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RESEARCH

Perpetual American Compound Fixed-Strike Lookback Options on Maxima Drawdowns

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Abstract

We present closed-form solutions to the problem of pricing of the perpetual American compound lookback options on the maximum drawdown with fixed strikes in the Black-Merton-Scholes model. It is shown that the optimal exercise times are the first times at which the underlying risky asset price process reaches either lower or upper stochastic boundaries depending on the current values of its running maximum and maximum drawdown processes. The proof is based on the reduction of the original double optimal stopping problem to a sequence of two single optimal stopping problems for the resulting three-dimensional continuous Markov process. The latter problems are solved as the equivalent free-boundary problems by means of the smooth-fit and normal-reflection conditions for the value functions at the optimal stopping boundaries and at the edges of the three-dimensional state space. We show that the optimal exercise boundaries are determined as the maximal and minimal solutions to the appropriate first-order nonlinear ordinary differential equations.

Keywords Perpetual American compound options · Double optimal stopping problem · Geometric Brownian motion · Running maximum and maximum drawdown · First hitting time · Free-boundary problem · A change-of-variable formula with local time on surfaces

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1 Introduction

In order to give a precise mathematical formulation of the problem, we consider a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$. Let us consider the process $S = (S_t)_{t \geq 0}$ defined by

$$S_t = s \, \exp\left(\left(r - \delta - \frac{\sigma^2}{2}\right) t + \sigma \, B_t\right) \tag{1.1}$$

which solves the stochastic differential equation

$$dS_t = (r - \delta) \, S_t \, dt + \sigma \, S_t \, dB_t \quad (S_0 = s) \tag{1.2}$$

where $r > 0$, $\delta > 0$, and $\sigma > 0$ are given constants, and $s > 0$ is fixed. The process S defined in Eqs. 1.1-1.2 can be interpreted as the price of a risky asset on a financial market, where r is the riskless interest rate, δ is the dividend rate paid to the asset holders, and σ is the volatility rate. The main aim of this paper is to compute closed-form expressions for the value (or price) of the following discounted double optimal stopping problem

$$P^* = \sup_{\tau \leq \zeta} \mathbb{E}\Big[e^{-r\tau} \Big(\max_{0 \leq t \leq \tau} S_t - K \Big) + e^{-r\zeta} \Big(L - \max_{0 \leq t \leq \zeta} S_t + \max_{0 \leq t \leq \zeta} \Big(\max_{0 \leq u \leq t} S_u - S_t \Big) \Big) \Big] \tag{1.3}$$

for some given constants $K > L > 0$, where the supremum in Eq. 1.3 is taken over all subsequently occurring stopping times τ and ζ with respect to the (Brownian) natural filtration $(\mathcal{G}_t)_{t \geq 0}$ of the process S defined by $\mathcal{G}_t = \sigma(S_u \,|\, 0 \leq u \leq t)$, for all $t \geq 0$, while the expectation there is taken with respect to the risk-neutral (or martingale) probability measure $\mathbb{Q}$. In this case, the value of Eq. 1.3 can be interpreted as the rational (or no-arbitrage) price of the perpetual American compound lookback option on the running maximum and the maximum drawdown with fixed strikes $K > L > 0$ in the Black-Merton-Scholes model (cf., e.g. Shiryaev 1999, Chapter VIII; Section 2a, Peskir and Shiryaev 2006, Chapter VII; Section 25, or Detemple 2006, for an extensive overview of other related results in the area). Note that, on the one hand, the holder of the option with the value of Eq. 1.3 is aware of the large values of the asset at the time of the first exercise at which the holder receives the (discounted) largest value of the asset so far, by paying the first fixed strike price K to the option writer as a compensation. On the other hand, the holder of the considered option is aware of the large falls of the asset after its historic maximum at the time of the second exercise at which the holder pays the (discounted) difference between the largest value of the asset so far and the maximum drawdown, by receiving the second fixed strike price L from the writer as a compensation. In this respect, the holder purchases such an option as a protection against the shortfall of the risky asset price after reaching its historic maximum value.

Compound options are financial contracts which give their holders the right (but not the obligation) to buy or sell some other options at certain times in the future by the strike prices agreed in advance. Such contingent claims and the related hedging strategies are widely used in various financial markets for the purpose of risk protection (cf., e.g. Geske 1977, 1979 and Hodges and Selby 1987 for the first applications of compound options of European type with fixed maturity times). Other important versions of such contracts are compound contingent claims of American type in which both the outer and inner options can be exercised by their holders at any random (stopping) times up to maturity. The rational pricing problems for these options can thus be embedded into double (two-step) optimal stopping problems for the underlying asset price processes. The latter problems are decomposed into the appropriate sequences of single (one-step) optimal stopping problems which can then be solved separately. Moreover, in the real financial world, a common application of such contracts is the hedging of suggestions for business opportunities which may or may not be accepted in the future, and which become available only after the previous ones are undertaken. This fact makes compound options an important example of the real options to undertake business decisions which can be expressed in the presented perspective (cf. Dixit and Pindyck 1994, Chapter X for an extensive introduction to the area).

Apart from the singular and impulse stochastic control problems, the multiple (multi-step) optimal stopping problems for one-dimensional diffusion processes have recently drawn a considerable attention in the related literature. Duckworth and Zervos (2000) studied an investment model with entry and exit decisions alongside a choice of the production rate for a single commodity. The initial valuation problem was reduced to a double (two-step) optimal stopping problem which was solved through the associated dynamic programming differential equation. Carmona and Touzi (2008) derived a constructive solution to the problem of pricing of perpetual swing contracts, the recall components of which could be viewed as contingent claims with multiple exercises of American type, by using the connection between optimal stopping problems and the associated with them Snell envelopes. Carmona and Dayanik (2008) then obtained a closed form solution to a multiple (multi-step) optimal stopping problem for a general linear regular diffusion process and a general payoff function. Algorithmic constructions of the related exercise boundaries were also proposed and illustrated with several examples of such optimal stopping problems for several linear and mean-reverting diffusions. Other infinite horizon optimal stopping problems with finite sequences of stopping times, which are related to hiring and firing options, have been recently considered by Egami and Xu (2008) among others.

Discounted optimal stopping problems for certain reward functionals depending on the running maxima and minima of continuous Markov (diffusion-type) processes were initiated by Shepp and Shiryaev (1993) and further developed by Pedersen (2000), Guo and Shepp (2001), Gapeev (2007), Guo and Zervos (2010), Peskir (2012, 2014), Glover et al. (2013), Rodosthenous and Zervos (2017), Gapeev (2019, 2020, 2022, 2025), Gapeev et al. (2021) among others. The main feature in the analysis of such optimal stopping problems was that the normal-reflection conditions hold for the value functions at the diagonals of the state spaces of the multi-dimensional continuous Markov processes having the initial processes and the running extrema as their components. It was shown, by using the established by Peskir (1998) maximality principle for solutions of optimal stopping problems, which is equivalent to the superharmonic characterisation of the value functions, that the optimal stopping boundaries are characterised by the appropriate extremal solutions of certain (systems of) first-order nonlinear ordinary differential equations. Other optimal stopping problems in models with spectrally negative Lévy processes and their running maxima were studied by Asmussen et al. (2003), Avram et al. (2004), Ott (2013), Kyprianou and Ott (2014) among others.

We further reduce the original double (two-step) problem of Eq. 1.3 decomposed into a sequence of two single (one-step) optimal stopping problems of Eqs. 2.5 and 2.6 for the three-dimensional continuous Markov process (*S*, *Y*, *Z*), having the underlying risky asset price *S* and its running maximum *Y* and the maximum drawdown *Z* as their state space components. The resulting problems turn out to be necessarily three-dimensional in the sense that they cannot be reduced to optimal stopping problems for Markov processes of lower dimensions. The resulting single optimal stopping problems are solved as the equivalent free-boundary problems for the value functions which satisfy the smooth-fit conditions at the optimal stopping boundaries and the normal-reflection conditions at the edges of the three-dimensional state space. A remarkable observation is that, in comparison with the previous considerations of the perpetual American compound options and other options with payoffs depending on the current maximum or minimum of the underlying process, there are no subsets of the state space of the three-dimensional process (*S*, *Y*, *Z*) which may have both lower and upper optimal stopping boundaries for the underlying risky asset process *S*, simultaneously. This property provides a brand new feature for the optimal stopping problems for maxima and maxima drawdown processes and the related pricing of the perpetual American compound lookback fixed-strike options.

Optimal stopping problems with the appropriate one-sided continuation regions in similar models based on the original diffusion-type processes with coefficients depending on the running maximum and the running maximum drawdown were considered in Gapeev and Rodosthenous (2014), Gapeev and Rodosthenous (2016a), Gapeev and Rodosthenous (2016b). Some distributional characteristics including the probability of a drawdown of a given size occurring before a drawup of a fixed size were earlier computed by Pospisil et al. (2009) in several one-dimensional diffusion models (cf. also Zhang 2018 for an extensive survey of models with stochastic drawdowns). The problem of pricing of American compound standard put and call options in the classical Black-Merton-Scholes model was explicitly solved in Gapeev and Rodosthenous Gapeev and Rodosthenous (2014). The same problem in the more general stochastic volatility framework was studied by Chiarella and Kang (2009), where the associated two-step free-boundary problems for partial differential equations were solved numerically, by means of a modified sparse grid approach. The perpetual American compound lookback options with floating strikes as well as running maxima and maxima drawdowns or maxima and minima of the underlying processes were recently considered in Gapeev (2022) and Gapeev et al. (2022), respectively. The perpetual American lookback call option on the maximum of the market depth in Eq. 2.6 was recently formulated and solved in Gapeev and Rodosthenous (2016b) in a diffusion-type extension of the Black-Merton-Scholes model with random coefficients. However, in this paper, we indicate some additional features for the analysis of the optimal stopping problem of Eq. 2.6 for the considered classical Black-Merton-Scholes model with constant coefficients.

