



Projections in Enlargements of Filtrations under Jacod's Absolute Continuity Hypothesis for Marked Point Processes

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Abstract

We consider the initial enlargement $\mathbb{F}^{(\zeta)}$ of a filtration $\mathbb{F}$ (called the reference filtration) generated by a marked point process with a random variable ζ . We assume Jacod's absolute continuity hypothesis, that is, the existence of a nonnegative conditional density for this random variable with respect to $\mathbb{F}$. Then, we derive explicit expressions for the coefficients that appear in the integral representation for the optional projection of an $\mathbb{F}^{(\zeta)}$ -(square integrable) martingale on $\mathbb{F}$. In the case in which ζ is strictly positive (called a random time in that case), we also derive explicit expressions for the coefficients, that appear in the related representation for the optional projection of an $\mathbb{F}^{(\zeta)}$ -martingale on $\mathbb{G}$, the reference filtration progressively enlarged by ζ . We also provide similar results for the $\mathbb{F}$ -optional projection of any martingale in $\mathbb{G}$. The arguments of the proof are built on the methodology that was developed in our paper (Gapeev et al. in *Electron J Probab* 26:1–24 2021) in the Brownian motion setting under the more restrictive Jacod's equivalence hypothesis.

Keywords Marked point process · Compensator random measure · Conditional probability density · Jacod's absolute continuity hypothesis · Initial and progressive enlargements of filtrations · Weak representation property · Changes of probability measures

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1 Introduction

In this paper, we consider the initial enlargement of a filtration $\mathbb{F}$ (called hereafter the reference filtration) generated by a marked point process with finite activity (MPP for short) with a random variable ζ , denoted by $\mathbb{F}^{(\zeta)}$. In the case in which ζ is strictly positive (called hereafter a random time), we consider $\mathbb{G}$, the progressive enlargement of $\mathbb{F}$ with this random time. We refer the reader to the monograph [1] for the results on enlargements of filtrations, to the monographs [6] and [26] for the studies on marked point processes, and to the monograph [5] and article [31] for the applications of models based on marked point processes in financial mathematics. The reason why we are working with such processes is that a marked point process in $\mathbb{F}$ remains a marked point process in any larger filtration, as a random measure, that is, the jump measure of a semimartingale with finite activity, with possibly a different compensator, and admits the weak representation property (cf. [20] and [8] for the properties with respect to the initial and progressive enlargements of the reference filtrations, respectively).

We assume in the whole paper that Jacod's absolute continuity hypothesis introduced in [2] and [18] holds (see Sect. 3 for details). We study the relationships between the integral representations of martingales in the initially (resp. progressively) enlarged filtration and their various optional projections. An important application of our results is presented in [4] for the study of the characteristics of semimartingales and their optional projections. Our results also play a crucial role for the comparison of optimal strategies of investors having different information flows (cf., e.g. [2] and [18]). Note that, without any difficulties, one can study models driven by independent Brownian motions and marked point processes, this would simply lead to longer formulae. Detailed studies of the weak representation property, when the process also has a continuous martingale part, are provided in [10] (cf. also [11, 12]). We do not study the optional projections of local martingales, since they may fail to be local martingales (cf., e.g. [36]). We recall that bounded (or positive) processes admit optional projections (cf., e.g. [9, Ch. VI, Th. 43]).

The paper is organised as follows: In Sect. 2, we recall standard results of stochastic analysis that we use in the paper. In Sect. 3, we recall some basic definitions and results related to the initial enlargement of a filtration $\mathbb{F}$ generated by a marked point process with a random variable ζ , denoted by $\mathbb{F}^{(\zeta)}$, and to $\mathbb{G}$, the progressive enlargement of $\mathbb{F}$ with a random time $\zeta = \tau$, in the case in which ζ is strictly positive, under Jacod's absolute continuity hypothesis, which is less restrictive than Jacod's equivalence hypothesis used in our previous paper [17]. In Sect. 4, we recall the well-known results that the weak representation property holds in the reference filtration $\mathbb{F}$ and in its initial enlargement $\mathbb{F}^{(\zeta)}$ with respect to the compensated random measure and, with an addition of another martingale, in the progressive enlargement $\mathbb{G}$, when $\zeta = \tau$ is a random time. In Sect. 5, we consider the optional projection of an $\mathbb{F}^{(\zeta)}$ -(square integrable) martingale on the filtration $\mathbb{F}$ as well as the optional projections of an $\mathbb{F}^{(\tau)}$ -(square integrable) martingale on the filtrations $\mathbb{G}$ and $\mathbb{F}$, in the case in which $\zeta = \tau$

is a random time. We derive explicit expressions for the coefficients in the integral representations of these projections in terms of the original $\mathbb{F}^{(\zeta)}$ -martingale and the components in its representation as a stochastic integral and give analogous results in the case of the $\mathbb{F}$ -optional projection of a $\mathbb{G}$ -martingale. In Sect. 6, we consider the corresponding optional projection of a strictly positive $\mathbb{F}^{(\zeta)}$ -(square integrable) martingale on $\mathbb{F}$, the optional projection of a strictly positive $\mathbb{F}^{(\tau)}$ -(square integrable) martingale on $\mathbb{G}$ and $\mathbb{F}$, in the case in which $\zeta = \tau$ is a random time, as well as the $\mathbb{F}$ -optional projection of a strictly positive (square integrable) $\mathbb{G}$ -martingale. We also describe the set of equivalent (local) martingale measures in the associated extension of the exponential model driven by a marked point process and enhanced with the random time $\zeta = \tau$. In particular, we show that the set of equivalent martingale probability measures in the model with the progressively enlarged filtration $\mathbb{G}$ is essentially larger than the one obtained by means of the optional projections on $\mathbb{G}$ of the Radon–Nikodym densities in the model with the initially enlarged filtration $\mathbb{F}^{(\tau)}$. Some technical proofs are presented in Appendix.

2 Preliminary Definitions and Results

We work on a standard complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$, on which there exists a sequence $(T_n, Z_n)_{n \geq 1}$, where $(T_n)_{n \geq 1}$ is a strictly increasing sequence of finite strictly positive integrable random variables with no accumulation points, and $(Z_n)_{n \geq 1}$ a sequence of (real-valued) random variables. We shall say that the sequence $(T_n, Z_n)_{n \geq 1}$ is a *marked point process* (MPP) (cf. [21]).

We denote by $\mathcal{B}(\mathbb{R}_+)$ (resp. $\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R} \setminus \{0\})$) the σ -algebra of the Borel sets on $\mathbb{R}_+ \equiv [0, \infty)$ (resp. $\mathbb{R}$ or $\mathbb{R} \setminus \{0\}$) and introduce the random measure on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ defined, for any set $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ and any $t \geq 0$, by

$$\mu(\omega; (0, t], A) = \sum_{n \geq 1} \mathbb{I}_{\{T_n(\omega) \leq t\}} \mathbb{I}_{\{Z_n(\omega) \in A\}}, \quad (1)$$

which is called the *random jump measure* of the MPP (cf., e.g. [19, Def. 11.3]). We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the *natural filtration* of the MPP given by

$$\mathcal{F}_t = \sigma(\mu(\cdot; (a, b], A), 0 \leq a < b \leq t, A \in \mathcal{B}(\mathbb{R} \setminus \{0\})), \quad \forall t \geq 0, \quad (2)$$

which is a right-continuous filtration (cf. Proposition 3.39 in [22]), so that $\mathcal{F}_0$ is trivial. We call $\mathbb{F}$ hereafter the *reference filtration* and note that all the jump times T_n , for $n \geq 1$, in the representation of (1) are $\mathbb{F}$ -stopping times. For any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, we define the *compensator random measure* $\nu \equiv \nu(\omega; (0, t], A)$ of $\mu \equiv \mu(\omega; (0, t], A)$ with respect to $\mathbb{F}$ from (2) as the unique predictable random measure

$$\nu(\omega; (0, t], A) = \int_0^t \int_A \nu(\omega; ds, dz), \quad \forall t \geq 0, \quad (3)$$

such that the compensated random measure $\tilde{\mu} \equiv \tilde{\mu}(\omega; (0, t], A)$ defined by

$$\tilde{\mu}(\omega; (0, t], A) := \mu(\omega; (0, t], A) - \nu(\omega; (0, t], A), \quad \forall t \geq 0, \quad (4)$$

induces an $\mathbb{F}$ -local martingale, for each $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed. More generally, if $\mathbb{K}$ is a filtration larger than $\mathbb{F}$, we say that a $\mathbb{K}$ -predictable random measure $\nu^{\mathbb{K}}$ is the $\mathbb{K}$ -compensator of μ . In this respect, the associated random compensated measure $\tilde{\mu}^{\mathbb{K}}$ defined by

$$\tilde{\mu}^{\mathbb{K}}(\omega; (0, t], A) := \mu(\omega; (0, t], A) - \nu^{\mathbb{K}}(\omega; (0, t], A), \quad \forall t \geq 0,$$

induces a $\mathbb{K}$ -local martingale $\tilde{\mu}^{\mathbb{K}}((0, \cdot], A) = (\tilde{\mu}^{\mathbb{K}}((0, t], A))_{t \geq 0}$, while the $\mathbb{K}$ -predictable random measure $\nu^{\mathbb{K}}(\omega; (0, t], A)$, for all $t \geq 0$, induces a $\mathbb{K}$ -predictable process $\nu^{\mathbb{K}}((0, \cdot], A) = (\nu^{\mathbb{K}}((0, t], A))_{t \geq 0}$, for each $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed.

Hypothesis 2.1 We assume, as in Chapter VIII, Definition D5, page 236 of [6], that the $\mathbb{F}$ -compensator random measure $\nu \equiv \nu(\omega; dt, dz)$ from (3) and (4) admits the representation

$$\nu(\omega; dt, dz) = \eta_t(\omega; dz) dt, \quad \forall t \geq 0, \quad (5)$$

where $\eta_t(dz) \equiv \eta_t(\omega; dz)$ is the intensity kernel of ν . Moreover, we assume that the intensity kernel $\eta_t(dz)$ from (5) is integrable, that is, the condition

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R} \setminus \{0\}} \eta_s(dz) ds \right] < \infty, \quad \forall t \geq 0, \quad (6)$$

holds. In this respect, the pair of $(\mathbb{P}, \mathbb{F})$ -local characteristics of $\nu \equiv \nu(dt, dz)$ is given by $(\eta_t(\mathbb{R} \setminus \{0\}), \eta_t(dz)/\eta_t(\mathbb{R} \setminus \{0\}))$, where $\eta_t(\mathbb{R} \setminus \{0\}) = (\eta_t(\mathbb{R} \setminus \{0\}))_{t \geq 0}$ is a nonnegative $\mathbb{F}$ -predictable process and $\eta_t(dz)/\eta_t(\mathbb{R} \setminus \{0\})$ is a probability transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+))$ into $(\mathbb{R} \setminus \{0\}, \mathcal{B}(\mathbb{R} \setminus \{0\}))$.

Note that, under the condition of (6), for any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed, the process $\tilde{\mu}((0, \cdot], A)$ is an $\mathbb{F}$ -martingale (cf. Chapter VIII, Corollary C4, page 235 in [6]).

As usual, we denote by $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{F})$) the predictable (resp. optional) σ -algebra on $\mathbb{F}$. For a family of processes $\xi(z) = (\xi_t(z))_{t \geq 0}$, parametrised by $z \in \mathbb{R} \setminus \{0\}$, we shall say that the function ξ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable, when the map $(t, \omega, z) \rightarrow \xi_t(\omega; z)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable, and we define other $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable functions in a similar way.

Recall that, under Hypothesis 2.1, if ξ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function such that the condition

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R} \setminus \{0\}} |\xi_s(z)| \eta_s(dz) ds \right] < \infty, \quad \forall t \geq 0, \quad (7)$$

holds, then the process $Y = (Y_t)_{t \geq 0}$ defined by

$$Y_t = Y_0 + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \xi_s(z) \tilde{\mu}(ds, dz), \quad \forall t \geq 0, \quad (8)$$

is an $\mathbb{F}$ -martingale (cf. Chapter VIII, Corollary C4, page 235 in [6]). Furthermore, any $\mathbb{F}$ -martingale Y admits a representation as in (8) with some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ξ such that

$$\int_0^t \int_{\mathbb{R} \setminus \{0\}} |\xi_s(z)| \eta_s(dz) ds < \infty, \quad \forall t \geq 0,$$

holds (cf. Chapter VIII, Theorem T8, page 239 in [6] and Theorem 2.2 in [31]). This property is referred to as the *weak representation property* (WRP) of the marked point process in the filtration $\mathbb{F}$ with respect to the *compensated jump measure* $\tilde{\mu} = \mu - \nu$ (cf. Theorem 4.37 in [23]). Such a representation is essentially unique ($\mathbb{P} \times \eta_t(dz) \times dt$ -a.e.).

Let $X = (X_t)_{t \geq 0}$ be a measurable process and $\mathbb{H}$ be a filtration satisfying the usual hypotheses of completeness and right continuity. We denote by ${}^{p, \mathbb{H}}X = ({}^{p, \mathbb{H}}X_t)_{t \geq 0}$ (resp. ${}^{o, \mathbb{H}}X = ({}^{o, \mathbb{H}}X_t)_{t \geq 0}$) the $\mathbb{H}$ -predictable (resp. $\mathbb{H}$ -optional) projection of the process X whenever they exist (cf. Chapter V, Th. 5.2 (resp. 5.1) in [19] or Section 1.3.1, page 15 in [1]).

3 Jacod's Absolute Continuity Hypothesis

In the whole paper, we work on a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ which supports a marked point process $(T_n, Z_n)_{n \geq 1}$ with a right-continuous and completed natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and a random variable ζ valued in $\mathbb{R}$. Note that the inclusion $\mathcal{F}_\infty \subset \mathcal{G}$ holds and, in general, this inclusion is strict. We recall that any $\mathbb{F}$ -martingale admits a càdlàg modification (cf. Corollary 2.48 in [19]).

Hypothesis 3.1 *We assume in the whole paper that Jacod's absolute continuity hypothesis holds, that is, the regular conditional distributions of ζ given $\mathcal{F}_t$ are absolutely continuous with respect to the measure $\rho \equiv \rho(\cdot)$, the unconditional law of the random variable ζ , so that the property*

$$\mathbb{P}(\zeta \in B \mid \mathcal{F}_t) \ll \mathbb{P}(\zeta \in B) = \rho(B), \quad \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)},$$

holds, for any $B \in \mathcal{B}(\mathbb{R})$ fixed.

