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Financial equilibrium with preference updating

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Abstract
Financial equilibrium models provide important information on the movement of asset prices in response to subjective beliefs and consumption patterns of economic agents. It is standard to assume that agents are rational and have fixed preferences from the outset—in particular, these do not depend on other agents' actions in the market. This work deviates from this assumption by allowing the subjective views and consumption clocks of an individual agent to depend on the whole history of the wealth and consumption distribution across agents in the economy. The updating mechanism is generic and may accommodate different behavioural models; for example, it can model herding. In order to analyse existence and uniqueness of equilibrium, we assume that agents have numeraire-invariant preferences, which are rich enough to render any observe agents' behaviour optimal. The market contains a borrowing and lending account in zero net supply, as well as a stock in positive net supply providing certain dividend stream, exogenously specified. A characterisation of existence and uniqueness of equilibrium in a Brownian setting is provided in terms of stochastic differential equations. The proposed framework naturally allows for equilibria where the risky asset in positive net supply is suboptimal to hold for investment.

Keywords Financial equilibrium · Preferences · Updating · Numéraire invariance

Introduction

Equilibrium models are ubiquitous in economic theory, providing the most natural ground in understanding the interactions between the underlying structure of the economy and the behaviour of agents. The premise is as simple as it is appealing, based on supply-demand balance: the collective effect of participants, individually acting towards optimising their own utility, is such that markets (of goods, services, financial instruments, etc) have to clear. Arrow-Debreu equilibrium, as proposed in [8] is the basis of all further development in general equilibrium theory. Specialised in a financial setting, it postulates that all possible scenarios, present and future, can be securitised and priced. With this understanding, Arrow-Debreu equilibrium offers in effect a "static" view of randomness, since all future uncertainty is resolved immediately at the beginning of time, upon creating a market for any possible future outcome. Of course, such extreme level of uncertainty resolution would be practically

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impossible; building on this, the concept of *Radner* financial equilibrium, introduced in [20], includes a time-information structure, under which financial instruments are created that dynamically span future uncertainly, rather than pricing individually any future realisation from the outset.

An important ingredient in any equilibrium model is the preference structure of the acting agents. Preferences are almost universally modelled through utility-function numerical representations. Axiomatic approaches have provided the underlying reason for doing so, since the early works of von Neumann and Morgenstern [23], as well as Savage [22]. Preferences of agents are typically assumed to be fixed from the outset, and to not depend on the actions of other agents, or the resulting equilibrium environment. In more mathematical terms, the optimisation criterion of the acting players in the financial economy does not depend on the actions of other players, or the state of equilibrium allocations. While such an assumption simplifies the quantitative analysis, it can be argued that it is not very reasonable; humans are behavioural beings, and observations of actions of others certainly affects their own actions. Therefore, models where such behavioural components are considered have appeared, where the utility of individual agents may depend on aggregate quantities related to all other agents, bringing equilibrium more aligned to an interpretation *a-la-Nash*. For example, in [3], individual utility depends on an aggregate consumption level, which can be interpreted as an *external* form of habit formation.

In the present work, we consider a Brownian continuous-time financial equilibrium model with a finite[1] number of agents and two investment opportunities: a risky asset in positive net supply providing a dividend, as well as a money market for borrowing and lending, in zero net supply. Agents are heterogeneous, and assumed to have *numeraire-invariant* preferences, which is morally equivalent to asking that they possess unit risk aversion. Heterogeneity involves different subjective views, modelled via local martingale density processes, as well as cumulative consumption impatience clocks, modelled via increasing processes with values in the unit interval. Compared to other works on heterogeneous agent equilibrium such as [5], the drawback of the present approach is that unit risk aversion has to be assumed for all agents. On the other hand, this restriction allows several desirable properties. Firstly, one can be completely general regarding the dynamics of primitive market inputs, such as dividends, and similarly completely general regarding density processes of subjective views and consumption impatience clocks of agents. There is no need to assume that the coefficients are constant, or of any other particular form. Secondly, and quite importantly, the preference structure of each agent is general enough to accommodate *any* observed investment-consumption behaviour as optimal; to that effect, see Remark 2.5. The latter is not the case when agents' risk aversion coefficient may vary, because such freedom necessarily implies heavy restrictions in the underlying model in order for it to be workable.

Instead of preferences being fixed from the outset, a *generic* updating mechanism for them is suggested. Such updating mechanisms are well suited to model behavioural patterns in financial markets, as is for example the well-documented *herding behaviour*, where economic agents mimic others (see, for instance, [2] in an equilibrium setting), or *keeping up with the Joneses*, where preferences depend on consumption relative to the aggregate one (see, for instance, [1] and [6], with the latter being in an equilibrium setting).

The main contribution of the present paper is that it provides an explicit (up to the solution of a system of potentially path-dependent, but otherwise standard, forward stochastic differential equations) characterisation of financial equilibrium when agents' preferences

[1] Extending the analysis to potentially infinite, including uncountable, number of agents is possible; the only technical modifications will involve measurability considerations.

may depend on full past history of the capital and consumption distributions across agents. Naturally, this framework extends the classical case where no updating happens, as well as when updating only depends on "sufficient statistics" of the capital and consumption distributions, such as averages. However, it should be stressed that the results of this paper are valid under completely generic updating mechanisms, opening the door to the creation of diverse behavioural models, a possibility left for future research. To the best of the author's knowledge, this is the first work where updating can be based on full past observations of the capital and consumption distributions among agents in the market. The emergence of the capital and consumption distributions as driving factors for preference updating is natural, and also not entirely ad-hoc, arising from a careful analysis of standard financial equilibrium. As a further observation discussed in the paper, preference updating can lead to certain inefficiency in the actions of financial agents, which in turn results in the emergence of bubbles in positive net supply assets, even though agents cannot be described as "irrational" and do not face investment constraints.

The analysis in this work is made on the basis of a resulting market completeness: it is assumed that uncertainly is generated by a single Brownian motion, and equilibrium prices should span all uncertainty. Incomplete market equilibria are notoriously harder to analyse and require strong assumptions on primitives; see, for example, [7]. The main difficulty with incompleteness is the inability to solve the individual investment-consumption problems in a "reasonably" explicit form; for instance, dual formulations in terms of risk-neutral probabilities (or, more generally, stochastic discount factors) as in [16] and [18], give good characterisations only when markets are complete, due to uniqueness of the dual variables. It is probable that these difficulties will not be an obstacle in the setting of this paper, due to the fact that solutions of the individual investment-consumption problem under numeraire-invariant preferences have explicit forms even under market incompleteness, without requiring passage to the dual formulation. That being said, there is no concious effort to generalise the set-up of the paper to market incompleteness, since the technicalities involved would outweigh any conceptual gain.

Structure of the paper

Section 1 introduces the market model structure, and discusses investment, consumption, and financeability. The problem of optimal investment and consumption for an economic agent with numéraire-invariant preferences is taken up in Section 2. The notion of financial market equilibrium is defined in Section 3, where some direct consequences of it are also discussed, while Section 4 contains a more in-depth analysis within equilibrium, including in particular dynamics for the movement of the cross-sectional capital and consumption distributions of the acting agents. All the previous analysis is synthesised in Section 5, where necessary and sufficient conditions for the existence and uniqueness of equilibrium with preference updating are presented, in terms of solutions to certain stochastic differential equations. Finally, certain technical proofs that would otherwise disrupt the text flow are deferred to Section A of the Appendix.

General notation

All stochastic processes in this paper are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t); t \geq 0), \mathbb{P})$. It will be assumed throughout that $(\mathcal{F}(t); t \geq 0)$ is the filtration generated by a standard one-dimensional Brownian motion $W \equiv (W(t); t \geq 0)$.

We shall be using Leb to denote Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$, where $\mathcal{B}_{\mathbb{R}_+}$ is the Borel sigma-algebra over $\mathbb{R}_+$. Several properties between processes will hold $(\mathbb{P} \otimes \mathsf{Leb})$-a.e., where $\mathbb{P} \otimes \mathsf{Leb}$ is the usual product measure on $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}_{\mathbb{R}_+})$.

For $n \in \{1, 2\}$, $\mathcal{I}^n$ shall denote the class of all real-valued predictable processes χ such that $\int_0^t |\chi(s)|^n \mathrm{d}s < \infty$ holds $\mathbb{P}$-a.e. for all $t \geq 0$. Furthermore, $\mathcal{I}^1_+$ denotes the class of all nonnegative processes in $\mathcal{I}^1$.

1 The Financial Market

1.1 The market model

We assume the existence of two investment opportunities. The first is the money market (which, in the equilibrium set-up later on, will be in zero net-supply), providing a continuously-compounded short rate process $r = (r(t);\ t \geq 0)$, satisfying

$$r \in \mathcal{I}^1. \tag{1.1}$$

For future reference, define the discounting process D, associated with the money market, via

$$D := \exp\left(-\int_0^\cdot r(t)\mathrm{d}t\right). \tag{1.2}$$

In the financial market there also exists a risky asset (which, in the equilibrium set-up later on, will be in normalised unit net-supply), paying an instantaneous dividend stream $\delta \equiv (\delta(t);\ t \geq 0)$ per share; δ is assumed to be a strictly positive Itô process satisfying

$$\frac{\mathrm{d}\delta(t)}{\delta(t)} = \nu(t)\mathrm{d}t + \xi(t)\mathrm{d}W(t); \quad t \geq 0, \tag{1.3}$$

where in order to ensure that δ is well-defined and strictly positive, we assume that

$$\delta(0) \in (0, \infty), \quad \nu \in \mathcal{I}^1, \quad \xi \in \mathcal{I}^2. \tag{1.4}$$

In the infinitesimal interval $(t, t + \mathrm{d}t]$ the risky asset pays $\delta(t)\mathrm{d}t$ to the holder of one share. The actual price $S \equiv (S(t);\ t \geq 0)$ of the risky asset is assumed to be a *strictly positive* Itô process. The coefficients of S are parametrised in a way that has immediate financial interpretation, writing dynamics

$$\frac{\mathrm{d}S(t) + \delta(t)\mathrm{d}t}{S(t)} = (r(t) + \sigma(t)\theta(t))\,\mathrm{d}t + \sigma(t)\mathrm{d}W(t); \quad t \geq 0, \tag{1.5}$$

where in order for the above equation to make sense and to avoid degeneracy in the market, it is assumed that

$$\theta \in \mathcal{I}^2, \quad \sigma \in \mathcal{I}^2, \quad (\mathbb{P} \otimes \mathsf{Leb})\,[\sigma = 0] = 0. \tag{1.6}$$

The process $\theta \equiv (\theta(t);\ t \geq 0)$ is the financial market's local (in time and uncertainty) *risk premium*, while the process $\sigma \equiv (\sigma(t);\ t \geq 0)$ is the risky asset's local *volatility rate*.

Remark 1.1 In view of the Cauchy-Schwartz inequality, the requirements $\theta \in \mathcal{I}^2$ and $\sigma \in \mathcal{I}^2$ of (1.6) imply $(\sigma\theta) \in \mathcal{I}^1$; using also (1.1), the linear stochastic integral equation (1.5) can be solved for S. In fact, in order to ensure that (1.5) is well defined with a unique solution, the conditions $(\sigma\theta) \in \mathcal{I}^1$ and $\sigma \in \mathcal{I}^2$ (together with $r \in \mathcal{I}^1$) are sufficient. However, the slightly more restrictive requirements in (1.6) are crucial in what follows. In fact, given that $\sigma \in \mathcal{I}^2$, the requirement $\theta \in \mathcal{I}^2$ is equivalent to a mild form of market viability; for more information, consult [13, Section 4], as well as [15, §2.2].