The rest of the paper is organised as follows. In Section 2, we embed the original optimal double stopping problem of Eq. 1.3 into the sequence of optimal (so-called *outer* and *inner*) single optimal stopping problems with the value function $P^*(s, y, z)$ in Eq. 2.5, which is equivalent to the one with $V^*(s, y, z)$ in Eq. 2.27, and the value function $U^*(s, y, z)$ in Eq. 2.6, for the three-dimensional continuous Markov process (*S*, *Y*, *Z*) defined in Eqs. 1.1-1.2 and 2.1. It is shown that the optimal exercise time $\eta^* = \eta^*(S, Y, Z)$ in the *inner* problem is the first time at which the process *S* reaches some upper boundary $h^*(Y, Z)$, while the optimal exercise time $\tau^* = \tau^*(S, Y, Z)$ in the *outer* problem is the first time at which the underlying risky asset process *S* exits an interval with the lower and upper boundaries $a^*(Y, Z)$ and $b^*(Y, Z)$ depending on the current values of the processes *Y* and *Z*. In

Section 3, we derive closed-form expressions for the candidate value functions $U^*(s, y, z)$ and $V^*(s, y, z)$ as solutions to the equivalent free-boundary problems and apply the normal-reflection conditions at the edges of the three-dimensional state spaces for (*S*, *Y*, *Z*) to characterise the lower and upper candidate optimal stopping boundaries for $a^*(Y, Z)$ and $b^*(Y, Z)$ with $h^*(Y, Z)$ as the maximal and minimal solutions to the appropriate first-order nonlinear ordinary differential equations, respectively. It follows from the structure of the solution to the *outer* free-boundary problem that there are no subsets of the state space of the three-dimensional process (*S*, *Y*, *Z*) which may have both the candidate stopping boundaries $a^*(Y, Z)$ and $b^*(Y, Z)$ for the process *S*, simultaneously. In Section 4, by applying the change-of-variable formula with local time on surfaces from Peskir (2007, Theorem 3.1), it is verified that the resulting solutions to the free-boundary problems provide the expressions for the value functions and the optimal stopping boundaries for the underlying asset price process in the original *inner* and *outer* single optimal stopping problems. The main results of the paper are stated in Theorems 4.1 and 4.2. The resulting optimal sequential exercise strategy is presented in Corollary 4.3 and described in Remark 4.4.

2 Preliminaries

In this section, we introduce the setting and notation of the three-dimensional double optimal stopping problem associated with the value of Eq. 1.3 and decompose it into an appropriate sequence of two single optimal stopping problems. We also specify the structure of the optimal exercise times and formulate the equivalent free-boundary problems.

2.1 The Double Optimal Stopping Problem

In order to proceed with the consideration of the problem in Eq. 1.3, we define the associated with *S* *running maximum* process $Y = (Y_t)_{t \geq 0}$ and the *running maximum drawdown* process $Z = (Z_t)_{t \geq 0}$ by

$$Y_t = y \vee \max_{0 \leq u \leq t} S_u \equiv \max\left\{y, \max_{0 \leq u \leq t} S_u\right\} \quad \text{and} \quad Z_t = z \vee \max_{0 \leq u \leq t} (Y_u - S_u) \tag{2.1}$$

for any arbitrary $0 < y - z \leq s \leq y$ fixed. In order to show the (strong) Markov property of the resulting (continuous time-homogeneous) three-dimensional process (*S*, *Y*, *Z*), we extend the arguments of Peskir (1998, Subsection 3.1), which proved that the process of the type (*S*, *Y*) is Markovian. More precisely, we observe that the triple $(S, Y, Z) = (S_t, Y_t, Z_t)_{t \geq 0}$ is a three-dimensional process with the state space $E = \{(s, y, z) \in \mathbb{R}^3_{++} \equiv (0, \infty)^3 \mid 0 < y - z \leq s \leq y\}$, which can change (increase) in the second coordinate only after hitting the plane $d_1 = \{(s, y, z) \in \mathbb{R}^3_{++} \mid 0 < y - z < s = y\}$, and change (increase) in the third coordinate only after hitting the plane $d_2 = \{(s, y, z) \in \mathbb{R}^3_{++} \mid 0 < y - z = s < y\}$, for any $s > 0$ fixed. Outside these planes, the process (*S*, *Y*, *Z*) changes only in the first coordinate and may be identified with the geometric Brownian motion *S*, which is a one-dimensional continuous time-homogeneous strong Markov process. Due to the form of the process (*S*, *Y*, *Z*) and its behaviour at the planes d_1 and d_2, the infinitesimal generator of (*S*, *Y*, *Z*) is thus specified as in Subsection 2.3 below.

In this case, the problem of Eq. 1.3 can naturally be embedded into the double optimal stopping problem for the three-dimensional (time-homogeneous continuous strong) Markov process $(S, Y, Z) = (S_t, Y_t, Z_t)_{t \geq 0}$ with the value

$$P^* = \sup_{\tau \leq \zeta} \mathbb{E}\big[e^{-r\tau}(Y_\tau - K) + e^{-r\zeta}(L - Y_\zeta + Z_\zeta)\big] \tag{2.2}$$

for some $K > L > 0$ fixed, where the supremum is taken over all subsequently occurring stopping times $\tau \leq \zeta$ with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}$. Then, by applying the tower property for conditional expectations, we get

$$\begin{aligned} &\mathbb{E}\big[e^{-r\tau}(Y_\tau - K) + e^{-r\zeta}(L - Y_\zeta + Z_\zeta)\big] \\ &= \mathbb{E}\big[e^{-r\tau}\big(Y_\tau - K + \mathbb{E}\big[e^{-r(\zeta-\tau)}(L - Y_{\tau+(\zeta-\tau)} + Z_{\tau+(\zeta-\tau)}) \,\big|\, \mathcal{F}_\tau\big]\big)\big] \end{aligned} \tag{2.3}$$

for any stopping times $\tau \leq \zeta$ with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}$. Hence, by applying arguments similar to the ones used in the proofs of the results of Carmona and Dayanik (2008, Propositions 3.1 and 3.2), we may conclude from the expression in Eq. 2.3 that the representation

$$\begin{aligned} &\sup_{\tau \leq \zeta} \mathbb{E}\big[e^{-r\tau}(Y_\tau - K) + e^{-r\zeta}(L - Y_\zeta + Z_\zeta)\big] \\ &= \sup_{\tau} \mathbb{E}\left[e^{-r\tau}\left(Y_\tau - K + \operatorname*{ess\,sup}_{\eta} \mathbb{E}\big[e^{-r\eta}(L - Y_{\tau+\eta} + Z_{\tau+\eta}) \,\big|\, \mathcal{F}_\tau\big]\right)\right] \end{aligned} \tag{2.4}$$

should hold, where the suprema are taken over all stopping times τ with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}$ and η with respect to $(\mathcal{F}_t^{(\tau)})_{t \geq 0}$ defined by $\mathcal{F}_t^{(\tau)} = \sigma(S_{\tau+u}/S_\tau \,|\, 0 \leq u \leq t)$, for all $t \geq 0$. Therefore, taking into account the expressions in Eqs. 2.3 and 2.4, by virtue of the strong Markov property of the process (S, Y, Z) (cf. also the proof of the result in Carmona and Dayanik 2008, Proposition 4.1 for another application), we may conclude that the original problem of Eqs. 1.3 and 2.2 can be decomposed into the sequence of the single optimal stopping problems with the *outer* value function

$$P^*(s, y, z) = \sup_{\tau} \mathbb{E}_{s,y,z}\big[e^{-r\tau}\, G(S_\tau, Y_\tau, Z_\tau)\big] \quad \text{with} \quad G(s, y, z) = y - K + U^*(s, y, z) \tag{2.5}$$

and the *inner* value function

$$U^*(s, y, z) = \sup_{\eta} \mathbb{E}_{s,y,z}\big[e^{-r\eta}(L - Y_\eta + Z_\eta)\big] \tag{2.6}$$

for some given constants $K > L > 0$, where the suprema are taken over all stopping times τ and η of the process (S, Y, Z). Here, we denote by $\mathbb{E}_{s,y,z}$ the expectation with respect to the probability measure $\mathbb{Q}_{s,y,z}$ under which the three-dimensional (time-homogeneous) continuous strong Markov process (S, Y, Z) starts at $(s, y, z) \in E$. Note that *inner* optimal stopping problem of Eq. 2.6 has been studied in Gapeev and Rodosthenous (2016b, Theorem 4.1) (cf. also Theorem 4.1 below), and we give the arguments of the proof for completeness, since some of the arguments will be used for the analysis of the *outer* optimal stopping problem of Eq. 2.5.

2.2 The Structure of the Optimal Inner Exercise Time

Let us now specify the structure of the optimal stopping time in the *inner* optimal stopping problem of Eq. 2.6. Note that the structure of the optimal stopping time in Eq. 2.6 has been earlier specified by means of the arguments of the proof of Gapeev and Rodosthenous (2016b, Lemma 2.1).