We assume that Hypothesis 3.1 holds, in order to be able to apply the existing results on the enlargements of filtrations and obtain explicit expressions for the coefficients of the (random) functions in the corresponding weak representations given the considered initially and progressively enlarged filtrations $\mathbb{F}^{(\zeta)}$ and $\mathbb{G}$, respectively. This assumption implies (cf. Lemma 2.3 in [15]) that there exists a family of nonnegative processes $p(u) = (p_t(u))_{t \geq 0}$, parametrised by $u \in \mathbb{R}$, such that the function $(\omega, t, u) \mapsto p_t(\omega; u)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ -measurable, and, for each $u \in \mathbb{R}$, the process $p(u) \equiv p(\omega; u)$ is a càdlàg $\mathbb{F}$ -martingale. Moreover, for any (bounded) Borel function f , the equality

$$\mathbb{E}[f(\zeta) \mid \mathcal{F}_t] = \int_{-\infty}^{\infty} f(u) p_t(u) \rho(du), \quad \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)}, \quad (9)$$

holds. The expression in (9) implies that the equality

$$\mathbb{P}(\zeta > s \mid \mathcal{F}_t) = \int_s^\infty p_t(u) \rho(du), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R} \text{ (}\mathbb{P}\text{-a.s.)},$$

is satisfied, so that the property

$$\int_{-\infty}^\infty p_t(u) \rho(du) = 1, \quad \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)},$$

holds, and $p_0(u) = 1$, for each $u \in \mathbb{R}$ fixed.

We shall call the family of $\mathbb{F}$ -optional processes $p(u)$, for each $u \in \mathbb{R}$, the *$\mathbb{F}$ -conditional density family* with respect to $\rho(du)$. Note that, in the case of a random time τ even if the process $p(u)$ is not strictly positive, for each $u \in \mathbb{R}_+$, but the processes $p(\tau)$ and $p_-(\tau)$ are strictly positive (cf. [22, Cor. 1.11] and [1, Equality 4.10]).

The following proposition is proved as a consequence of the weak representation property in the filtration $\mathbb{F}$ of (8) in Sect. 2 justified by Chapter VIII, Theorem T8, p. 239, in [6].

Proposition 3.2 *For each $u \in \mathbb{R}$ fixed, the $\mathbb{F}$ -martingale $p(u)$ admits the representation*

$$dp_t(u) = \int_{\mathbb{R} \setminus \{0\}} f_t(u, z) \tilde{\mu}(dt, dz), \quad \forall t \geq 0, \quad p_0(u) = 1, \quad (10)$$

for an $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function f (inducing the parametrised process $f(u, z) = (f_t(u, z))_{t \geq 0}$, for each $u \in \mathbb{R}$ and $z \in \mathbb{R} \setminus \{0\}$ fixed) such that the condition

$$\int_0^t \int_{\mathbb{R} \setminus \{0\}} |f_s(u, z)| \eta_s(dz) ds < \infty, \quad \forall t \geq 0, \quad \forall u \in \mathbb{R} \text{ (}\mathbb{P}\text{-a.s.)}, \quad (11)$$

holds.

In the case in which $\zeta = \tau$ is a *random time* (a strictly positive random variable), let us denote by $H = (H_t)_{t \geq 0}$ with $H_t = \mathbb{1}_{\{\tau \leq t\}}$, for all $t \geq 0$, the *indicator process*. Moreover, since H is a $\mathbb{G}$ -adapted càdlàg process bounded by 0 and 1, we can introduce the $\mathbb{F}$ -supermartingale $G = (G_t)_{t \geq 0}$ defined by $G = {}^{o, \mathbb{F}}(1 - H)$, that is, the $\mathbb{F}$ -optional projection of the process $1 - H$ satisfying the property

$$G_t = \mathbb{P}(\tau > t \mid \mathcal{F}_t), \quad \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)}, \quad (12)$$

which, according to the equality (9), can be represented in the form

$$G_t = \int_t^\infty p_t(u) \rho(du), \quad \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)}.$$

Note that $G_0 = 1$, since τ is assumed to be strictly positive. The $\mathbb{F}$ -supermartingale G is called the *Azéma supermartingale* of the random time τ .

4 Enlargements of Filtrations and Martingales

For a random variable ζ , we consider the initial enlargement of $\mathbb{F}$ obtained by adding the σ -algebra $\sigma(\zeta)$ at time 0 and denoted by $\mathbb{F}^{(\zeta)}$. In the case in which ζ is strictly positive, we will also consider the progressive enlargement of $\mathbb{F}$ obtained by progressively adding information of $\sigma(\zeta \wedge t)$ at time $t \geq 0$, or, more precisely, the smallest right-continuous filtration $\mathbb{G}$ containing $\mathbb{F}$ and turning out ζ into a stopping time. In the latter case, we will use the traditional notation $\zeta =: \tau$ and call it a *random time*.

The aim of the paper is to explicitly compute the components in the integral representations of the optional projections of the $\mathbb{F}^{(\zeta)}$ -martingales and the $\mathbb{G}$ -martingales. In this section, we recall some well-known results on the initial and progressive enlargements of filtrations generated by marked point processes. In particular, we give the form of the $\mathbb{F}^{(\zeta)}$ -semimartingale decomposition and the $\mathbb{G}$ -semimartingale decomposition of the martingale-valued random measure defined in (4) as well as the $\mathbb{G}$ -semimartingale decomposition of H . We underline that the martingale part $\tilde{\mu}^{(\tau)}((0, \cdot], A)$ of the $\mathbb{F}^{(\tau)}$ -semimartingale decomposition of the process $\tilde{\mu}((0, \cdot], A)$ enjoys the $\mathbb{F}^{(\tau)}$ -weak representation property, while the couple $(\tilde{\mu}^{\mathbb{G}}((0, \cdot], A), M^{\mathbb{G}})$ of the martingale parts of the $\mathbb{G}$ -semimartingale decompositions of the processes $\tilde{\mu}((0, \cdot], A)$, for any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed, and H enjoy the $\mathbb{G}$ -weak representation property, where the stochastic integral with respect to this couple is understood componentwise.

4.1 The Initially Enlarged Filtration

As in the introduction, let us denote by $\mathbb{F}^{(\zeta)} = (\mathcal{F}_t^{(\zeta)})_{t \geq 0} = (\mathcal{F}_t \vee \sigma(\zeta))_{t \geq 0}$ the initial enlargement of the filtration $\mathbb{F}$ with the random variable ζ , so that $\mathcal{F}_0^{(\zeta)} = \sigma(\zeta)$ holds. We recall that, under Hypothesis 3.1, any $\mathbb{F}$ -local martingale is an $\mathbb{F}^{(\zeta)}$ -special semimartingale (cf., e.g., Theorem 2.1 in [21] or Proposition 5.30, page 116 in [1]).

Note that, according to Proposition 4.20 in [1], the filtration $\mathbb{F}^{(\zeta)}$ is right-continuous.

Notation 4.1 We further denote $\mathbb{F}^{(\zeta)}$ -adapted processes with the superscript (ζ) as in $Y^{(\zeta)}$. We denote $\mathbb{F}$ -adapted processes by capital letters as X , or lower case x , or φ , or even x^0 .

We recall that, for any $t \geq 0$ fixed, any $\mathcal{F}_t^{(\zeta)}$ -measurable random variable $Y_t^{(\zeta)}$ is of the form $Y_t(\omega; \zeta(\omega))$, for some $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$ -measurable function $(\omega, u) \mapsto Y_t(\omega; u)$ (cf., e.g., Proposition 2.7, part (i) in [7]). In particular, any $\mathcal{F}_0^{(\zeta)}$ -measurable random variable is a Borel function of ζ . We also recall that any $\mathbb{F}^{(\zeta)}$ -predictable process can be represented in the form $Y_t(\omega; \zeta(\omega))$, for all $t \geq 0$, where the mapping $(\omega, t, u) \mapsto Y_t(\omega; u)$ defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ and valued in $\mathbb{R}$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ -measurable. Moreover, under Hypothesis 3.1, any $\mathbb{F}^{(\zeta)}$ -optional process $Y^{(\zeta)} = (Y_t^{(\zeta)})_{t \geq 0}$ can be written as $Y_t^{(\zeta)} = Y_t(\zeta)$, for all $t \geq 0$, where the parametrised process $Y = (Y_t(u))_{t \geq 0}$, for each $u \in \mathbb{R}$, is associated with an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ -measurable function Y (cf. Theorem 6.9 in [35]).

As an immediate consequence of Hypothesis 3.1, we observe that, for each $t \geq 0$ fixed, if the $\mathcal{F}_t^{(\zeta)}$ -measurable random variable $Y_t^{(\zeta)}$ is integrable, then the representation

$$\mathbb{E}[Y_t^{(\zeta)} | \mathcal{F}_t] = \int_{-\infty}^{\infty} Y_t(u) p_t(u) \rho(du), \quad \forall t \geq 0,$$

holds (cf., e.g., Proposition 4.18 (b), page 85 in [1]).

In the following proposition, we give the $\mathbb{F}^{(\zeta)}$ -semimartingale decomposition of the process $\tilde{\mu}((0, \cdot], A) = (\tilde{\mu}((0, t], A))_{t \geq 0}$, defined in (4), for each $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed.

Proposition 4.2 *We assume that, for any $t \geq 0$, together with (6), the condition*

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R} \setminus \{0\}} \left| \frac{f_s(\zeta, z)}{p_{s-}(\zeta)} \right| \eta_s(dz) ds \right] < \infty \quad (13)$$

holds, where the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function f is given by (10) and satisfies (11). Then, for any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed, the $\mathbb{F}$ -martingale $\tilde{\mu}((0, \cdot], A)$ is decomposed as

$$\tilde{\mu}((0, t], A) = \tilde{\mu}^{(\zeta)}((0, t], A) + \int_0^t \int_A \frac{f_s(\zeta, z)}{p_{s-}(\zeta)} \eta_s(dz) ds, \quad \forall t \geq 0,$$

where $\tilde{\mu}^{(\zeta)}((0, \cdot], A)$ is an $\mathbb{F}^{(\zeta)}$ -martingale. In other terms, $(T_n, Z_n)_{n \geq 1}$ is an $\mathbb{F}^{(\zeta)}$ -marked point process, where the $\mathbb{F}^{(\zeta)}$ -predictable (finite) random measure

$$\nu^{(\zeta)}(dt, dz) = \left(\frac{f_t(\zeta, z)}{p_{t-}(\zeta)} + 1 \right) \eta_t(dz) dt, \quad \forall t \geq 0, \forall z \in \mathbb{R} \setminus \{0\}, \quad (14)$$

is the $\mathbb{F}^{(\zeta)}$ -compensator of the random jump measure μ . In particular, we have $1 + f_t(\zeta, z)/p_{t-}(\zeta) \geq 0$, for all $t \geq 0$ and each $z \in \mathbb{R} \setminus \{0\}$.

Proof From the results on the initial enlargements of filtrations¹ (cf. also Chapter VIII, Corollary C4, page 235 in [6]), for any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed, the process $\tilde{\mu}^{(\zeta)}((0, \cdot], A)$ defined by

$$\tilde{\mu}^{(\zeta)}((0, t], A) = \tilde{\mu}((0, t], A) - \int_0^t \frac{d\langle \tilde{\mu}((0, \cdot], A), p(u) \rangle_s^{\mathbb{F}}}{p_{s-}(u)} \Big|_{u=\zeta}, \quad \forall t \geq 0,$$

¹ One applies Theorem 2.1 in [21], which states that, under Hypothesis 3.1, for any $\mathbb{F}$ -local martingale $X = (X_t)_{t \geq 0}$, the process $X^{(\zeta)} = (X_t(\zeta))_{t \geq 0}$ defined by

$$X_t(\zeta) = X_t - \int_0^t \frac{d\langle X, p(u) \rangle_s^{\mathbb{F}}}{p_{s-}(u)} \Big|_{u=\zeta}, \quad \forall t \geq 0,$$

is an $\mathbb{F}^{(\zeta)}$ -local martingale.

is an $\mathbb{F}^{(\zeta)}$ -martingale. In order to compute the predictable covariation of the processes $\tilde{\mu}((0, \cdot], A)$ and $p(u)$, for any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ and $u \in \mathbb{R}$ fixed, we start by computing their quadratic covariation process (cf. [19, Def. 8.2]). From the result of [29, Pro. 1.1.6], for any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed, for the martingale $\tilde{\mu}((0, \cdot], A)$ being of finite variation, we have

$$[\tilde{\mu}((0, \cdot], A), p(u)]_t = \int_0^t \int_A f_s(u, z) \mu(ds, dz), \quad \forall t \geq 0, \quad \forall u \in \mathbb{R},$$

and hence, since the predictable covariation process is the dual predictable projection of the quadratic covariation process (cf., e.g., [19, Theorem 6.28, part 2]), we obtain

$$\langle \tilde{\mu}((0, \cdot], A), p(u) \rangle_t^{\mathbb{F}} = \int_0^t \int_A f_s(u, z) \eta_s(dz) ds, \quad \forall t \geq 0, \quad \forall u \in \mathbb{R}.$$

It follows that, for any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed, the process

$$\tilde{\mu}^{(\zeta)}((0, t], A) = \mu((0, t], A) - \int_0^t \int_A \left(\frac{f_s(\zeta, z)}{p_{s-}(\zeta)} + 1 \right) \eta_s(dz) ds, \quad \forall t \geq 0,$$

is an $\mathbb{F}^{(\zeta)}$ -martingale, and the $\mathbb{F}^{(\zeta)}$ -compensator of the measure μ is

$$\nu^{(\zeta)}(dt, dz) = \left(\frac{f_t(\zeta, z)}{p_{t-}(\zeta)} + 1 \right) \eta_t(dz) dt, \quad \forall t \geq 0, \quad \forall z \in \mathbb{R} \setminus \{0\},$$

since the process $\tilde{\mu}^{(\zeta)}((0, \cdot], A) = (\tilde{\mu}^{(\zeta)}((0, t], A))_{t \geq 0}$ is an $\mathbb{F}^{(\zeta)}$ -martingale, that completes the proof. $\square$

Note that the weak representation property for the marked point processes holds in $\mathbb{F}^{(\zeta)}$, since the MPP property is stable under the initial enlargements of the reference filtrations (cf. [20, Th. 5.4]).

Proposition 4.3 *Each $(\mathbb{P}, \mathbb{F}^{(\zeta)})$ -martingale $Y^{(\zeta)} = (Y_t(\zeta))_{t \geq 0}$ admits a representation of the form*

$$Y_t^{(\zeta)} = Y_t(\zeta) = Y_0(\zeta) + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \psi_s(\zeta, z) \tilde{\mu}^{(\zeta)}(ds, dz), \quad \forall t \geq 0, \quad (15)$$

for some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ψ (inducing the parametrised process $\psi(u, z) = (\psi_t(u, z))_{t \geq 0}$, for each $u \in \mathbb{R}$ and $z \in \mathbb{R} \setminus \{0\}$ fixed), satisfying the condition

$$\int_0^t \int_{\mathbb{R} \setminus \{0\}} |\psi_s(\zeta, z)| \nu^{(\zeta)}(ds, dz) < \infty, \quad \forall t \geq 0 \text{ } (\mathbb{P}\text{-a.s.}),$$

where the $\mathbb{F}^{(\zeta)}$ -predictable compensator measure $\nu^{(\zeta)}$ is defined in (14).