1.2 Investment and consumption

Consider an economic agent in the market as described in §1.1. Starting with capital $x \in (0, \infty)$, investing according to a predictable process $h \equiv (h(t);\ t \geq 0)$ representing the units of the risky asset invested in the portfolio, and consuming according to a *nonnegative* predictable process $c \equiv (c(t);\ t \geq 0)$, with the interpretation of consumption rate per unit of time, the resulting investment-consumption process $X^{x;h,c}$ satisfies $X_0^{x;h,c} = x$, and dynamics

$$\begin{aligned} \mathrm{d}X^{x;h,c}(t) &= h(t)\left(\mathrm{d}S(t) + \delta(t)\mathrm{d}t\right) + \Big(X^{x;h,c}(t) - h(t)S(t)\Big) r(t)\mathrm{d}t - c(t)\mathrm{d}t \\ &= X^{x;h,c}(t)r(t)\mathrm{d}t + h(t)S(t)\sigma(t)\left(\theta(t)\mathrm{d}t + \mathrm{d}W(t)\right) - c(t)\mathrm{d}t; \quad t \geq 0. \end{aligned}$$

In order for the above stochastic differential equation to make sense, we require that

$$(hS\sigma) \in \mathcal{I}^2, \ c \in \mathcal{I}^1_+. \tag{1.7}$$

(Note that $(hS\sigma) \in \mathcal{I}^2$ and $\theta \in \mathcal{I}^2$ combined imply $(hS\sigma\theta) \in \mathcal{I}^1$.) Under the force of conditions (1.7), the previous linear stochastic differential equation in $X^{x;h,c}$ has a unique solution, which deflated by the discounting process D of (1.2) satisfies

$$DX^{x;h,c} = x + \int_0^{\cdot} h(t)D(t)S(t)\sigma(t)\left(\theta(t)\mathrm{d}(t) + \mathrm{d}W(t)\right) - \int_0^{\cdot} D(t)c(t)\mathrm{d}t. \tag{1.8}$$

Credit constraints have to be enforced in order to avoid so-called *doubling strategies*. We shall require that wealth never becomes negative. Define the class of nonnegative investment-consumption processes with initial capital $x \in (0, \infty)$ via

$$\mathcal{X}(x) := \left\{X^{x;h,c} \mid (h,c) \text{ satisfies (1.7)}, \ X^{x;h,c} \text{ is given by (1.8)}, \ \text{and } X^{x;h,c} \geq 0\right\}.$$

Define also the set $\mathcal{X} := \bigcup_{x \in (0,\infty)} \mathcal{X}(x)$ of all nonnegative investment-consumption processes.

Definition 1.2 Fix $x \in (0, \infty)$. A process $X \in \mathcal{X}(x)$ is said to *finance* a consumption stream $c \in \mathcal{I}^1_+$ if $X \equiv X^{x;h,c}$ for some predictable process h such that (1.7) hold.

We introduce the minimal capital required to finance a consumption stream:

$$x(c) := \inf\{x \in (0,\infty) \mid \exists X \in \mathcal{X}(x) \text{ which finances } c\}; \quad c \in \mathcal{I}^1_+. \tag{1.9}$$

(If the last set is empty for some $c \in \mathcal{I}^1_+$, the convention $x(c) = \infty$ is used.) We also introduce the class of all consumption rate processes c that can be financed starting with capital $x \in (0, \infty)$; namely,

$$\mathcal{C}(x) := \left\{c \in \mathcal{I}^1_+ \mid x(c) \leq x\right\}. \tag{1.10}$$

Note that $\mathcal{C}(x_1) \subseteq \mathcal{C}(x_2)$ holds whenever $0 < x_1 < x_2 < \infty$.

It is a consequence of Theorem 1.3 below that, whenever $x(c) < \infty$, the infimum in the definition of $x(c)$ in (1.9) is actually attained. Furthermore, $x(c)$ has a natural representation in terms of present value of the consumption stream. In order to make headway with the previous facts, first note that (1.1) and (1.6) imply that the process

$$Y := \exp\left(-\int_0^{\cdot}\left(r(t)+\frac{1}{2}\theta^2(t)\right)\mathrm{d}t - \int_0^{\cdot}\theta(t)\mathrm{d}W(t)\right) \tag{1.11}$$

is well defined and $\mathbb{P}$-a.e. strictly positive. Recalling the discounting process D of (1.2), it follows that $D^{-1}Y$ is a strictly positive local martingale. If it is a uniformly integrable martingale, one may define a probability measure $\mathbb{Q}$ such that $D^{-1}Y$ is the density process of $\mathbb{Q}$ with respect to $\mathbb{P}$, and $\mathbb{Q}$ would be a *risk-neutral probability*, under which fair valuation of streams would be made. In general, $D^{-1}Y$ may fail to be an actual martingale; however, a consistent valuation theory may still be developed, directly using the *stochastic discount factor* Y and the original probability $\mathbb{P}$. The following is the important result in this direction; for its validity in the generality of the current set-up, see [15]. (Of course, several versions of this folklore result are abundant; for example, the first statement of Theorem 1.3 is discussed in [12], valid even when a risk-neutral probability fails to exist.)

Theorem 1.3 *Assume the financial market setting of §1.1, and define Y via (1.11). Fix $c \in \mathcal{I}^1_+$.*

(1) The quantity $x(c)$ of (1.9) satisfies

$$x(c) = \mathbb{E}\left[\int_0^\infty Y(u)c(u)\mathrm{d}u\right].$$

Assuming that $x(c) < \infty$, define a process $X^{[c]}$ via

$$X^{[c]}(t) = \mathbb{E}\left[\int_t^\infty \frac{Y(u)}{Y(t)}c(u)\mathrm{d}u \;\middle|\; \mathcal{F}(t)\right], \quad t \geq 0. \tag{1.12}$$

Then, $X^{[c]} \in \mathcal{X}(x(c))$ and $X^{[c]}$ finances c.

(2) Suppose that $X \in \mathcal{X}$ finances c. (Necessarily, $x(c) < \infty$.) Then, the process $Y(X - X^{[c]})$ is a nonnegative local martingale; in particular, it is a nonnegative supermartingale.

Not that Theorem 1.3 implies a “dual” representation for feasible consumption stream sets:

$$\mathcal{C}(x) = \left\{c \in \mathcal{I}^1_+ \;\middle|\; \mathbb{E}\left[\int_0^\infty Y(u)c(u)\mathrm{d}u\right] \leq x\right\}; \quad x \in (0,\infty). \tag{1.13}$$

Remark 1.4 In the setting of Theorem 1.3, and when $x(c) < \infty$, it follows that $X^{[c]}$ is the *minimal* investment-consumption process financing $c \in \mathcal{I}^1_+$. Furthermore, the fact that a nonnegative supermartingale remains forever at zero once it reaches it [17, Problem 1.3.29] implies that, whenever $X \in \mathcal{X}$ finances $c \in \mathcal{I}^1_+$, $X = X^{[c]}$ is actually equivalent to $X(0) = x(c) = X^{[c]}(0)$. In particular, this last fact implies uniqueness of the minimal financing investment-consumption process.

Remark 1.5 Assume that $X \in \mathcal{X}$ finances $c \in \mathcal{I}^1_+$ (with $x(c) < \infty$). Denote by N^c the uniformly integrable martingale such that $N^c(t) = \mathbb{E}\left[\int_0^\infty Y(u)c(u)\mathrm{d}u \mid \mathcal{F}(t)\right]$ holds for all $t \geq 0$. By Theorem 1.3, it follows that the process

$$YX + \int_0^{\cdot} Y(t)c(t)\mathrm{d}t = YX^{[c]} + \int_0^{\cdot} Y(t)c(t)\mathrm{d}t + Y(X - X^{[c]}) = N^c + Y(X - X^{[c]})$$

is a non-negative local martingale. The above discussion shows in particular that the process YX is a (nonnegative) supermartingale for all $X \in \mathcal{X}$.

Remark 1.6 The price-process of the risky asset itself is a particular case of an investment-consumption process; in fact, $S = X^{S(0);1,\delta}$, which in particular implies that S finances the dividend stream δ. Theorem 1.3 implies that we can write $S = \widetilde{S} + B_S$, where

$$\widetilde{S}(t) = \mathbb{E}\left[\int_t^\infty \frac{Y(u)}{Y(t)}\delta(u)\mathrm{d}u \,\Big|\, \mathcal{F}(t)\right], \quad t \geq 0, \tag{1.14}$$

and B_S is a nonnegative continuous process with the property that YB_S is a local martingale. The process $\widetilde{S}$ is the present value of the future dividend stream for each $t \geq 0$, which may be thought as representing the *fundamental value* of the risky asset. The second term is used in the literature to model potential price *bubble*; see, for example, [19] and [4].

Before abandoning this remark, note that non-existence of a bubble in the price of the risky asset does not necessarily imply that the latter asset is desirable in terms of pure investment; in fact, Example 3.6 will present a case where investing all the dividend proceedings back to the risky asset without bubble produces a pure investment process that is suboptimal.

2 Optimal Investment and Consumption via Numéraire-Invariant Preferences

2.1 Numéraire-invariant preferences

Given initial capital $x \in (0, \infty)$, an economic agent's problem will be to optimally pick a consumption stream in the set $\mathcal{C}(x)$ of (1.10). We shall assume that the agent has so-called *numéraire-invariant* preferences, which are essentially (but not exactly) equivalent to maximising expected logarithmic consumption rate under certain subjective probability and certain "consumption clock". In order to motivate these preference structures, we briefly discuss first the latter log-utility optimisation problem.

Assume that the probability $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ represents subjective views of an agent, where $\widehat{\mathbb{P}}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}(t)$ for all $t \geq 0$. Furthermore, let K be an adapted, continuous, nondecreasing $[0, 1]$-valued process such that $K(0) = 0$ and $K(\infty) := \lim_{t\to\infty} K(t) = 1$ hold in the $\widehat{\mathbb{P}}$-a.e. sense. The agent wishes to choose a consumption stream such that the functional[2]

$$\mathcal{C}(x) \ni c \mapsto \widehat{\mathbb{E}}\left[\int_0^\infty \log(c(t))\mathrm{d}K(t)\right] \tag{2.1}$$

is maximised, where "$\widehat{\mathbb{E}}$" denotes expectation under $\widehat{\mathbb{P}}$. Since $\mathcal{C}(x)$ is convex, first-order conditions for optimality of $\widehat{c} \in \mathcal{C}(x)$ formally read $\widehat{\mathbb{E}}\left[\int_0^\infty (1/\widehat{c}(t))\,(c(t) - \widehat{c}(t))\,\mathrm{d}K(t)\right] \leq 0$ for all $c \in \mathcal{C}(x)$; furthermore, since $\widehat{\mathbb{P}}[K(\infty) - K(0) = 1] = 1$, the latter conditions are equivalent to:

$$\widehat{\mathbb{E}}\left[\int_0^\infty \left(\frac{c(t)}{\widehat{c}(t)}\right)\mathrm{d}K(t)\right] \leq 1; \quad c \in \mathcal{C}(x). \tag{2.2}$$

Let L be the strictly positive $\mathbb{P}$-martingale that is the density process of $\widehat{\mathbb{P}}$ with respect to $\mathbb{P}$. Using a combination of the integration-by-parts formula and the monotone convergence theorem, it is straightforward to check that $\widehat{\mathbb{E}}\left[\int_0^\infty V(t)\mathrm{d}K(t)\right] = \mathbb{E}\left[\int_0^\infty V(t)L(t)\mathrm{d}K(t)\right]$

[2] As this discussion serves as a heuristic guide to the introduction of numéraire-invariant preferences in the next paragraph, technical issues such as existence of expectations are swept under the carpet.

holds for any nonnegative predictable process V. In particular, $\widehat{\mathbb{P}}[K(0) = 0,\ K(\infty) = 1]$ is equivalent to $\mathbb{P}[K(0) = 0] = 1$ and

$$\mathbb{E}\left[\int_0^\infty L(t)\mathrm{d}K(t)\right] = 1; \tag{2.3}$$

furthermore, the first order conditions in (2.2) for optimality of $\widehat{c} \in \mathcal{C}(x)$ equivalently read

$$\mathbb{E}\left[\int_0^\infty \left(\frac{c(t)}{\widehat{c}(t)}\right) L(t)\mathrm{d}K(t)\right] \leq 1; \quad c \in \mathcal{C}(x). \tag{2.4}$$

The set-up described above has certain drawbacks; particularly, the maximisation problem involving the functional (2.1) may be ill-posed, with the possibility that the value is infinite (in which case a continuum of solutions exists). However, if one uses the first-order conditions (2.4) as a starting point, such concerns disappear. Indeed, Theorem 2.2 establishes existence of consumption streams $\widehat{c} \in \mathcal{C}(x)$ such that (2.4) hold with no further assumptions apart from the ones made in Section 1. In fact, defining optimality via these first-order conditions is economically justified in terms of so-called *numéraire-invariant*[3] preferences—for the axiomatic characterisation of these and more discussion on what follows, refer to [14].