(i) Following the arguments of Subsection 2.2 above, we apply Itô's formula (cf., e.g. Liptser and Shiryaev 2001, Chapter IV, Theorem 4.4 or Revuz and Yor 1999, Chapter II, Theorem 3.2) to the process $(e^{-rt}(L - Y_t + Z_t))_{t\geq 0}$ to obtain the representation

$$e^{-rt}\,(L - Y_t + Z_t) = L - y + z + \int_0^t e^{-ru}\,r\,(Y_u - Z_u - L)\,du - \int_0^t e^{-ru}\,d(Y_u - Z_u) \tag{2.7}$$

for all $t \geq 0$, and each $0 < y - z \leq s \leq y$ fixed. Then, inserting the optimal stopping time η^* in place of *t* and applying Doob's optional sampling theorem (cf., e.g. Liptser and Shiryaev 2001, Chapter III, Theorem 3.6 or Revuz and Yor 1999, Chapter II, Theorem 3.2) to the expression in Eq. 2.7, we get that the equality

$$\begin{aligned}&\mathbb{E}_{s,y,z}\big[e^{-r\eta^*}\,(L - Y_{\eta^*} + Z_{\eta^*})\big] = L - y + z\\&+\mathbb{E}_{s,y,z}\bigg[\int_0^{\eta^*} e^{-ru}\,r\,(Y_u - Z_u - L)\,du - \int_0^{\eta^*} e^{-ru}\,d(Y_u - Z_u)\bigg]\end{aligned} \tag{2.8}$$

holds. Hence, it follows from the structure of the integrand in the first integral of Eq. 2.8 and the fact that the second integral there increases, whenever the equality $S_t = Y_t - Z_t$ holds, that it is not optimal to exercise the *inner* part of the contract, that is, exercise the compound option for the *second* time, when either the inequalities $L \leq Y_t - Z_t \leq S_t < Y_t$ hold, for any $t \geq 0$, respectively. In other words, these facts mean that the set $C_2' = \{(s, y, z) \in E \,|\, L \leq y - z \leq s < y\}$ as well as the plane $d_2 = \{(s, y, z) \in E \,|\, 0 < s = y - z < y\}$ belong to the continuation region C_2^*, which, according to the general theory of optimal stopping problems for Markov processes (cf., e.g. Peskir and Shiryaev 2006, Chapter I, Section 2.2), is given by

$$C_2^* = \big\{(s, y, z) \in E \;\big|\; U^*(s, y, z) > L - y + z\big\} \tag{2.9}$$

while the corresponding stopping region D_2^* has the form

$$D_2^* = \big\{(s, y, z) \in E \;\big|\; U^*(s, y, z) = L - y + z\big\}. \tag{2.10}$$

It is seen from the results of Theorem 4.1 formulated below that the value function $U^*(s, y, z)$ is continuous, so that the set C_2^* in Eq. 2.9 is open and the set D_2^* in Eq. 2.10 is closed.

(ii) We now observe that it follows from the definition of the process (*S*, *Y*, *Z*) in Eqs. 1.1 and 2.1 and the structure of the reward in Eq. 2.6 that, for each $0 < y - z < L$

fixed, there may also exist a sufficiently large $s > 0$ such that the point (s, y, z) belongs to the stopping region D_2^*. By virtue of arguments similar to the ones applied in Dubins et al. (1993, Subsection 3.3) and Peskir (1998, Subsection 3.3), these properties can be explained by the facts that the costs of waiting until the process S coming from such a large $s > 0$ decreases to the current value of the process $Y - Z$ may be too large due to the presence of the discounting factors in the reward functional of Eq. 2.6. Furthermore, by virtue of properties of the running maximum Y and the running maximum drawdown Z from Eq. 2.1 of the generalised geometric Brownian motion S from Eqs. 1.1-1.2, it follows that the reward functionals in Eq. 2.6 infinitesimally either decrease or increase particularly when either $S_t = Y_t$ or $S_t = Y_t - Z_t$ holds, for each $t \geq 0$, that is, the process is located either at the plane $d_1 = \{(s, y, z) \in E \,|\, 0 < y - z < s = y\}$ or at the plane $d_2 = \{(s, y, z) \in E \,|\, 0 < s = y - z < y\}$, respectively (cf., e.g. Dubins et al. 1993, Subsection 3.3 for similar arguments applied to the running maxima of the Bessel processes and Peskir 1998, Proposition 2.1 for the running maxima of a general diffusion process).

(iii) We now clarify the structure of the continuation and stopping regions C_2^* and D_2^* in Eqs. 2.9-2.10, respectively. The existence of such regions is shown in Parts (i) and (ii) of this subsection above. For the ease of presentation, in this part of the section, we also indicate by $(S^{(s)}, Y^{(y,s)}, Z^{(z,y,s)})$ the dependence of the process (S, Y, Z) defined in Eqs. 1.1 and 2.1 from its starting point $(s, y, z) \in E$. We also remind that the process S is a geometric Brownian motion explicitly given by Eq. 1.1, so that its sample paths $S^{(s)} = (S_t^{(s)})_{t \geq 0}$ started at different points $s > 0$ do not intersect each other over the whole infinite time interval.

Let us now take some point $(s, y, z) \in C_2^*$ from Eq. 2.9 and consider the optimal stopping time $\eta^* = \eta^*(s, y, z)$ for the problem Eq. 2.6 indicating the starting point (s, y, z) of the process (S, Y, Z) from Eqs. 1.1 and 2.1. On the one hand, taking into account the structure of the running maximum $Y^{(y,s)}$ and the running maximum drawdown $Z^{(z,y,s)}$ of the process $S^{(s)}$ as well as the fact that the difference process $Y^{(y,s)} - Z^{(z,y,s)}$ is actually increasing in s, for any other starting point (s_2, y, z) of the process (S, Y, Z) such that $y - z \leq s_3 < s \leq y$ holds, we obtain that the inequalities

$$\begin{aligned} U^*(s_3, y, z) &\geq \mathbb{E}\big[e^{-r\eta^*}\,\big(L - Y_{\eta^*}^{(y,s_3)} + Z_{\eta^*}^{(z,y,s_3)}\big)\big] \\ &\geq \mathbb{E}\big[e^{-r\eta^*}\,\big(L - Y_{\eta^*}^{(y,s)} + Z_{\eta^*}^{(z,y,s)}\big)\big] = U^*(s, y, z) > L - y + z \end{aligned} \tag{2.11}$$

hold, so that $(s_3, y, z) \in C_2^*$ too. On the other hand, if we take some $(s', y', z') \in D_2^*$ from Eq. 2.10 and follow the arguments applied in the expression Eq. 2.11, then we get for any other starting point (s_4, y', z') of the process (S, Y, Z) such that $y' - z' \leq s' < s_4 \leq y'$ holds that the inequalities

$$\begin{aligned} U^*(s_4, y', z') &\leq \mathbb{E}\big[e^{-r\eta^*}\,\big(L - Y_{\eta^*}^{(y',s_4)} + Z_{\eta^*}^{(z',y',s_4)}\big)\big] \\ &\leq \mathbb{E}\big[e^{-r\eta^*}\,\big(L - Y_{\eta^*}^{(y',s')} + Z_{\eta^*}^{(z',y',s')}\big)\big] = U^*(s', y', z') = L - y' + z' \end{aligned} \tag{2.12}$$

are satisfied, so that $(s_4, y', z') \in D_2^*$ too. Therefore, we may conclude that there exists a function $h^*(y, z)$ satisfying the inequality $h^*(y, z) > y - z$, for $0 < y - z < L$, such that the continuation and stopping regions C_2^* and D_2^* in Eqs. 2.9-2.10 for the optimal stopping problem of Eq. 2.6 have the form

$$C_2^* = \big\{(s, y, z) \in E \;\big|\; s < h^*(y, z)\big\} \cup C_2' \quad \text{and} \quad D_2^* = \big\{(s, y, z) \in E \;\big|\; s \geq h^*(y, z)\big\}. \tag{2.13}$$

(iv) We now specify the behaviour of the boundary $h^*(y, z)$ in the variables *y* and *z*. On the one hand, we may take some $(s, y, z) \in C_2^*$ from Eq. 2.13 again and consider the appropriate optimal stopping time $\eta^* = \eta^*(s, y, z)$ for the problem of Eq. 2.6. Then, by virtue of the structure and properties of the running maximum $Y^{(y,s)}$ and the running maximum drawdown $Z^{(z,y,s)}$ of the process $S^{(s)}$ in Eqs. 1.1 and 2.1 as well as because the linear structure of the payoff in Eq. 2.6, for another starting point (s, y_3, z_3) of the process (S, Y, Z) such that $y - z < y - z_3 \leq s \leq y < y_3$ holds, we obtain that the inequalities

$$\begin{aligned} &U^*(s, y_3, z_3) - (L - y_3 + z_3) \geq \mathbb{E}\big[e^{-r\eta^*}\,\big(L - Y_{\eta^*}^{(y_3,s)} + Z_{\eta^*}^{(z_3,y_3,s)}\big)\big] - (L - y_3 + z_3) \\ &\geq \mathbb{E}\big[e^{-r\eta^*}\,\big(L - Y_{\eta^*}^{(y,s)} + Z_{\eta^*}^{(z,y,s)}\big)\big] - (L - y + z) = U^*(s, y, z) - (L - y + z) > 0 \end{aligned} \tag{2.14}$$

hold, so that $(s, y_3, z_3) \in C_2^*$ too. On the other hand, if we take some $(s', y', z') \in D_2^*$ from Eq. 2.13 and apply arguments similar to the ones used by the derivation of the expression in Eq. 2.14 above, then we get that, for another starting point (s', y_4, z_4) of the process (S, Y, Z) such that $y_4 - z_4 < y_4 - z' \leq s' \leq y_4 < y'$ holds, the inequalities

$$\begin{aligned} &U^*(s', y_4, z_4) - (L - y_4 + z_4) \leq \mathbb{E}\big[e^{-r\eta^*}\,\big(L - Y_{\eta^*}^{(y_4,s')} + Z_{\eta^*}^{(z_4,y_4,s')}\big)\big] - (L - y_4 + z_4) \\ &\leq \mathbb{E}\big[e^{-r\eta^*}\,\big(L - Y_{\eta^*}^{(y',s')} + Z_{\eta^*}^{(z',y',s')}\big] - (L - y' + z') = U^*(s', y', z') - (L - y' + z') = 0 \end{aligned} \tag{2.15}$$

are satisfied, so that $(s', y_4, z_4) \in D_2^*$ too. Therefore, we may conclude that the upper optimal stopping boundary $h^*(y, z)$ for the process *S* in Eq. 2.13 is increasing in *y* but decreasing in *z* on $0 < y - z \leq L$.

