THE SHORT-RUN APPROACH TO LRMC PRICING FOR MULTIPLE OUTPUTS WITH NONDIFFERENTIABLE COSTS*

by

Anthony Horsley and Andrew J Wrobel
London School of Economics and Political Science

Contents:

Discussion Paper
No. TE/00/393
June 2000

The Suntory Centre
Suntory and Toyota International Centres
for Economics and Related Disciplines
London School of Economics and Political Science
Houghton Street
London WC2A 2AE
Tel.: 020-7405 7686

---

* Financial support of the ESRC is gratefully acknowledged (grant R000232822).

Abstract

Using convex calculus, we extend the Wong-Viner Theorem to nondifferentiable costs by equating the capital inputs' rental prices to their profit-imputed marginal values. Thus extended, the short-run approach to LRMC pricing is applied to peak-load pricing with storage.

Keywords: Wong-Viner theorem; multiple outputs; peak-load pricing; energy storage.

JEL Nos.: D24, D46, D58, L94.

1. Introduction

As taught, the short-run approach to long-run marginal cost (LRMC) pricing is based on the equality of LRMC to SRMC at the optimum of those inputs which are fixed in the short run. But on inspection, both in practice and theoretically—e.g., in multiple-output problems with capacity constraints such as peak-load pricing—the Wong-Viner Theorem turns out to be inapplicable because the relevant cost functions are nondifferentiable. And although the calculus can usually be extended by using a generalised, multi-valued derivative such as the subdifferential ∂C of a convex function C, a direct transcription of the Wong-Viner Theorem fails because it only shows that, with marginal costs formalised as subgradients, every LRMC is an SRMC (i.e., if $p \in \partial_y C_{\mathrm{LR}}$ then $p \in \partial_y C_{\mathrm{SR}}$); whereas what one needs is a result which identifies the circumstances in which, conversely, an SRMC is necessarily an LRMC—i.e., finds an additional condition on a $p \in \partial_y C_{\mathrm{SR}}(y,k)$ to ensure that $p \in \partial_y C_{\mathrm{LR}}(y,r)$. As we show, the required condition is that the rental prices of the fixed inputs be equal to their efficiency rents, defined as *profit*-imputed marginal values—i.e., that $r = \nabla_k \Pi_{\mathrm{SR}}(p,k)$, or that $r \in \partial_k \Pi_{\mathrm{SR}}$ should the ordinary gradient of the operating profit Π_{SR} fail to exist. In other words, the operating profits must cover the capital costs, on the margin and hence also in total.[1] As usual, C_{LR}, C_{SR}, and Π_{SR} denote the long-run cost, short-run cost, and profit, as functions of: the output bundle y and its price system p, the fixed-input quantities k and their prices r. The variable-input prices, w, are fixed.

For a convex technology, this result—that $p \in \partial_y C_{\mathrm{LR}}(y,r)$ if $p \in \partial_y C_{\mathrm{SR}}(y,k)$ and $r \in \partial_k \Pi_{\mathrm{SR}}(p,k)$—is an application of a fundamental principle of convex calculus, viz., the equivalence of the generalised gradients for saddle functions and their bivariate convex counterparts (Theorem 5.3 and Corollary 5.4). It extends the Wong-Viner Theorem to nondifferentiable costs by strengthening its input optimality assumption to the valuation condition $r \in \partial_k \Pi_{\mathrm{SR}}$: this implies that $r \in -\partial_k C_{\mathrm{SR}}$, which is equivalent to $k \in \partial_r C_{\mathrm{LR}}$ (by conjugate duality) and hence also to the optimality of k (by Shephard's Lemma). When C_{SR} *is* differentiable, our result reduces to Wong and Viner's because $\nabla_k \Pi_{\mathrm{SR}} = -\nabla_k C_{\mathrm{SR}}$ in that case. But $\nabla_k \Pi_{\mathrm{SR}}$ can exist also when $\nabla_k C_{\mathrm{SR}}$ does not: indeed, this is so in peak-load pricing (Formulae (6.2) and (6.14)–(6.15)). And even when $\nabla_k \Pi_{\mathrm{SR}}$ fails to exist, $\partial_k \Pi_{\mathrm{SR}}$ will always serve the purpose, whereas $\partial_k C_{\mathrm{SR}}$ will not do: this is because $\partial_k \Pi_{\mathrm{SR}}(p,k) \subseteq -\partial_k C_{\mathrm{SR}}(y,k)$, when $p \in \partial_y C_{\mathrm{SR}}$ (Lemma 5.2).

The extension differs from the original Wong-Viner Theorem not only in its applicability but also in the concepts and methods employed. It uses the SR profit function, and this makes it possible to formulate the assumption in terms of *partial*

[1] With constant returns to scale for the long run, $r = \nabla_k \Pi_{\mathrm{SR}}$ implies that $r \cdot k = \Pi_{\mathrm{SR}}(k)$ by Euler's Theorem. In conjunction with $p \in \partial_y C_{\mathrm{SR}}$, it also implies that $C_{\mathrm{LR}}(y) = p \cdot y$, i.e., LR cost recovery from sales at p (since this follows from $p \in \partial_y C_{\mathrm{LR}}$). But aggregate cost recovery cannot, even when k is optimal, replace the assumption that $r = \nabla_k \Pi_{\mathrm{SR}}$ (except when k is one-dimensional).

subdifferentials, as $p \in \partial_y C_{\mathrm{SR}}$ and $r \in \partial_k \Pi_{\mathrm{SR}}$. There is, of course, an equivalent condition in terms of the SR cost alone—viz., $(p, -r) \in \partial_{y,k} C_{\mathrm{SR}}$—but it does not lend itself to further analysis because the joint subdifferential $\partial_{y,k}$ does *not* factorise into the Cartesian product of partials ∂_y and ∂_k. This is a major difference between a bivariate convex function such as C_{SR} and a saddle function such as Π_{SR} (which is convex in p and concave in k). The extension is thus based on the duality between biconvex and saddle functions, and not merely on the definitional "envelope" relationship (3.1) between the LR and SR costs (which suffices in the differentiable case but not generally, since it yields only that $\partial_y C_{\mathrm{LR}} \subseteq \partial_y C_{\mathrm{SR}}$ without equality unless $\nabla_y C_{\mathrm{SR}}$ exists).