Motivated by the previous discussion, the problem of an economic agent with initial capital $x \in (0, \infty)$ acting in a market as described in Section 1 is to choose $\widehat{c} \in \mathcal{C}(x)$ such that (2.4) hold, where the pair (K, L) satisfies the following properties:

(P1) K is an adapted, nondecreasing, $[0, 1)$-valued process, such that $K(0) = 0$ and almost every (modulo $\mathbb{P}$) path is absolutely continuous with respect to Lebesgue measure.
(P2) L is a $\mathbb{P}$-a.e. *strictly positive* local martingale with $L(0) = 1$.

Assume that (K, L) is any pair of processes satisfying (P1) and (P2). Using integration-by-parts, the process $L(1 - K) + \int_0^\cdot L(t)\mathrm{d}K(t) = 1 + \int_0^\cdot (1 - K(t))\mathrm{d}L(t)$ is a nonnegative local martingale; in particular, it is a nonnegative supermartingale. The optional sampling theorem for nonnegative supermartingales implies that

$$\mathbb{E}\left[\int_0^\infty L(t)\mathrm{d}K(t)\right] \leq 1. \tag{2.5}$$

Contrast this last inequality with the actual equality in (2.3); we shall see in Theorem 2.2 below that failure of (2.3) is related to inefficiency in choosing "maximal" consumption streams. We shall actually allow for such possibility in our setting, especially in relation to financial equilibrium with preference updating—in this respect, see Remark 2.4. Note that failure of (2.3) is related to appearance of bubbles in the risky asset price—see Proposition 3.5 later on.

In view of the local martingale representation theorem [17, §3.4.D], pairs (K, L) satisfying (P1) and (P2) are in one-to-one correspondence with pairs of processes (κ, λ) such that

$$\kappa \in \mathcal{I}^1_+, \quad \lambda \in \mathcal{I}^2, \tag{2.6}$$

via the identifications

$$K = 1 - \exp\left(-\int_0^\cdot \kappa(t)\mathrm{d}t\right), \quad L = \exp\left(-\frac{1}{2}\int_0^\cdot \lambda^2(t)\mathrm{d}t + \int_0^\cdot \lambda(t)\mathrm{d}W(t)\right). \tag{2.7}$$

[3] The appellation stems from the fact that a choice of $\widehat{c} \in \mathcal{C}(x)$ that satisfies (2.4) is invariant under changes of numéraire: if consumption is counted in units of some strictly positive wealth process V, in the sense that $c_V(t) := V^{-1}(t)c(t)$ is the value of consumption at time $t \geq 0$, then $\widehat{c}_V \equiv V^{-1}\widehat{c}$ will be such that $\mathbb{E}\left[\int_0^\infty (c_V(t)/\widehat{c}_V(t))\, L(t)\mathrm{d}K(t)\right] \leq 1$ holds for all $c \in \mathcal{C}(x)$, since $c_V/\widehat{c}_V = c/\widehat{c}$ is true for all $c \in \mathcal{C}(x)$.

The pair of processes (κ, λ) has direct financial interpretation: κ represents the agent's local impatience rate, while λ represents a "correction" in the market price of risk as perceived (or believed) by the agent; in this respect, see also Remark 2.3.

2.2 A family of examples

Whenever L is an actual martingale, in which case $\mathbb{E}[L(t)] = 1$ holds for all $t \geq 0$, any *deterministic* process K that satisfies (P1) and is such $K(\infty) = 1$ will result in a pair (K, L) satisfying (2.3). When L is a *strict local martingale* in the terminology of [10], this is no longer the case—however, stochastic choices for K are certainly available. Given a certain process L such that (P2) holds and $\mathbb{P}[L(\infty) = 0] = 1$, the next example provides a couple generic ways for constructing K such that (P1) and (2.3) hold.

Example 2.1 Let $\lambda \in \mathcal{I}^2$ be *any* process such that $\mathbb{P}\left[\int_0^\infty \lambda^2(t)\mathrm{d}t = \infty\right] = 1$ holds. Then if L is the local martingale as in (2.7), and since $\int_0^\cdot \lambda^2(t)\mathrm{d}t$ is the quadratic variation process of the stochastic logarithm of L, it follows that $\mathbb{P}[L(\infty) = 0] = 1$. The nondecreasing process $G := \exp\left(\int_0^\cdot \lambda^2(t)\mathrm{d}t\right)$ will be used in the sequel. Clearly, $G(0) = 1$; furthermore, $\mathbb{P}[G(\infty) = \infty] = 1$.

As a first example, let $K := 1 - 1/G$. It is obvious that K satisfies (P1); additionally, note that $\mathbb{P}[K(\infty) = 1] = 1$. In §A.1 of the Appendix, we show that (2.3) is also always satisfied.

In addition to the previous case, it is possible to construct processes K such that (P1) and (2.3) hold, but where $\mathbb{P}[K(\infty) < 1] = 1$. (In particular, under (P1) and (P2), (2.3) is possible even if L is a strict local martingale *and* $\mathbb{P}[K(\infty) < 1] = 1$.) Define the nondecreasing process $\overline{L} := \sup_{t\in[0,\cdot]} L(t)$, and set $K := (1/G)\int_0^\cdot \left(1 - 1/\overline{L}(t)\right)\mathrm{d}G(t)$. Note that K is nondecreasing, in view of the fact that both G and $\overline{L}$ are nondecreasing. It is straightforward to check that (P1) holds for K. Furthermore, since $\overline{L}$ is $\mathbb{P}$-a.e. eventually constant and $\mathbb{P}[G(\infty) = \infty] = 1$, it follows that $\mathbb{P}\left[K(\infty) = 1 - 1/\overline{L}(\infty)\right] = 1$. Since $\mathbb{P}\left[\overline{L}(\infty) < \infty\right] = 1$, we have that $\mathbb{P}[K(\infty) < 1] = 1$. In §A.1 of the Appendix, we establish that (K, L) is such that (2.3) holds.

2.3 Optimal investment and consumption

In the previous setting, for $x \in (0, \infty)$ there exists a closed-form expression for a consumption stream $\widehat{c} \in \mathcal{C}(x)$ satisfying (2.4).

Theorem 2.2 *In the financial market described in Section 1, define Y via (1.11), let the pair (κ, λ) satisfy (2.6) and define the pair (K, L) via (2.7). Fix $x \in (0, \infty)$, and define processes*

$$\widehat{X} := x\frac{L}{Y}(1 - K), \quad \widehat{h} := \frac{\widehat{X}}{S}\left(\frac{\theta + \lambda}{\sigma}\right), \quad \widehat{c} := \widehat{X}\kappa. \tag{2.8}$$

Then, the following are true:

(1) $\widehat{X} = X^{x;\widehat{h},\widehat{c}}$ in the notation of (1.8). In particular, $\widehat{X} \in \mathcal{X}(x)$, $\widehat{c} \in \mathcal{C}(x)$ and $\widehat{X}$ finances $\widehat{c}$.
(2) The consumption stream $\widehat{c}$ is such that (2.4) holds.
(3) The pair (K, L) satisfies (2.3) if and only if $\widehat{X} = X^{[\widehat{c}]}$, in the notation of Theorem 1.3. In this case, if (2.4) also holds with $\overline{c} \in \mathcal{I}^1_+$ in place of $\widehat{c}$, then $\overline{c} = \widehat{c}$ holds in the $(\mathbb{P} \otimes \mathsf{Leb})$-a.e. sense.

Theorem 2.2 is proved in §A.2 of the Appendix. Below, remark on some of its statements.

Remark 2.3 Recalling the discussion in the beginning of §2.1, assume that there exists a probability $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ such that L is the density process of $\widehat{\mathbb{P}}$ with respect to $\mathbb{P}$. In this case, Girsanov's theorem implies that $\widehat{W} := W - \int_0^\cdot \lambda(t)\mathrm{d}t$ is a Brownian motion under $\widehat{\mathbb{P}}$; therefore, defining the process $\widehat{\theta} := \theta + \lambda$, (1.5) implies that

$$\frac{\mathrm{d}S(t) + \delta(t)\mathrm{d}t}{S(t)} = \left(r(t) + \sigma(t)\widehat{\theta}(t)\right)\mathrm{d}t + \sigma(t)\mathrm{d}\widehat{W}(t).$$

In particular, λ models the agent's digression in belief from the actual market risk premium.

It is well-known that $\widehat{\theta}/\sigma$ is the optimal (myopic) fraction of wealth that should be invested in the risky asset by an agent possessing logarithmic utility and subjective probability $\widehat{\mathbb{P}}$ (given, of course, that the latter log-optimal problem is well-defined); for example, one can check [11], where a more general problem is addressed. Note that $\widehat{\theta}/\sigma = (\theta + \lambda)/\sigma$, and compare with the definition of $\widehat{h}$ in (2.8). In general, a probability $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ such that L is the density process of $\widehat{\mathbb{P}}$ with respect to $\mathbb{P}$ may not exist; however, the previous discussion and (2.8) imply that λ encodes the probabilistic subjective views of the economic agent.

Further to the above discussion regarding λ, note that the equation $\widehat{c} = \widehat{X}\kappa$ implies that κ, further to its interpretation of local discount rate, also coincides with the *marginal propensity to consume* of an agent that has numéraire-invariant preferences represented by the pair (K, L).[4]

In particular, the pair (κ, λ) separates the problems of consumption and investing: κ affects only consumption through the agent's marginal propensity to consume, while λ affects the agent's investment by affecting locally and additively the perceived risk premium.