2.3 The Inner Free-Boundary Problem

By means of standard arguments based on an application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}$ of the process (S, Y, Z) from Eqs. 1.1-1.2 and 2.1 has the form

$$\mathbb{L} = (r - \delta)\, s\, \partial_s + \frac{\sigma^2 s^2}{2}\, \partial_{ss} \quad \text{in} \quad 0 < y - z < s < y \tag{2.16}$$

$$\partial_y = 0 \quad \text{at} \quad 0 < y - z < s = y \quad \text{and} \quad \partial_z = 0 \quad \text{at} \quad 0 < s = y - z < y \tag{2.17}$$

under the probability measure $\mathbb{Q}$ (cf., e.g. Peskir 1998, Subsection 3.1 and Gapeev and Rodosthenous 2014, 2016a, b). In order to find analytic expressions for the unknown value function $U^*(s, y, z)$ from Eq. 2.6 and the unknown boundary $h^*(y, z)$ from Eq. 2.13, we use the results of general theory of optimal stopping problems for Markov processes (cf., e.g. Peskir and Shiryaev 2006, Chapter IV, Section 8) as well as optimal stopping problems for maximum processes (cf., e.g. Peskir and Shiryaev 2006, Chapter V, Sections 15-20 and references therein). More precisely, we formulate the equivalent free-boundary problem

$$(\mathbb{L}U - rU)(s, y, z) = 0 \quad \text{for} \quad s < h(y, z) \wedge y \tag{2.18}$$

$$U(s, y, z)\big|_{s=h(y,z)-} = L - y + z, \quad \partial_s U(s, y, z)\big|_{s=h(y,z)-} = 0 \tag{2.19}$$

$$\partial_z U(s, y, z)\big|_{s=(y-z)+} = 0, \quad \partial_y U(s, y, z)\big|_{s=y-} = 0 \tag{2.20}$$

$$U(s, y, z) = L - y + z \quad \text{for} \quad s \geq h(y, z) \tag{2.21}$$

$$U(s, y, z) > L - y + z \quad \text{for} \quad s < h(y, z) \wedge y \tag{2.22}$$

$$(\mathbb{L}U - rU)(s, y, z) < 0 \quad \text{for} \quad s > h(y, z) \wedge y \tag{2.23}$$

where the conditions of Eq. 2.19 are satisfied, when $y - z < h(y, z) \leq y$ holds, and the left-hand condition of Eq. 2.20 is satisfied, when $y - z < h(y, z)$ holds, while the right-hand condition of Eq. 2.20 is satisfied, when $h(y, z) > y$ holds, for all $0 < z < y$. Observe that the superharmonic characterisation of the value function (cf., e.g. Peskir and Shiryaev 2006, Chapter IV, Section 9) implies that $U^*(s, y, z)$ is the smallest function satisfying the equations in Eqs. 2.18-2.19 and 2.21-2.22 with the boundary $h^*(y, z)$.

2.4 The Equivalent Outer Optimal Stopping Problem

Let us now transform the reward in the expression of Eq. 2.2 with the aim to formulate the equivalent *outer* optimal stopping problem. For this purpose, we first recall from the results of Gapeev and Rodosthenous (2016b, Theorem 4.1) on the value function $U^*(s, y, z)$, which has the expression of Eq. 4.1 and solves the free-boundary problem in Eqs. 2.18-2.23 above, that the process $(e^{-rt} G(S_t, Y_t, Z_t))_{t \geq 0}$ with $G(s, y, z)$ given by Eq. 2.5 admits the representation

$$\begin{aligned}
e^{-rt}\, G(S_t, Y_t, Z_t) &= G(s, y, z) + \int_0^t e^{-ru}\, H(S_u, Y_u, Z_u)\, I\big(Y_u - Z_u < S_u < Y_u\big)\, du \\
&+ \int_0^t e^{-ru}\, I\big(S_u = Y_u < h^*(Y_u, Z_u)\big)\, dY_u + \int_0^t e^{-ru}\, I\big(S_u = Y_u - Z_u \geq h^*(Y_u, Z_u)\big)\, dZ_u \\
&+ \int_0^t e^{-ru}\, \partial_s G(S_u, Y_u, Z_u)\, I\big(Y_u - Z_u < S_u < Y_u\big)\, \sigma\, S_u\, dB_u
\end{aligned} \tag{2.24}$$

with

$$\begin{aligned}
H(s, y, z) &= (\mathbb{L}G - r\, G)(s, y, z) \\
&\equiv r\,(K - y)\, I\big(s < h^*(y, z)\big) + r\,(K - L - z)\, I\big(s \geq h^*(y, z)\big)
\end{aligned} \tag{2.25}$$

for each $0 < y - z < s < y$, and all $t \geq 0$, where $I(\cdot)$ denotes the indicator function. Observe that, since the partial derivative $\partial_s G(s, y, z) \equiv \partial_s U^*(s, y, z)$ is a continuous bounded function on the state space E of the process (S, Y, Z), it follows that the stochastic integral process in the third line of the expression in Eq. 2.24 is a (continuous) square integrable martingale, and hence, it is a uniformly integrable martingale under the probability measure $\mathbb{Q}$. Note that the processes Y and Z may change their values only at the times when $S_t = Y_t$ and $S_t = Y_t - Z_t$ holds, for $t \geq 0$, respectively, and such times accumulated over the infinite horizon form the sets of the Lebesgue measure zero, so that the appropriate indicators in the first and second lines of the expression in Eq. 2.24 can be ignored (cf. also Proof of Theorem 4.2 below for more explanations and references). Moreover, since the boundary $h^*(y, z)$ satisfies the inequalities $h^*(y, z) > y - z$, for all $0 < y - z \leq L$ (see the formulation of Theorem 4.1 below), we may conclude that the second integral in the second line of the expression in Eq. 2.24 turns out to be zero. Then, inserting τ in place of t and applying Doob's optional sampling theorem to the expression in Eq. 2.24, we get that the equality

$$
\begin{aligned}
&\mathbb{E}_{s,y,z}\big[e^{-r\tau}\, G(S_\tau, Y_\tau, Z_\tau)\big] = G(s, y, z) \\
&+ \mathbb{E}_{s,y,z}\bigg[\int_0^\tau e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^\tau e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u\bigg]
\end{aligned}
\tag{2.26}
$$

holds, for any stopping time τ with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$. Hence, taking into account the expression in Eq. 2.26, we conclude that the optimal stopping problem with the value of Eq. 2.5 is equivalent to the optimal stopping problem with the value function

$$
V^*(s, y, z) = \sup_\tau \mathbb{E}_{s,y,z}\bigg[\int_0^\tau e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^\tau e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u\bigg] \tag{2.27}
$$

where the function $H(s, y, z)$ is defined in Eq. 2.25, for all $(s, y, z) \in E$. Note that, since the process Y may change its values only at the times when $S_t = Y_t$ holds, for $t \geq 0$, the second integral in Eq. 2.27 makes a (strictly) positive contribution into the reward only when the running maximum process Y is located below the stochastic boundary $h^*(Y, Z)$ within the state space E of the process (S, Y, Z).

We further derive a solution to the optimal stopping problem of Eq. 2.27 with the value function $V^*(s, y, z)$, which is equivalent to the optimal stopping problem of Eq. 2.5 with the value function $P^*(s, y, z) = G(s, y, z) + V^*(s, y, z)$, and thus, the latter value function gives the solution to the original double optimal stopping problem in Eq. 1.3 under $y = s$ and $z = 0$.

2.5 The Structure of the Optimal Outer Exercise Time

Let us now specify the structure of the optimal stopping times in the *outer* optimal stopping problem of Eq. 2.27.

(i) It follows from the structure of the second integral in Eq. 2.27 as well as the fact that the process Y is increasing that it is not optimal to exercise the *outer* part of the contract, that is, exercise the compound option for the *first* time, whenever the appropriate integrand is positive. In other words, the points of the sets $c_1 = \{(s, y, z) \in E \,|\, 0 < y - z < s = y < h^*(y, z)\}$ and $c_2 = \{(s, y, z) \in E \,|\, 0 < h^*(y, z) \leq s = y - z < y\}$ belong to the continuation region

C_1^*, which, according to the general theory of optimal stopping problems for Markov processes, together with the corresponding stopping region D_1^*, is given by

$$C_1^* = \big\{(s, y, z) \in E \;\big|\; V^*(s, y, z) > 0\big\} \quad \text{and} \quad D_1^* = \big\{(s, y, z) \in E \;\big|\; V^*(s, y, z) = 0\big\} \tag{2.28}$$

respectively. It is seen from the results of Theorem 4.2 proved below that the value function $V^*(s, y, z)$ is continuous, so that the set C_1^* is open but the set D_1^* is closed in Eq. 2.28. Moreover, it follows from the structure of the first integral in Eq. 2.27 that it is also not optimal to exercise the *outer* part of the contract, when the inequality $H(S_t, Y_t, Z_t) \geq 0$ holds, which is equivalent to $0 < Y_t \leq K$ with $S_t < h^*(Y_t, Z_t)$, and $0 < Z_t \leq K - L$ (whenever the latter inequalities hold) with $S_t \geq h^*(Y_t, Z_t)$, for all $t \geq 0$. In other words, these facts mean that the points of the sets $C_1' = \{(s, y, z) \in E \,|\, 0 < y \leq K, s < h^*(y, z)\}$ and $C_1'' = \{(s, y, z) \in E \,|\, 0 < z \leq K - L, s \geq h^*(y, z)\}$ (whenever the latter set is nonempty) belong to the continuation region C_1^* in Eq. 2.28.