Dispensing with differentiability means that the SR approach is extended to the case of complementary fixed and variable inputs, which cannot be substituted for each other. By contrast, the original Wong-Viner Theorem relies on input substitution: this ensures that, at the optimum, an extra unit of output can be produced as cheaply by varying a particular input as by varying any other or indeed all inputs (so SRMC equals LRMC). This idea obviously fails with complementary inputs: for example, if the output is at the capacity constraint, it cannot be increased in the short run at all. Convex analysis copes with this case by regarding the SR cost of an infeasible output as infinite. So, at full capacity, the SRMC is partly indeterminate: in addition to the unit variable cost it includes a "capacity premium", which cannot be quantified in pure SR cost calculations, but *is* quantified by the valuation condition ($\nabla_k \Pi_{\mathrm{SR}} = r$).

The simplest case of this condition can be found in Boiteux's work on thermal electricity generation with constant coefficients, as expounded by Drèze [3, pp. 8–17, esp. (8) and (11)]. In the one-station case there is a single capacity k, and an SRMC can exceed the unit running cost w by an indeterminate capacity premium $\kappa(t)$ whenever the output rate $y(t)$ equals k. In the long run the total capacity charge over the cycle must equal the unit capacity cost r (so that the plant breaks even), i.e., an SRMC price function $p(t) = w + \kappa(t)$ is an LRMC if $\int_0^T \kappa(t)\,\mathrm{d}t = r$. This is a special case of our result (Corollary 5.4) because $\Pi_{\mathrm{SR}}(p,k) = k \int_0^T (p(t) - w)^+ \,\mathrm{d}t$.[2] It readily extends to the case of independent fixed inputs, in which the technology consists of production techniques using a single fixed input each. But our analysis applies also to technologies with interdependent capacities; and as an example we show how it allows the inclusion of pumped storage in the SR approach to peak-load pricing for electricity. See also [7] for a similar application to hydro-thermal generation.

2. Technology and the commodity and price spaces

The technology is taken to produce an output bundle y from a fixed-input bundle k and a variable-input bundle v, with constant returns to scale in the long run. The corresponding price systems for outputs and for the fixed and the variable inputs are denoted by p, r and w. One way to specify a technology is by its LR production set

[2] When $y(t) = 0$, $p(t) - w$ may be negative but $(p(t) - w)^+ = 0$.

$\mathbb{Y}$, which consists of the feasible input-output bundles $(y, -k, -v)$. Equivalent dual descriptions are the short-run (variable) cost function $C_{\mathrm{SR}}(y, k, w)$, the long-run cost $C_{\mathrm{LR}}(y, r, w)$, and the operating or short-run profit $\Pi_{\mathrm{SR}}(p, k, w)$.

The *commodity spaces* for outputs, fixed and variable inputs are denoted by Y, K and V. Each of these is taken to be a *dual Banach lattice*, $(Y, \|\cdot\|, \leq)$, etc. The Banach dual of Y is denoted by $(Y^*, \|\cdot\|^*, \leq)$, and the Banach predual of Y is $(Y', \|\cdot\|', \leq)$, with $Y' \subseteq Y^* = Y'^{**}$. In general $Y' \neq Y^*$; and either space can serve as the price space, depending on the price representation required. The nonnegative cone in Y is denoted by Y_+, etc. For Banach lattices, see, e.g., [2, XV.12].

In our application to continuous-time peak-load pricing (Section 6) the input spaces (K and V) are finite-dimensional, but the output space Y is $L^\infty[0, T]$, the commodity space of all essentially bounded functions, in which case Y' equals $L^1[0, T]$, the price space of all integrable functions on the interval $[0, T]$ of the real line $\mathbb{R}$. The larger price space $L^{\infty *}$ is also of interest, and the lack of a tractable mathematical representation can be side-stepped when the equilibrium allocation lies in the space of continuous functions $\mathcal{C}[0, T] \subset L^\infty[0, T]$. Then the restriction, to $\mathcal{C}$, of a linear functional $p \in L^{\infty *}$ has a Riesz representation by a *countably* additive measure. This can have a singular part as well as a density part: for example, Dirac measures are needed to represent capacity charges in the case of point peaks—see [5].

Given any $w \in V'_+$, the SR cost

$$C_{\mathrm{SR}}(y, k) := \inf_v \{\langle w, v\rangle : (y, -k, -v) \in \mathbb{Y}\} \tag{2.1}$$

is taken to be a (jointly) convex and weakly* lower semicontinuous function from $Y \times K$ into $\mathbb{R}_+ \cup \{+\infty\}$ which is nonincreasing in $k \in K$, and nondecreasing in y on the set

$$\mathrm{proj}_Y(\mathbb{Y}) := \{y \in Y : \exists (k, v) \ \ (y, -k, -v) \in \mathbb{Y}\}.$$

Comments:

1. The SR cost is $+\infty$ if the output cannot be produced, because of capacity constraints or for other reasons. The set $\mathrm{proj}_Y(\mathbb{Y})$ captures any constraints on the output other than input scarcity. This set need not be comprehensive downwards unless unlimited free disposal of output is assumed. For example, in Section 6, the projection of the storage technology set $\mathbb{Y}_{\mathrm{PS}}$ onto the output space $L^\infty[0, T]$ is $\left\{y \in L^\infty : \int_0^T y(t)\,\mathrm{d}t = 0\right\}$. Unlike the inputs $k \geq 0$ and $v \geq 0$, the "output" bundle can in general be signed: $y = y^+ - y^-$, with $y^\pm$ denoting the nonnegative and nonpositive parts. This is convenient when y represents a single but dated good, and the dated commodities cannot *a priori* be classed as net inputs or net outputs.
2. See [4] for conditions on the production set $\mathbb{Y}$ which guarantee the required continuity and monotonicity properties of C_{SR}.