Remark 2.4 Fix $x \in (0, \infty)$ and define $\widehat{c}$ as in (2.8). The quantities appearing in (2.8) give

$$\mathbb{E}\left[\int_0^\infty Y(t)\widehat{c}(t)\mathrm{d}t\right] = x\mathbb{E}\left[\int_0^\infty L(t)\mathrm{d}K(t)\right],$$

which actually establishes the first claim in statement (3) of Theorem 2.2. If (2.3) fails, $\widehat{c}$ is strictly dominated by other consumption streams in $\mathcal{C}(x)$—for a concrete example, note

[4] The fact that κ represents marginal propensity to consume is reinforced by noting that the optimal consumption rate per unit of wealth is always κ, regardless of the fraction of wealth invested in the risky asset. To wit, fix $\eta \equiv (\eta(t);\ t \geq 0)$, where $\eta(t)$ represents the fraction of wealth investment in the risky asset at time $t \geq 0$; the process η should be such that $(\eta\sigma) \in \mathcal{I}^2$. With initial capital $x \in (0, \infty)$, for any process $\gamma \in \mathcal{I}^1_+$ representing consumption rate per unit of wealth, the corresponding consumption stream $c^{x;\eta,\gamma}$ satisfies

$$c^{x;\eta,\gamma} := x \exp\left(\int_0^\cdot \left(r(t) + \eta(t)\sigma(t)\theta(t) - \frac{1}{2}\eta^2(t)\sigma^2(t)\right)\mathrm{d}t + \int_0^\cdot \eta(t)\sigma(t)\mathrm{d}W(t)\right)\exp\left(-\int_0^\cdot \gamma(t)\mathrm{d}t\right)\gamma.$$

For fixed $\gamma \in \mathcal{I}^1_+$, define $\Gamma := 1 - \exp\left(-\int_0^\cdot \gamma(t)\mathrm{d}t\right)$, and note that

$$\frac{c^{x;\eta,\gamma}}{c^{x;\eta,\kappa}} = \frac{\exp\left(-\int_0^\cdot \gamma(t)\mathrm{d}t\right)\gamma}{\exp\left(-\int_0^\cdot \kappa(t)\mathrm{d}t\right)\kappa} = \frac{(1-\Gamma)\gamma}{(1-K)\kappa};$$

therefore, since $\mathrm{d}\Gamma(t) = (1 - \Gamma(t))\gamma(t)\mathrm{d}t$ and $\mathrm{d}K(t) = (1 - K(t))\kappa(t)\mathrm{d}t$ holds for all $t \geq 0$, it follows that

$$\mathbb{E}\left[\int_0^\infty \left(\frac{c^{x;\eta,\gamma}(t)}{c^{x;\eta,\kappa}(t)}\right) L(t)\mathrm{d}K(t)\right] = \mathbb{E}\left[\int_0^\infty \left(\frac{(1-\Gamma(t))\gamma(t)}{(1-K(t))\kappa(t)}\right) L(t)\mathrm{d}K(t)\right] = \mathbb{E}\left[\int_0^\infty L(t)\mathrm{d}\Gamma(t)\right] \leq 1,$$

where the last inequality follows exactly as (2.5). In other words, with any fixed strategy η representing the fraction of wealth invested in the risky asset, it is optimal to use κ as the consumption rate per unit of wealth.

that $\mathbb{E}\left[\int_0^\infty L(t)\mathrm{d}K(t)\right]^{-1}\widehat{c} \in \mathcal{C}(x)$. Therefore, the interpretation of "optimality" for $\widehat{c}$ is unsatisfactory. Indeed, given that an agent has fixed preference structure given by (L, K), failure of (2.3) is certainly nonsensical in terms of rationality. However, one of the main points of the present paper is to introduce the idea of preference updating as time evolves in a multi-agent environment—see Section 5. The point to keep in mind is that the pair (κ, λ) for a specific agent will be determined as time evolves, by observation of the capital distribution and consumption patterns of all agents in the economy. In particular, preferences will not be initially fixed, but will be formed as part of the equilibrium path. For this reason, it may not be possible to ensure that (2.3) holds, as this is a condition that involves all possible realisations, and would require the agent to think of all possible outcomes of equilibrium paths with every possible combination of heterogeneous agents that themselves update preferences in time.

Remark 2.5 Numéraire-invariant preferences are general enough to accommodate as "optimal" any observed behaviour: for any pair of investment and consumption strategies (as long as they result in a strictly positive investment-consumption process), one can associate a pair (K, L) satisfying conditions (P1) and (P2) which makes the prescribed pair of investment and consumption strategies optimal under the specific numéraire-invariant preferences. Indeed, suppose that the market is described as in Section 1. Let $X \equiv X^{x,h,c} \in \mathcal{X}(x)$ for some $x \in (0, \infty)$ and arbitrary (h, c) that satisfies (1.7). Suppose additionally that X is $\mathbb{P}$-a.e. strictly positive. Define then a pair (κ, λ) via

$$\kappa := Xc, \quad \lambda := \frac{hS\sigma}{X} - \theta. \tag{2.9}$$

Note that $\kappa \in \mathcal{I}^1_+$ and $\lambda \in \mathcal{I}^2$ follow from (1.7), the $\mathbb{P}$-a.e. continuity of the paths of S and X, and the $\mathbb{P}$-a.e. strict positivity of X. Theorem 2.2 implies that (h, c) is optimal for an agent with numéraire-invariant preferences given by the pair (K, L) defined via (3.2), with (κ, λ) given in (2.9).

In conjunction with Remark 2.4, note that the pair (K, L) that will be constructed using the previous discussion may fail to satisfy (2.3). In effect, the way that numéraire-invariant preferences have been presently defined also allows for encoding certain suboptimal behaviour.

3 Equilibrium: Definition and First Consequences

3.1 Agents

Let J be a finite index-set, with each $j \in J$ corresponding to a distinct agent (or group of agents with the same characteristics) in a market as described in Section 1. For each $j \in J$, let (κ_j, λ_j) be a pair of processes such that

$$\kappa_j \in \mathcal{I}^1_+, \quad \lambda_j \in \mathcal{I}^2; \quad j \in J, \tag{3.1}$$

and define the pair (K_j, L_j) via

$$K_j = 1 - \exp\left(-\int_0^\cdot \kappa_j(t)\mathrm{d}t\right), \quad L = \exp\left(-\frac{1}{2}\int_0^\cdot \lambda_j^2(t)\mathrm{d}t + \int_0^\cdot \lambda_j(t)\mathrm{d}W(t)\right); \quad j \in J. \tag{3.2}$$

Agent $j \in J$ will have numéraire-invariant preferences with associated pair (K_j, L_j), as in §2.1; note that (3.1) implies that each pair (K_j, L_j) for $j \in J$ satisfies conditions (P1) and (P2) of §2.1.

Each agent $j \in J$ is endowed with an initial fraction $e_j \in (0, 1)$ of the risky asset; the vector $e \equiv \left(e_j;\ j \in J\right)$ satisfies $\sum_{i \in J} e_i = 1$. Given the risky asset price $S(0)$ at time zero, this endowment translates to a cash equivalent of

$$x_j := e_j S(0); \quad j \in J. \tag{3.3}$$

According to Theorem 2.2, an optimal investment-consumption process $X_j \in \mathcal{X}(x_j)$ and the associated pair (h_j, c_j) such that $X_j = X^{x_j, h_j, c_j}$ for agent $j \in J$ (we drop all "hats" from notation in order to help the reading) are given by

$$X_j := x_j \frac{L_j}{Y}(1 - K_j), \quad h_j := \frac{X_j}{S}\left(\frac{\theta + \lambda_j}{\sigma}\right), \quad c_j := X_j \kappa_j; \quad j \in J. \tag{3.4}$$

In view of (3.4), a straightforward application of Itô's formula implies the following dynamics for the investment-consumption processes:

$$\frac{\mathrm{d}X_j(t)}{X_j(t)} = \left(r(t) + \theta(t)\left(\theta(t) + \lambda_j(t)\right) - \kappa_j(t)\right)\mathrm{d}t + \left(\theta(t) + \lambda_j(t)\right)\mathrm{d}W(t); \quad t \geq 0,\ j \in J. \tag{3.5}$$

These dynamics will be useful later on, in establishing Theorem 5.2.

3.2 Equilibrium

The financial market will be in equilibrium when supply equals demand. The risky asset is assumed to be in positive net-supply, which we simply normalise to be unit. This implies the requirement $\sum_{i \in J} h_i = 1$. Furthermore, in equilibrium, the money market account will clear, in the sense that total borrowing amongst agents should equal total lending. Since the quantity invested in the money market for each agent $j \in J$ is $X_j - h_j S$, the latter requirement translates to $\sum_{i \in J} (X_i - h_i S) = 0$. Given the previous requirement $\sum_{i \in J} h_i = 1$, $\sum_{i \in J} (X_i - h_i S) = 0$, is actually equivalent to $\sum_{i \in J} X_i = S$. We reach the next definition.

Definition 3.1 With the above notation, we say that there is *financial market equilibrium* if the following clearing conditions hold (up to $\mathbb{P}$-evanescence):

$$\sum_{i \in J} X_i = S, \tag{E1}$$

$$\sum_{i \in J} h_i = 1. \tag{E2}$$

3.3 First consequences of equilibrium

As it turns out, the clearing conditions of Definition 3.1 additionally imply that $\sum_{i \in J} c_i = \delta$, which can be thought as clearing in the "goods" market. This condition typically appears in the definition of financial equilibrium; Proposition 3.2 implies it is superfluous.

Proposition 3.2 *Condition (E1) of Definition 3.1 implies that,* $(\mathbb{P} \otimes \mathsf{Leb})$*-a.e.,*

$$\sum_{i \in J} c_i = \delta. \tag{E3}$$

Proof According to Remark 1.5, the process $YX_j + \int_0^\cdot Y(t)c_j(t)\mathrm{d}t$ is a local martingale for each $j \in J$. Similarly, a combination of Remark 1.5 with Remark 1.6 imply that $YS + \int_0^\cdot Y(t)\delta(t)\mathrm{d}t$ is a local martingale. If $\sum_{i\in J} X_i = S$, it follows that the process

$$\int_0^\cdot Y(t)\left(\delta(t) - \sum_{i\in J} c_i(t)\right)\mathrm{d}t = YS + \int_0^\cdot Y(t)\delta(t)\mathrm{d}t - \sum_{i\in J}\left(YX_i + \int_0^\cdot Y(t)c_i(t)\mathrm{d}t\right)$$

is a local martingale with continuous paths. Since $\int_0^\cdot Y(t)\left(\delta(t) - \sum_{i\in J} c_i(t)\right)\mathrm{d}t$ is a process of finite variation, it follows that, $(\mathbb{P}\otimes \mathsf{Leb})$-a.e., $Y\left(\delta - \sum_{i\in J} c_i\right) = 0$ holds. Finally, since $Y > 0$ holds $(\mathbb{P}\otimes \mathsf{Leb})$-a.e., (E3) follows in the $(\mathbb{P}\otimes \mathsf{Leb})$-a.e. sense. □

Remark 3.3 In Definition 3.1, conditions (E1) and (E2) could have been enforced in the $(\mathbb{P}\otimes \mathsf{Leb})$-a.e. sense. Since (E1) involves processes with $\mathbb{P}$-a.e. continuous paths, asking that it holds in the $(\mathbb{P}\otimes \mathsf{Leb})$-a.e. sense would imply that it holds up to $\mathbb{P}$-evanescent sets. We ask that (E2) also holds up to $\mathbb{P}$-evanescent sets, which is stronger but more natural, and does not affect subsequent results. In the setting of Proposition 3.2, one may only conclude that (E3) holds in the $(\mathbb{P}\otimes \mathsf{Leb})$-a.e. sense. However, note that δ is a strictly positive Itô process, and so are the processes X_j for all $j \in J$. Given that $c_j = X_j\kappa_j$ for all $j \in J$, (E3) implies that the processes $(\kappa_j;\ j \in J)$ have to have special structure in order to achieve equilibrium. Indeed, we shall eventually ask that they are strictly positive Itô processes. In this case, the fact that (E3) holds in the $(\mathbb{P}\otimes \mathsf{Leb})$-a.e. sense will also imply that it holds up to $\mathbb{P}$-evanescent sets.

The following definition is motivated by Remark 1.4, in conjunction with statement (3) of Theorem 2.2.

Definition 3.4 Agent $j \in J$ will be said to *act efficiently* if $\mathbb{E}\left[\int_0^\infty L_j(t)\mathrm{d}K_j(t)\right] = 1$ holds.

In financial market equilibrium as described in Definition 3.1, one can characterise the cases where there is no price bubble (as described in Remark 1.6) in the risky asset: it is exactly when all agents act efficiently.

Proposition 3.5 *Suppose that the market is in financial equilibrium. Then, the equality*

$$S(t) = \mathbb{E}\left[\int_t^\infty \frac{Y(u)}{Y(t)}\delta(u)\mathrm{d}u \;\middle|\; \mathcal{F}(t)\right] =: \widetilde{S}(t); \quad t \ge 0$$

holds if and only if every agent in J acts efficiently.