(ii) We now observe that it follows from the definition of the process (S, Y, Z) in Eqs. 1.1 and 2.1 and the structure of the reward in Eq. 2.27 that, for any $y > K$ or $0 \vee (K - L) < z < y$ fixed, there may exist a sufficiently small $s < h^*(y, z)$ or a sufficiently large $s' > h^*(y, z)$ such that the points (s, y, z) and (s', y, z) belong to the stopping region D_1^* in Eq. 2.28. By virtue of arguments similar to the ones applied in Dubins et al. (1993, Subsection 3.3) and Peskir (1998, Subsection 3.3), these properties can be explained by the facts that the costs of waiting until the process S coming from either such a small $s > 0$ increases to the current value of the running maximum process Y or S coming from such a large $s > 0$ decreases to the current value of the difference process $Y - Z$ may be too large due to the presence of the discounting factors in the reward functional of Eq. 2.27. Furthermore, by virtue of properties of the running maximum Y and the running maximum drawdown Z from Eq. 2.1 of the geometric Brownian motion S from Eqs. 1.1-1.2, it follows that the reward functional in Eq. 2.27 infinitesimally increase particularly when $S_t = Y_t$, for each $t \geq 0$ (cf., e.g. Dubins et al. 1993, Subsection 3.3 for similar arguments applied to the running maxima of the Bessel processes and Peskir 1998, Proposition 2.1 for the running maxima of a general diffusion process). Note that these facts are also implied directly from the arguments of the proof of Theorem 4.2 below.

(iii) We now clarify the structure of the left-hand and right-hand parts of the continuation and stopping regions C_1^* and D_1^* in Eq. 2.28, which are separated by the boundary $h^*(y, z)$ being specified in Theorem 4.1 below. The existence of such regions is shown in Parts (i) and (ii) of this subsection above. For the ease of presentation, in this part of the section, we indicate by $(S^{(s)}, Y^{(y,s)}, Z^{(z,y,s)})$ the dependence of the process (S, Y, Z) defined in Eqs. 1.1 and 2.1 from its starting point $(s, y, z) \in E$. We also remark that the optimal exercise boundary $h^*(y, z)$, which separates the continuation and stopping regions C_2^* and D_2^* from Eqs. 2.9-2.10 for the *inner* option optimal stopping problem with the value function $U^*(s, y, z)$ from Eq. 2.6 below, is increasing (non-decreasing) in y but decreasing (non-increasing) in z (see Subsection 2.2 above). In order to simplify the arguments below, we note that the value function in Eq. 2.27 admits the representation

$$\begin{aligned} V^*(s,y,z) &= \mathbb{E}_{s,y,z}\bigg[\int_0^{\tau^*} e^{-ru}\, H(S_u,Y_u,Z_u)\,du + \int_0^{\tau^*} e^{-ru}\, I\big(Y_u < h^*(Y_u,Z_u)\big)\,dY_u\bigg] \\ &\equiv \mathbb{E}\bigg[\int_0^{\tau^*} e^{-ru}\, H\big(S_u^{(s)},Y_u^{(y,s)},Z_u^{(z,y,s)}\big)\,du + \int_0^{\tau^*} e^{-ru}\, I\big(Y_u^{(y,s)} < h^*\big(Y_u^{(y,s)},Z_u^{(z,y,s)}\big)\big)\,dY_u^{(y,s)}\bigg] \end{aligned} \tag{2.29}$$

where the function $H(s,y,z)$ is defined in Eq. 2.25, for all $(s,y,z) \in E$, and $\tau^* = \tau^*(s,y,z)$ denotes the optimal stopping time for the problem of Eq. 2.27 under the assumption that the process (S, Y, Z) defined in Eq. 1.1 and Eq. 2.1 starts at any point $(s,y,z) \in E$.

On the one hand, we can take some point $(s,y,z) \in C_1^*$ from Eq. 2.28 such that either $s < h^*(y,z)$ or $s > h^*(y,z)$ holds. Then, taking into account the dependence of the running maximum $Y^{(y,s)}$ and the running maximum drawdown $Z^{(z,y,s)}$ of the process $S^{(s)}$ on the starting point (s,y,z) as well as the structure of the reward functional in Eq. 2.29 together with the form of the function $H(s,y,z)$ defined in Eq. 2.25, for any other starting point (s_1,y,z) such that either $0 < y-z \le s < s_1 \le h^*(y,z) \wedge y$ or $0 < y-z < h^*(y,z) \le s_1 < s \le y$ holds, respectively, we obtain that the inequalities

$$\begin{aligned} V^*(s_1,y,z) &\ge \mathbb{E}_{s_1,y,z}\bigg[\int_0^{\tau^*} e^{-ru}\, H(S_u,Y_u,Z_u)\,du + \int_0^{\tau^*} e^{-ru}\, I\big(Y_u < h^*(Y_u,Z_u)\big)\,dY_u\bigg] \\ &\ge \mathbb{E}_{s,y,z}\bigg[\int_0^{\tau^*} e^{-ru}\, H(S_u,Y_u,Z_u)\,du + \int_0^{\tau^*} e^{-ru}\, I\big(Y_u < h^*(Y_u,Z_u)\big)\,dY_u\bigg] = V^*(s,y,z) > 0 \end{aligned} \tag{2.30}$$

are satisfied, so that $(s_1,y,z) \in C_1^*$ too. Here, we have used the facts that the process (S, Y, Z) started at (s, y, z) may reach the point (s,y,z'), for some $0 < y-z' \le y-z \le y$, before hitting the upper plane $d_1 = \{(s,y,z) \in E \,|\, 0 < y-z \le s = y\}$ at which the running maximum process $Y^{(y,s)}$ has an increase, but may also reach the point (s,y',z), for some $0 < z \le y \le y'$, before hitting the lower plane $d_2 = \{(s,y,z) \in E \,|\, 0 < s = y-z < y\}$ at which the running maximum drawdown process $Z^{(z,y,s)}$ has an increase, as well as come to the set $c_1 = \{(s,y,z) \in E \,|\, 0 < y-z < s = y < h^*(y,z)\}$ at which the increase of $Y^{(y,s)}$ yields an increase of the whole reward functional in Eq. 2.29.

On the other hand, we can take some point $(s',y',z') \in D_1^*$ from Eq. 2.28 such that either $s' \le h^*(y',z')$ or $s' \ge h^*(y',z')$ holds. Then, we may follow the arguments applied by the derivation of the expression in Eq. 2.30 above to get that, for another starting point (s_2,y',z') of the process (S, Y, Z) such that either $0 < y'-z' \le s_2 < s' \le h^*(y',z') \wedge y'$ or $0 < y'-z' < h^*(y',z') \le s' < s_2 \le y'$ holds, the inequalities

$$\begin{aligned} V^*(s_2,y',z') &\le \mathbb{E}_{s_2,y',z'}\bigg[\int_0^{\tau^*} e^{-ru}\, H(S_u,Y_u,Z_u)\,du + \int_0^{\tau^*} e^{-ru}\, I\big(Y_u < h^*(Y_u,Z_u)\big)\,dY_u\bigg] \\ &\le \mathbb{E}_{s',y',z'}\bigg[\int_0^{\tau^*} e^{-ru}\, H(S_u,Y_u,Z_u)\,du + \int_0^{\tau^*} e^{-ru}\, I\big(Y_u < h^*(Y_u,Z_u)\big)\,dY_u\bigg] = V^*(s',y',z') = 0 \end{aligned} \tag{2.31}$$

are satisfied, so that $(s_2,y',z') \in D_1^*$ too. Therefore, we may conclude that there exist functions $a^*(y,z)$ and $b^*(y,z)$ satisfying the inequalities $a^*(y,z) \le h^*(y,z) \wedge y$ and $b^*(y,z) \ge h^*(y,z) > y-z$, for $y > K$ and $K-L < z < y$, respectively, such that the continuation and stopping regions C_1^* and D_1^* in Eq. 2.28 have the form

$$C_1^* = \Big\{(s,y,z) \in E \;\Big|\; a^*(y,z) < s < b^*(y,z)\Big\} \cup C_1' \cup C_1'' \tag{2.32}$$

and

$$D_1^* = \{(s, y, z) \in E \mid s \le a^*(y, z) \ \text{ or } \ s \ge b^*(y, z)\}. \tag{2.33}$$

(iv) We now specify the behaviour of the lower and upper stopping boundaries $a_*(y, z)$ and $b^*(y, z)$ from Eqs. 2.32-2.33 for the process S in the variables *y* and *z*. On the one hand, we can take some point $(s, y, z) \in C_1^*$ from Eq. 2.28 such that either $s < h^*(y, z)$ or $s > h^*(y, z)$ holds. Hence, we may follow the arguments applied by the derivation of the expression in Eq. 2.30 above to get that, for another starting point (s, y_1, z_1) of the process (*S*, *Y*, *Z*) such that either $y_1 - z_1 < y_1 - z \le s \le y_1 < y$ or $y - z < y - z_1 \le s \le y < y_1$ holds, the inequalities

$$\begin{aligned} V^*(s, y_1, z_1) &\ge \mathbb{E}_{s,y_1,z_1}\left[\int_0^{\tau^*} e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^{\tau^*} e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u\right] \\ &\ge \mathbb{E}_{s,y,z}\left[\int_0^{\tau^*} e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^{\tau^*} e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u\right] = V^*(s, y, z) > 0 \end{aligned} \tag{2.34}$$

are satisfied, so that $(s, y_1, z_1) \in C_1^*$ too. On the other hand, let us now fix some $(s', y', z') \in D_1^*$ from Eq. 2.28 such that either $s' < h^*(y', z')$ or $s' \ge h^*(y', z')$ holds. Then, using the arguments applied by the derivation of the expression in Eq. 2.34 above, for another starting point (s', y_2, z_2) of the process (*S*, *Y*, *Z*) such that either $y' - z' < y' - z_2 \le s' \le y' < y_2$ or $y_2 - z_2 < y_2 - z' \le s' \le y_2 < y'$ holds, we obtain that the inequalities

$$\begin{aligned} V^*(s', y_2, z_2) &\le \mathbb{E}_{s',y_2,z_2}\left[\int_0^{\tau^*} e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^{\tau^*} e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u\right] \\ &\le \mathbb{E}_{s',y',z'}\left[\int_0^{\tau^*} e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^{\tau^*} e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u\right] = V^*(s', y', z') = 0 \end{aligned} \tag{2.35}$$

are satisfied, so that $(s', y_2, z_2) \in D_1^*$ too. Therefore, we may conclude that the upper and lower boundaries $a^*(y, z)$ and $b^*(y, z)$ for the process *S* in Eqs. 2.32 and 2.33 are increasing (non-decreasing) in *y* on $y > K$ but decreasing (non-increasing) in *z* on $K - L < z < y$.