3. Conjugacy relationships between cost and profit functions

With a variable-input price system $w \in V'_+$ fixed, costs and profit are viewed as functions of the two dual pairs of vector variables (y, p) and (k, r). Both $(p, k) \mapsto \Pi_{\mathrm{SR}}(p, k)$ and $(y, r) \mapsto C_{\mathrm{LR}}(y, r)$ are saddle (convex-concave) functions, derived as partial conjugates from the jointly convex function $(y, k) \mapsto C_{\mathrm{SR}}(y, k)$. More precisely, $\Pi_{\mathrm{SR}}(\cdot, k)$ is the conjugate of $C_{\mathrm{SR}}(\cdot, k)$ in the convex sense, whilst $C_{\mathrm{LR}}(y, \cdot)$ is the conjugate of $-C_{\mathrm{SR}}(y, \cdot)$ in the concave sense: for $r \in K^*_+$ and $p \in Y^*$,

$$C_{\mathrm{LR}}(y, r) := \inf_k \{\langle r, k\rangle + C_{\mathrm{SR}}(y, k)\} \tag{3.1}$$

$$\Pi_{\mathrm{SR}}(p, k) := \sup_y \{\langle p, y\rangle - C_{\mathrm{SR}}(y, k)\}. \tag{3.2}$$

Each of these definitional relationships is next inverted to represent C_{SR} as a partial conjugate.

Lemma 3.1. $C_{\mathrm{SR}}(y, k) = \sup_r \{C_{\mathrm{LR}}(y, r) - \langle r, k\rangle : r \in K^*_+\}$. *The supremum remains the same when taken over* $r \in K'$ *instead of* K^*.[3]

Lemma 3.2. $C_{\mathrm{SR}}(y, k) = \sup_p \{\langle p, y\rangle - \Pi_{\mathrm{SR}}(p, k) : p \in Y^*\}$. *The supremum remains the same when taken over* $p \in Y'$ *instead of* Y^*.

4. Shephard-Hotelling Lemmas

For a conjugate pair of functions, the subdifferential correspondences are inverses of each other. Furthermore, the subdifferential of the one function equals the set of points realising the maximum (or minimum) that defines this function as the conjugate of the other. Applied to the relevant partial subdifferentials of cost or profit, this yields the Shephard-Hotelling Lemmas, which are spelt out next (along with their dual versions). The set of all fixed-input bundles that minimise the LR cost is denoted by $\check{K}(y, r)$. Similarly $\hat{Y}(p, k)$ consists of all the output bundles that maximise SR profit.

For the infinite-dimensional case we adopt the algebraic concept of the sub- or super-differential ∂W of a convex or concave function W on a vector space Y. So $\partial W(y)$ is in general a (convex) subset of the algebraic dual of Y, which is larger than the norm-dual Y^*. But actually $\partial W(y) \subset Y^*$ if W is norm-continuous or monotone (and Y is a Banach lattice). So $-\partial_k C_{\mathrm{SR}}(y, k) \subseteq K^*_+$ and $\partial_y C_{\mathrm{SR}}(y, k) \subseteq Y^*_+$ (with free disposal); and similarly $\partial_k \Pi_{\mathrm{SR}}(p, k) \subseteq K^*_+$ and $\partial_y C_{\mathrm{LR}}(y, r) \subseteq Y^*_+$.

Corollary 4.1 (Shephard)**.** *The following conditions are equivalent:*

1. $k \in \check{K}(y, r)$, *i.e., k yields the infimum in (3.1).*
2. $k \in \partial_r C_{\mathrm{LR}}(y, r) \cap K$.
3. $-r \in \partial_k C_{\mathrm{SR}}(y, k)$.
4. r *yields the supremum in Lemma 3.1.*

[3] This is because C_{SR} is weakly* (and not only weakly) l.s.c. in k.

Corollary 4.2 (Hotelling). *The following conditions are equivalent:*

1. $y \in \hat{Y}(p,k)$, *i.e., y yields the supremum in (3.2).*
2. $y \in \partial_p \Pi_{\mathrm{SR}}(p,k) \cap Y$.
3. $p \in \partial_y C_{\mathrm{SR}}(y,k)$.
4. *p yields the supremum in Lemma 3.2.*

Proof. See, e.g., [9, Corollary 12A] or [1, 4.4.4 and 4.4.5]. ∎

5. Extended Wong-Viner Theorem

For a pair of functions partially conjugate to each other, the partial subdifferentials in their common, non-conjugated argument are related by an inclusion which is generally strict. Applied to $\partial_y C_{\mathrm{SR}}$ and $\partial_y C_{\mathrm{LR}}$, this merely spells out the difficulty: C_{SR} is typically "less smooth" than C_{LR}. But when the same result is applied to $\partial_k C_{\mathrm{SR}}$ and $\partial_k \Pi_{\mathrm{SR}}$, a strict inclusion means that the SR profit is "smoother" than the SR cost (so the use of Π_{SR} can help).

Lemma 5.1. *If k and r are mutually optimal (given y)—i.e., if (k,r) meets one of the equivalent conditions of Corollary 4.1—then $\partial_y C_{\mathrm{LR}}(y,r) \subseteq \partial_y C_{\mathrm{SR}}(y,k)$.*

Lemma 5.2. *If y and p are mutually optimal (given k)—i.e., if (y,p) meets one of the equivalent conditions of Corollary 4.2—then $\partial_k \Pi_{\mathrm{SR}}(p,k) \subseteq -\partial_k C_{\mathrm{SR}}(y,k)$.*

For example, in the peak-load pricing problem the gradient $\nabla_k \Pi_{\mathrm{SR}}$ exists, i.e., $\partial_k \Pi_{\mathrm{SR}}$ is a singleton whereas $\partial_k C_{\mathrm{SR}}$ is an unbounded set (Section 6).