Proof Recall from Theorem 1.3 that, for fixed $j \in J$,

$$X_j(t) = \mathbb{E}\left[\int_t^\infty \frac{Y(u)}{Y(t)}c_j(u)\mathrm{d}u \;\middle|\; \mathcal{F}(t)\right] + B_j(t); \quad t \ge 0,$$

where B_j is a nonnegative process, and $B_j \equiv 0$ holds if and only if agent $j \in J$ acts efficiently. Adding up over all agents, and using (E1) and (E3), we obtain $S = \widetilde{S} + \sum_{j\in J} B_j$. The result now readily follows. □

3.4 Analysis of single-agent equilibrium

Here, financial equilibrium in the case where there is only a single agent will be discussed; mathematically, assume that $J = \{0\}$. Although a rather uninteresting case economically speaking, it will prove useful in the multi-agent analysis of Section 4 (see §4.2), through a *representative agent* framework.

When $J = \{0\}$, condition (E1) of Definition 3.1 becomes $X_0 = S$. Therefore, in financial equilibrium it holds that $c_0 = X_0 \kappa_0 = S\kappa_0$; using (E3), we conclude that

$$S = \frac{\delta}{\kappa_0}, \tag{3.6}$$

in the $(\mathbb{P} \otimes \mathsf{Leb})$-a.e. sense. In particular, κ_0 must $(\mathbb{P} \otimes \mathsf{Leb})$-a.e. coincide with a strictly positive Itô process in equilibrium. Therefore, we enforce κ_0 in the sequel to have dynamics

$$\frac{\mathrm{d}\kappa_0(t)}{\kappa_0(t)} = \alpha_0(t)\mathrm{d}t + \beta_0(t)\mathrm{d}W(t); \quad t \geq 0, \tag{3.7}$$

where the initial value $\kappa_0(0)$ and the processes α_0 and β_0 satisfy

$$\kappa_0(0) \in (0, \infty), \quad \alpha_0 \in \mathcal{I}^1, \quad \beta_0 \in \mathcal{I}^2. \tag{3.8}$$

By (3.4), $h_0 = (X_0/S)(\theta + \lambda_0)/\sigma$. In financial equilibrium, conditions (E1) and (E2) give $X_0 = S$ and $h_0 = 1$; therefore

$$\theta = \sigma - \lambda_0. \tag{3.9}$$

From (3.6), we obtain $\delta = S\kappa_0$. Using (1.5) and (3.7), as well as the fact that $\delta/S = \kappa_0$, straightforward applications of Itô's formula give the following dynamics for $\kappa_0 S$:

$$\frac{\mathrm{d}\left(\kappa_0(t)S(t)\right)}{\kappa_0(t)S(t)} = \left(\alpha_0(t) + r(t) - \kappa_0(t) + \sigma(t)(\theta(t) + \beta_0(t))\right)\mathrm{d}t + \left(\beta_0(t) + \sigma(t)\right)\mathrm{d}W(t); \quad t \geq 0.$$

Comparing the above with (1.3), once again using the fact that $\delta = S\kappa_0$ holds in equilibrium, we obtain the process-equalities (in the $(\mathbb{P} \otimes \mathsf{Leb})$-a.e. sense) $\nu = \alpha_0 + r + \sigma\theta - \kappa_0 + \beta_0\sigma$ and $\xi = \beta_0 + \sigma$. Using also (3.9), we have a complete description of r, θ and σ. In particular, the equilibrium interest rate is given by

$$r = \nu - \alpha_0 + \kappa_0 - (\lambda_0 - \xi)(\beta_0 - \xi), \tag{3.10}$$

while the processes θ and σ associated to the risky asset are given by

$$\theta = \xi - \beta_0 - \lambda_0, \quad \sigma = \xi - \beta_0. \tag{3.11}$$

The point of the above discussion was to provide a full description of equilibrium, in the sense that all market parameters are expressed in terms of the dividend stream δ and the agent's pair (κ_0, λ_0), together with all the coefficients that appear in the dynamics of δ and κ_0. The description of financial equilibrium is indeed complete, as long as we ensure that there is no degeneracy in the market: the requirement $(\mathbb{P} \otimes \mathsf{Leb})\,[\sigma = 0] = 0$ of (1.6) translates into the necessary condition

$$\beta_0 \neq \xi, \quad (\mathbb{P} \otimes \mathsf{Leb})\text{-a.e.} \tag{3.12}$$

Example 3.6 Suppose that L_0 is the unique strong solution of the stochastic differential equation $\mathrm{d}L_0(t) = -L_0^2(t)\mathrm{d}W(t)$ for $t \geq 0$, where $L_0(0) = 1$. In this case, $\lambda_0 \equiv -L_0$. It is well known that L_0 is the reciprocal of a three-dimensional Bessel process, which is the prototypical example of a strict local martingale—see [10]. By the transient property of the three-dimensional Bessel process, it follows that $\mathbb{P}\,[L_0(\infty) = 0] = 1$; since $\int_0^\cdot \lambda_0^2(t)\mathrm{d}t$ is the quadratic variation process of the stochastic logarithm of L_0, $\mathbb{P}\,[L_0(\infty) = 0] = 1$ is actually equivalent to $\mathbb{P}\left[\int_0^\infty \lambda_0^2(t)\mathrm{d}t = \infty\right] = 1$. It follows that we are in the setting of Example

2.1. Recall from the latter example the nondecreasing process $G_0 := \exp\left(\int_0^\cdot \lambda_0^2(t)\mathrm{d}t\right)$. Set $\kappa_0 := \lambda_0^2 = L_0^2$, and define $K_0 = 1 - \exp\left(-\int_0^\cdot \kappa_0(t)\mathrm{d}t\right) = 1 - 1/G_0$. Note that (P1) and (P2) are valid for the pair (K_0, L_0); furthermore, in view of Example 2.1, (the unique) agent $0 \in J$ acts efficiently. In the notation of (3.7), straightforward computations involving the dynamics of $\kappa_0 = L_0^2$ give $\alpha_0 = L_0^2$ and $\beta_0 = -2L_0$.

Define $\delta := L_0^3/G_0$. In the notation of (1.3), an application of Itô's formula gives $\nu = 2L_0^2$ and $\xi = -3L_0$. Condition (3.12) is equivalent to $L_0 \neq 0$ in the $(\mathbb{P} \otimes \mathsf{Leb})$-a.e. sense, which certainly holds since $\mathbb{P}[L_0(t) > 0,\ \forall t \geq 0] = 1$. It follows from (3.10) that the equilibrium interest rate is $r = 0$. Furthermore, (3.6) gives $S = \delta/\kappa_0 = L_0/G_0$ and (3.11) give $\theta = 0$ and $\sigma = -L_0$. In view of the fact that the (unique) agent acts efficiently, Proposition 3.5 implies that the risky asset contains no bubble in this equilibrium model. As has already been hinted out in Remark 1.6, this does not imply that the risky asset is a "good" investment—in fact, reinvesting all dividend proceedings back in the risky asset will be strictly suboptimal, if one is interested about wealth accumulation up to a certain point $T \in (0, \infty)$ in time, as we shall see in the next paragraph.

Assume that an investor starts with initial capital $x = 1$, and at all times invests all capital (including all proceedings from dividends) in the risky asset. In this case, the agent's pure investment wealth process $Z \in \mathcal{X}(1)$ will satisfy the stochastic differential equation

$$\frac{\mathrm{d}Z(t)}{Z(t)} = \frac{\mathrm{d}S(t) + \delta(t)\mathrm{d}t}{S(t)} = (r(t) + \sigma(t)\theta(t))\,\mathrm{d}t + \sigma(t)\mathrm{d}W(t) = -L_0(t)\mathrm{d}W(t) = \frac{\mathrm{d}L_0(t)}{L_0(t)},$$

for $t \geq 0$. In view of $Z(0) = 1 = L_0(0)$, we conclude that $Z = L_0$. Since the dynamics $\mathrm{d}S(t) + \delta(t)\mathrm{d}t = -\left(L_0^3(t)/G(t)\right)\mathrm{d}W(t)$ hold for all $t \geq 0$ and $\mathbb{P}\left[L_0^3(t)/G(t) \neq 0,\ \forall t \geq 0\right] = 1$, market completeness implies that $\mathcal{X}(1)$ already contains all nonnegative stochastic integrals with respect to W. In particular, for fixed $T \in (0, \infty)$ there exists $X \in \mathcal{X}(1)$ such that[5] $X(T) = (1/\mathbb{E}[L_0(T)])L_0(T)$, and since $\mathbb{E}[L_0(T)] < 1$ and $Z = L_0$ hold, it follows that $\mathbb{P}[X(T) > Z(T)] = 1$, which implies that Z is suboptimal as far as pure investment with financial planning horizon $T \in (0, \infty)$ is concerned.

4 Analysis Within Equilibrium

We shall now delve deeper in the analysis of financial equilibrium. In §4.2 we shall extend in a multi-agent setting the analysis of single-agent financial equilibrium that was carried in §3.4: all the elements of equilibrium will be expressed in terms of market primitives, namely, the dividend process δ, as well as the pairs (κ_j, λ_j) which represent the numéraire-invariant preferences of each individual agent $j \in J$. In order to do so, major role will be played by two processes that will act as "factors" in the description of financial equilibrium; these are the capital and consumption distribution amongst agents, and will be analysed in §4.1.

[5] In fact, since L_0 is the reciprocal of a three-dimensional Bessel process, for fixed $T \in (0, \infty)$, the process $X \in \mathcal{X}(1)$ with $X(T) = (1/\mathbb{E}[L_0(T)])L_0(T)$ can be seen to satisfy

$$X(t) = \frac{1}{\mathbb{E}[L_0(T)]}\mathbb{E}[L_0(T) \mid \mathcal{F}(t)] = \frac{2\Phi\left(1/L_0(t)\sqrt{T-t}\right) - 1}{2\Phi\left(1/\sqrt{T}\right) - 1}L_0(t); \quad t \in [0, T],$$

where Φ is the standard normal cumulative distribution function. For similar calculations, see [15, Exercise 2.9]. Also, [9] has further information on financial connections with the three-dimensional Bessel process.

4.1 Capital distribution and consumption fractions

Let J be an arbitrary finite index set. Given the quantities (3.4) associated with each agent $j \in J$, define the processes

$$p_j := \frac{X_j}{\sum_{i \in J} X_i}, \quad q_j := \frac{c_j}{\sum_{i \in J} c_i}; \quad j \in J. \tag{4.1}$$

The process $p \equiv \big(p_j;\ j \in J\big)$ represents the *capital distribution* amongst agents in the economy. Furthermore, the process $q \equiv \big(q_j;\ j \in J\big)$ represents the agent's consumption fractions.[6] Both processes p and q are $\triangle_J$-valued, where

$$\triangle_J := \left\{ z \equiv \big(z_j;\ j \in J\big) \in \mathbb{R}^J \;\middle|\; z_j \geq 0 \text{ for all } j \in J, \text{ and } \sum_{j \in J} z_j = 1 \right\}. \tag{4.2}$$

Define also the relative interior $\triangle_J^\circ$ of the convex set $\triangle_J$ via

$$\triangle_J^\circ := \left\{ z \equiv \big(z_j;\ j \in J\big) \in \mathbb{R}^J \;\middle|\; z_j > 0 \text{ for all } j \in J, \text{ and } \sum_{j \in J} z_j = 1 \right\}. \tag{4.3}$$

In view of (3.4), and recalling the initial endowment vector $e \equiv \big(e_j;\ j \in J\big) \in \triangle_J^\circ$ which implies by (3.3) that $x_j = e_j S(0)$ holds for all $j \in J$, it follows that

$$p_j = \frac{e_j L_j (1 - K_j)}{\sum_{i \in J} e_i L_i (1 - K_i)}, \quad q_j = \frac{e_j L_j (1 - K_j) \kappa_j}{\sum_{i \in J} e_i L_i (1 - K_i) \kappa_i}; \quad j \in J. \tag{4.4}$$

Note that the above facts only rely on the optimality relations (3.4) for the individual agents; the equilibrium relations of Definition 3.1 were not used at all. Furthermore, note that if the initial endowment vector $e \in \triangle_J^\circ$ and the pairs (κ_j, λ_j) for $j \in J$ are exogenously given (which implies that (K_j, L_j) for $j \in J$ are also specified via (3.2)), the processes p and q are completely specified via (4.4); in particular, they are both $\triangle_J^\circ$-valued. In Section 5, we shall discuss equilibrium where the pairs (κ_j, λ_j) for $j \in J$ are not given exogenously, but what is rather given is a mechanism to specify them within equilibrium. There, the dynamics of p and q will be indispensable in establishing the characterisation of financial equilibrium; in order to present these dynamics, we introduce a notation convention that will greatly simplify reading.