4.2 The Progressively Enlarged Filtration

In this subsection, we assume that ζ is a strictly positive random variable called a *random time* and denoted by τ , and we suppose that Hypothesis 3.1 holds. We denote by $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ the progressive enlargement of $\mathbb{F}$ with τ , that is, the right-continuous version of $\mathcal{G}^0$, where

$$\mathcal{G}_t^0 = \bigcap_{s > t} (\mathcal{F}_s \vee \sigma(\tau \wedge s)), \quad \forall t \geq 0. \quad (16)$$

Hypothesis 4.4 *For simplicity of presentation, we further assume that the process G defined in (12) and its $\mathbb{F}$ -predictable projection G_- are strictly positive. In that case, we shall be able to divide by their current values in the expressions below.*

This hypothesis is studied in [33, 34] and [16] where specific cases are given.

Note that τ is a $\mathbb{G}$ -stopping time and that, according to the hypothesis that the random variable τ is strictly positive, the σ -algebra $\mathcal{G}_0$ is trivial, so that the initial value of a $\mathbb{G}$ -adapted process is a constant one. Observe that, under Hypothesis 3.1, any $\mathbb{F}$ -martingale is a $\mathbb{G}$ -semimartingale (cf., e.g., Proposition 5.30, page 116 in [1] or Theorem 3.1 in [24]).

We observe that the completion of the two enlargements $\mathbb{G}$ and $\mathbb{F}^{(\tau)}$ follows from the inclusion $\mathcal{F}_\infty \subset \mathcal{G}_\infty = \mathcal{F}_\infty \vee \sigma(\tau)$. Note that the inclusion is strict, unless τ is an $\mathcal{F}_\infty$ -measurable random time.

Notation 4.5 *We indicate with the superscript $\mathbb{G}$ the processes which are $\mathbb{G}$ -adapted, as $Y^{\mathbb{G}}$, as we shall do now for the $\mathbb{G}$ -adapted process $H^{\mathbb{G}} = (H_t^{\mathbb{G}})_{t \geq 0}$ defined by $H_t^{\mathbb{G}} = \mathbb{1}_{\{\tau \leq t\}}$, for all $t \geq 0$.*

We recall that, under Hypothesis 3.1, any $\mathbb{G}$ -optional process $Y^{\mathbb{G}}$ can be written as:

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau > t\}} Y_t^0 + \mathbb{1}_{\{\tau \leq t\}} Y_t^1(\tau), \quad \forall t \geq 0, \quad (17)$$

where the process Y^0 is $\mathbb{F}$ -optional and the function $Y^1 = (Y^1(t, u), t \geq 0, u \in \mathbb{R}_+)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable (cf. Theorem 6.9 in [35]). A particular case occurs, when $Y^{\mathbb{G}}$ is the optional projection of an $\mathbb{F}^{(\tau)}$ -adapted process $Y^{(\tau)} = (Y_t(\tau))_{t \geq 0}$, where the function $(\omega, u) \mapsto Y_t(\omega; u)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable, while the function Y is not necessarily $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable. In that case, one has

$$Y_t^0 = \frac{1}{G_t} \int_t^\infty Y_t(u) p_t(u) \rho(du), \quad \forall t \geq 0, \quad \text{and} \\ Y_t^1(u) = Y_t(u), \quad \forall t, u \geq 0 \text{ such that } t \geq u,$$

where the process G is defined in (12). Here, Y^0 is called the $\mathbb{F}$ -optional reduction of $Y^{\mathbb{G}}$. We also recall that, under Hypothesis 3.1, any $\mathbb{G}$ -predictable process $Y^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t \geq 0}$ can be written as

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau \geq t\}} \widehat{Y}_t^0 + \mathbb{1}_{\{\tau < t\}} \widehat{Y}_t^1(\tau), \quad \forall t \geq 0,$$

where the process $\widehat{Y}^0$ is $\mathbb{F}$ -predictable and the function $\widehat{Y}^1$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable (cf., e.g., Proposition 2.11, page 36 in [1]). In this case, $\widehat{Y}^0$ is called the $\mathbb{F}$ -predictable reduction of $Y^{\mathbb{G}}$. Note that the càg process $(\mathbb{I}_{\{\tau < t\}})_{t \geq 0}$ is $\mathbb{G}$ -predictable.

As it follows from the Doob–Meyer decomposition of the supermartingale $H^{\mathbb{G}}$ and the fact that any $\mathbb{G}$ -predictable process is equal, on the set $\{\tau \geq t\}$, to an $\mathbb{F}$ -predictable process, there exists an $\mathbb{F}$ -predictable increasing process $\Lambda = (\Lambda_t)_{t \geq 0}$ such that the process $M^{\mathbb{G}} = (M_t^{\mathbb{G}})_{t \geq 0}$ defined by

$$M_t^{\mathbb{G}} = H_t^{\mathbb{G}} - \Lambda_{t \wedge \tau}, \quad \forall t \geq 0, \quad (18)$$

is a $\mathbb{G}$ -martingale. It is known that, under Hypothesis 3.1, the process Λ admits the representation

$$\Lambda_t = \int_0^t \frac{p_s(s)}{G_s} \rho(ds) \equiv \int_0^t \frac{p_{s-}(s)}{G_{s-}} \rho(ds), \quad \forall t \geq 0, \quad (19)$$

where we have also taken into account the fact that the measure ρ has no atoms to replace the term $p_s(s)/G_s$ by $p_{s-}(s)/G_{s-}$, for any $s \geq 0$, in order to obtain the second equality (cf. Corollary 5.27 (b), page 114 in [1]). We recall the following result that, under Hypothesis 3.1, the random time τ avoids $\mathbb{F}$ -stopping times, that is, $\mathbb{P}(\tau = T_0 < \infty) = 0$, for any $\mathbb{F}$ -stopping time T_0 (cf. [13]). In particular, ρ is non-atomic, since constants are stopping times. In this respect, the process $\lambda = (\lambda_t)_{t \geq 0}$ defined by

$$\lambda_t = \frac{p_{t-}(t)}{G_{t-}}, \quad \forall t \geq 0, \quad (20)$$

is the *intensity rate* of τ with respect to the measure $\rho(dt)$ (cf. Proposition 2.15, page 37 in [1]).

Hypothesis 4.6 *We assume in the rest of the paper that the $\mathbb{F}$ -predictable increasing process $\Lambda = (\Lambda_t)_{t \geq 0}$ defined in (19) is of finite expectation, that is, $\mathbb{E}[\Lambda_t] < \infty$, for all $t \geq 0$.*

The Doob–Meyer decomposition of the Azéma supermartingale can be given explicitly, and its multiplicative decomposition is as follows.

Proposition 4.7 *The Doob–Meyer decomposition of the Azéma supermartingale G is given by*

$$G_t = 1 - \int_0^t G_s \lambda_s \rho(ds) + \int_{s=0}^t \int_{\mathbb{R} \setminus \{0\}} \int_{u=s}^{\infty} f_s(u, z) \tilde{\mu}(ds, dz) \rho(du), \quad \forall t \geq 0, \quad (21)$$

or, in a simplified form, by

$$G_t = 1 - \int_0^t G_s \lambda_s \rho(ds) + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \varphi_s(z) \tilde{\mu}(ds, dz), \quad \forall t \geq 0,$$

where the process λ is defined in (20) above and the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function φ (inducing the parametrised process $\varphi(z) = (\varphi_t(z))_{t \geq 0}$, for each $z \in \mathbb{R} \setminus \{0\}$) is given by

$$\varphi_t(z) = \int_t^\infty f_t(u, z) \rho(du), \quad \forall t \geq 0, \forall z \in \mathbb{R} \setminus \{0\}. \quad (22)$$

Therefore, the martingale part $m = (m_t)_{t \geq 0}$ of the Doob–Meyer decomposition of the process G is given by:

$$m_t = \int_0^t \int_{\mathbb{R} \setminus \{0\}} \varphi_s(z) \tilde{\mu}(ds, dz), \quad \forall t \geq 0.$$

Proof The Doob–Meyer decomposition of G is obtained by using Itô–Ventzell’s formula, as it is developed in Theorem 3.1 in [27], to the parametrised process $G(u) = (G_t(u))_{t \geq 0}$, for each $u \in \mathbb{R}$, given by

$$\begin{aligned} G_t(u) &= \mathbb{P}(\tau > u \mid \mathcal{F}_t) = \int_u^\infty p_t(v) \rho(dv) \\ &= \int_u^\infty p_0(v) \rho(dv) + \int_{s=0}^t \int_{\mathbb{R} \setminus \{0\}} \int_{u=v}^\infty f_s(v, z) \tilde{\mu}(ds, dz) \rho(dv), \quad \forall t \geq 0, \end{aligned}$$

where the forward integral (with respect to the compensated measure) in [27] is the usual stochastic integral in our setting since we integrate predictable processes. Then (with the notation of [27], $\gamma = 0$, $H(s, v, z) = \int_v^\infty f_s(w, z) \rho(dw)$), we have

$$G_t = G_t(t) = 1 - \int_0^t G_s \lambda_s \rho(ds) + \int_{s=0}^t \int_{\mathbb{R} \setminus \{0\}} \int_{u=s}^\infty f_s(u, z) \rho(du) \tilde{\mu}(ds, dz), \quad \forall t \geq 0.$$

□

In the following proposition, for any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed, we give the semimartingale decomposition of the process $\tilde{\mu}((0, \cdot], A)$ defined in (4) in the filtration $\mathbb{G}$.

Proposition 4.8 *We assume that, for any $t \geq 0$ fixed, together with (6) and (13), the condition*

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R} \setminus \{0\}} \left| \frac{\varphi_s(z)}{G_{s-}} \right| \eta_s(dz) ds \right] < \infty \quad (23)$$

holds, where the process $\varphi(z)$, for each $z \in \mathbb{R} \setminus \{0\}$, is defined in (22), and the process G is given by (12). Then, for any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed, the $\mathbb{G}$ -semimartingale decomposition of the martingale $\tilde{\mu}((0, \cdot], A)$ is given by:

$$\begin{aligned} \tilde{\mu}((0, t], A) &= \tilde{\mu}^{\mathbb{G}}((0, t], A) + \int_0^{t \wedge \tau} \int_A \frac{\varphi_s(z)}{G_{s-}} \eta_s(dz) ds \\ &\quad + \int_{t \wedge \tau}^t \int_A \frac{f_s(\tau, z)}{p_{s-}(\tau)} \eta_s(dz) ds, \quad \forall t \geq 0, \end{aligned} \quad (24)$$

where the process $\tilde{\mu}^{\mathbb{G}}((0, \cdot], A)$ is a $\mathbb{G}$ -martingale, and the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function f is defined in (10) and satisfies (11). Moreover, since we assume that the conditions of (23) and (13) hold with (6), the $\mathbb{G}$ -predictable (finite) random measure

$$\nu^{\mathbb{G}}(dt, dz) = \left[\mathbb{1}_{\{\tau \geq t\}} \left(\frac{\varphi_t(z)}{G_{t-}} + 1 \right) + \mathbb{1}_{\{\tau < t\}} \left(\frac{f_t(\tau, z)}{p_{t-}(\tau)} + 1 \right) \right] \eta_t(dz) dt, \quad \forall t \geq 0, \quad \forall z \in \mathbb{R} \setminus \{0\}, \quad (25)$$

is the $\mathbb{G}$ -compensator of the random jump measure μ .

Proof For any $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ fixed, the $\mathbb{G}$ -semimartingale decomposition² of the $\mathbb{F}$ -martingale $\tilde{\mu}((0, \cdot], A)$ is given by

$$\begin{aligned} \tilde{\mu}((0, t], A) &= \tilde{\mu}^{\mathbb{G}}((0, t], A) + \int_0^{t \wedge \tau} \frac{d\langle \tilde{\mu}((0, \cdot], A), m \rangle_s^{\mathbb{F}}}{G_{s-}} \\ &\quad + \int_{t \wedge \tau}^t \frac{d\langle \tilde{\mu}((0, \cdot], A), p(u) \rangle_s^{\mathbb{F}}}{p_{s-}(u)} \Big|_{u=\tau} \\ &= \tilde{\mu}^{\mathbb{G}}((0, t], A) + \int_0^{t \wedge \tau} \int_A \frac{\varphi_s(z)}{G_{s-}} \eta_s(dz) ds \\ &\quad + \int_{t \wedge \tau}^t \int_A \frac{f_s(\tau, z)}{p_{s-}(\tau)} \eta_s(dz) ds, \quad \forall t \geq 0, \end{aligned}$$

where the process $\tilde{\mu}^{\mathbb{G}}((0, \cdot], A)$ forms a $\mathbb{G}$ -martingale. It thus follows that the $\mathbb{G}$ -compensator of μ is given by (25). $\square$

Remark 4.9 Note that, for any $A \in \mathcal{B}(\mathbb{R})$ fixed, the $\mathbb{G}$ -predictable process $\nu^{\mathbb{G}}((0, \cdot], A) = (\nu^{\mathbb{G}}((0, t], A))_{t \geq 0}$, which is associated with the $\mathbb{G}$ -predictable compensator measure $\nu^{\mathbb{G}}$ of the random jump measure μ given by (25), is increasing. Moreover, the value $f_t(\tau, z)/p_{t-}(\tau) + 1$ is nonnegative ($\mathbb{P} \times \eta_t(dz) \times dt$ -a.e.), and the value $\varphi_t(z)/G_{t-} + 1$ is nonnegative too, for all $t \geq 0$ and each $z \in \mathbb{R} \setminus \{0\}$ fixed.

Introducing the MPP $(T'_n, Z'_n)_{n \geq 1}$ where $(T'_n)_{n \geq 1}$ is the ordered family of $\mathbb{G}$ stopping times $(T_n, \tau)_{n \geq 1}$ (recall that τ avoids $(T_n)_{n \geq 1}$) and setting the jump size associated with τ equal to 1, the following result is a consequence of [3, Th. 3.1].

² Since the random time τ avoids all $\mathbb{F}$ -stopping times, the dual optional projection of H is continuous and equal to the dual predictable projection of H , denoted by H^p (cf. Proposition 1.48 (a), page 22 in [1]). Therefore, the martingale m which appears in the general formulae of the semimartingale decomposition (see Proposition 5.30, page 116 in [1]) is equal to the martingale part of the Doob–Meyer decomposition of G , that is, one has $G = m - H^p$ with an $\mathbb{F}$ -martingale $m = (m_t)_{t \geq 0}$. In particular, the predictable projection of G is ${}^pG = {}^p m - H^p = m_- - H^p = G_-$. One can apply Theorem 5.30, page 116, in [1] to deduce that, for any $\mathbb{F}$ -martingale X , the process $X^{\mathbb{G}} = (X_t^{\mathbb{G}})_{t \geq 0}$ defined by

$$X_t^{\mathbb{G}} = X_t - \int_0^{t \wedge \tau} \frac{d\langle X, m \rangle_s^{\mathbb{F}}}{G_{s-}} - \int_{t \wedge \tau}^t \frac{d\langle X, p(u) \rangle_s^{\mathbb{F}}}{p_{s-}(u)} \Big|_{u=\tau}, \quad \forall t \geq 0,$$

is a $\mathbb{G}$ -martingale.