The *joint* subdifferential $\partial_{y,k}$ of the biconvex function C_{SR} is equivalent—through a permutation of the four variables in the correspondence $(y,k) \mapsto (p,-r) \in \partial_{y,k} C_{\mathrm{SR}}$—to the Cartesian product of the *partial* sub/super-differentials for either of the saddle functions C_{LR} and Π_{SR}.

Theorem 5.3. *For every $(y,p;k,r) \in Y \times Y^* \times K_+ \times K_+^*$, the following conditions are equivalent:*

1. $(p,-r) \in \partial_{y,k} C_{\mathrm{SR}}(y,k)$.
2. $(p,k) \in \partial_y C_{\mathrm{LR}}(y,r) \times \partial_r C_{\mathrm{LR}}(y,r)$.
3. $(y,r) \in \partial_p \Pi_{\mathrm{SR}}(p,k) \times \partial_k \Pi_{\mathrm{SR}}(p,k)$.

Proof. See, e.g., [1, 4.4.14]. ∎

There is no equivalent in terms of the partials of C_{SR} alone: SRMC pricing ($p \in \partial_y C_{\mathrm{SR}}$) and input optimality ($r \in -\partial_k C_{\mathrm{SR}}$) do *not* imply that $(p,-r) \in \partial_{y,k} C_{\mathrm{SR}}$. But using $\partial_k \Pi_{\mathrm{SR}}$ instead of $-\partial_k C_{\mathrm{SR}}$ does give an equivalent condition.

Corollary 5.4 (Extended Wong-Viner Theorem). *For every $(y,p;k,r) \in Y \times Y^* \times K_+ \times K_+^*$, if $p \in \partial_y C_{\mathrm{SR}}(y,k)$ and $r \in \partial_k \Pi_{\mathrm{SR}}(p,k)$, then $p \in \partial_y C_{\mathrm{LR}}(y,r)$ and $k \in \partial_r C_{\mathrm{LR}}(y,r)$; and vice versa.*

Proof. By Corollary 4.2, the conjunction of $p \in \partial_y C_{\mathrm{SR}}$ and $r \in \partial_k \Pi_{\mathrm{SR}}$ is equivalent to Condition 3, and therefore also to Condition 2, of Theorem 5.3. ∎

6. Peak-load pricing with storage by the short-run approach

The preceding analysis is next applied to electricity supply from thermal generation with pumped storage. Unlike the *purely* thermal case, a direct implementation of the long-run solution with storage is hampered by the lack of explicit formulae for either the LRMC or the optimal plant system. However, the short-run problems are tractable; and the relevant results of [6] on profit-based valuation and plant operation are summarised here and "fed into" the extended Wong-Viner Theorem.

6.1. **Thermal technology of electricity generation: plant valuation and operation, and short-run marginal costs.** A multi-station technology of thermal electricity generation is a finite set Θ of techniques corresponding to the various types of thermal generating station, each with a capacity cost and a running cost. The LR production set of technique $\theta \in \Theta$ is

$$\mathbb{Y}_\theta := \left\{ (y_\theta; -k_\theta, -v_\theta) \in L^\infty_+ [0,T] \times \mathbb{R}^2_- : 0 \le y_\theta \le k_\theta, \int_0^T y_\theta (t) \, \mathrm{d}t \le v_\theta \right\}, \tag{6.1}$$

where: T is the length of a cycle, $y_\theta (t)$ is the output rate at any time $t \in [0,T]$ from the generating capacity k_θ of type θ, and v_θ is the fuel input of type θ. Whilst capacity and output rate are measured in units of power (kW), fuel is measured in energy units (kWh), and it can simply be measured in terms of generated electricity (on the assumption that different types of station use different fuels). Then the price of fuel for station type θ is the same as the station's unit variable cost w^θ (in \$/kWh).

The rental values of the thermal capacities, $k_{\mathrm{Th}} = (k_\theta)_{\theta \in \Theta}$, can be calculated from explicit formulae for the optimal output and the operating profit $\Pi^{\mathrm{Th}}_{\mathrm{SR}}$, which is a linear function of k_{Th} (abbreviated to k). Under the simplifying assumptions (of fixed coefficients, no start-up or shutdown costs and no transmission constraints), profit-maximising operation takes, essentially, the "bang-bang" form spelt out below.

With $\Theta = \{1, 2\}$ for simplicity, the SR profit is $\Pi^{\mathrm{Th}}_{\mathrm{SR}} (p, k) = \Pi^1_{\mathrm{SR}} (p, k_1) + \Pi^2_{\mathrm{SR}} (p, k_2)$, where $\Pi^\theta_{\mathrm{SR}} (p, k_\theta)$ is the SR profit of technique θ. Therefore, with $p \in L^1 [0,T]$ denoting a time-of-use (TOU) electricity price (in \$/kWh), the unit rent of a thermal station of type θ is

$$\frac{\partial \Pi^{\mathrm{Th}}_{\mathrm{SR}}}{\partial k_\theta} (p, k, w) = \frac{\partial \Pi^\theta_{\mathrm{SR}}}{\partial k_\theta} \left(p, k_\theta, w^\theta\right) = \frac{\Pi^\theta_{\mathrm{SR}}}{k_\theta} = \int_0^T \left(p(t) - w^\theta\right)^+ \mathrm{d}t, \tag{6.2}$$

in \$/kW. This is because the SR profit-maximising output is $y_\theta (t) = k_\theta$ if $p(t) > w^\theta$ and $y_\theta (t) = 0$ if $p(t) < w^\theta$.

The SR cost function of generating an output $y_\theta \in L^\infty_+ [0,T]$ from a station of type θ is

$$C^\theta_{\mathrm{SR}} \left(y_\theta; k_\theta, w^\theta\right) = \begin{cases} w^\theta \int_0^T y_\theta (t) \, \mathrm{d}t & \text{if } 0 \le y_\theta \le k_\theta \\ +\infty & \text{otherwise} \end{cases} . \tag{6.3}$$

This is obviously nondifferentiable at every (y_θ, k_θ) with $k_\theta = \operatorname{ess\,sup}_{t \in [0,T]} y_\theta (t)$.

