))) \quad (A.25)$$

using (A.21) and (A.23). Thus, if the future transition rule is fixed at (s*(i),s*(-i)) then it is also optimal to choose s*(i) for a present choice.

We are now in a position to reconsider the Nash equilibrium. Take i odd. Then (A.19) reduces to (A.20) with $\xi = (i+1)/2$ so (A.25) implies (5): s*(i) is a median voter rule when membership is of size i. When i is even, either both m(i) = i/2 and m(i) = i/2 + 1 weakly prefer s*(i) to s(i) or one has a strict preference in which case (A.25) holds weakly for both or strictly for one; whatever, (5) is implied which tells us that for all i, s*(i) is a median voter rule. ■

Proposition 15. Assume SQC. A Markov Voting Equilibrium exists in pure strategies.

Proof. We construct a non-stochastic median voter rule $\tilde{y}$. We determine $\mu(x)$ uniquely by assuming, for x even, that μ is the average of the club sizes most preferred by the two median voters; $\mu(x)$ may not be an integer. Partition $X = \{1,\ldots,\bar{x}\}$ into intervals where either $\mu(x) \geq x$ for all x in an interval - an up-interval - or $\mu(x) \leq$ for all x in an interval - a down-interval. Adjacent up-intervals or adjacent down-intervals can be combined into a single interval so that different up-intervals will be separated by down-intervals and vice-versa. Assume that, as a matter of convention, the upper-bound of an interval cannot be set any higher, e.g. if $[x_1,x_2]$ is an up-interval then $\mu(x_2+1)<x_2+1$.

Consider two adjacent intervals, an up-interval $[x_1,x_2]$ and a down-interval $[x_2+1,x_3]$. Then $\mu(x_2) \geq x_2$ and $\mu(x_2+1) < x_2+1$. If x_2 is odd, then these imply that individual $(x_2+1)/2$ most prefers x_2 and $\mu(x_2) = x_2$. If x_2 is even then they imply that both individuals $x_2/2$ and $(x_2/2+1)$ most prefer x_2 and $\mu(x_2) = x_2$. Thus, x_2 is an extrinsic steady state and $\tilde{y}(x_2)=x_2$ is a median voter transition.

Median voter transitions, moving away from x_2, can now be constructed. We will impose the extra restriction that, starting from any x in the interval $[x_1,x_3]$, only club sizes that lie between x and x_2 can be chosen. Later, we show that this restriction can be dropped. We proceed by induction. Assume that non-stochastic median voter transitions can be constructed for all membership sizes in $[x_1, x_3]$ that are within r of x_2 This is true for r=0 ($\tilde{y}(x_2)=x_2$). Now consider x'= x_2 - r-1 (club sizes in the down-interval are treated similarly). The median voter at x' is restricted to choosing a transition from $[x', x_2]$ and the dynamic path generated under the restricted median voter rule will then remain in that interval (and be defined by the non-stochastic rule assumed to exist by the induction hypothesis). The median voter value functions V exist and $\tilde{y}$(x') can be set to a club size that maximises V when x' odd, or maximises V for x'/2 from the set that maximises V for (x'/2)+1, when x' is even. This defines $\tilde{y}$(x'). This completes the induction hypothesis: a restricted median voter rule exists which is non-stochastic for all x in $[x_1, x_2]$. This argument applies to all intervals where an up-interval lies below a down-interval. The only other possibilities are a down-interval of small club sizes $[1, x_1]$, in which case a similar argument applies moving away from a club size of unity (which is a corner steady state) and an up-interval $[x_3, \overline{x}]$, in which case the argument applies moving away from a club size of $\overline{x}$ (which is a corner steady state).