Proposition 4.10 Each $(\mathbb{P}, \mathbb{G})$ -martingale $Y^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t \geq 0}$ can be represented as

$$Y_t^{\mathbb{G}} = Y_0^{\mathbb{G}} + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \alpha_s^{\mathbb{G}}(z) \tilde{\mu}^{\mathbb{G}}(ds, dz) + \int_0^t \beta_s^0 dM_s^{\mathbb{G}}, \quad \forall t \geq 0, \quad (26)$$

for some $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\alpha^{\mathbb{G}}$ (inducing the $\mathbb{G}$ -predictable parametrised process $\alpha^{\mathbb{G}}(z)$, for each $z \in \mathbb{R} \setminus \{0\}$), satisfying the condition

$$\int_0^t \int_{\mathbb{R} \setminus \{0\}} |\alpha_s^{\mathbb{G}}(z)| v^{\mathbb{G}}(ds, dz) < \infty, \quad \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)},$$

where the $\mathbb{G}$ -predictable compensator measure $v^{\mathbb{G}}$ of the random jump measure μ is given by (25), and β^0 is an $\mathbb{F}$ -predictable process, satisfying the condition

$$\int_0^t |\beta_s^0| d\Lambda_s < \infty, \quad \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)}.$$

Here, the function $\alpha^{\mathbb{G}}$ is of the form

$$\alpha_t^{\mathbb{G}}(z) = \mathbb{1}_{\{\tau \geq t\}} \alpha_t^0(z) + \mathbb{1}_{\{\tau < t\}} \alpha_t(\tau, z), \quad \forall t \geq 0, \quad \forall z \in \mathbb{R} \setminus \{0\}, \quad (27)$$

where α^0 is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function and α is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function.

Remark 4.11 Note that, if the $\mathbb{G}$ -predictable process $\beta^{\mathbb{G}}$, satisfying the condition

$$\int_0^t |\beta_s^{\mathbb{G}}| d\Lambda_s < \infty, \quad \forall t \geq 0 \text{ (}\mathbb{P}\text{-a.s.)},$$

admits the representation

$$\beta_t^{\mathbb{G}} = \mathbb{1}_{\{\tau \geq t\}} \beta_t^0 + \mathbb{1}_{\{\tau < t\}} \beta_t^1(\tau), \quad \forall t \geq 0,$$

with some $\mathbb{F}$ -predictable process $\beta^0 = (\beta_t^0)_{t \geq 0}$ and $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable function β^1 , then the equality

$$\int_0^t \beta_s^{\mathbb{G}} dM_s^{\mathbb{G}} = \int_0^t \beta_s^0 dM_s^{\mathbb{G}}, \quad \forall t \geq 0,$$

holds, for any choice of the corresponding induced $\mathbb{F}$ -predictable parametrised process $\beta^1(u)$, for each $u \geq 0$, since the martingale $M^{\mathbb{G}}$ defined in (18) is flat after τ , that is, the equality $M_t^{\mathbb{G}} = M_{t \wedge \tau}^{\mathbb{G}}$ holds, for all $t \geq 0$.

5 Optional Projections of Square Integrable Martingales

Our goal in this section is to define the projections of (square integrable) martingales with respect to a larger filtration into a smaller filtration. We recall from the arguments on pages 113 and 118 of [9] that the process Y is the $\mathbb{K}$ -optional projection of X , if the equality $\mathbb{E}[X_T | \mathcal{K}_T] = Y_T$ ($\mathbb{P}$ -a.s.) holds, for any finite $\mathbb{K}$ -stopping time T . It is known that any positive process admits an optional projection.

A $\mathbb{K}$ -square integrable martingale $X^{\mathbb{K}}$, which satisfies the condition $\sup_t \mathbb{E}[(X_t^{\mathbb{K}})^2] < \infty$, also satisfies the condition $\sup_t \mathbb{E}[|X_t^{\mathbb{K}}|] < \infty$. Then, according to [36, Th. 2.3], which states that any $\mathbb{K}$ -martingale $X^{\mathbb{K}}$ bounded in the space of integrable random variables admits a decomposition $X^{\mathbb{K}} = X^{\mathbb{K},1} - X^{\mathbb{K},2}$, into two positive (local) martingales $X^{\mathbb{K},i} = (X_t^{\mathbb{K},i})_{t \geq 0}$, for $i = 1, 2$, one obtains that square integrable martingales admit optional projections on smaller filtrations.

In this section, we will give explicit integral representations of $\mathbb{H}$ -optional projections of $\mathbb{K}$ -square integrable martingales in a large filtration $\mathbb{K}$ on a smaller filtration $\mathbb{H}$. For the simplicity of presentation, we assume that the $\mathbb{K}$ -martingale is square integrable with respect to a larger filtration, so that its optional projection on a smaller filtration is square integrable too. In our setting, any square integrable $\mathbb{F}$ -martingale Y admits the representation (8) with some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ξ satisfying the condition

$$\int_0^t \int_{\mathbb{R} \setminus \{0\}} \xi_s^2(z) \eta_s(dz) ds \equiv \int_0^t \int_{\mathbb{R} \setminus \{0\}} \xi_s^2(z) \nu(ds, dz) < \infty, \quad \forall t \geq 0 \quad (28)$$

(cf. Chapter VIII, Theorem T8, page 239, in [6]).

We further denote by $\mathcal{T}(\mathbb{F})$ and $\mathcal{T}(\mathbb{G})$ the sets of all *finite* stopping times with respect to the filtrations $\mathbb{F}$ and $\mathbb{G}$, respectively.

5.1 The Optional Projections of $\mathbb{F}^{(\zeta)}$ -Martingales on $\mathbb{F}$

In this subsection, we consider the optional projections of (square integrable) martingales in the initially enlarged filtration into the reference one. More precisely, we study the $\mathbb{F}$ -optional projection Y (an $\mathbb{F}$ -martingale) of the process $Y^{(\zeta)}$ (an $\mathbb{F}^{(\zeta)}$ -martingale).

Proposition 5.1 *For a random variable ζ , let $Y^{(\zeta)}$ be a square integrable $\mathbb{F}^{(\zeta)}$ -martingale with the representation given by equality (15). Then, the $\mathbb{F}$ -optional projection $Y = (Y_t)_{t \geq 0}$ of $Y^{(\zeta)}$ admits the representation of (8), with $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ξ satisfying the condition of (28) and being of the form*

$$\begin{aligned} \xi_t(z) &= \int_{\mathbb{R} \setminus \{0\}} \left(\psi_t(u, z) (f_t(u, z) + p_{t-}(u)) + Y_{t-}(u) f_t(u, z) \right) \rho(du), \\ &(\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \end{aligned} \quad (29)$$

where the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ψ is given by (15), while the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function f is defined in (10) and satisfies (11).

Proof The square integrability of $Y^{(\zeta)}$ implies that the function ψ given by (15) satisfies

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R} \setminus \{0\}} \psi_s^2(\zeta, z) v^{(\zeta)}(ds, dz) \right] < \infty, \quad \forall t \geq 0. \quad (30)$$

Consider a bounded $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ε (inducing the parametrised process $\varepsilon(z) = (\varepsilon_t(z))_{t \geq 0}$, for each $z \in \mathbb{R} \setminus \{0\}$ fixed) and define the process $K = (K_t)_{t \geq 0}$ by

$$K_t = \int_0^t \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \tilde{\mu}(ds, dz), \quad \forall t \geq 0, \quad (31)$$

which is a square integrable $\mathbb{F}$ -martingale. In this case, the square integrable random variable $Y_T = \mathbb{E}[Y_T(\zeta) | \mathcal{F}_T]$, which is the value at time T of the optional projection of $Y(\zeta)$ on $\mathbb{F}$, is the only $\mathcal{F}_T$ -measurable random variable such that

$$\mathbb{E}[Y_T(\zeta) K_T] = \mathbb{E}[Y_T K_T], \quad \forall T \in \mathcal{T}(\mathbb{F}), \quad (32)$$

holds, for any process K of the form (31). The equality in (32) is equivalent to the following equality

$$\begin{aligned} & \mathbb{E} \left[Y_T(\zeta) \int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \tilde{\mu}(ds, dz) \right] \\ &= \mathbb{E} \left[Y_T \int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \tilde{\mu}(ds, dz) \right], \quad \forall T \in \mathcal{T}(\mathbb{F}). \end{aligned} \quad (33)$$

On the one hand, one has

$$\begin{aligned} & \mathbb{E} \left[Y_T(\zeta) \int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \tilde{\mu}(ds, dz) \right] \\ &= \mathbb{E} \left[Y_T(\zeta) \int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \tilde{\mu}^{(\zeta)}(ds, dz) \right. \\ & \quad \left. + Y_T(\zeta) \int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) (v^{(\zeta)}(ds, dz) - \eta_s(dz) ds) \right], \\ & \quad \forall T \in \mathcal{T}(\mathbb{F}), \end{aligned}$$

where the $\mathbb{F}^{(\zeta)}$ -predictable compensator measure $v^{(\zeta)}$ of the random jump measure μ is defined in (14). Integrating by parts on the random interval $[0, T]$ the product of the two $\mathbb{F}^{(\zeta)}$ -martingales $Y^{(\zeta)}$ and $\Phi^{(\zeta)} = (\Phi_t(\zeta))_{t \geq 0}$ defined by

$$\Phi_t(\zeta) = \int_0^t \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \tilde{\mu}^{(\zeta)}(ds, dz), \quad \forall t \geq 0,$$

by using the appropriate square integrability assumptions, we get that

$$\begin{aligned}\mathbb{E}[Y_T^{(\zeta)} \Phi_T(\zeta)] &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \psi_s(\zeta, z) v^{(\zeta)}(ds, dz)\right] \\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \psi_s(\zeta, z) \left(\frac{f_s(\zeta, z)}{p_{s-}(\zeta)} + 1\right) \eta_s(dz) ds\right], \forall T \in \mathcal{T}(\mathbb{F}).\end{aligned}$$

Integrating by parts on the random interval $[0, T]$ the product of $Y^{(\zeta)}$ and the process of bounded variation $\Delta^{(\zeta)} = (\Delta_t(\zeta))_{t \geq 0}$ defined by

$$\Delta_t(\zeta) = \int_0^t \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) (v^{(\zeta)}(ds, dz) - \eta_s(dz) ds), \forall t \geq 0,$$

by using the equality (14), we obtain that

$$\mathbb{E}[Y_T(\zeta) \Delta_T(\zeta)] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) Y_{s-}(\zeta) \frac{f_s(\zeta, z)}{p_{s-}(\zeta)} \eta_s(dz) ds\right], \forall T \in \mathcal{T}(\mathbb{F}).$$

Hence, we have

$$\begin{aligned}\mathbb{E}\left[Y_T(\zeta) \int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \tilde{\mu}(ds, dz)\right] \\ = \mathbb{E}[Y_T(\zeta) \Phi_T(\zeta)] + \mathbb{E}[Y_T(\zeta) \Delta_T(\zeta)], \forall T \in \mathcal{T}(\mathbb{F}).\end{aligned}$$

On the other hand, one has

$$\begin{aligned}\mathbb{E}\left[Y_T \int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \tilde{\mu}(ds, dz)\right] &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \xi_s(z) \eta_s(dz) ds\right], \\ \forall T \in \mathcal{T}(\mathbb{F}).\end{aligned}$$

Finally, taking into account the existence of the $\mathbb{F}$ -conditional density family of processes $p(u)$, for each $u \in \mathbb{R}$, we see that, since the expression in (32) implies that, for any $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ε , the equality

$$\begin{aligned}\mathbb{E}\left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \int_0^\infty \left(\psi_s(u, z) (f_s(u, z) + p_{s-}(u)) + Y_{s-}(u) f_s(u, z)\right) \rho(du) \eta_s(dz) ds\right] \\ = \mathbb{E}\left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \varepsilon_s(z) \xi_s(z) \eta_s(dz) ds\right], \forall T \in \mathcal{T}(\mathbb{F}),\end{aligned}$$

holds, we obtain the expression (29). $\square$

5.2 The Optional Projections of $\mathbb{F}^{(\tau)}$ -Martingales on $\mathbb{G}$

In this subsection, we consider the optional projections of (square integrable) martingales in the initially enlarged filtration into the progressively enlarged one. More precisely, we study the $\mathbb{G}$ -optional projection $Y^{\mathbb{G}}$ (a $\mathbb{G}$ -martingale) of the process $Y^{(\tau)}$ (an $\mathbb{F}^{(\tau)}$ -martingale).

Proposition 5.2 *For a random time τ , let $Y^{(\tau)}$ be an $\mathbb{F}^{(\tau)}$ -square integrable martingale with the representation (15) above, for some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ψ , satisfying the condition (30). Then, the $\mathbb{G}$ -optional projection $Y^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t \geq 0}$ of $Y^{(\tau)}$ admits the representation of (26) with the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\alpha^{\mathbb{G}}$ of the form (27) (inducing the $\mathbb{G}$ -predictable parametrised process $\alpha^{\mathbb{G}}(z)$, for each $z \in \mathbb{R} \setminus \{0\}$ fixed), where the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function α^0 , the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function α , and the $\mathbb{F}$ -predictable process β^0 are given by*

$$\alpha_t^0(z) = \frac{1}{\varphi_t(z) + G_{t-}} \quad (34)$$

$$\times \int_t^\infty \left((\psi_t(u, z) + Y_{t-}(u)) (f_t(u, z) + p_{t-}(u)) - Y_{t-}(u) \left(\frac{\varphi_t(z)}{G_{t-}} + 1 \right) p_{t-}(u) \right) \rho(du),$$

$$(\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}),$$

$$\alpha_t(u, z) = \psi_t(u, z), \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \quad (35)$$

$$\beta_t^0 = {}^{p, \mathbb{F}}(Y_{t-}(t)) - \widehat{Y}_t^0, \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \quad (36)$$

while the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function φ has the form (22), the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ψ is given by (15), the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function f is defined by (10) and satisfies (11), and the process $\widehat{Y}^0 = (Y_t^0)_{t \geq 0}$ is the $\mathbb{F}$ -predictable reduction of $Y^{\mathbb{G}}$ given by

$$\widehat{Y}_t^0 = \frac{1}{G_{t-}} \int_t^\infty Y_{t-}(u) p_{t-}(u) \rho(du), \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \quad (37)$$

with the supermartingale G given by (12).