By looking ahead, we also remark from the arguments of Subsection 3.4 below that there could be no region of the state space E of the three-dimensional process (*S*, *Y*, *Z*) in which we would have both the lower and upper optimal stopping boundaries $a^*(y, z)$ and $b^*(y, z)$ for the component *S*, simultaneously. In other words, in will be shown below that only the situations $0 < a^*(y, z) < b^*(y, z) = \infty$ or $0 = a^*(y, z) < b^*(y, z) < \infty$ or $0 = a^*(y, z) < b^*(y, z) = \infty$ can occur in the expressions for C_1^* and D_1^* in Eqs. 2.32-2.33, for any $y > K$ and $K - L < z < y$ fixed, respectively.

2.6 The Outer Free-Boundary Problem

In order to find analytic expressions for the unknown value function $V^*(s, y, z)$ from Eq. 2.27 with the unknown boundaries boundaries $a^*(y, z)$ and $b^*(y, z)$ from Eqs. 2.32 and 2.33, we apply the results of general theory of optimal stopping problems for Markov processes to reduce the optimal stopping problem of Eq. 2.27 to the equivalent free-boundary problem

$$(\mathbb{L}V - r\,V)(s, y, z) = -H(s, y, z) \quad \text{for} \quad (y-z) \vee a(y,z) < s < b(y,z) \wedge y \tag{2.36}$$

$$V(s, y, z)\big|_{s=a(y,z)+} = 0, \quad V(s, y, z)\big|_{s=b(y,z)-} = 0 \tag{2.37}$$

$$\partial_s V(s, y, z)\big|_{s=a(y,z)+} = 0, \quad \partial_s V(s, y, z)\big|_{s=b(y,z)-} = 0 \tag{2.38}$$

$$\partial_z V(s, y, z)\big|_{s=(y-z)+} = 0, \quad \partial_y V(s, y, z)\big|_{s=y-} = -I(y < h^*(y,z)) \tag{2.39}$$

$$V(s, y, z) = 0 \quad \text{for} \quad s \le a(y,z) \;\; \text{and} \;\; s \ge b(y,z) \tag{2.40}$$

$$V(s, y, z) > 0 \quad \text{for} \quad (y-z) \vee a(y,z) < s < b(y,z) \wedge y \tag{2.41}$$

$$(\mathbb{L}V - r\,V)(s, y, z) < -H(s, y, z) \quad \text{for} \quad s \le a(y,z) \quad \text{and} \quad s \ge b(y,z) \tag{2.42}$$

where the function $H(s, y, z)$ is defined in Eq. 2.25, the left-hand conditions of Eqs. 2.37-2.38 are satisfied, when $y - z \le a(y,z) < y$ holds, and the right-hand conditions of Eqs. 2.37-2.38 are satisfied, when $y - z < b(y,z) \le y$ holds, as well as the left-hand condition of Eq. 2.39 is satisfied, when $a(y,z) < y - z < b(y,z) \le y$ holds, and the right-hand condition of Eq. 2.39 is satisfied, when $y - z \le a(y,z) < y < b(y,z)$ holds, for $0 < z < y$. Observe that the superharmonic characterisation of the value function implies that $V^*(s, y, z)$ is the smallest function satisfying the the equations in Eqs. 2.36-2.37 and 2.40-2.41 with the boundaries $a^*(y,z)$ and $b^*(y,z)$.

3 Solutions to the Free-Boundary Problems

In this section, we obtain closed-form expressions for the value functions $U^*(s, y, z)$ in Eq. 2.6 and $V^*(s, y, z)$ in Eq. 2.27 of the perpetual American compound lookback fixed-strike put option on the maximum drawdown. We also derive arithmetic as well as first-order nonlinear ordinary differential equations for the optimal exercise boundaries $h^*(y,z)$ in Eq. 2.13 and $a^*(y,z)$ and $b^*(y,z)$ in Eqs. 2.32-2.33 providing solutions to the free-boundary problems in Eqs. 2.18-2.23 and 2.36-2.42 above, respectively.

3.1 The Candidate Inner Value Function

We first observe that the general solution of the second-order ordinary differential equation in Eq. 2.18 with Eq. 2.16 has the form

$$U(s, y, z) = D_1(y,z)\, s^{\gamma_1} + D_2(y,z)\, s^{\gamma_2} \tag{3.1}$$

for $0 < y - z \leq s \leq y$, where $D_j(y,z)$, for $j = 1, 2$, are some (arbitrary) continuously differentiable functions, and the roots of the corresponding (quadratic) characteristic equation γ_j, for $j = 1, 2$, are given by

$$\gamma_j = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{3.2}$$

so that $\gamma_2 < 0 < 1 < \gamma_1$ holds. Then, by applying the conditions from Eqs. 2.19-2.20 to the function in Eq. 3.1, we get that the equalities

$$D_1(y,z)\, h^{\gamma_1}(y,z) + D_2(y,z)\, h^{\gamma_2}(y,z) = L - y + z \tag{3.3}$$

$$D_1(y,z)\, \gamma_1\, h^{\gamma_1}(y,z) + D_2(y,z)\, \gamma_2\, h^{\gamma_2}(y,z) = 0 \tag{3.4}$$

$$\partial_z D_1(y,z)\, (y-z)^{\gamma_1} + \partial_z D_2(y,z)\, (y-z)^{\gamma_2} = 0 \tag{3.5}$$

$$\partial_y D_1(y,z)\, y^{\gamma_1} + \partial_y D_2(y,z)\, y^{\gamma_2} = 0 \tag{3.6}$$

hold, for some boundary $h(y, z)$, for $0 < z < y$. Here, the conditions of Eqs. 3.3-3.4 are satisfied, when $y - z < h(y,z) \leq y$ holds, while the condition of Eq. 3.5 is satisfied, when $y - z < h(y,z)$ holds, and the condition of Eq. 3.6 is satisfied, when $h(y,z) > y$ holds, for $0 < z < y$.

Hence, by solving the system of equations in Eqs. 3.3+3.4, we obtain that the candidate value function admits the representation

$$U(s,y,z;h(y,z)) = D_1(y,z;h(y,z))\, s^{\gamma_1} + D_2(y,z;h(s,y))\, s^{\gamma_2} \tag{3.7}$$

holds, for $0 < y - z \leq s < h(y,z) \leq y$, where we set

$$D_j(y,z;h(y,z)) = \frac{\gamma_{3-j}(L-y+z)}{(\gamma_{3-j}-\gamma_j)h^{\gamma_j}(y,z)} \tag{3.8}$$

for $0 < y - z < h(y,z) \leq y$, for every $j = 1, 2$. Also, taking into account the conditions of Eqs. 3.3-3.6, we obtain that the candidate value function admits the representation

$$U(s,y,z;y(z),z(y)) = D_1(y,z;y(z),z(y))\, s^{\gamma_1} + D_2(y,z;y(z),z(y))\, s^{\gamma_2} \tag{3.9}$$

for $0 < y - z \leq s \leq y < h(y,z)$. Here, the functions $D_j(y,z;y(z),z(y))$, for $j = 1, 2$, provide a solution to the two-dimensional coupled system of first-order linear partial differential equations

$$\partial_z D_1(y,z;y(z),z(y))\, (y-z)^{\gamma_1} + \partial_z D_2(y,z;y(z),z(y))\, (y-z)^{\gamma_2} = 0 \tag{3.10}$$

$$\partial_y D_1(y,z;y(z),z(y))\, y^{\gamma_1} + \partial_y D_2(y,z;y(z),z(y))\, y^{\gamma_2} = 0 \tag{3.11}$$

for $0 < z < y$, satisfying the boundary conditions

$$D_1(\underline{y}(z)-, z; y(z), z(y))\,(\underline{y}(z))^{\gamma_1} + D_2(\underline{y}(z)-, z; y(z), z(y))\,(\underline{y}(z))^{\gamma_2} = L - y + z \tag{3.12}$$

and

$$D_1(\underline{y}(z)-, z; y(z), z(y))\,\gamma_1\,(\underline{y}(z))^{\gamma_1} + D_2(\underline{y}(z)-, z; y(z), z(y))\,\gamma_2\,(\underline{y}(z))^{\gamma_2} = 0 \tag{3.13}$$