For the rest of the analysis it can be assumed that $w^1 < w^2$. The SR cost of generating an output y from a system $k = (k_1, k_2)$ is the convex integral functional

$$C_{\mathrm{SR}}^{\mathrm{Th}}(y, k, w) = \int_0^T c_{\mathrm{SR}}(y(t), k, w)\, \mathrm{d}t, \tag{6.4}$$

where, with 1_A denoting the 0-1 indicator of the set A,

$$\begin{aligned} c_{\mathrm{SR}}(\mathsf{y}, k, w) &:= \int_0^{\mathsf{y}} \left(w^1 1_{[0,k_1]}(\mathsf{y}) + w^2 1_{[k_1, k_1+k_2]}(\mathsf{y})\right) \mathrm{d}\mathsf{y} \\ &= w^1 \mathsf{y} + \left(w^2 - w^1\right)(\mathsf{y} - k_1)^+ \end{aligned} \tag{6.5}$$

if $0 \le \mathsf{y} \le k_1 + k_2$ (with $c_{\mathrm{SR}} = +\infty$ otherwise). The integrand c_{SR} is the instantaneous SR cost per unit time (in \$/h); and it is an increasing, convex and piecewise linear function of $\mathsf{y} \in \mathbb{R}_+$, with $c_{\mathrm{SR}}(0) = 0$. The thermal SRMC, as a function of time over the cycle, is simply a trajectory of the instantaneous SRMC: if y lies between 0 and $k_1 + k_2$, then $p \in \partial_y C_{\mathrm{SR}}^{\mathrm{Th}}(y, k) \cap L^1$ if and only if $\int_0^T |p(t)|\, \mathrm{d}t < +\infty$ and $p(t) \in \partial_{\mathsf{y}} c_{\mathrm{SR}}(y(t), k, w)$ for almost every $t \in [0, T]$. When $k_1 > 0$ and $k_2 > 0$,[4] in the two-station model with $w^1 < w^2$, this means by (6.5) that $p \in \partial_y C_{\mathrm{SR}}^{\mathrm{Th}}(y, k) \cap L^1[0, T]$ if and only if $p \in L^1$ and, for a.e. t,

$$p(t) \in \partial_{\mathsf{y}} c_{\mathrm{SR}}(y(t), k) = \begin{cases} (-\infty, w^1] & \text{if } y(t) = 0 \\ \{w^1\} & \text{if } y(t) \in (0, k_1) \\ [w^1, w^2] & \text{if } y(t) = k_1 \\ \{w^2\} & \text{if } y(t) \in (k_1, k_1 + k_2) \\ [w^2, +\infty) & \text{if } y(t) = k_1 + k_2 \end{cases}. \tag{6.6}$$

Comment: With a finite set Θ, the SRMC curve—the graph (in $\mathbb{R}^2$) of the correspondence $\mathsf{y} \mapsto \partial c(\mathsf{y})$ is a "right-angled" broken line. In a model with a "continuum" of types of station it is a general nondecreasing curve, but the continuum model cannot make C_{SR} differentiable: the SRC curve will still have a kink at the peak and, typically, also offpeak kinks [7, Remark 4].

6.2. **Pumped-storage technology: plant valuation and operation.** In pumped storage the stock is an intermediate good, viz., a storable form of energy produced from electricity. The outflow from the reservoir, $-\dot{s}(t) = -\mathrm{d}s/\mathrm{d}t$, is a signed, bounded function of time in the cycle, $t \in [0, T]$. Energy is moved in and out of storage with a converter, which is taken to be perfectly efficient and symmetrically reversible: this means that in a unit time a unit converter can either turn a unit of the marketed good (electricity) into a unit of the stocked intermediate good (a storable form of energy), or *vice versa*.[5] On this simplifying assumption, $-\dot{s}(t)$ equals the net output rate for the good, $y(t) = (y^+ - y^-)(t)$. The converter's capacity is denoted by k_{Co} (measured in kW). The reservoir's capacity is k_{St} (in kWh). There is *no* variable input, i.e., the

[4] When k is not strictly positive, obvious changes are needed in (6.6).

[5] See [6] for the case of imperfect conversion with a round-trip conversion efficiency $\eta < 1$.

stock can be held in storage at no running cost (or loss of stock). Formally, the LR production set for pumped storage is

$$\mathbb{Y}_{\mathrm{PS}} := \left\{ (y, -k_{\mathrm{St}}, -k_{\mathrm{Co}}) \in L^{\infty} \times \mathbb{R}^2_{-} : -k_{\mathrm{Co}} \le y \le k_{\mathrm{Co}}, \int_0^T y(t)\,\mathrm{d}t = 0 \right.$$
$$\left. \text{and } \exists \mathsf{s} \in \mathbb{R}\ \forall t \in [0,T]\ \ 0 \le s(t) := \mathsf{s} - \int_0^t y(\tau)\,\mathrm{d}\tau \le k_{\mathrm{St}} \right\}.$$

The profit-maximising operation problem is to maximise $\langle p, y \rangle$ over y subject to $(y, -k_{\mathrm{PS}}) \in \mathbb{Y}_{\mathrm{PS}}$, where $k_{\mathrm{PS}} = (k_{\mathrm{St}}, k_{\mathrm{Co}})$ means the two capacities of a storage plant (and p is a given TOU electricity tariff). This problem can be formulated as the following doubly infinite linear programme, in which p is taken to be a continuous function on $[0,T]$ with $p(0) = p(T)$.