Proof In the first step, we determine the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\alpha^{\mathbb{G}}$ (inducing the $\mathbb{G}$ -predictable parametrised process $\alpha^{\mathbb{G}}(z)$, for each $z \in \mathbb{R} \setminus \{0\}$ fixed), while, in the second step, we determine the $\mathbb{F}$ -predictable process β^0 . We introduce the sign $\overset{\text{TP}}{=}$ to indicate that the tower property for conditional expectations is applied.

We note that any square integrable $\mathbb{G}$ -martingale $Y^{\mathbb{G}}$ admits the representation (26) with some $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\alpha^{\mathbb{G}}$ satisfying

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R} \setminus \{0\}} (\alpha_s^{\mathbb{G}}(z))^2 \nu^{\mathbb{G}}(ds, dz) \right] < \infty, \quad \forall t \geq 0, \quad (38)$$

and $\mathbb{F}$ -predictable process β^0 , where the $\mathbb{G}$ -predictable random measure $\nu^{\mathbb{G}}$ is defined in (25) (cf. Chapter VIII, Theorem T8, page 239, in [6]).

First step: We assume that the $\mathbb{F}^{(\tau)}$ -martingale $Y^{(\tau)}$ is square integrable, so that the $\mathbb{G}$ -martingale $Y^{\mathbb{G}}$ is square integrable too. Then, consider some $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable bounded function $\gamma^{\mathbb{G}}$ (inducing the $\mathbb{G}$ -predictable parametrised bounded process $\gamma^{\mathbb{G}}(z) = (\gamma_t^{\mathbb{G}}(z))_{t \geq 0}$, for each $z \in \mathbb{R} \setminus \{0\}$ fixed), as well as some $\mathbb{F}$ -predictable bounded process $\theta^0 = (\theta_t^0)_{t \geq 0}$, and define the process $K^{\mathbb{G}} = (K_t^{\mathbb{G}})_{t \geq 0}$ by

$$K_t^{\mathbb{G}} = \int_0^t \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \tilde{\mu}^{\mathbb{G}}(ds, dz) + \int_0^t \theta_s^0 dM_s^{\mathbb{G}}, \quad \forall t \geq 0, \quad (39)$$

where the process $M^{\mathbb{G}} = (M_t^{\mathbb{G}})_{t \geq 0}$ is defined in (18). It is seen that the process $K^{\mathbb{G}}$ is a square integrable $\mathbb{G}$ -martingale, since the parametrised process $\gamma^{\mathbb{G}}(z)$, for each $z \in \mathbb{R} \setminus \{0\}$, satisfies the condition

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R} \setminus \{0\}} (\gamma_s^{\mathbb{G}}(z))^2 \nu^{\mathbb{G}}(ds, dz) \right] < \infty, \quad \forall t \geq 0, \quad (40)$$

where the $\mathbb{G}$ -predictable compensator measure $\nu^{\mathbb{G}}$ of the random jump measure μ is given by (25) and the process θ^0 is $\mathbb{F}$ -predictable and bounded. In this case, the square integrable random variable $Y_T^{\mathbb{G}} = \mathbb{E}[Y_T(\tau) | \mathcal{G}_T]$ is the only $\mathcal{G}_T$ -measurable random variable such that the expression

$$\mathbb{E}[Y_T(\tau) K_T^{\mathbb{G}}] = \mathbb{E}[Y_T^{\mathbb{G}} K_T^{\mathbb{G}}], \quad \forall T \in \mathcal{T}(\mathbb{G}), \quad (41)$$

holds. Thus, the equality in (41) is equivalent to the system of two following equalities

$$\begin{aligned} & \mathbb{E} \left[Y_T(\tau) \int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \tilde{\mu}^{\mathbb{G}}(ds, dz) \right] \\ &= \mathbb{E} \left[Y_T^{\mathbb{G}} \int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \tilde{\mu}^{\mathbb{G}}(ds, dz) \right], \quad \forall T \in \mathcal{T}(\mathbb{G}), \end{aligned} \quad (42)$$

and

$$\mathbb{E} \left[Y_T(\tau) \int_0^T \theta_s^0 dM_s^{\mathbb{G}} \right] = \mathbb{E} \left[Y_T^{\mathbb{G}} \int_0^T \theta_s^0 dM_s^{\mathbb{G}} \right], \quad \forall T \in \mathcal{T}(\mathbb{G}), \quad (43)$$

for any $\mathbb{G}$ -predictable parametrised bounded process $\gamma^{\mathbb{G}}(z)$, for each $z \in \mathbb{R} \setminus \{0\}$ fixed, as well as any $\mathbb{F}$ -predictable bounded process θ^0 .

We now determine the functions α^0 and α from the equality (42). On the one hand, one has

$$\begin{aligned} & \mathbb{E} \left[Y_T(\tau) \int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{\mathbb{G}}(ds, dz)) \right] \\ &= \mathbb{E} \left[Y_T(\tau) \left(\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) (\mu(ds, dz) - \nu^{(\tau)}(ds, dz)) \right) \right] \end{aligned}$$

$$+ \int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \left(\nu^{(\tau)}(ds, dz) - \nu^{\mathbb{G}}(ds, dz) \right) \Big], \quad \forall T \in \mathcal{T}(\mathbb{G}),$$

where the $\mathbb{F}^{(\tau)}$ -predictable measure $\nu^{(\tau)}$ is defined in (14). By integrating by parts on the random interval $[0, T]$ the product of the two $\mathbb{F}^{(\tau)}$ -martingales, $Y^{(\tau)}$ and $\Upsilon = (\Upsilon_t)_{t \geq 0}$ defined by

$$\Upsilon_t = \int_0^t \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \tilde{\mu}^{(\tau)}(ds, dz), \quad \forall t \geq 0, \quad (44)$$

and taking into account the fact proved in Appendix below that the terms $\Upsilon_{t-} dY_t(\tau)$ and $Y_{t-}(\tau) d\Upsilon_t$ correspond to the true martingales, by applying Doob's optional stopping theorem from Chapter I, Theorem 1.39, of [23] to those martingales stopped at T , we get

$$\begin{aligned} & \mathbb{E} \left[Y_T(\tau) \int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \tilde{\mu}^{(\tau)}(ds, dz) \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \psi_s(\tau, z) \nu^{(\tau)}(ds, dz) \right], \quad \forall T \in \mathcal{T}(\mathbb{G}). \end{aligned}$$

Now, integrating by parts on the random interval $[0, T]$ the product of the martingale $Y^{(\tau)}$ and the bounded variation process $\Gamma^{(\tau)} = (\Gamma_t(\tau))_{t \geq 0}$ defined by

$$\Gamma_t(\tau) = \int_0^t \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \left(\nu^{(\tau)}(ds, dz) - \nu^{\mathbb{G}}(ds, dz) \right), \quad \forall t \geq 0,$$

we obtain

$$\begin{aligned} \mathbb{E}[Y_T(\tau) \Gamma_T(\tau)] &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} Y_{s-}(\tau) \gamma_s^{\mathbb{G}}(z) \left(\nu^{(\tau)}(ds, dz) - \nu^{\mathbb{G}}(ds, dz) \right) \right], \\ &\quad \forall T \in \mathcal{T}(\mathbb{G}). \end{aligned}$$

On the other hand, integrating by parts again, we have

$$\begin{aligned} & \mathbb{E} \left[Y_T^{\mathbb{G}} \int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \tilde{\mu}^{(\tau)}(ds, dz) \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \alpha_s^{\mathbb{G}}(z) \nu^{\mathbb{G}}(ds, dz) \right], \quad \forall T \in \mathcal{T}(\mathbb{G}). \end{aligned}$$

Finally, for any $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable bounded function $\gamma^{\mathbb{G}}$ (inducing the parametrised process $\gamma^{\mathbb{G}}(z)$, for each $z \in \mathbb{R} \setminus \{0\}$), satisfying the condition of (40),

the equality (42) is equivalent to

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \left(\psi_s(\tau, z) v^{(\tau)}(ds, dz) + Y_{s-}(\tau) \left(v^{(\tau)}(ds, dz) - v^{\mathbb{G}}(ds, dz) \right) \right) \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^{\mathbb{G}}(z) \alpha_s^{\mathbb{G}}(z) v^{\mathbb{G}}(ds, dz) \right], \quad \forall T \in \mathcal{T}(\mathbb{G}). \end{aligned} \quad (45)$$

Furthermore, for any $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable bounded function $\gamma^{\mathbb{G}}$ such that $\gamma_t^{\mathbb{G}}(z) = \mathbb{1}_{\{\tau \geq t\}} \gamma_t^0(z)$, for all $t \geq 0$ and $z \in \mathbb{R} \setminus \{0\}$, where the function γ^0 is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable, by using the identities in (14) and (25), we have

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^0(z) \mathbb{1}_{\{\tau \geq s\}} \left((\psi_s(\tau, z) + Y_{s-}(\tau)) \left(\frac{f_s(\tau, z)}{p_{s-}(\tau)} + 1 \right) \right. \right. \\ & \quad \left. \left. - Y_{s-}(\tau) \left(\frac{\varphi_s(z)}{G_{s-}} + 1 \right) \right) \eta_s(dz) ds \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^0(z) \mathbb{1}_{\{\tau \geq s\}} \alpha_s^0(z) \left(\frac{\varphi_s(z)}{G_{s-}} + 1 \right) \eta_s(dz) ds \right], \quad \forall T \in \mathcal{T}(\mathbb{G}). \end{aligned} \quad (46)$$

Then, taking the conditional expectation with respect to $\mathcal{F}_s$ inside the integrals in (46), by using the tower property as well as the existence of the conditional density, setting $F_t(\zeta, z) = f_t(\zeta, z)/p_{t-}(\zeta) + 1$ and $\Phi_t(z) = \varphi_t(z)/G_{t-} + 1$, for all $t \geq 0$ and each $z \in \mathbb{R} \setminus \{0\}$, we obtain that the left-hand side of the expression (46) is equal to

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^0(z) \mathbb{1}_{\{\tau \geq s\}} \left((\psi_s(\tau, z) + Y_{s-}(\tau)) F_s(\tau, z) - Y_{s-}(\tau) \Phi_s(z) \right) \eta_s(dz) ds \right] \\ & \stackrel{\text{TP}}{=} \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^0(z) \int_{u=s}^{\infty} \left((\psi_s(u, z) + Y_{s-}(u)) F_s(\tau, z) \right. \right. \\ & \quad \left. \left. - Y_{s-}(u) \Phi_s(z) \right) p_{s-}(u) \rho(du) \eta_s(dz) ds \right], \quad \forall T \in \mathcal{T}(\mathbb{G}), \end{aligned} \quad (47)$$

where, in the last equality, we have used the fact that the $\mathbb{F}$ -predictable projection of $p(u)$ is $p_{-}(u)$, for the process $p(u)$ being a martingale, for each $u \geq 0$ fixed.

We also note that, by using the fact that G_{-} is the $\mathbb{F}$ -predictable projection of G , the right-hand side of (46) is equal to

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^0(z) \mathbb{1}_{\{\tau \geq s\}} \alpha_s^0(z) \Phi_s(z) \eta_s(dz) ds \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^0(z) G_s \alpha_s^0(z) \Phi_s(z) \eta_s(dz) ds \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s^0(z) G_{s-} \alpha_s^0(z) \Phi_s(z) \eta_s(dz) ds \right], \quad \forall T \in \mathcal{T}(\mathbb{G}). \end{aligned} \quad (48)$$

It follows from the expression in (46) that the right-hand sides of the expressions in (47) and (48) are equal, for any $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable bounded function γ^0 , and hence, we have

$$\begin{aligned} & \int_t^\infty \left((\psi_t(u, z) + Y_{t-}(u)) F_t(u, z) - Y_{t-}(u) \Phi_t(z) \right) p_{t-}(u) \rho(du) \\ &= \int_t^\infty \left((\psi_t(u, z) + Y_{t-}(u)) \left(\frac{f_t(u, z)}{p_{t-}(u)} + 1 \right) - Y_{t-}(u) \left(\frac{\varphi_t(z)}{G_{t-}} + 1 \right) \right) p_{t-}(u) \rho(du) \\ &= G_{t-} \alpha_t^0(z) \Phi_t(z) = \alpha_t^0(z) (\varphi_t(z) + G_{t-}), \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \end{aligned}$$

so that the expression in (34) holds.

By using the identities in (14) and (25), for the $\mathbb{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable bounded function $\gamma^\mathbb{G}$ of the form $\gamma_t^\mathbb{G}(z) = \gamma_t(\tau, z) \mathbb{1}_{\{\tau < t\}}$, for all $t \geq 0$, for some $\mathbb{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable bounded function γ , the equality (45) leads to

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s(\tau, z) \mathbb{1}_{\{\tau < s\}} \psi_s(\tau, z) F_s(\tau, z) \eta_s(dz) ds \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \gamma_s(\tau, z) \mathbb{1}_{\{\tau < s\}} \alpha_s(\tau, z) F_s(\tau, z) \eta_s(dz) ds \right], \quad \forall T \in \mathcal{T}(\mathbb{G}), \end{aligned}$$

and we can choose $\alpha = \psi$ on the event $\{\tau < t\}$, so that the expression in (35) holds.

Second step: In the second step, we compute the value of β^0 , from the expression (43). It is straightforward to see that, for any $\mathbb{F}$ -predictable bounded process θ^0 , we have

$$\begin{aligned} \mathbb{E} \left[Y_T^\mathbb{G} \int_0^t \theta_s^0 dM_s^\mathbb{G} \right] &= \mathbb{E} \left[\int_0^T \beta_s^0 \theta_s^0 \lambda_s \mathbb{1}_{\{\tau > s\}} \rho(ds) \right] \\ &\stackrel{\text{TP}}{=} \mathbb{E} \left[\int_0^T \beta_s^0 \theta_s^0 \lambda_s G_s \rho(ds) \right], \quad \forall T \in \mathcal{T}(\mathbb{F}). \end{aligned}$$

It follows from the definition of the process $M^\mathbb{G}$ in (18), that

$$\begin{aligned} \mathbb{E} \left[Y_T(\tau) \int_0^T \theta_s^0 dM_s^\mathbb{G} \right] &= \mathbb{E} \left[Y_T(\tau) \left(\mathbb{1}_{\{\tau \leq T\}} \theta_\tau^0 - \int_0^T \mathbb{1}_{\{\tau > s\}} \theta_s^0 \lambda_s \rho(ds) \right) \right] \\ &\stackrel{\text{TP}}{=} \mathbb{E} \left[\int_0^T Y_T(s) \theta_s^0 p_T(s) \rho(ds) - \int_0^T \theta_s^0 \lambda_s \mathbb{E}[Y_s^{(\tau)} \mathbb{1}_{\{\tau > s\}} | \mathcal{F}_s] \rho(ds) \right] \\ &= \mathbb{E} \left[\int_0^T Y_{s-}(s) p_{s-}(s) \theta_s^0 \rho(ds) - \int_0^T \theta_s^0 \lambda_s \left(\int_{u=s}^\infty Y_{s-}(u) p_{s-}(u) \rho(du) \right) \rho(ds) \right] \\ &= \mathbb{E} \left[\int_0^T p_{\Sigma_s} p_{s-}(s) \theta_s^0 \rho(ds) - \int_0^T \theta_s^0 \lambda_s \left(\int_{u=s}^\infty Y_{s-}(u) p_{s-}(u) \rho(du) \right) \rho(ds) \right], \\ &\quad \forall T \in \mathcal{T}(\mathbb{F}), \end{aligned}$$

where we have used in the third equality the fact that the process $Y(u)p(u)$ is an $\mathbb{F}$ -martingale as in [1, Pro. 4.33] with the predictable projection $Y_{-}(u)p_{-}(u)$, for each $u \geq 0$ fixed, and the process $\Sigma = (\Sigma_t)_{t \geq 0}$ defined by $\Sigma_t = Y_{t-}(t)$, for all $t \geq 0$. We are not able to give the conditions under which the process Σ is predictable, since we do not have regularity of the process $Y_{-}(u) = (Y_{t-}(u))_{t \geq 0}$ with respect to the variable u , for each $u \geq 0$ fixed, and that is why we have to take its predictable projection.