for $0 < \underline{y}(z) - z < L$, where we set $\underline{y}(z) = \sup\{z < y < z + L \,|\, h(y,z) \le y\}$, representing the value of the y-coordinate of the point on the curve at which the surface $\{(s,y,z) \in E \,|\, s = h(y,z)\}$ intersects the plane $d_1 = \{(s,y,z) \in E \,|\, 0 < y - z < s = y\}$, by either entering or leaving (this depends on the point of view of the consideration of the resulting picture) the state space E, for $z > 0$ fixed. Moreover, for the functions $D_j(y, z; y(z), z(y))$, for $j = 1, 2$, from Eq. 3.9, we have

$$\begin{aligned} &D_1(y, \underline{z}(y)+; y(z), z(y))\,(y - \underline{z}(y))^{\gamma_1} + D_2(y, \underline{z}(y)+; y(z), z(y))\,(y - \underline{z}(y))^{\gamma_2} \\ &= D_1(y, \underline{z}(y)-; y(z), z(y))\,(y - \underline{z}(y))^{\gamma_1} + D_2(y, \underline{z}(y)-; y(z), z(y))\,(y - \underline{z}(y))^{\gamma_2} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\partial_z D_1(y, \underline{z}(y)+; y(z), z(y))\,(y - \underline{z}(y))^{\gamma_1} + \partial_z D_2(y, \underline{z}(y)+; y(z), z(y))\,(y - \underline{z}(y))^{\gamma_2} \\ &= \partial_z D_1(y, \underline{z}(y)-; y(z), z(y))\,(y - \underline{z}(y))^{\gamma_1} + \partial_z D_2(y, \underline{z}(y)-; y(z), z(y))\,(y - \underline{z}(y))^{\gamma_2} \end{aligned} \tag{3.15}$$

for $0 < \underline{z}(y) < y$, where we set $\underline{z}(y) = \inf\{0 < z < y \,|\, h(y,z) \le y\}$, representing the value of the z-coordinate of the point on the curve at which the surface $\{(s,y,z) \in E \,|\, s = h(y,z)\}$ intersects the plane $d_1 = \{(s,y,z) \in E \,|\, 0 < y - z < s = y\}$, by either entering or leaving (this depends on the point of view of the consideration of the resulting picture) the state space E, for $y > 0$ fixed (see Gapeev and Rodosthenous 2016b, Section 3).

3.2 The Candidate Inner Stopping Boundary

Furthermore, assuming that the candidate boundary function $h(y, z)$ is continuously differentiable, we apply the right-hand condition of Eq. 2.20 to the function $U(s, y, z; h(y, z))$ in Eq. 3.7 with $D_j(y, z; h(y, z))$, for $j = 1, 2$, in Eq. 3.8 to conclude that the candidate boundary $h(y, z)$ satisfies the first-order nonlinear ordinary differential equation (with a parameter)

$$\partial_z h(y,z) = \frac{\gamma_2((y-z)/h(y,z))^{\gamma_1} - \gamma_1((y-z)/h(y,z))^{\gamma_2}}{\gamma_1\gamma_2(L - y + z)(((y-z)/h(y,z))^{\gamma_1} - ((y-z)/h(y,z))^{\gamma_2})} \tag{3.16}$$

for $0 < y - z < L$. Note that the right-hand side of the expression in Eq. 3.16 is (locally) continuous in $(y, z, h(y, z))$ and (locally) Lipschitz in $h(y, z)$, for each $0 < y - z < L$ fixed. Thus, by means of the classical results on the existence and uniqueness of solutions for first-order nonlinear ordinary differential equations, the equation in Eq. 3.16 admits a (locally) unique solution. Then, it is shown by means of standard arguments (similar to the ones applied in Subsection 3.5 below) that one can construct a (continuous) *minimal admissible* solution $h^*(y, z)$ to the the equation in Eq. 3.16 such that the inequalities $h^*(y, z) > y - z > 0$ hold, for all $0 < y - z < L$. It also follows from the structure of the ordinary differential equation in Eq. 3.16 as well as the fact proved in Part (iv) of Subsection 2.2 above that the boundary $h^*(y,z)$ is increasing in y but decreasing in z (whenever it exists) that $h^*(y,z)$ is located in the region $\{(s,y,z) \in E \,|\, 0 < y-z \le L\}$. Observe from the structure of the equation in Eq. 3.16 that the boundary $h^*(y,z)$ can also be characterised as an increasing (non-decreasing) function of the difference $h^*(y,z) \equiv \widetilde{h}^*(y-z)$, for all $0 < y-z \le L$.

3.3 The Candidate Outer Value Function

We now follow straightforward calculations similar to the ones used in Subsection 3.1 above to show that the general solution of the second-order ordinary differential equation in Eq. 2.36 with Eq. 2.16 has the form

$$V(s,y,z) = C_1(y,z)\, s^{\gamma_1} + C_2(y,z)\, s^{\gamma_2} - F(s,y,z) \tag{3.17}$$

for $0 < y-z \le s \le y$, where $C_j(y,z)$, for $j=1,2$, are some (arbitrary) continuously differentiable functions, the function $F(s,y,z)$ is the appropriate particular solution given by

$$F(s,y,z) = (y-K)\, I\big(s < h^*(y,z)\big) + (L-K+z)\, I\big(s \ge h^*(y,z)\big) \tag{3.18}$$

for $0 < y-z \le s \le y$, and $\gamma_2 < 0 < 1 < \gamma_1$ are defined in Eq. 3.2 above. Then, by applying the conditions from Eqs. 2.37-2.39 to the function in Eq. 3.17, we get that the equalities

$$C_1(y,z)\, a^{\gamma_1}(y,z) + C_2(y,z)\, a^{\gamma_2}(y,z) = y-K \tag{3.19}$$

$$C_1(y,z)\, b^{\gamma_1}(y,z) + C_2(y,z)\, b^{\gamma_2}(y,z) = L-K+z \tag{3.20}$$

$$C_1(y,z)\, \gamma_1\, a^{\gamma_1}(y,z) + C_2(y,z)\, \gamma_2\, a^{\gamma_2}(y,z) = 0 \tag{3.21}$$

$$C_1(y,z)\, \gamma_1\, b^{\gamma_1}(y,z) + C_2(y,z)\, \gamma_2\, b^{\gamma_2}(y,z) = 0 \tag{3.22}$$

$$\partial_z C_1(y,z)\, (y-z)^{\gamma_1} + \partial_z C_2(y,z)\, (y-z)^{\gamma_2} = 0 \tag{3.23}$$

$$\partial_y C_1(y,z)\, y^{\gamma_1} + \partial_y C_2(y,z)\, y^{\gamma_2} = 0 \tag{3.24}$$

hold, for some boundaries $a(y,z) < h^*(y,z)$ and $b(y,z) \ge h^*(y,z)$, for $0 < z < y$. Here, the conditions of Eqs. 3.19 and 3.21 are satisfied, when $y-z \le a(y,z) < y$ holds, and the conditions of Eqs. 3.20 and 3.22 are satisfied, when $y-z < b(y,z) \le y$ holds, while the condition of Eq. 3.23 is satisfied, when $a(y,z) < y-z < b(y,z) \le y$ holds, and the condition of Eq. 3.24 is satisfied, when $y-z \le a(y,z) < y < b(y,z)$ holds, for $0 < z < y$.

Hence, by solving the system of equations in Eqs. 3.19+3.21, we obtain that the candidate value function admits the representation

$$V(s,y,z;a(y,z)) = C_1(y,z;a(y,z))\, s^{\gamma_1} + C_2(y,z;a(y,z))\, s^{\gamma_2} - F(s,y,z) \tag{3.25}$$

for $0 < y-z \le a(y,z) < s \le y < b(y,z)$, where we set

$$C_j(y,z;a(y,z)) = \frac{\gamma_{3-j}(y-K)}{(\gamma_{3-j}-\gamma_j)a^{\gamma_j}(y,z)} \tag{3.26}$$

when $0 < y - z \le a(y,z) < y < b(y,z)$ holds, for every $j = 1, 2$. Also, by solving the system of equations in Eqs. 3.20+3.22, we obtain that the candidate value function admits the representation

$$V(s,y,z;b(y,z)) = C_1(y,z;b(y,z))\,s^{\gamma_1} + C_2(y,z;b(y,z))\,s^{\gamma_2} - F(s,y,z) \tag{3.27}$$

for $0 < a(y,z) < y - z \le s < b(y,z) \le y$, where we set

$$C_j(y,z;b(y,z)) = \frac{\gamma_{3-j}(L-K+z)}{(\gamma_{3-j}-\gamma_j)b^{\gamma_j}(y,z)} \tag{3.28}$$

when $0 < a(y,z) < y - z < b(y,z) \le y$ holds, for $K - L < z < y$ and every $j = 1, 2$.

Moreover, by means of straightforward computations, it can be deduced from the expressions in Eqs. 3.25 and 3.27 with Eq. 3.18 that the first-order partial derivative $\partial_s V(s,y,z;a(y,z),b(y,z))$ (of either $V(s,y,z;a(y,z))$ or $V(s,y,z;b(y,z))$) takes the form

$$\partial_s V(s,y,z;a(y,z),b(y,z)) = \sum_{j=1}^{2} C_j(y,z;a(y,z),b(y,z))\,\gamma_j\,s^{\gamma_j-1} \tag{3.29}$$

(with $C_j(y,z;a(y,z),b(y,z))$ for either $C_j(y,z;a(y,z))$ or $C_j(y,z;b(y,z))$) and the appropriate second-order partial derivative $\partial_{ss} V(s,y,z;a(y,z),b(y,z))$ is given by

$$\partial_{ss} V(s,y,z;a(y,z),b(y,z)) = \sum_{j=1}^{2} C_j(y,z;a(y,z),b(y,z))\,\gamma_j\,(\gamma_j-1)\,s^{\gamma_j-2} \tag{3.30}$$

on the interval $(y-z) \vee a(y,z) < s < b(y,z) \wedge y$, for each $K - L < z < y$, respectively.