(6.7) Given $(p, k_{\mathrm{St}}, k_{\mathrm{Co}}) \in \mathcal{C}^{\mathrm{Per}}[0,T] \times \mathbb{R}^2_{++}$

(6.8) maximise $\int_0^T p(t)\, y(t)\,\mathrm{d}t$ over $y \in L^{\infty}[0,T]$ and $\mathsf{s} \in \mathbb{R}$

(6.9) subject to: $0 \le s(t) := \mathsf{s} - \int_0^t y(\tau)\,\mathrm{d}\tau \le k_{\mathrm{St}}$ for every t

(6.10) $-k_{\mathrm{Co}} \le y(t) \le k_{\mathrm{Co}}$ for almost every t

(6.11) $\int_0^T y(t)\,\mathrm{d}t = 0.$

The optimal value of the primal programme (6.7)–(6.11) is $\Pi^{\mathrm{PS}}_{\mathrm{SR}}(p, k_{\mathrm{PS}})$, the SR profit of the pumped-storage plant. It is sublinear in $k_{\mathrm{PS}} = (k_{\mathrm{St}}, k_{\mathrm{Co}})$, but not linear. Unlike the case of $\Pi^{\mathrm{Th}}_{\mathrm{SR}}$, no explicit formulae for $\Pi^{\mathrm{PS}}_{\mathrm{SR}}$ are available; and both rental valuation and optimal operation of a pumped-storage plant are best approached through the dual programme.

Originally the dual is also an LP, and it consists in shadow pricing the fixed resources k_{PS} to minimise their total value. By expressing the unit values of the reservoir and converter capacities in terms of a shadow price ψ for the energy stock (and in terms of p), the dual is next reformulated as a convex programme for optimal stock valuation, i.e., for finding a TOU stock price function ψ that minimises the plant's value: see [6] for details. The dual decision variable ψ is a function of bounded variation; and ψ can be taken to be periodically continuous on $[0,T]$ when p is. Such ψ's form the space $\mathrm{CBV}^{\mathrm{Per}}[0,T]$. The *total positive variation* (a.k.a. upper variation) of ψ is denoted by $\mathrm{Var}^{+}(\psi)$; informally, this is the sum of all rises of ψ. In these terms, the dual programme is:

(6.12) Given $(p, k_{\mathrm{St}}, k_{\mathrm{Co}}) \in \mathcal{C}^{\mathrm{Per}}[0,T] \times \mathbb{R}^2_{++}$

(6.13) minimise $k_{\mathrm{St}}\,\mathrm{Var}^{+}(\psi) + k_{\mathrm{Co}} \int_0^T |p(t) - \psi(t)|\,\mathrm{d}t$ over $\psi \in \mathrm{CBV}^{\mathrm{Per}}[0,T]$.

It has a *unique* solution, denoted by $\hat{\psi}(p, k_{\mathrm{PS}})$. It follows that $\Pi_{\mathrm{SR}}^{\mathrm{PS}}$ is differentiable in k_{PS}, and the unit rents of the reservoir and of the converter are

$$\frac{\partial \Pi_{\mathrm{SR}}^{\mathrm{PS}}}{\partial k_{\mathrm{St}}}(p, k_{\mathrm{St}}, k_{\mathrm{Co}}) = \mathrm{Var}_{\mathrm{c}}^{+}\left(\hat{\psi}(p, k_{\mathrm{PS}})\right) \tag{6.14}$$

$$\frac{\partial \Pi_{\mathrm{SR}}^{\mathrm{PS}}}{\partial k_{\mathrm{Co}}}(p, k_{\mathrm{St}}, k_{\mathrm{Co}}) = \int_0^T \left| p(t) - \hat{\psi}(p, k_{\mathrm{PS}})(t) \right| \mathrm{d}t. \tag{6.15}$$

Comments:

1. The two interdependent capacities are perfect complements (in the sense that the flow y to be generated from storage fully determines the capacity requirements: see [6] or [4] for explicit formulae. It is noteworthy that separate values can be imputed to such capacities (i.e., $\nabla_k \Pi_{\mathrm{SR}}^{\mathrm{PS}}(p, k_{\mathrm{PS}})$ exists) if $p \in \mathcal{C}[0, T]$.
2. The above form of the dual is derived from a more general formulation with an arbitrary (integrable) price function $p \in L^1[0, T]$, in which case the dual variable ψ must range over $\mathrm{BV}(0, T)$, the space of all functions of bounded variation on $(0, T)$, and $\mathrm{Var}^{+}(\psi)$ must be replaced by $\mathrm{Var}_{\mathrm{c}}^{+}(\psi) := \mathrm{Var}^{+}(\psi) + (\psi(0+) - \psi(T-))^{+}$. The simplification to (6.13) is made possible by a number of results in [6], viz., that the dual solution $\hat{\psi}$ is unique and belongs to $\mathcal{C}[0, T]$ if $p \in \mathcal{C}[0, T]$, and that $\hat{\psi}(0) = \hat{\psi}(T)$ if additionally $p(0) = p(T)$. When $p \notin \mathcal{C}[0, T]$, dual solutions can be nonunique (and then $\nabla_k \Pi_{\mathrm{SR}}^{\mathrm{PS}}(p, k_{\mathrm{PS}})$ fails to exist).

The primal problem (6.7)–(6.11) has a solution for any $p \in L^1[0, T]$. If p has no plateaux (i.e., meas $\{t : p(t) = \mathsf{p}\} = 0$ for every $\mathsf{p} \in \mathbb{R}$), then there is a *unique* solution $\hat{y}_{\mathrm{PS}}(p, k_{\mathrm{PS}})$. When additionally $p \in \mathcal{C}^{\mathrm{Per}}[0, T]$, the unique dual solution $\hat{\psi}$ determines $\hat{y}_{\mathrm{PS}}$ by a "bang-coast-bang" formula: $\hat{y}_{\mathrm{PS}}(t)$ equals k_{Co}, 0 or $-k_{\mathrm{Co}}$ if, respectively, $p(t) > \hat{\psi}(t)$, $p(t) = \hat{\psi}(t)$ or $p(t) < \hat{\psi}(t)$.