Therefore, since the equality

$$\mathbb{E} \left[\int_0^T \beta_s^0 \theta_s^0 \lambda_s G_s \rho(ds) \right] = \mathbb{E} \left[\int_0^T {}^{p, \mathbb{F}} \Sigma_s p_{s-}(s) \theta_s^0 \rho(ds) - \int_0^T \theta_s^0 \lambda_s \left(\int_{u=s}^\infty Y_{s-}(u) p_{s-}(u) \rho(du) \right) \rho(ds) \right], \quad \forall T \in \mathcal{T}(\mathbb{F}),$$

holds, for any $\mathbb{F}$ -predictable bounded process θ^0 , it follows that the expression

$$\begin{aligned} \beta_t^0 &= \frac{1}{\lambda_t G_{t-}} \left({}^{p, \mathbb{F}} \Sigma_t p_{t-}(t) - \lambda_t \int_t^\infty Y_{t-}(u) p_{t-}(u) \rho(du) \right) \\ &= {}^{p, \mathbb{F}} \Sigma_t - \frac{1}{G_{t-}} \int_t^\infty Y_{t-}(u) p_{t-}(u) \rho(du), \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \end{aligned} \quad (49)$$

holds, where we have used the equality of (20). The expression in (49) implies the fact that the expression in (36) holds with (37). $\square$

5.3 The Optional Projections of $\mathbb{G}$ -Martingales on $\mathbb{F}$

In this last subsection, we consider the optional projections of (square integrable) martingales in the initially enlarged filtration into the reference one. More precisely, we study the $\mathbb{F}$ -optional projection Y (an $\mathbb{F}$ -martingale) of the process $Y^{\mathbb{G}}$ (a $\mathbb{G}$ -martingale).

Proposition 5.3 *Let $Y^{\mathbb{G}}$ be a $\mathbb{G}$ -square integrable martingale with the representation given by the equality in (26) and the decomposition given in (17) above. Then, the $\mathbb{F}$ -optional projection Y of $Y^{\mathbb{G}}$ is given by (8) above, where the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ξ (inducing the parametrised process $\xi(z) = (\xi_t(z))_{t \geq 0}$, for each $z \in \mathbb{R} \setminus \{0\}$ fixed), satisfying the condition of (28), is given by*

$$\begin{aligned} \xi_t(z) &= \alpha_t^0(z) (\varphi_t(z) + G_{t-}) + \widehat{Y}_{t-}^0 \varphi_t(z) \\ &\quad + \int_0^t \left(\alpha_t(u, z) (f_t(u, z) + p_{t-}(u)) + Y_{t-}(u) f_t(u, z) p_{t-}(u) \right) \rho(du), \\ &(\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \end{aligned} \quad (50)$$

with the supermartingale G given by (12), the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function φ having the form (22), the $\mathbb{F}$ -predictable reduction $\widehat{Y}^0$ of $Y^{\mathbb{G}}$ given by (37), the $\mathcal{P}(\mathbb{F}) \otimes$

$\mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function α^0 given by (34), the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function α given by (35), the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable process f defined by (10) and satisfying (11).

Proof As before, for any $\mathbb{G}$ -predictable bounded process $\theta^{\mathbb{G}}$, we consider the equality satisfied by Y such that

$$\mathbb{E} \left[Y_T \int_0^T \int_{\mathbb{R} \setminus \{0\}} \theta_s^{\mathbb{G}} \tilde{\mu}(ds, dz) \right] = \mathbb{E} \left[Y_T^{\mathbb{G}} \int_0^T \int_{\mathbb{R} \setminus \{0\}} \theta_s^{\mathbb{G}} \tilde{\mu}(ds, dz) \right], \quad \forall T \in \mathcal{T}(\mathbb{F}). \quad (51)$$

Then, for each $T \in \mathcal{T}(\mathbb{F})$, the left-hand side of (51) is equal to

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \xi_s(z) \theta_s^{\mathbb{G}} \eta_s(dz) ds \right],$$

while the right-hand side of (51) is equal to

$$\begin{aligned} & \mathbb{E} \left[Y_T^{\mathbb{G}} \int_0^T \int_{\mathbb{R} \setminus \{0\}} \theta_s^{\mathbb{G}} (\tilde{\mu}^{\mathbb{G}}(ds, dz) + \nu_s^{\mathbb{G}}(ds, dz) - \eta_s(dz) ds) \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \alpha_s^{\mathbb{G}} \theta_s^{\mathbb{G}} \nu_s^{\mathbb{G}}(ds, dz) + \int_0^T \int_{\mathbb{R} \setminus \{0\}} \theta_s^{\mathbb{G}} Y_{s-}^{\mathbb{G}} (\nu_s^{\mathbb{G}}(ds, dz) - \eta_s(dz) ds) \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \theta_s^{\mathbb{G}} \left(\alpha_s^0(z) \left(\frac{\varphi_s(z)}{G_{s-}} + 1 \right) + Y_{s-}^0 \frac{\varphi_s(z)}{G_{s-}} \right) \eta_s(dz) \mathbb{1}_{\{\tau > s\}} ds \right] \\ &\quad + \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \theta_s^{\mathbb{G}} \left(\int_{u=0}^{u=s} \left(\alpha_s(u, z) (f_s(u, z) + p_{s-}(u)) \right. \right. \right. \\ &\quad \left. \left. \left. + Y_{s-}(u) f_s(u, z) \right) \rho(du) \right) \eta_s(dz) ds \right] \\ &\stackrel{\text{TP}}{=} \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \theta_s^0 \left(\alpha_s^0(z) (\varphi_s(z) + G_s) + Y_{s-}^0 \varphi_s(z) \right) \eta_s(dz) ds \right] \\ &\quad + \mathbb{E} \left[\int_0^T \int_{\mathbb{R} \setminus \{0\}} \theta_s^{\mathbb{G}} \left(\int_{u=0}^{u=s} \left(\alpha_s(u, z) (f_s(u, z) + p_s(u)) \right. \right. \right. \\ &\quad \left. \left. \left. + Y_{s-}(u) f_s(u, z) \right) \rho(du) \right) \eta_s(dz) ds \right], \end{aligned}$$

where the $\mathbb{G}$ -predictable compensator measure $\nu^{\mathbb{G}}$ is given by (25). Hence, since the equality in (51) holds, for any $\mathbb{G}$ -predictable bounded process $\theta^{\mathbb{G}}$, we get that the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function ξ (inducing the parametrised process $\xi(z) = (\xi_t(z))_{t \geq 0}$, for each $z \in \mathbb{R} \setminus \{0\}$ fixed) has the form of (50), that completes the proof. $\square$

6 Changes of Probability Measures and Applications

In this section, as an example of application of the results from the previous section, we consider the relationships between strictly positive $\mathbb{F}^{(\zeta)}$ -(square integrable) martingales (or $\mathbb{G}$ -martingales) and their optional projections. We then apply the results in a financial market framework to study the set of equivalent martingale measures in various filtrations.

A probability measure $\mathbb{Q}$ is said to be *locally equivalent* to $\mathbb{P}$ on the filtration $\mathbb{K}$, if there exists a strictly positive $\mathbb{K}$ -martingale $N = (N_t)_{t \geq 0}$ such that

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{K}_t} = N_t, \text{ and } \mathbb{E}_{\mathbb{P}}[N_t] = 1, \forall t \geq 0.$$

The martingale N is called the *Radon–Nikodym density* of $\mathbb{Q}$ with respect to $\mathbb{P}$. The “locally” terminology is needed, since, as in [2], we cannot define the new probability measure $\mathbb{Q}$ on $\mathcal{K}_{\infty}$, because the density process N is not necessarily a uniformly integrable martingale on $\mathbb{K}$.

6.1 The Optional Projections of Strictly Positive Martingales

We suppose throughout this section that the following assumption holds.

Hypothesis 6.1 *We assume through the whole section that all the random jump times T_n , for $n \geq 1$, in (1) admit their densities on $\mathbb{R}_+$, which particularly implies that $\nu(\omega; \{t\}, A) = 0$, for all $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ in (3).*

- For a random variable ζ , let $L^{(\zeta)}$ be a strictly positive $\mathbb{F}^{(\zeta)}$ -(square integrable) martingale. Then, in particular, we have $L_0^{(\zeta)} > 0$ ($\mathbb{P}$ -a.s.). Moreover, by applying Proposition 4.3 for $Y^{(\zeta)} = L^{(\zeta)}$, and, defining the $\mathcal{P}(\mathbb{F}^{(\zeta)}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\Theta_t(u, z)$ by the equation $L_t(u)(\Theta_t(u, z) - 1) = \psi_t(u, z)$, for all $t \geq 0$ and $u \in \mathbb{R}$, and for each $z \in \mathbb{R} \setminus \{0\}$, we can write $L^{(u)} = (L_t(u))_{t \geq 0}$ in the form

$$L_t(u) = L_0(u) + \int_0^t L_{s-}(u) \int_{\mathbb{R} \setminus \{0\}} (\Theta_s(u, z) - 1) \tilde{\mu}^{(\zeta)}(ds, dz), \quad \forall t \geq 0, \forall u \in \mathbb{R}, \quad (52)$$

where, according to the Hypothesis 6.1 and the result of Theorem 3 in Chapter VII, Sect. 3 g in [32], we have that $\Theta > 0$ (actually following directly from the fact that the expression $\Theta_{T_n}(u, Z_n) = L_{T_n}(u)/L_{T_n-}(u)$ holds, for every $n \geq 1$ and all $u \in \mathbb{R}_+$) as well as the condition

$$\int_0^t L_{s-}(u) \int_{\mathbb{R} \setminus \{0\}} |\Theta_s(u, z) - 1| \nu^{(\zeta)}(ds, dz) < \infty, \quad \forall t \geq 0, \forall u \in \mathbb{R},$$

holds (cf., e.g., [8, Pro. 8.20]). In this case, the $\mathbb{F}$ -optional projection L of $L^{(\zeta)}$ is a strictly positive $\mathbb{F}$ -martingale, which admits the integral representation

$$L_t = \mathbb{E}[L_0^{(\zeta)}] + \int_0^t L_{s-} \int_{\mathbb{R} \setminus \{0\}} \chi_s(z) \tilde{\mu}(ds, dz), \quad \forall t \geq 0, \quad (53)$$

with some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\chi > -1$ (inducing the parametrised process $\chi(z) = (\chi_t(z))_{t \geq 0}$, for each $z \in \mathbb{R} \setminus \{0\}$ fixed) such that the condition

$$\int_0^t L_{s-} \int_{\mathbb{R} \setminus \{0\}} |\chi_s(z)| \eta_s(dz) ds < \infty, \quad \forall t \geq 0, \quad (54)$$

holds.

Corollary 6.2 *For a random variable ζ , let $L^{(\zeta)} = (L_t(\zeta))_{t \geq 0}$ be a strictly positive (square integrable) martingale of the form (52) and such that the process $Y^{(\zeta)} = L^{(\zeta)}$ satisfies the conditions of Proposition 5.1. Then, the $\mathbb{F}$ -optional projection L of $L^{(\zeta)}$ admits the integral representation of (53) with the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function χ which satisfies the condition of (54) and is given by*

$$\chi_t(z) = \frac{1}{L_{t-}} \int_0^\infty L_{t-}(u) \left((\Theta_t(u, z) - 1) (f_t(u, z) + p_{t-}(u)) + f_t(u, z) \right) \rho(du),$$

$$(\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}),$$

where the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function f is defined by (10) and satisfies (11).

- We now consider the case in which $\tau := \zeta$ is a random time and the process $L^{(\tau)} = (L_t(\tau))_{t \geq 0}$ admits the representation of (52). Note that, if $\mathbb{E}[L_0^{(\tau)}] = 1$ holds, then we can associate to the strictly positive $\mathbb{F}^{(\tau)}$ -martingale $L^{(\tau)} = (L_t(\tau))_{t \geq 0}$ the probability measure $\tilde{\mathbb{P}}$ locally equivalent to $\mathbb{P}$ on the filtration $\mathbb{F}^{(\tau)}$ defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t^{(\tau)}} = L_t(\tau), \quad \mathbb{E}[L_t(\tau) | \mathcal{F}_0^{(\tau)}] = L_0(\tau), \quad \forall t \geq 0, \quad \text{and} \quad \mathbb{E}[L_0(\tau)] = 1.$$

Let us now consider the $\mathbb{G}$ -optional projection $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t \geq 0}$ of the strictly positive (square integrable) martingale $L^{(\tau)}$. In this case, by applying Proposition 4.10 with $Y^{\mathbb{G}} = L^{\mathbb{G}}$, and, defining the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\varkappa^{\mathbb{G}}$ by the equation $L_t^{\mathbb{G}}(\varkappa_t^{\mathbb{G}}(z) - 1) = \alpha_t^{\mathbb{G}}(z)$, for all $t \geq 0$ and $z \in \mathbb{R} \setminus \{0\}$, we see that $L^{\mathbb{G}}$ admits the representation

$$L_t^{\mathbb{G}} = L_0^{\mathbb{G}} + \int_0^t L_{s-}^{\mathbb{G}} \int_{\mathbb{R} \setminus \{0\}} (\varkappa_s^{\mathbb{G}}(z) - 1) \tilde{\mu}^{\mathbb{G}}(ds, dz) + \int_0^t L_{s-}^{\mathbb{G}} \xi_s^0 dM_s^{\mathbb{G}}, \quad \forall t \geq 0, \quad (55)$$

where the second stochastic integral of some $\mathbb{F}$ -predictable process ξ^0 exists according to Hypothesis 4.6 and, according to the Hypothesis 6.1 and the result of Theorem 3 in Chapter VII, Section 3 g in [32], the arguments similar to the ones used above lead to the conclusion that $\varkappa^{\mathbb{G}} > 0$ and $\xi^0 > -1$, as well as the condition

$$\int_0^t L_{s-}^{\mathbb{G}} \int_{\mathbb{R} \setminus \{0\}} |\varkappa_s^{\mathbb{G}}(z) - 1| \nu^{\mathbb{G}}(ds, dz) + \int_0^t L_{s-}^{\mathbb{G}} |\xi_s^0| d\Lambda_s < \infty, \quad \forall t \geq 0, \quad (56)$$

is satisfied, where the $\mathbb{G}$ -predictable random measure $\nu^{\mathbb{G}}$ is defined in (25). Here, the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\varkappa^{\mathbb{G}}$ (inducing the parametrised process $\varkappa^{\mathbb{G}}(z)$, for each $z \in \mathbb{R} \setminus \{0\}$ fixed) is of the form

$$\varkappa_t^{\mathbb{G}}(z) = \mathbb{1}_{\{\tau \geq t\}} \varkappa_t^0(z) + \mathbb{1}_{\{\tau < t\}} \varkappa_t(\tau, z), \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}),$$

with some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\varkappa^0$ and a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\varkappa$.