Notation 4.1 Let $\chi \equiv \big(\chi_j;\ j \in J\big)$ be a $\mathbb{R}^J$-valued process, and $\pi \equiv \big(\pi_j;\ j \in J\big)$ be a $\triangle_J$-valued process. Define new processes χ_π and $\chi_{j|\pi}$ for all $j \in J$ via

$$\chi_\pi := \sum_{i \in J} \pi_i \chi_i, \quad \text{and} \quad \chi_{j|\pi} := \chi_j - \sum_{i \in J} \pi_i \chi_i = \chi_j - \chi_\pi; \quad j \in J.$$

Furthermore, whenever $\psi \equiv \big(\psi_j;\ j \in J\big)$ is another $\mathbb{R}^J$-valued process, define new processes $(\chi \cdot \psi)_\pi$ and $(\chi \cdot \psi)_{j|\pi}$ for all $j \in J$ via

$$(\chi \cdot \psi)_\pi := \sum_{i \in J} \pi_i \chi_i \psi_i; \quad (\chi \cdot \psi)_{j|\pi} := \chi_j \psi_j - \sum_{i \in J} \pi_i \chi_i \psi_i = \chi_j \psi_j - (\chi \cdot \psi)_\pi; \quad j \in J.$$

[6] Within equilibrium, condition (E3) of Proposition 3.2 implies that $q_j = c_j / \delta$ holds for all $j \in J$; in other words, q_j becomes the fraction of dividend consumption for each agent $j \in J$.

Given the above notation convention, straightforward computations using Itô's formula on (4.4) gives the following dynamics for p:

$$\frac{\mathrm{d}p_j(t)}{p_j(t)} = -\lambda_p(t)\lambda_{j|p}(t)\mathrm{d}t - \kappa_{j|p}(t)\mathrm{d}t + \lambda_{j|p}(t)\mathrm{d}W(t); \quad t \geq 0, \ j \in J, \tag{4.5}$$

with initial conditions $p(0) \equiv \left(e_j;\ j \in J\right) \in \triangle_J^\circ$. In order to write down dynamics for q, certain Itô-process structure has to be enforced on the processes $(\kappa_j;\ j \in J)$. Recalling the discussion in §3.4 (see also Remark 3.3), such structure is necessary if financial equilibrium is to exist. Similar to (3.7) and (3.8), we assume that

$$\kappa_j = k_j + \int_0^\cdot \kappa_j(t)\alpha_j(t)\mathrm{d}t + \int_0^\cdot \kappa_j(t)\beta_j(t)\mathrm{d}W(t); \quad j \in J, \tag{4.6}$$

where, in order for the processes $\left(\kappa_j;\ j \in J\right)$ to be well-defined and $\mathbb{P}$-a.e. strictly positive, the parameters $k \equiv \left(k_j;\ j \in J\right)$, $\alpha \equiv \left(\alpha_j;\ j \in J\right)$ and $\beta \equiv \left(\beta_j;\ j \in J\right)$ satisfy

$$k_j \in (0, \infty), \quad \alpha_j \in \mathcal{I}^1, \quad \beta_j \in \mathcal{I}^2; \quad j \in J. \tag{4.7}$$

Under the force of (4.6), straightforward (albeit, somewhat lengthy) computations using Itô's formula on (4.4) give the following dynamics for q:[7]

$$\begin{aligned}\frac{\mathrm{d}q_j(t)}{q_j(t)} &= \left(\alpha_{j|q}(t) + (\lambda\cdot\beta)_{j|q}(t) - \left(\lambda_q(t) + \beta_q(t)\right)\left(\lambda_{j|q}(t) + \beta_{j|q}(t)\right)\right)\mathrm{d}t - \kappa_{j|q}(t)\mathrm{d}t \\ &\quad + \left(\lambda_{j|q}(t) + \beta_{j|q}(t)\right)\mathrm{d}W(t); \quad t \geq 0, \ j \in J,\end{aligned} \tag{4.8}$$

with initial conditions $q(0) = \left(e_j k_j / \sum_{i\in J} e_i k_i;\ j \in J\right) \in \triangle_J^\circ$.

The following result, interesting in its own right, will prove essential in questions of existence and uniqueness of financial equilibrium with preference updating in Section 5.

Proposition 4.2 *Suppose that (κ_j, λ_j) satisfy (3.1) and define the pairs (K_j, L_j) via (3.2), for all $j \in J$. Furthermore, suppose that $(\kappa_j;\ j \in J)$ satisfy (4.6), with the parameter restrictions (4.7). Then, the processes p and q defined in (4.4) are the unique $\triangle_J^\circ$-valued Itô processes satisfying the stochastic differential equations (4.5) and (4.8) with initial conditions $p(0) = (e_j;\ j \in J)$ and $q(0) = \left(e_j k_j / \sum_{i\in J} e_i k_i;\ j \in J\right)$, respectively.*

Proof Suppose that p is *any* $\triangle_J^\circ$-valued process which satisfies (4.5) with initial conditions $p(0) = (e_j;\ j \in J)$. Define a new $\triangle_J^\circ$-valued process $\overline{p}$ via

$$\overline{p}_j := \frac{p_j / L_j(1 - K_j)}{\sum_{i\in J}\left(p_i / L_i(1 - K_i)\right)}; \quad j \in J. \tag{4.9}$$

Observe that one can recover p from $\overline{p}$ via the relations

$$p_j := \frac{\overline{p}_j L_j(1 - K_j)}{\sum_{i\in J}\left(\overline{p}_i L_i(1 - K_i)\right)}; \quad j \in J. \tag{4.10}$$

Lengthy, but otherwise straightforward, applications of Itô's formula on (4.9) show that $\mathrm{d}\overline{p}_j(t) = 0$ holds for all $t \geq 0$. Since $\overline{p}_j(0) = p_j(0) = e_j$ for all $j \in J$, it follows that $\overline{p}(t) = (e_j;\ j \in J)$, for all $t \geq 0$. In this case, (4.10) shows that p *has* to be given as in (4.4).

[7] Attempting to write down in full the dynamics of (4.8) should help in appreciating Notation 4.1.

Suppose now that that q is *any* $\triangle_J^\circ$-valued Itô process which satisfies the dynamics (4.8), and such that $q(0) = \left(e_j k_j / \sum_{i \in J} e_i k_i;\ j \in J\right)$. Define the $\triangle_J^\circ$-valued process p via

$$p_j = \frac{q_j/\kappa_j}{\sum_{i \in J} (q_i/\kappa_i)}; \quad j \in J, \tag{4.11}$$

and observe that one can recover q from p via the relations

$$q_j = \frac{p_j \kappa_j}{\sum_{i \in J} p_i \kappa_i}; \quad j \in J. \tag{4.12}$$

An application of Itô's formula, using also the dynamics (4.6), shows that p as defined in (4.11) satisfies the dynamics (4.5) with initial conditions $p(0) = (e_j;\ j \in J)$. It follows that p is given as in (4.4). Then, (4.12) implies that q is also given as in (4.4). □

4.2 Analysis of multi-agent equilibrium

Armed with the discussion in §3.4 and §4.1, we move on to a full description of the market parameters in financial equilibrium, given market primitives. As in §4.1, assume that J is an arbitrary finite index set. Each agent $j \in J$ will be associated with a pair (κ_j, λ_j) satisfying (3.1); furthermore, for each agent $j \in J$ the processes κ_j are assumed to be of the form (4.6) with the coefficients (k_j, α_j, β_j) satisfying the restrictions in (4.7).

In addition to the above agents, we shall also consider a *fictitious* agent indexed by “0” (which is *not* assumed to be an element of J), with corresponding pair (κ_0, λ_0) given by $\kappa_0 = \kappa_p$ and $\lambda_0 = \lambda_p$, where $p = (p_j;\ j \in J)$ is given in (4.4). In the sequel, it will become apparent that agent 0 plays the role of a representative agent in the financial market. As is familiar in complete-market financial equilibrium literature, such representative agent has preferences that are a “weighted average” of all the agents' preferences; in the present numéraire-invariant preferences setting, the weights are dynamic and stochastic, and given by the capital distribution $p \equiv (p_j;\ j \in J)$, which will of course be endogenously determined within equilibrium.

A combination of (E1) in Definition 3.1 with (4.1) implies that $X_j = p_j S$ holds for all $j \in J$. It follows that $c_j = X_j \kappa_j = (p_j \kappa_j) S$ holds for all $j \in J$; therefore, by (E3) in Proposition 3.2, $\kappa_p S = \delta$ holds in the $(\mathbb{P} \otimes \mathsf{Leb})$-a.e. sense, where conventions of Notation 4.1 are being used throughout. We have thus established that the equilibrium price of the risky asset equals

$$S = \frac{\delta}{\kappa_p}. \tag{4.13}$$

Note the this corresponds exactly to (3.6) for our representative agent with $\kappa_0 \equiv \kappa_p$. Given the dynamics (4.5) and (4.6), straightforward applications of Itô's formula imply that the process κ_0 satisfies (3.7) with the corresponding processes α_0 and β_0 given by

$$\alpha_0 := \alpha_q + (\lambda \cdot \beta)_q - \lambda_p \beta_q - \lambda_p \left(\lambda_q - \lambda_p\right) - \left(\kappa_q - \kappa_p\right), \quad \beta_0 := \beta_q + \lambda_q - \lambda_p. \tag{4.14}$$

Note that requirements (3.8) are indeed satisfied in view of (3.1) and (4.7), combined with the fact that p and q are $\triangle_J$-valued processes.

Continuing, (3.4) gives $h_j = (X_j/S)(\theta + \lambda_j)/\sigma$ for all $j \in J$; combined with $X_j/S = p_j$ for all $j \in J$ that follows from condition (E1) of Definition 3.1, condition (E2) of Definition 3.1 reads $\sum_{i \in J} p_i (\theta + \lambda_i)/\sigma = 1$ or, equivalently, $(\theta + \lambda_p)/\sigma = 1$; rearranging,

$$\theta = \sigma - \lambda_p.$$

Again, note the this exactly corresponds to (3.6) for our representative agent with $\lambda_0 \equiv \lambda_p$. A careful look at the way that (3.10) and (3.11) were obtained reveals that we may use the same formulas (3.10) and (3.11) in the multi-agent case, as long as one replaces in the latter two equations the pair (κ_0, λ_0) with (κ_p, λ_p) and plugs in the values for the pair (α_0, β_0) given in (4.14). Straightforward algebra gives the values of r, θ and σ in equilibrium; in particular, with $\mathbf{1}_J$ denoting the $\mathbb{R}^J$-valued vector with all unit entries, the equilibrium interest rate satisfies

$$r = \nu - \alpha_q + \kappa_q - ((\beta - \xi \mathbf{1}_J) \cdot (\lambda - \xi \mathbf{1}_J))_q \,, \tag{4.15}$$

while θ and σ are given by

$$\theta = \xi - \beta_q - \lambda_q, \quad \sigma = \xi - \beta_q - \left(\lambda_q - \lambda_p\right). \tag{4.16}$$

Remark 4.3 The equalities (3.10) and (4.15) for the equilibrium interest rate in the single-agent and multi-agent setting are very similar. In fact, (4.15) can be also written as

$$r = \sum_{i \in J} q_i \left(\nu - \alpha_i + \kappa_i - (\beta_i - \xi)(\lambda_i - \xi)\right), \tag{4.17}$$

which allows more direct comparison with (3.10); indeed, (4.17) is nothing but a "weighted" version of (3.10), with the dynamic weights given by the process q.