So if the TOU electricity tariff p is both continuous and plateau-less, then the two programmes, of pumped-storage plant operation and of shadow pricing of stock, have unique solutions $\hat{y}_{\mathrm{PS}}$ and $\hat{\psi}$.[6]

6.3. **LRMC pricing by the SR approach with generation and storage.** The system we consider consists of thermal plants (of types $\theta = 1, 2$) and one pumped-storage plant; and our objective is to give a set of conditions that involve only the SR functions and ensure that:

1. p is an LRMC electricity tariff, for a system output y_{ThPS}, based on the input prices, viz., on the thermal stations' unit fuel costs (w^1, w^2) and on the unit capacity costs of the thermal stations, the reservoir and the converter

$$r^{\mathrm{ThPS}} = \left(r^{\mathrm{Th}}, r^{\mathrm{PS}}\right) = \left(r^1, r^2; r^{\mathrm{St}}, r^{\mathrm{Co}}\right).$$

[6] In [6] we also show how to find $\hat{y}_{\mathrm{PS}}$ and $\hat{\psi}$ in terms of (p, k_{PS}) when p is piecewise strictly monotone on $[0, T]$.

2. For the output y_{ThPS}, the generation-and-storage system

$$k_{\text{ThPS}} = (k_{\text{Th}}, k_{\text{PS}}) = (k_1, k_2; k_{\text{St}}, k_{\text{Co}})$$

is optimal.
3. y_{ThPS} is split optimally (i.e., to minimise the thermal fuel cost) into the sum of the thermal output y_{Th} (from the system $k_{\text{Th}} = (k_1, k_2)$) and the pumped-storage output y_{PS} (from the storage plant with capacities $k_{\text{PS}} = (k_{\text{St}}, k_{\text{Co}})$).

The SR approach is needed because the LRMC pricing and system optimality conditions (6.16)–(6.17) cannot be expanded for want of an explicit formula for $C_{\text{LR}}^{\text{ThPS}}$, the LR cost function for the combined technology. For the same reason, the LR formulation (6.18) of cost-minimising output scheduling is ineffective. The SR problem of cost-minimising despatch (6.20) is tractable, but a direct construction of its solution would be complex. We therefore deal with this question indirectly, by deducing cost-optimality from simpler SR conditions.

In formal terms, $C_{\text{LR}}^{\text{ThPS}}$ is the LR cost function derived from the algebraic sum $\mathbb{Y}_{\text{ThPS}}$ of the production sets $\mathbb{Y}_{\text{Th}}$ and $\mathbb{Y}_{\text{PS}}$; and the objective is to give—entirely in terms of the SR functions—a set of conditions equivalent to:

$$p \in \partial_y C_{\text{LR}}^{\text{ThPS}} \left(y_{\text{ThPS}}, r^{\text{ThPS}}\right) \tag{6.16}$$

$$k_{\text{ThPS}} \in \partial_r C_{\text{LR}}^{\text{ThPS}} \left(y_{\text{ThPS}}, r^{\text{ThPS}}\right) \tag{6.17}$$

$$C_{\text{LR}}^{\text{ThPS}} \left(y_{\text{ThPS}}, r^{\text{ThPS}}\right) = C_{\text{LR}}^{\text{Th}} \left(y_{\text{Th}}, r^{\text{Th}}\right) + C_{\text{LR}}^{\text{PS}} \left(y_{\text{PS}}, r^{\text{PS}}\right) \tag{6.18}$$

$$y_{\text{ThPS}} = y_{\text{Th}} + y_{\text{PS}}. \tag{6.19}$$

Given an optimal supply system k_{ThPS}, splitting the output in a LR cost-minimising way is equivalent to SR cost-minimising despatch, i.e., to finding a thermal output

$$y_{\text{Th}} \in \operatorname*{argmin}_y \left\{ C_{\text{SR}}^{\text{Th}} \left(y, k_{\text{Th}}, w\right) : \left(y_{\text{ThPS}} - y, -k_{\text{PS}}\right) \in \mathbb{Y}_{\text{PS}} \right\}. \tag{6.20}$$

This approach leads to the following set of SR conditions which are necessary and sufficient for LRMC pricing, system optimality and optimal despatch. Since storage uses no variable input,[7] we choose to recast the corresponding SRMC pricing condition, $p \in \partial_y C_{\text{SR}}^{\text{PS}}$, in terms of SR profit maximisation.

Theorem 6.1 (SR approach to electricity supply with storage)**.** *The set of conditions (6.16), (6.17), either (6.18) or equivalently (6.20), and (6.19) on: the system output $y_{\text{ThPS}} \in L^\infty [0, T]$, the pumped-storage output y_{PS}, the thermal output y_{Th}, a time-continuous electricity tariff $p \in \mathcal{C} [0, T]$ with no plateaux,[8] the thermal capacities $k_\theta > 0$ (for each θ), the storage capacity $k_{\text{St}} > 0$, the conversion capacity $k_{\text{Co}} > 0$,*

[7]Formally, $C_{\text{SR}}^{\text{PS}}$ is the 0-∞ indicator function of the production set $\mathbb{Y}_{\text{PS}}$; and an indicator's subdifferential, at y, consists of the outward normal vectors p: see, e.g., [9, p. 35].