Corollary 6.3 *For a random time τ , let $L^{(\tau)} = (L_t(\tau))_{t \geq 0}$ be a strictly positive (square integrable) martingale of the form (52) and such that the process $Y^{(\tau)} = L^{(\tau)}$ satisfies the conditions of Proposition 5.2. Then, the $\mathbb{G}$ -optional projection $L^{\mathbb{G}}$ of $L^{(\tau)}$ admits the representation of (55) with the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\varkappa^{\mathbb{G}}$ (inducing the $\mathbb{G}$ -predictable parametrised process $\varkappa^{\mathbb{G}}(z)$, for each $z \in \mathbb{R} \setminus \{0\}$), satisfying the condition of (56), the $\mathbb{F}$ -predictable process ξ^0 , and the $\mathbb{F}$ -predictable reduction $\widehat{L}^0$ of $L^{\mathbb{G}}$ given by*

$$\begin{aligned} \varkappa_t^{\mathbb{G}}(z) - 1 &= \frac{\mathbb{1}_{\{\tau \geq t\}}}{\widehat{L}_t^0(\varphi_t(z) + G_{t-})} \\ &\quad \times \int_t^\infty L_{t-}(u) \left(\Theta_t(u, z) \left(\frac{f_t(u, z)}{p_{t-}(u)} + 1 \right) - \left(\frac{\varphi_t(z)}{G_{t-}} + 1 \right) \right) p_{t-}(u) \rho(du) \\ &\quad + \mathbb{1}_{\{\tau < t\}} (\Theta_t(\tau, z) - 1), \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \end{aligned} \quad (57)$$

$$\xi_t^0 = \frac{{}^{p, \mathbb{F}}(L_{t-}(t))}{\widehat{L}_t^0} - 1, \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \quad (58)$$

$$\widehat{L}_{t-}^0 = \frac{1}{G_{t-}} \int_t^\infty L_{t-}(u) p_{t-}(u) \rho(du), \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \quad (59)$$

where the supermartingale G is given by (12), the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function φ has the form (22), while the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function f is defined by (10) and satisfies (11).

Proof Consider the $\mathbb{F}^{(\tau)}$ -martingale $L^{(\tau)}$ given by the equality in (52). In this case, its $\mathbb{G}$ -optional projection $L^{\mathbb{G}}$ has the form of (55). Then, Proposition 5.2 is applied with $Y^{(\tau)} = L^{(\tau)}$ and $L_{t-}(u)(\Theta_t(u, z) - 1) = \psi_t(u, z)$, for all $t, u \geq 0$ and each $z \in \mathbb{R} \setminus \{0\}$, and therefore, the equalities $L_t^{\mathbb{G}}(\varkappa_t^{\mathbb{G}}(z) - 1) = \alpha_t^{\mathbb{G}}(z)$ hold, for all $t \geq 0$ and each $z \in \mathbb{R} \setminus \{0\}$. That is, $\widehat{L}_t^0(\varkappa_t^0(z) - 1) = \alpha_t^0(z)$ and $L_{t-}(u)(\varkappa_t(u, z) - 1) = \alpha_t(u, z)$,

for all $t, u \geq 0$ such that $t \geq u$ and for each $z \in \mathbb{R} \setminus \{0\}$, while $\widehat{L}_t^0 \xi_t^0 = \beta_t^0$ holds, for all $t \geq 0$, as well as the expressions in (57)–(59) are satisfied. $\square$

- By means of arguments similar to the ones used above, it follows that its $\mathbb{F}$ -optional projection $L = (L_t)_{t \geq 0}$ admits the integral representation

$$L_t = L_0^{\mathbb{G}} + \int_0^t L_{s-} \int_{\mathbb{R} \setminus \{0\}} (\sigma_s(z) - 1) \widetilde{\mu}(ds, dz), \quad \forall t \geq 0, \quad (60)$$

with some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function σ (inducing the $\mathbb{F}$ -predictable parametrised process $\sigma(z) = (\sigma_t(z))_{t \geq 0}$, for each $z \in \mathbb{R} \setminus \{0\}$), where, according to the Hypothesis 6.1 and the result of Theorem 3 in Chapter VII, Sect. 3 g in [32], we have that $\sigma > 0$ and the condition

$$\int_0^t L_{s-} \int_{\mathbb{R} \setminus \{0\}} |\sigma_s(z) - 1| \eta_s(dz) ds < \infty, \quad \forall t \geq 0, \quad (61)$$

holds. We also recall that any strictly positive $\mathbb{G}$ -martingale $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t \geq 0}$ admits the equivalent representation of (55) and, being a $\mathbb{G}$ -optional process, it admits the decomposition

$$L_t^{\mathbb{G}} = \mathbb{1}_{\{\tau > t\}} L_t^0 + \mathbb{1}_{\{\tau \leq t\}} L_t^1(\tau), \quad \forall t \geq 0,$$

with some $\mathbb{F}$ -optional process L^0 and some $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable function L^1 .

Corollary 6.4 *Let $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t \geq 0}$ be a strictly positive (square integrable) martingale of the form (55) and such that the process $Y^{\mathbb{G}} = L^{\mathbb{G}}$ satisfies the conditions of Proposition 5.3. Then, the $\mathbb{F}$ -optional projection L of $L^{\mathbb{G}}$ admits the representation of (60) with the strictly positive $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function σ which satisfies the condition of (61) and is given by*

$$\begin{aligned} \sigma_t(z) - 1 &= \frac{1}{L_{t-}} \left(L_{t-}^0 \left(\varkappa_t^0(z) (\varphi_t(z) + G_{t-}) - G_{t-} \right) \right. \\ &\quad \left. + \int_0^t L_{t-}^1(u) \left((\varkappa_t(u, z) - 1) (f_t(u, z) + p_{t-}(u)) + f_t(u, z) \right) \rho(du) \right), \\ &\quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \end{aligned} \quad (62)$$

where the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable process f is defined by (10) and satisfies (11).

Proof In order to derive the expression of (62) for σ , it suffices to apply Proposition 5.3 with $Y^{\mathbb{G}} = L^{\mathbb{G}}$. In this case, we have that $(L_{t-}^{\mathbb{G}} (\varkappa_t^{\mathbb{G}}(z) - 1))^0 = \alpha_t^0(z)$, $L_{t-}^1(u) (\varkappa_t(u, z) - 1) = \alpha_t(u, z)$, for all $t, u \geq 0$ such that $t \geq u$ and for each $z \in \mathbb{R} \setminus \{0\}$, and $L_{t-} \sigma_t(z) = \xi_t(z)$ holds for all $t \geq 0$ and for each $z \in \mathbb{R} \setminus \{0\}$, and therefore, we have $Y^0 = L^0$ and $Y = L$. The equality $(L_{t-}^{\mathbb{G}} (\varkappa_t^{\mathbb{G}}(z) - 1))^0 = L_{t-}^0 (\varkappa_t^0(z) - 1)$, for all $t \geq 0$ and each $z \in \mathbb{R} \setminus \{0\}$, follows from the definition of optional reduction. $\square$

6.2 The Equivalent (Local) Martingale Measures

Let us now consider a model of a financial market in which the risky asset price process $S = (S_t)_{t \geq 0}$ follows the stochastic differential equation

$$dS_t = S_{t-} \delta_t dt + S_{t-} \int_{\mathbb{R} \setminus \{0\}} h_t(z) \tilde{\mu}(dt, dz), \quad (63)$$

with an $\mathbb{F}$ -predictable *bounded* process $\delta = (\delta_t)_{t \geq 0}$ and an $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function h (inducing the parametrised process $h(z) = (h_t(z))_{t \geq 0}$) such that $h_t(z) > -1$, for all $t \geq 0$ and each $z \in \mathbb{R} \setminus \{0\}$ (to satisfy the positivity of S), and such that the condition

$$\int_0^t S_{s-} \int_{\mathbb{R} \setminus \{0\}} |h_s(z)| \eta_s(dz) ds < \infty, \quad \forall t \geq 0, \quad (64)$$

holds. We formally suppose that the riskless asset has a zero interest rate and recall that such a model does not admit arbitrage opportunities (cf., e.g., Chapter VII, Sect. 2a in [32]).

The change of probability in $\mathbb{F}$: A change of the probability measure in $\mathbb{F}$ has a Radon–Nikodym density process $L = (L_t)_{t \geq 0}$ started at $L_0 = 1$ and satisfying the stochastic differential equation given from the weak representation property of Proposition 2.1 in [28] by

$$dL_t = L_{t-} \int_{\mathbb{R} \setminus \{0\}} (\pi_t(z) - 1) \tilde{\mu}(dt, dz), \quad L_0 = 1, \quad (65)$$

for a strictly positive $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function π (inducing the parametrised process $\pi(z) = (\pi_t(z))_{t \geq 0}$, for each $z \in \mathbb{R} \setminus \{0\}$) such that the condition

$$\int_0^t L_{s-} \int_{\mathbb{R} \setminus \{0\}} |\pi_s(z) - 1| \eta_s(dz) ds < \infty, \quad \forall t \geq 0, \quad (66)$$

holds. This Radon–Nikodym density process L will correspond to an equivalent (local) martingale measure $\mathbb{Q}(\mathbb{F})$, when the process SL is an $\mathbb{F}$ -(local) martingale. We will denote by $\mathcal{M}(\mathbb{F})$ the set of all such martingale measures in the corresponding model of financial markets with the information flow expressed by the filtration $\mathbb{F}$.

It follows from the straightforward calculations based on the application of the integration-by-parts formula and performed in Subsection 2.2 in [28] that the process SL is an $\mathbb{F}$ -(local) martingale if and only if the function $\pi > 0$ from (65) satisfies the equality

$$\delta_t + \int_{\mathbb{R} \setminus \{0\}} h_t(z) (\pi_t(z) - 1) \eta_t(dz) = 0, \quad \forall t \geq 0, \quad (67)$$

for a given $\mathbb{F}$ -predictable bounded process δ and a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $h > -1$, satisfying the condition of (64). Observe that the expression in

(67) represents a Fredholm-type integral equation of the first kind, which may admit multiple (infinitely many) solutions for a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\pi > 0$ (cf., e.g., Chapter II in [37]), except in the case when $\eta_t(dz)$ represents a Dirac measure at one given point, in which there exists exactly one solution to the equation in (67). Note that a nontrivial solution to the equation of (67) is given by the expressions of Corollary 2.2 in [28], under certain additional assumptions related to the existence of the corresponding *equivalent minimal martingale measure* (cf. [14] for the introduction of this notion). Such functions $\pi > 0$, satisfying the condition of (66), generate the stochastic exponentials L solving the equation of (65) and identifying the corresponding equivalent (local) martingale measures in the class $\mathcal{M}(\mathbb{F})$.

The change of probability in $\mathbb{G}$: It follows from Proposition 4.8 that, according to the equality in (25), the process S from (63) admits the following representation

$$\begin{aligned} dS_t &= S_{t-} \delta_t dt + S_{t-} \int_{\mathbb{R} \setminus \{0\}} h_t(z) (v^{\mathbb{G}}(dt, dz) - v(dt, dz)) \\ &\quad + S_{t-} \int_{\mathbb{R} \setminus \{0\}} h_t(z) \tilde{\mu}^{\mathbb{G}}(dt, dz) \\ &= S_{t-} \left(\delta_t + \int_{\mathbb{R} \setminus \{0\}} h_t(z) \left(\mathbb{1}_{\{\tau \geq t\}} \frac{\varphi_t(z)}{G_{t-}} + \mathbb{1}_{\{\tau < t\}} \frac{f_t(\tau, z)}{p_{t-}(\tau)} \right) \eta_t(dz) \right) dt \\ &\quad + S_{t-} \int_{\mathbb{R} \setminus \{0\}} h_t(z) \tilde{\mu}^{\mathbb{G}}(dt, dz), \end{aligned} \quad (68)$$

in the filtration $\mathbb{G}$, where δ is an $\mathbb{F}$ -predictable bounded process and the last term is a $\mathbb{G}$ -(local) martingale, when the condition

$$\int_0^t S_{s-} \int_{\mathbb{R} \setminus \{0\}} |h_s(z)| \left(\mathbb{1}_{\{\tau \geq s\}} \frac{\varphi_s(z)}{G_{s-}} + \mathbb{1}_{\{\tau < s\}} \frac{f_s(\tau, z)}{p_{s-}(\tau)} + 1 \right) \eta_s(dz) ds < \infty, \quad \forall t \geq 0, \quad (69)$$

holds.

We recall the fact from Proposition 4.10, which particularly implies the representation in (55) under the condition of (56) that any positive (square integrable) $\mathbb{G}$ -martingale $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t \geq 0}$ started at $L_0^{\mathbb{G}} = 1$ admits the representation

$$dL_t^{\mathbb{G}} = L_{t-}^{\mathbb{G}} \left(\int_{\mathbb{R} \setminus \{0\}} (\varkappa_t^{\mathbb{G}}(z) - 1) \tilde{\mu}^{\mathbb{G}}(dt, dz) + \xi_t^0 dM_t^{\mathbb{G}} \right), \quad L_0^{\mathbb{G}} = 1, \quad (70)$$

for some strictly positive $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\varkappa^{\mathbb{G}} > 0$ and some $\mathbb{R}$ predictable process $\xi^0 > -1$ such that the condition

$$\int_0^t L_{s-}^{\mathbb{G}} \int_{\mathbb{R} \setminus \{0\}} |\varkappa_s^{\mathbb{G}}(z) - 1| v^{\mathbb{G}}(ds, dz) + \int_0^t L_{s-}^{\mathbb{G}} |\xi_s^0| d\Lambda_s < \infty, \quad \forall t \geq 0, \quad (71)$$

holds (see Proposition 4.10). In this case, $L^{\mathbb{G}}$ is the Radon-Nikodym density process corresponding to an equivalent martingale measure $\mathbb{Q}(\mathbb{G})$, when the process $SL^{\mathbb{G}}$ is an

$\mathbb{G}$ -(local) martingale. We will denote by $\mathcal{M}(\mathbb{G})$ the set of all such martingale measures in the corresponding model of financial markets with the information flow expressed by the filtration $\mathbb{G}$.