Equations (4.13), (4.15), (4.16) indeed provide a full description of equilibrium given the dividend rate process and the agents' preference structure, modulo the non-degeneracy requirement $(\mathbb{P} \otimes \mathsf{Leb})\,[\sigma = 0] = 0$ of (1.6), which translates to

$$\xi \neq \beta_q + \lambda_q - \lambda_p, \quad (\mathbb{P} \otimes \mathsf{Leb})\text{-a.e.} \tag{4.18}$$

Remark 4.4 In the development above, we have introduced a representative agent, whose preferences representation pair is equal to $(\kappa_0, \lambda_0) = (\kappa_p, \lambda_p)$. Since p is a $\triangle_J$-valued process, the conditions (3.1) imply that $\kappa_0 \in \mathcal{I}^1_+$ and $\lambda \in \mathcal{I}^2$. Note that the pair (K_0, L_0) is given by

$$K_0 = 1 - \exp\left(-\int_0^{\cdot} \kappa_p(t)\mathrm{d}t\right), \quad L_0 = \exp\left(-\frac{1}{2}\int_0^{\cdot} \lambda_p^2(t)\mathrm{d}t + \int_0^{\cdot} \lambda_p(t)\mathrm{d}W(t)\right) \tag{4.19}$$

Recall the concept of agents' efficient acting in the economy, introduced in Definition 3.4. Since the equilibrium financial market is the same when viewed from both perspectives of either the representative agent or the collection of all agents in J, Proposition 3.5 implies that the representative agent acts efficiently in the market if, *and only if*, all agents in J act efficiently. In mathematical terms, and with (4.19) defining (K_0, L_0), it follows that

$$\mathbb{E}\left[\int_0^{\infty} L_0(t)\mathrm{d}K_0(t)\right] = 1 \iff \mathbb{E}\left[\int_0^{\infty} L_j(t)\mathrm{d}K_j(t)\right] = 1; \quad j \in J.$$

Interestingly, the validity of the equivalence above does not seem to follow from any straightforward mathematical considerations.

Remark 4.5 Even though we shall not attempt to do so here, we note that the very explicit formulas derived above can be used in comparative statics, i.e., in sensitivity analysis of how market primitives affect equilibrium.

5 Equilibrium with Preference Updating

5.1 Mechanism of preference updating

As opposed to having preferences fixed from the beginning of time, we shall be assuming that agents will be updating their preference structures (including subjective views and consumption profiles), taking into account the behaviour and performance of the other participants in the market. More precisely, each agent will be observing the capital and consumption distribution in the economy, and will be updating his preferences accordingly. Below, we describe in mathematical terms the mechanics of this update.

Notation 5.1 Denote by $C(\mathbb{R}_+; \triangle_J)$ the class of continuous functions from $\mathbb{R}_+$ to $\triangle_J$, where the latter set is defined in (4.2). For $n \in \{1, 2\}$, $\mathcal{P}^n$ shall denote the family of functionals

$$\chi : \Omega \times \mathbb{R}_+ \times C(\mathbb{R}_+; \triangle_J) \times C(\mathbb{R}_+; \triangle_J) \mapsto \mathbb{R}$$

with the property that, for any $\triangle_J$-valued Itô processes π and ρ, the process $(\chi(t, \pi, \rho);\ t \geq 0)$ (dependence on $\omega \in \Omega$ is suppressed, as usual) belongs to $\mathcal{I}^n$. (In particular, note that $\mathcal{P}^n$ for $n \in \{1, 2\}$ consist of "predictable" functionals.)

To model the evolution of the consumption profile κ_j of agent $j \in J$, recall from §3.4 and §4.1 that an Itô-process structure has to be imposed on κ_j, $j \in J$, if financial equilibrium is to exist. Consider a vector $k \equiv \big(k_j;\ j \in J\big)$ and functionals $a \equiv \big(a_j;\ j \in J\big)$ and $b \equiv \big(b_j;\ j \in J\big)$ such that

$$k_j \in (0, \infty), \quad a_j \in \mathcal{P}^1, \quad b_j \in \mathcal{P}^2; \quad j \in J. \tag{5.1}$$

Given the previous quantities, as well as p and q of (4.1), we assume that the process κ_j that will specify the consumption propensity in the preference pair of agent $j \in J$ will satisfy

$$\kappa_j = k_j + \int_0^\cdot \kappa_j(t) a_j(t, p, q) \mathrm{d}t + \int_0^\cdot \kappa_j(t) b_j(t, p, q) \mathrm{d}W(t); \quad j \in J.$$

In order to model updating of subjective views, consider functionals $\ell \equiv \big(\ell_j;\ j \in J\big)$ such that

$$\ell_j \in \mathcal{P}^2; \quad j \in J. \tag{5.2}$$

These functionals will be used in the following way: given the capital and consumption distribution processes p and q of (4.1), the process λ_j that will specify the local martingale L_j via (3.2) in the preference pair of agent $j \in J$ will satisfy $\lambda_j(t) = \ell_j(t, p, q)$ for all $t \geq 0$.

For each agent $j \in J$, define the updating mechanism

$$\mathcal{U}_j := \big((k_j, a_j, b_j), \ell_j\big), \text{ where } (k_j, a_j, b_j) \text{ satisfies (5.1) and } \ell_j \text{ satisfies (5.2)}; \quad j \in J. \tag{5.3}$$

Together with the initial endowment vector $e \equiv \big(e_j;\ j \in J\big) \in \triangle^\circ_J$, this will constitute a full mathematical description of the agents. The exogenously specified dividend process δ as in (1.3) such that (1.4) hold will complete the financial market primitives.

5.2 Existence and uniqueness of equilibrium

Taking into view the updating mechanism for the agents described in §5.1 and the dynamics for p and q given in (4.5) and (4.5) in §4.1, it follows that in equilibrium with preference updating, it necessarily holds that

$$\frac{\mathrm{d}\kappa_j(t)}{\kappa_j(t)} = a_j(t,p,q)\mathrm{d}t + b_j(t,p,q)\mathrm{d}W(t); \quad t \geq 0,\ j \in J, \tag{5.4}$$

$$\frac{\mathrm{d}p_j(t)}{p_j(t)} = -\left(\ell_p \ell_{j|p}\right)(t,p,q)\mathrm{d}t - \kappa_{j|p}(t)\mathrm{d}t + \ell_{j|p}(t,p,q)\mathrm{d}W(t); \quad t \geq 0,\ j \in J, \tag{5.5}$$

$$\begin{aligned}\frac{\mathrm{d}q_j(t)}{q_j(t)} &= \left(a_{j|q} + (\ell \cdot b)_{j|q} - \left(\ell_q + b_q\right)\left(\ell_{j|q} + b_{j|q}\right)\right)(t,p,q)\mathrm{d}t - \kappa_{j|q}(t)\mathrm{d}t \\ &\quad + \left(\ell_{j|q} + b_{j|q}\right)(t,p,q)\mathrm{d}W(t); \quad t \geq 0,\ j \in J.\end{aligned} \tag{5.6}$$

with initial conditions

$$\kappa_j(0) = k_j, \quad p_j(0) = e_j, \quad q_j(0) = \frac{k_j e_j}{\sum_{i \in J} k_i e_i}; \quad j \in J. \tag{5.7}$$

The next result gives the corresponding sufficient criterion for financial equilibrium with preference updating. A form of "reverse engineering" is essentially used in its proof.

Theorem 5.2 *Let $\left((\mathcal{U}_j;\ j \in J), (e_j;\ j \in J), \delta\right)$ denote the market primitives, where $(\mathcal{U}_j;\ j \in J)$ satisfy (5.3), $(e_j;\ j \in J) \in \triangle_J^{\circ}$, and δ is as in (1.3), with (1.4) holding. Assume that (κ, p, q) is a $(0,\infty)^J \times \triangle_J^{\circ} \times \triangle_J^{\circ}$-valued process satisfying (5.4), (5.5) and (5.6), with initial conditions (5.7). Define processes $(\alpha_j;\ j \in J)$, $(\beta_j;\ j \in J)$ and $(\lambda_j;\ j \in J)$ via $\alpha_j(t) = a_j(t,p,q)$, $\beta_j(t) = b_j(t,p,q)$ and $\lambda_j(t) = \ell_j(t,p,q)$ for all $t \geq 0$ and $j \in J$. Assume further that condition (4.18) is valid.*

Define S via (4.13), as well as the processes r via (4.15) and (θ,σ) via (4.16). Then, r satisfies (1.1), the pair (θ,σ) satisfies (1.6), and (1.5) holds. With (K_j, L_j) defined as in (3.2) and X_j defined as in (3.4) for $j \in J$, the equalities (4.1) and (4.4) hold. Furthermore, the market as described above is in equilibrium.

Proof To begin with, note that conditions (3.1) and (4.7) are satisfied, which comes as a consequence of (5.1) and (5.2). Since r is defined via (4.15) and (θ,σ) via (4.16), the fact that p and q are $\triangle_J^{\circ}$-valued implies in a straightforward way that (1.1) and (1.6) are satisfied.

Proceeding, note that r defined via (4.15) is assumed to be the interest rate in the economy. Straightforward, but somewhat lengthy, applications of Itô's formula on the process $S := \delta/\kappa_p$ show that S indeed satisfies (1.5).

In the market as described above, the optimal investment-consumption processes X_j are defined as in (3.4). With the definitions of $(\alpha_j;\ j \in J)$, $(\beta_j;\ j \in J)$ and $(\lambda_j;\ j \in J)$ via $\alpha_j(t) = a_j(t,p,q)$, $\beta_j(t) = b_j(t,p,q)$ and $\lambda_j(t) = \ell_j(t,p,q)$ for all $t \geq 0$ and $j \in J$, it follows that the process p is actually a solution to the stochastic differential equation (4.5) with initial conditions $p(0) = (e_j;\ j \in J)$. The uniqueness result of Proposition 4.2 implies that p satisfies

$$p_j = \frac{X_j}{\sum_{i \in J} X_i} = \frac{e_j L_j (1 - K_j)}{\sum_{i \in J} e_i L_i (1 - K_i)}; \quad j \in J,$$

where the processes $(X_j;\ j \in J)$ are given in (3.4).

We proceed in showing (E1). Note that the processes $(X_j;\ j \in J)$ satisfy the dynamics (3.5). Note that

$$\begin{aligned}\frac{\mathrm{d}\left(\sum_{i\in J} X_i(t)\right)}{\sum_{i\in J} X_i(t)} &= \sum_{i\in J} p_i(t)\frac{\mathrm{d}X_i(t)}{X_i(t)}\\ &= \left(r(t) + \theta(t)\left(\theta(t) + \lambda_p(t)\right) - \kappa_p(t)\right)\mathrm{d}t + \left(\theta(t) + \lambda_p(t)\right)\mathrm{d}W(t)\\ &= \left(r(t) + \theta(t)\sigma(t) - \kappa_p(t)\right)\mathrm{d}t + \sigma(t)\mathrm{d}W(t); \quad t \geq 0,\end{aligned}$$

where the last equality follows from $\theta + \lambda_p = \sigma$, which in turn follows from (4.16). Recalling that $\kappa_p = \delta/S$ and comparing with (1.5), we conclude that

$$\frac{\mathrm{d}\left(\sum_{i\in J} X_i(t)\right)}{\sum_{i\in J} X_i(t)} = \frac{\mathrm{d}S(t)}{S(t)}; \quad t \geq 0.$$

Combined with (3.3) that imply $\sum_{i\in J} X_i(0) = S(0)$, we conclude that $\sum_{i\in J} X_i = S$, which is (E1). Given that (E1) holds, and in view of (3.4), (E2) is equivalent to $\theta = \sigma - \lambda_p$, which holds in view of (4.16). It follows that the market is in equilibrium, which completes the proof. □

Remark 5.3 In view of Theorem 5.2, questions of existence and uniqueness of equilibrium with preference updating correspond to the equivalent questions regarding the system (5.4), (5.5), (5.6) of stochastic differential equations with initial conditions (5.7). In this respect, standard results regarding existence and uniqueness of strong solutions of stochastic differential equations (for example, [21, Chapter IX]) can be used.