[8]The no-plateau assumption on p is restrictive: leading to y_{PS} that takes only the three values in (6.23), it cannot hold in a general equilibrium with a continuous output trajectory. Such an equilibrium is made possible only by intervals on which $0 < s(t) < k_{\text{St}}$ and $p = \hat{\psi} = \text{const.}$: being multi-valued, the instantaneous optimum is then compatible with a $y_{\text{PS}}(t)$ that gradually changes

and the corresponding rental prices $r^\theta \geq 0$*,* $r^{\mathrm{St}} \geq 0$ *and* $r^{\mathrm{Co}} \geq 0$*, with thermal fuel prices* w*, is equivalent to the following set of conditions:*

$$y_{\mathrm{ThPS}} = y_{\mathrm{Th}} + y_{\mathrm{PS}} \tag{6.21}$$

$$p(t) \in \partial_{\mathrm{y}} c_{\mathrm{SR}}^{\mathrm{Th}} (y_{\mathrm{Th}}(t), k_{\mathrm{Th}}, w), \tag{6.22}$$

where $\partial_{\mathrm{y}} c$ *is the scalar subdifferential spelt out in (6.6), and*

$$y_{\mathrm{PS}}(t) = \begin{cases} k_{\mathrm{Co}} & \text{if } p(t) > \psi(t) \\ 0 & \text{if } p(t) = \psi(t) \\ -k_{\mathrm{Co}} & \text{if } p(t) < \psi(t) \end{cases} \tag{6.23}$$

$$r^{\mathrm{St}} = \mathrm{Var}_{\mathrm{c}}^{+} \left(\hat{\psi}(p, k_{\mathrm{PS}}) \right) \tag{6.24}$$

$$r^{\mathrm{Co}} = \int_0^T \left| p(t) - \hat{\psi}(p, k_{\mathrm{PS}})(t) \right| \mathrm{d}t, \tag{6.25}$$

where $\hat{\psi}$ *is the unique solution to (6.12)–(6.13), and (for* $\theta = 1, 2$*)*

$$r^\theta = \int_0^T \left(p(t) - w^\theta \right)^+ \mathrm{d}t. \tag{6.26}$$

Proof. By Corollary 5.4, Conditions (6.16)–(6.17) are equivalent to the conjunction of

$$p \in \partial_y C_{\mathrm{SR}}^{\mathrm{ThPS}} (y_{\mathrm{ThPS}}, k_{\mathrm{ThPS}}, w) \tag{6.27}$$

$$r^{\mathrm{ThPS}} \in \partial_k \Pi_{\mathrm{SR}}^{\mathrm{ThPS}} (p, k_{\mathrm{ThPS}}, w). \tag{6.28}$$

And under (6.17), Condition (6.18) is equivalent to (6.20), as has been pointed out. Therefore what one needs to analyse further is the set of conditions (6.27), (6.28), (6.20) and (6.19).

Under (6.19), the pair of conditions (6.20) and (6.27) is equivalent—by subdifferentiating $C_{\mathrm{SR}}^{\mathrm{ThPS}}(\cdot, k_{\mathrm{ThPS}})$ as the infimal convolution of $C_{\mathrm{SR}}^{\mathrm{Th}}(\cdot, k_{\mathrm{Th}})$ and $C_{\mathrm{SR}}^{\mathrm{PS}}(\cdot, k_{\mathrm{PS}})$: see, e.g., [8, 6.6.3 and 6.6.4]—to the conjunction of $p \in \partial_y C_{\mathrm{SR}}^{\mathrm{Th}}(y_{\mathrm{Th}}, k_{\mathrm{Th}}, w)$ and $p \in \partial_y C_{\mathrm{SR}}^{\mathrm{PS}}(y_{\mathrm{PS}}, k_{\mathrm{PS}})$. The first of these conditions means (6.22), whilst the second is equivalent to (6.23). Furthermore, since $\Pi_{\mathrm{SR}}^{\mathrm{ThPS}}(k_{\mathrm{ThPS}}) = \Pi_{\mathrm{SR}}^{\mathrm{Th}}(k_{\mathrm{Th}}) + \Pi_{\mathrm{SR}}^{\mathrm{PS}}(k_{\mathrm{PS}})$, Condition (6.28) can be reformulated as: $r^{\mathrm{Th}} \in \partial_{k_{\mathrm{Th}}} \Pi_{\mathrm{SR}}^{\mathrm{Th}}(p, k_{\mathrm{Th}}, w)$ and $r^{\mathrm{PS}} \in \partial_{k_{\mathrm{PS}}} \Pi_{\mathrm{SR}}^{\mathrm{PS}}(p, k_{\mathrm{PS}})$. The storage rent condition can be spelt out as (6.24)–(6.25), by (6.14)–(6.15). And the thermal rent condition is (6.26) for each θ, by (6.2). ∎

from 0 to $\pm k_{\mathrm{Co}}$. Without this assumption, the operation problem (6.7)–(6.11) may have multiple solutions instead of (6.23), although $y_{\mathrm{PS}}(t)$ still equals $\pm k_{\mathrm{Co}}$ at any t with $p(t) \neq \hat{\psi}(t)$.

References

[1] Aubin, J. P., and I. Ekeland (1984): *Applied nonlinear analysis*. New York: Wiley.

[2] Birkhoff, G. (1967): *Lattice theory*. Providence, RI: American Mathematical Society.

[3] Drèze, J. H. (1964): "Some postwar contributions of French economists to theory and public policy", *American Economic Review*, 54 (supplement, June 1964), 1–64.

[4] Horsley, A., and A. J. Wrobel (1996): "The Wong-Viner Envelope Theorem for nonsmooth joint costs, rental valuation and the short-run approach to long-run equilibrium", STICERD Discussion Paper TE/96/299, LSE.

[5] Horsley, A., and A. J. Wrobel (1996): "Uninterruptible consumption, concentrated charges, and equilibrium in the commodity space of continuous functions", STICERD Discussion Paper TE/96/300, LSE.

[6] Horsley, A., and A. J. Wrobel (1996): "Efficiency rents of storage plants in peak-load pricing, I: pumped storage", STICERD Discussion Paper TE/96/301, LSE.

[7] Horsley, A., and A. J. Wrobel (1999): "Efficiency rents of hydroelectric storage plants in continuous-time peak-load pricing", in *The Current State of Economic Science* (Volume 1, pp. 453–480), ed. by S. B. Dahiya. Spellbound Publications: Rohtak.

[8] Laurent, P.-J. (1972): *Approximation and optimisation*. Paris: Hermann.

[9] Rockafellar, R. T. (1974): *Conjugate duality and optimization*. Philadelphia, PA: SIAM.

(Anthony Horsley and Andrew J. Wrobel) Department of Economics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom

E-mail address: `LSEecon123@msn.com`