It follows from the straightforward calculations based on the application of the integration-by-parts formula and performed in Subsection 2.2 in [28] that the process $SL^{\mathbb{G}}$ is a $\mathbb{G}$ -(local) martingale if and only if the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\varkappa^{\mathbb{G}} > 0$ from (70) satisfies the equality

$$\delta_t + \int_{\mathbb{R} \setminus \{0\}} h_t(z) \left(\varkappa_t^{\mathbb{G}}(z) \left(\mathbb{I}_{\{\tau \geq t\}} \frac{\varphi_t(z)}{G_{t-}} + \mathbb{I}_{\{\tau < t\}} \frac{f_t(\tau, z)}{p_{t-}(\tau)} + 1 \right) - 1 \right) \eta_t(dz) = 0, \quad \forall t \geq 0, \quad (72)$$

for a given $\mathbb{F}$ -predictable bounded process δ and an $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $h > -1$, satisfying the condition of (64). Then, we conclude by virtue of arguments similar to the ones applied in the previous part that the equality in (72) admits infinitely many solutions for a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\varkappa^{\mathbb{G}} > 0$. Such functions $\varkappa^{\mathbb{G}} > 0$, satisfying the conditions of (71), which together with the arbitrarily chosen process $\xi^0 > -1$ generate the stochastic exponentials $L^{\mathbb{G}}$ solving the equation of (70) and identifying the corresponding equivalent (local) martingale measures in the class $\mathcal{M}(\mathbb{G})$.

The change of probability in $\mathbb{F}^{(\zeta)}$: It follows from Proposition 4.2 that, according to the equality in (25), the process S from (63) admits the following representation

$$\begin{aligned} dS_t &= S_{t-} \delta_t dt + S_{t-} \int_{\mathbb{R} \setminus \{0\}} h_t(z) (v^{(\zeta)}(dt, dz) - v(dt, dz)) \\ &\quad + S_{t-} \int_{\mathbb{R} \setminus \{0\}} h_t(z) \tilde{\mu}^{(\zeta)}(dt, dz) \\ &= S_{t-} \left(\delta_t + \int_{\mathbb{R} \setminus \{0\}} h_t(z) \frac{f_t(\zeta, z)}{p_{t-}(\zeta)} \eta_t(dz) \right) dt + S_{t-} \int_{\mathbb{R} \setminus \{0\}} h_t(z) \tilde{\mu}^{(\zeta)}(dt, dz), \end{aligned} \quad (73)$$

in the filtration $\mathbb{F}^{(\zeta)}$, where δ is an $\mathbb{F}$ -predictable bounded process and the last term is an $\mathbb{F}^{(\zeta)}$ -(local) martingale, when the condition

$$\int_0^t S_{s-} \int_{\mathbb{R} \setminus \{0\}} |h_s(z)| \left(\frac{f_s(\zeta, z)}{p_{s-}(\zeta)} + 1 \right) \eta_s(dz) ds < \infty, \quad \forall t \geq 0, \quad (74)$$

holds.

As it is seen from Proposition 4.3, a change of a probability measure in $\mathbb{F}^{(\zeta)}$ has the Radon–Nikodym density process $L^{(\zeta)} = (L_t(\zeta))_{t \geq 0}$ started at a strictly positive random variable $L_0(\zeta) = \ell(\zeta)$, with a strictly positive Borel function ℓ such that $\mathbb{E}[\ell(\zeta)] = 1$ holds, and satisfying the stochastic differential equation given by (52),

which can also be written in the form:

$$dL_t(\zeta) = L_{t-}(\zeta) \int_{\mathbb{R} \setminus \{0\}} (\Theta_t(\zeta, z) - 1) \tilde{\mu}^{(\zeta)}(dt, dz), \quad L_0(\zeta) = \ell(\zeta), \quad (75)$$

with a strictly positive $\mathcal{P}(\mathbb{F}^{(\zeta)}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\Theta > 0$ such that the condition

$$\int_0^t L_{s-}(\zeta) \int_{\mathbb{R} \setminus \{0\}} |\Theta_s(\zeta, z) - 1| \nu^{(\zeta)}(ds, dz) < \infty, \quad \forall t \geq 0, \quad (76)$$

is satisfied. (see Proposition 4.3). In this case, $L^{(\zeta)}$ is the Radon–Nikodym density process corresponding to an equivalent martingale measure $\mathbb{Q}(\mathbb{F}^{(\zeta)})$, when the process $SL^{(\zeta)}$ is an $\mathbb{F}^{(\zeta)}$ -(local) martingale. We will denote by $\mathcal{M}(\mathbb{F}^{(\zeta)})$ the set of all such martingale measures in the corresponding model of financial markets with the information flow expressed by the filtration $\mathbb{F}^{(\zeta)}$.

It follows from the straightforward calculations based on the application of the integration-by-parts formula and performed in Subsection 2.2 in [28] that the process $SL^{(\zeta)}$ is an $\mathbb{F}^{(\zeta)}$ -(local) martingale if and only if the $\mathcal{P}(\mathbb{F}^{(\zeta)}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\Theta > 0$ satisfies the equality

$$\delta_t + \int_{\mathbb{R} \setminus \{0\}} h_t(z) \Theta_t(\zeta, z) \frac{f_t(\zeta, z)}{p_{t-}(\zeta)} \eta_t(dz) = 0, \quad \forall t \geq 0, \quad (77)$$

for a given $\mathbb{F}$ -predictable bounded process δ and an $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $h > -1$, satisfying the condition of (64). Then, we conclude from arguments similar to the ones applied in the previous part that the equality in (77) admits infinitely many solutions for a $\mathcal{P}(\mathbb{F}^{(\zeta)}) \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ -measurable function $\Theta > 0$. Such functions $\Theta > 0$, satisfying the condition of (76), generate the stochastic exponentials $L^{(\zeta)}$ solving the equation of (75) and identifying the corresponding equivalent (local) martingale measures $\mathcal{M}(\mathbb{F}^{(\zeta)})$.

Remark 6.5 Let $\mathcal{M}^*$ be the set of the equivalent (local) martingale measures provided by the $\mathbb{G}$ -optional projections $L^{*,\mathbb{G}}$ of the density processes $L^{*,(\tau)}$, which admit the representation of (55), where the function $\varkappa^{\mathbb{G}} > 0$ and the process $\xi^0 > -1$ are given by the equalities in (57) and (58). More precisely, one has

$$\begin{aligned} \varkappa_t^{\mathbb{G}}(z) - 1 &= \mathbb{I}_{\{\tau \geq t\}} \frac{1}{\widehat{L}_{t-}^{*,0}(\varphi_t(z) + G_{t-})} \\ &\times \int_t^\infty L_{t-}^{*,\mathbb{F}}(u) \left(\Theta_t(u, z) (f_t(u, z) + p_{t-}(u)) - \left(\frac{\varphi_t(z)}{G_{t-}} + 1 \right) p_{t-}(u) \right) \rho(du) \\ &+ \mathbb{I}_{\{\tau < t\}} (\Theta_t(\tau, z) - 1), \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \end{aligned} \quad (78)$$

$$\xi_t^0 = \frac{p_{t-}(\mathbb{F}(L_{t-}^*(t)))}{\widehat{L}_{t-}^{*,0}} - 1, \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \quad (79)$$

$$\widehat{L}_{t-}^{*,0} = \frac{1}{G_{t-}} \int_t^\infty L_{t-}^{*,\mathbb{F}}(u) p_{t-}(u) \rho(du), \quad (\mathbb{P} \times \eta_t(dz) \times dt\text{-a.e.}), \quad (80)$$

where the process $\widehat{L}^{*,0}$ is the $\mathbb{F}$ -predictable reduction of $L^{*,\mathbb{G}}$. Here, each element of $\mathcal{M}^*$ is a (locally) equivalent martingale measure on $\mathbb{G}$. Note that $\mathcal{H}^{\mathbb{G}}$ does not depend on the choice of L_0^* (see (78)), whereas ξ^0 depends on it. Since in the stochastic differential equation of (70) there are only the regularity constraints of (71) on the process $\xi^0 > -1$, the set $\mathcal{M}(\mathbb{G})$ is strictly larger than $\mathcal{M}^*$.

7 Appendix

By using the same methodology as in [17], we prove the martingale property of the two local martingales used in the proof of Proposition 5.2.

- We first prove that the $\mathbb{F}^{(\zeta)}$ -local martingale $\tilde{M}^{(\zeta)} = (\tilde{M}_t(\zeta))_{t \geq 0}$ defined by

$$\tilde{M}_t(\zeta) = \int_0^t \Upsilon_{s-} dY_s(\zeta), \quad \forall t \geq 0,$$

is a true martingale. This will be the case when, for any $T^* > 0$ fixed, the property

$$\mathbb{E} \left[\sup_{0 \leq t \leq T^*} |\tilde{M}_t(\zeta)| \right] < \infty$$

holds (cf. Chapter I, Theorem 51, page 38 in [30]). By Burkholder–Davis–Gundy’s inequality³, this condition is satisfied if the condition

$$\mathbb{E} \left[[\tilde{M}(\zeta)]_{T^*}^{1/2} \right] < \infty$$

holds. Note that we have

$$\begin{aligned} \mathbb{E} \left[[\tilde{M}^{(\zeta)}]_{T^*}^{1/2} \right] &= \mathbb{E} \left[\left(\int_0^{T^*} \Upsilon_{s-}^2 \psi_s^2(\zeta, z) \mu(ds, dz) \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq s \leq T^*} |\Upsilon_s| \left(\int_0^{T^*} \int_{\mathbb{R} \setminus \{0\}} \psi_s^2(\zeta, z) \mu(ds, dz) \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq s \leq T^*} |\Upsilon_s|^2 \right] + \mathbb{E} \left[\int_0^{T^*} \int_{\mathbb{R} \setminus \{0\}} \psi_s^2(\zeta, z) \mu(ds, dz) \right] \\ &= \mathbb{E} \left[\sup_{0 \leq s \leq T^*} |\Upsilon_s|^2 \right] + \mathbb{E} \left[\int_0^{T^*} \int_{\mathbb{R} \setminus \{0\}} \psi_s^2(\zeta, z) \nu^{(\zeta)}(ds, dz) \right], \end{aligned}$$

³ Burkholder–Davis–Gundy’s inequality states that, if M is a local martingale, for any $p \geq 1$, then the expression

$$\mathbb{E} \left[\sup_{0 \leq t \leq T^*} |M_t|^p \right] \leq C_p \mathbb{E} [[M]_{T^*}^{p/2}]$$

holds, for some $C_p > 0$ depending on p only (cf., e.g., Chapter IV, Section 4, Theorem 48, page 195, in [30]).

where we have used the fact that $|ab| \leq (a^2 + b^2)$, for any $a, b \in \mathbb{R}$. Then, by applying Burkholder–Davis–Gundy’s inequality to the process Υ defined in (44) which is a martingale, we obtain that

$$\mathbb{E} \left[\sup_{0 \leq s \leq T^*} |\Upsilon_s|^2 \right] \leq \tilde{C} \mathbb{E} \left[\int_0^{T^*} (\gamma_s^{\mathbb{G}})^2 \nu^{(\zeta)}(ds, dz) \right] < \infty,$$

for some constant $\tilde{C} > 0$. Moreover, by the assumption of square integrability of the $\mathbb{F}^{(\zeta)}$ -martingale $Y(\zeta)$, we have

$$\mathbb{E} \left[\int_0^{T^*} \int_{\mathbb{R} \setminus \{0\}} \psi_s^2(\zeta, z) \nu^{(\zeta)}(ds, dz) \right] < \infty,$$

so that the process $\tilde{M}(\zeta)$ is a martingale.

- We now prove that the $\mathbb{F}^{(\zeta)}$ -local martingale $\hat{M}^{(\zeta)} = (\hat{M}_t(\zeta))_{t \geq 0}$ defined by

$$\hat{M}_t(\zeta) = \int_0^t Y_s(\zeta) d\Upsilon_s, \quad \forall t \geq 0,$$

is a true martingale. As above, by Burkholder–Davis–Gundy’s inequality, this will be the case when, for any $T^* > 0$ fixed, the condition

$$\mathbb{E} \left[[\hat{M}_{T^*}^{(\zeta)}]^{1/2} \right] < \infty,$$

holds. Note that we have

$$\begin{aligned} \mathbb{E} \left[[\hat{M}_{T^*}^{(\zeta)}]^{1/2} \right] &= \mathbb{E} \left[\left(\int_0^{T^*} Y_s^2(\zeta) (\gamma_s^{\mathbb{G}}(z))^2 \mu(ds, dz) \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq s \leq T^*} |Y_s(\zeta)| \left(\int_0^{T^*} \int_{\mathbb{R} \setminus \{0\}} (\gamma_s^{\mathbb{G}}(z))^2 \mu(ds, dz) \right)^{1/2} \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq s \leq T^*} |Y_s(\zeta)|^2 \right] + \mathbb{E} \left[\int_0^{T^*} \int_{\mathbb{R} \setminus \{0\}} (\gamma_s^{\mathbb{G}}(z))^2 \mu(ds, dz) \right] \\ &= \mathbb{E} \left[\sup_{0 \leq s \leq T^*} |Y_s(\zeta)|^2 \right] + \mathbb{E} \left[\int_0^{T^*} \int_{\mathbb{R} \setminus \{0\}} (\gamma_s^{\mathbb{G}}(z))^2 \nu^{(\zeta)}(ds, dz) \right]. \end{aligned}$$

It follows, by means of Burkholder–Davis–Gundy’s inequality, that

$$\mathbb{E} \left[\sup_{0 \leq s \leq T^*} |Y_s(\zeta)|^2 \right] \leq \hat{C} \mathbb{E} \left[\int_0^{T^*} \psi_s^2(\zeta, z) \nu^{(\zeta)}(ds, dz) \right] < \infty,$$

for some constant $\hat{C} > 0$. Moreover, by the assumption of square integrability of the $\mathbb{F}^{(\zeta)}$ -martingale Υ , we have

$$\mathbb{E} \left[\int_0^{T^*} \int_{\mathbb{R} \setminus \{0\}} (\gamma_s^{\mathbb{G}})^2 \nu^{(\zeta)}(ds, dz) \right] < \infty,$$

so that the process $\hat{M}^{(\zeta)}$ is a martingale.

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