The system (5.4), (5.5), (5.6) of stochastic differential equations with initial conditions (5.7) appears $(3|J|)$-dimensional, where $|J|$ denotes the cardinality of J. However, noting that p and q are really $(|J| - 1)$-dimensional, and that, given κ and p, q is completely specified by $q_j = p_j\kappa_j / \sum_{i\in J} p_i\kappa_i$ for all $j \in J$, we conclude that the system (5.4), (5.5), (5.6) with initial conditions (5.7) is effectively $(2|J| - 1)$-dimensional.

Appendix A Certain Technical Proofs

A.1 Proof of claims in Example 2.1

Recalling the set-up in Example 2.1, note that $\log G = \int_0^\cdot \lambda^2(t)\mathrm{d}t$. For any $s \geq 0$, define $\tau(s) := \inf\{t \geq 0 \mid \log G(t) > s\}$; then, $(\tau(s);\ s \geq 0)$ is a nondecreasing family of stopping times indexed by s. The assumption $\mathbb{P}[G(\infty) = \infty] = 1$ implies that $\mathbb{P}[\tau(s) < \infty,\ \forall s \geq 0] = 1$. In fact, τ is the inverse of $\log G$, in the sense that $\log G(\tau(s)) = s$ holds $\mathbb{P}$-a.e. for all $s \geq 0$. As a consequence of the result of Dambis, Dubins and Schwarz [17, §3.4.B], there exists a standard Brownian motion Z such that

$$M(s) := L(\tau(s)) = \exp(Z(s) - s/2); \quad s \geq 0.$$

For the purposes of § A.1, processes time-indexed by $s \geq 0$ are adapted to $(\mathcal{G}(s);\ s \geq 0)$, where $\mathcal{G}(s) = \mathcal{F}(\tau(s))$, and should be regarded in this filtration. Note that the process M is a true martingale; in particular, $\mathbb{E}[M(s)] = 1$ holds for all $s \geq 0$.

Consider the first case. In the above notation, $K = 1 - 1/G$ implies that $K(\tau(s)) = 1 - \exp(-s)$ holds for all $s \geq 0$. Using the method of time-change, we compute

$$\int_0^{\tau(s)} L(t)\mathrm{d}K(t) = \int_0^s L(\tau(u))\mathrm{d}K(\tau(u)) = \int_0^s M(u)\exp(-u)\mathrm{d}u; \quad s \geq 0.$$

Taking expectations on both sides of the previous equality, we obtain that

$$\mathbb{E}\left[\int_0^\infty L(t)\mathrm{d}K(t)\right] \geq \mathbb{E}\left[\int_0^{\tau(s)} L(t)\mathrm{d}K(t)\right] = 1 - \exp(-s); \quad s \geq 0;$$

upon sending s to infinity, we infer $\mathbb{E}\left[\int_0^\infty L(t)\mathrm{d}K(t)\right] = 1$.

Consider now the second case, where $K = (1/G)\int_0^\cdot \left(1 - 1/\overline{L}(t)\right)\mathrm{d}G(t)$. With the notation $\overline{M} := \sup_{s \in [0,\cdot]} M(s)$, a standard time-change argument implies that

$$K(\tau(s)) = \exp(-s)\int_0^s \exp(u)\left(1 - \frac{1}{\overline{M}(u)}\right)\mathrm{d}u; \quad s \geq 0.$$

For all $s \geq 0$, define on $(\Omega, \mathcal{G}(s))$ the probability $\check{\mathbb{P}}^s$ with density $M(s)$ with respect to $\mathbb{P}$. Using time-change and a straightforward integration-by-parts argument, it follows that

$$\mathbb{E}\left[\int_0^{\tau(s)} L(t)\mathrm{d}K(t)\right] = \mathbb{E}\left[\int_0^s M(u)\mathrm{d}K(\tau(s))\right] = \check{\mathbb{E}}^s\left[K(\tau(s))\right]; \quad s \geq 0.$$

where $\check{\mathbb{E}}^s$ denotes expectation on the probability space $(\Omega, \mathcal{G}(s), \check{\mathbb{P}}^s)$. In view of Girsanov's theorem [17, §3.5], the law of the process $(M(u);\ u \in [0,s])$ under $\check{\mathbb{P}}^s$ is the same as the law of the process $(1/M(u);\ u \in [0,s])$ under $\mathbb{P}$. Therefore, the law of $\left(\overline{M}(u);\ u \in [0,s]\right)$ under $\check{\mathbb{P}}^s$ is the same as the law of $\left(\underline{M}(u);\ u \in [0,s]\right)$ under $\mathbb{P}$, where $\underline{M} := \inf_{s \in [0,\cdot]} M(s)$. Defining the nondecreasing process $\check{K}$ via

$$\check{K}(s) = \exp(-s)\int_0^s \exp(u)\left(1 - \underline{M}(u)\right)\mathrm{d}u; \quad s \geq 0,$$

it follows that $\check{\mathbb{E}}^s\left[K(\tau(s))\right] = \mathbb{E}\left[\check{K}(s)\right]$. Putting everything together, the monotone convergence theorem applied twice gives

$$\mathbb{E}\left[\int_0^\infty L(t)\mathrm{d}K(t)\right] = \lim_{s\to\infty}\mathbb{E}\left[\int_0^{\tau(s)} L(t)\mathrm{d}K(t)\right] = \lim_{s\to\infty}\mathbb{E}\left[\check{K}(s)\right] = \mathbb{E}\left[\check{K}(\infty)\right].$$

Note that the nondecreasing process $\check{K}$ is $[0,1]$-valued, and that $\mathbb{P}\left[\check{K}(\infty) = 1\right] = 1$, as follows from the fact that $\mathbb{P}\left[M(\infty) = 0\right] = 1$. Therefore, $\mathbb{E}\left[\int_0^\infty L(t)\mathrm{d}K(t)\right] = 1$ follows.

A.2 Proof of Theorem 2.2

Since Y remains strictly positive $\mathbb{P}$-a.e., it follows that $\widehat{X}$ has $\mathbb{P}$-a.e. continuous paths. Recall from (2.6) that $\kappa \in \mathcal{I}^1_+$; since $\widehat{X}$ has continuous paths, we obtain that $\widehat{c} \in \mathcal{I}^1_+$. Since $\widehat{h}S\sigma = \widehat{X}(\theta + \lambda)$, $\lambda \in \mathcal{I}^2$, $\theta \in \mathcal{I}^2$, and $\widehat{X}$ has $\mathbb{P}$-a.e. continuous paths, we conclude that $(\widehat{h}S\sigma) \in \mathcal{I}^2$. We have established that $(\widehat{h}, \widehat{c})$ satisfies (1.7). The fact that $\widehat{X} = X^{x;\widehat{h},\widehat{c}}$ can be seen by straightforward use of Itô's formula, using (1.8), (1.11) and (2.7). (Here, the requirement $(\mathbb{P} \otimes \mathsf{Leb})\left[\sigma = 0\right] = 0$ in (1.6) is crucial.) This establishes part (1) of Theorem 2.2.

Let $c \in \mathcal{C}(x)$; by (1.13), $\mathbb{E}\left[\int_0^\infty Y(t)c(t)\mathrm{d}t\right] \le x$. Given the definition of $\widehat{c}$, note that

$$\frac{1}{\widehat{c}(t)} L(t)\mathrm{d}K(t) = \frac{Y(t)}{x}\frac{1}{\kappa(t)}\frac{\mathrm{d}K(t)}{1-K(t)} = \frac{Y(t)}{x}\mathrm{d}t; \quad t \ge 0;$$

it then follows that

$$\mathbb{E}\left[\int_0^\infty \frac{c(t)}{\widehat{c}(t)} L(t)\mathrm{d}K(t)\right] = \mathbb{E}\left[\int_0^\infty \frac{Y(t)}{x} c(t)\mathrm{d}t\right] \le 1,$$

which establishes part (2) of Theorem 2.2.

Continuing, recall from Remark 1.4 that $\widehat{X} = X^{[\widehat{c}]}$ is equivalent to $\widehat{X}(0) = x(\widehat{c})$ or, equivalently, $\mathbb{E}\left[\int_0^\infty Y(t)\widehat{c}(t)\mathrm{d}t\right] = x$. Note that $Y(t)\widehat{c}(t)\mathrm{d}t = xL(t)(1-K(t))\kappa(t)\mathrm{d}t = xL(t)\mathrm{d}K(t)$ holds for all $t \ge 0$. It follows now that $\widehat{X} = X^{[\widehat{c}]}$ is equivalent to (2.3). Suppose now that (2.3) holds, and let $\overline{c} \in \mathcal{C}(x)$ be such that $\mathbb{E}\left[\int_0^\infty (\widehat{c}(t)/\overline{c}(t))\, L(t)\mathrm{d}K(t)\right] \le 1$ holds, along with $\mathbb{E}\left[\int_0^\infty (\overline{c}(t)/\widehat{c}(t))\, L(t)\mathrm{d}K(t)\right] \le 1$. In that case, (2.3) implies that

$$\mathbb{E}\left[\int_0^\infty \left(\frac{(\overline{c}(t)-\widehat{c}(t))}{\widehat{c}(t)}\right) L(t)\mathrm{d}K(t)\right] \le 0, \quad \mathbb{E}\left[\int_0^\infty \left(\frac{(\widehat{c}(t)-\overline{c}(t))}{\overline{c}(t)}\right) L(t)\mathrm{d}K(t)\right] \le 0.$$

Adding up the previous two inequalities, we obtain

$$\mathbb{E}\left[\int_0^\infty \left(\frac{(\overline{c}(t)-\widehat{c}(t))^2}{\widehat{c}(t)\overline{c}(t)}\right) L(t)\mathrm{d}K(t)\right] \le 0.$$

Since $L(t)\mathrm{d}K(t) = (1-K(t))L(t)\kappa(t)\mathrm{d}t$ holds for all $t \ge 0$, L is a $\mathbb{P}$-a.e. strictly positive processes and (P1) implies that $\mathbb{P}[1-K(t) > 0, \forall t \ge 0] = 1$, it follows that $\{\overline{c} \neq \widehat{c},\ \kappa > 0\}$ has zero $(\mathbb{P} \otimes \mathsf{Leb})$-measure. Now, note that $\{\widehat{c} > 0,\ \kappa = 0\}$ has zero $(\mathbb{P} \otimes \mathsf{Leb})$-measure; therefore, if $\{\overline{c} \neq \widehat{c},\ \kappa = 0\}$ had nonzero $(\mathbb{P} \otimes \mathsf{Leb})$-measure, it would follow that $\mathbb{E}\left[\int_0^\infty Y(t)\overline{c}(t)\mathrm{d}u\right] > \mathbb{E}\left[\int_0^\infty Y(t)\widehat{c}(t)\mathrm{d}u\right] = x$, which is impossible in view of (1.13), since $\overline{c} \in \mathcal{C}(x)$. We infer that $(\mathbb{P} \otimes \mathsf{Leb})\,[\overline{c} \neq \widehat{c}] = 0$ holds under (2.3), which concludes the proof.

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