The argument is completed by showing that the restricted median voter transition rule $\tilde{y}$ that has been constructed is also an unrestricted median voter transition rule. Consider any x' and assume that it belongs to an up-interval $[x_1, x_2]$, with a similar argument applying if x' belongs to a down-interval. Thus μ(x') $\geq$ x' and strict increasing differences implies that μ(x') $\leq$ x_2. If the median voter is permitted to choose any club size then a choice above x_2, x", will lead to a dynamic path generated by $\tilde{y}$ which stays in the interval including x" forever. By SQC, this path is worse for the median voter than x_2 being chosen forever - the median voter will not wish to choose a club size exceeding x_2. Similarly, a choice below x' will lead to a dynamic path (y_t) that is always weakly below x_2. By SQC this is dominated for the median voter by a 'tracking' path that stays at x' until the path (y_t) moves above x' and then replicates it - this is a feasible option for the median voter. Thus, a choice of club size below x' is not optimal for the median voter and $\tilde{y}$(x') is an unrestricted median voter transition. This applies for all x' so that $\tilde{y}$ as a non-stochastic median voter rule and, by Proposition 6, $\tilde{y}$ is a non-stochastic majority voting equilibrium.

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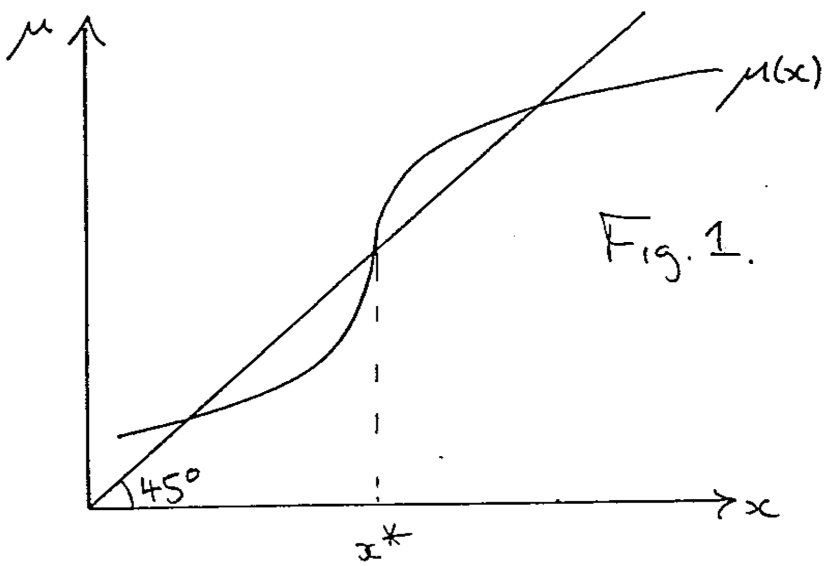

Fig. 1.

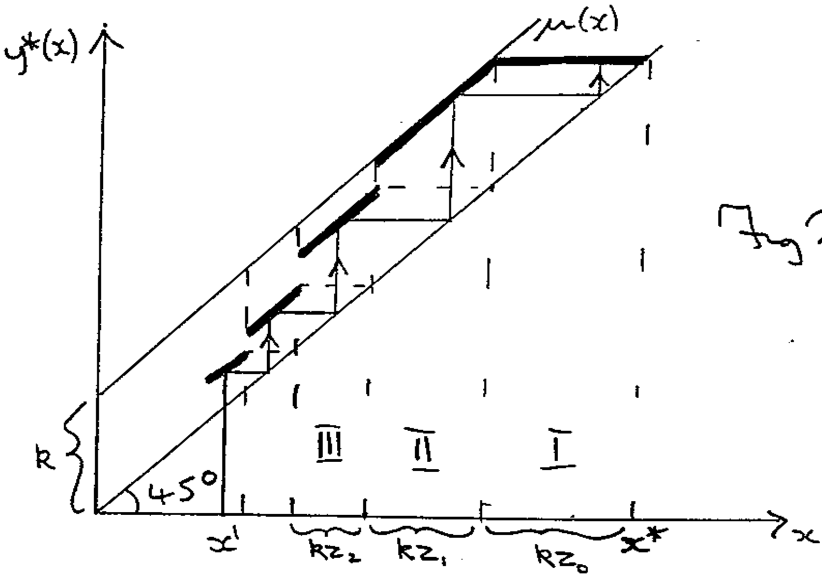

Fig 2.