Furthermore, taking into account the conditions of Eqs. 3.19-3.24, we obtain that the candidate value function admits the representation

$$V(s,y,z;y(z),z(y)) = C_1(y,z;y(z),z(y))\,s^{\gamma_1} + C_2(y,z;y(z),z(y))\,s^{\gamma_2} - F(s,y,z) \tag{3.31}$$

for $0 < a(y,z) < y - z \le s \le y < b(y,z)$. Here, the functions $C_j(y,z;y(z),z(y))$, for $j = 1, 2$, provide a solution to the two-dimensional coupled system of first-order linear partial differential equations

$$\partial_z C_1(y,z;y(z),z(y))\,(y-z)^{\gamma_1} + \partial_z C_2(y,z;y(z),z(y))\,(y-z)^{\gamma_2} = 0 \tag{3.32}$$

$$\partial_y C_1(y,z;y(z),z(y))\,y^{\gamma_1} + \partial_y C_2(y,z;y(z),z(y))\,y^{\gamma_2} = 0 \tag{3.33}$$

for $0 < z < y$, satisfying the boundary conditions

$$C_1(y, \widetilde{z}(y)-; y(z), z(y))\,(y - \widetilde{z}(y))^{\gamma_1} + C_2(y, \widetilde{z}(y)-; y(z), z(y))\,(y - \widetilde{z}(y))^{\gamma_2} = y - K \tag{3.34}$$

and

$$C_1(y, \widetilde{z}(y)-; y(z), z(y))\,\gamma_1\,(y - \widetilde{z}(y))^{\gamma_1} + C_2(y, \widetilde{z}(y)-; y(z), z(y))\,\gamma_2\,(y - \widetilde{z}(y))^{\gamma_2} = 0 \tag{3.35}$$

for $0 < \widetilde{z}(y) < y$, where we set $\widetilde{z}(y) = \sup\{0 < z < y \,|\, a(y, z) \geq y - z\}$, representing the value of the z-coordinate of the point on the curve at which the surface $\{(s, y, z) \in E \,|\, s = a(y, z)\}$ intersects the plane d_2 by either entering or leaving the state space E, for $y > 0$ fixed. Moreover, for the functions $C_j(y, z; y(z), z(y))$, for $j = 1, 2$, from Eq. 3.31, we have

$$\begin{aligned} &C_1(\widetilde{y}(z)-, z; y(z), z(y))\,(\widetilde{y}(z))^{\gamma_1} + C_2(\widetilde{y}(z)-, z; y(z), z(y))\,(\widetilde{y}(z))^{\gamma_2} \\ &= C_1(\widetilde{y}(z)+, z; y(z), z(y))\,(\widetilde{y}(z))^{\gamma_1} + C_2(\widetilde{y}(z)+, z; y(z), z(y))\,(\widetilde{y}(z))^{\gamma_2} \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} &\partial_y C_1(\widetilde{y}(z)-, z; y(z), z(y))\,(\widetilde{y}(z))^{\gamma_1} + \partial_y C_2(\widetilde{y}(z)-, z; y(z), z(y))\,(\widetilde{y}(z))^{\gamma_2} \\ &= \partial_y C_1(\widetilde{y}(z)+, z; y(z), z(y))\,(\widetilde{y}(z))^{\gamma_1} + \partial_y C_2(\widetilde{y}(z)+, z; y(z), z(y))\,(\widetilde{y}(z))^{\gamma_2} \end{aligned} \tag{3.37}$$

for $0 < z < \widetilde{y}(z)$, where we set $\widetilde{y}(z) = \inf\{z < y < z + L \,|\, a(y, z) \geq y - z\}$, representing the value of the y-coordinate of the point on the curve at which the surface $\{(s, y, z) \in E \,|\, s = a(y, z)\}$ intersects the plane d_2 by either entering or leaving the state space E, for $z > 0$ fixed.

Finally, we observe that the functions $C_j(y, z; y(z), z(y))$, for $j = 1, 2$, from Eq. 3.31 satisfy the boundary conditions

$$C_1(\overline{y}(z)-, z; y(z), z(y))\,(\overline{y}(z))^{\gamma_1} + C_2(\overline{y}(z)-, z; y(z), z(y))\,(\overline{y}(z))^{\gamma_2} = L - K + z \tag{3.38}$$

and

$$C_1(\overline{y}(z)-, z; y(z), z(y))\,\gamma_1\,(\overline{y}(z))^{\gamma_1} + C_2(\overline{y}(z)-, z; y(z), z(y))\,\gamma_2\,(\overline{y}(z))^{\gamma_2} = 0 \tag{3.39}$$

for $0 < \overline{y}(z) - z < L$, where we set $\overline{y}(z) = \sup\{z < y < z + L \,|\, b(y, z) \leq y\}$, representing the value of the y-coordinate of the point on the curve at which the surface $\{(s, y, z) \in E \,|\, s = b(y, z)\}$ intersects the plane d_1 by either entering or leaving the state space E, for $z > 0$ fixed. Moreover, for the functions $C_j(y, z; y(z), z(y))$, for $j = 1, 2$, from Eq. 3.31, we have

$$\begin{aligned} &C_1(y, \overline{z}(y)+; y(z), z(y))\,(y - \overline{z}(y))^{\gamma_1} + C_2(y, \overline{z}(y)+; y(z), z(y))\,(y - \overline{z}(y))^{\gamma_2} \\ &= C_1(y, \overline{z}(y)-; y(z), z(y))\,(y - \overline{z}(y))^{\gamma_1} + C_2(y, \overline{z}(y)-; y(z), z(y))\,(y - \overline{z}(y))^{\gamma_2} \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} &\partial_z C_1(y, \overline{z}(y)+; y(z), z(y))\,(y - \overline{z}(y))^{\gamma_1} + \partial_z C_2(y, \overline{z}(y)+; y(z), z(y))\,(y - \overline{z}(y))^{\gamma_2} \\ &= \partial_z C_1(y, \overline{z}(y)-; y(z), z(y))\,(y - \overline{z}(y))^{\gamma_1} + \partial_z C_2(y, \overline{z}(y)-; y(z), z(y))\,(y - \overline{z}(y))^{\gamma_2} \end{aligned} \tag{3.41}$$

for $0 < \underline{z}(y) < y$, where we set $\underline{z}(y) = \inf\{0 < z < y \mid b(y, z) \leq y\}$, representing the value of the z-coordinate of the point on the curve at which the surface $\{(s, y, z) \in E \mid s = b(y, z)\}$ intersects the plane d_1 by either entering or leaving the state space E, for $y > 0$ fixed (Figs. 1, 2, 3, 4, 5, and 6).

3.4 The Candidate Outer Exercise Boundaries

We now note that, if we equate the functions $C_j(y, z; a(y, z))$ and $C_j(y, z; b(y, z))$, for $j = 1, 2$, in Eqs. 3.26 and 3.28, then we obtain the equalities

$$\frac{y - K}{L - K + z} = \left(\frac{a(y, z)}{b(y, z)} \right)^{\gamma_j} \tag{3.42}$$

for every $j = 1, 2$ and all $0 < y - z \leq a(y, z) < h^*(y, z) \leq b(y, z) \leq y$ such that $y > K$ and $K - L < z < y$ holds. However, we can see that the system in Eq. 3.42 has no solution, which yields that there could not be any region in the state space E of the three-dimensional process (S, Y, Z) in which we have both the lower and upper candidate stopping boundaries $a^*(y, z)$ and $b^*(y, z)$ for the component S, simultaneously.

Furthermore, assuming that the candidate boundary functions $a(y, z)$ and $b(y, z)$ are continuously differentiable, we apply the condition of Eq. 3.24 to the functions $C_i(y, z; a(y, z))$, for $i = 1, 2$, in Eq. 3.26 to get that the candidate boundary $a(y, z)$ satisfies the ordinary differential equation (with a parameter)

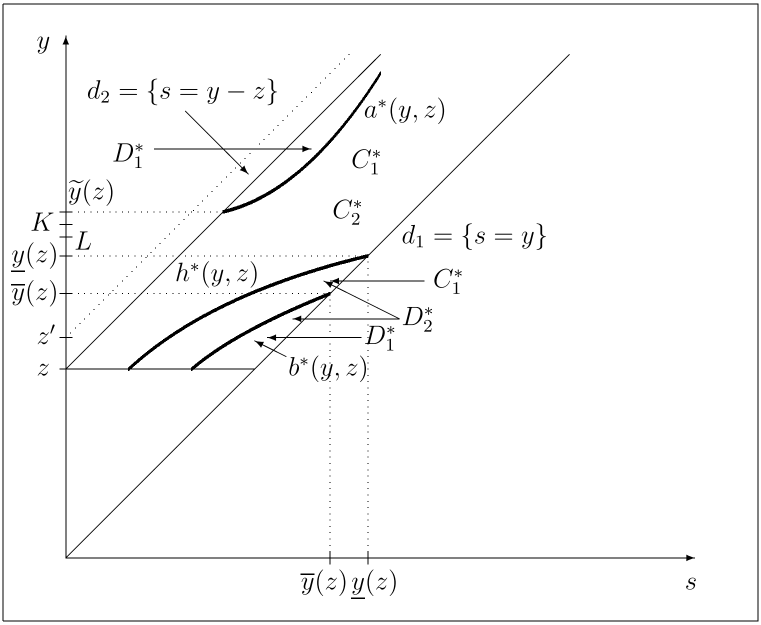

Fig. 1 A computer drawing of the optimal exercise boundaries $a^*(y, z)$, $b^*(y, z)$ and $h^*(y, z)$, for each $0 < z < L < K$ fixed

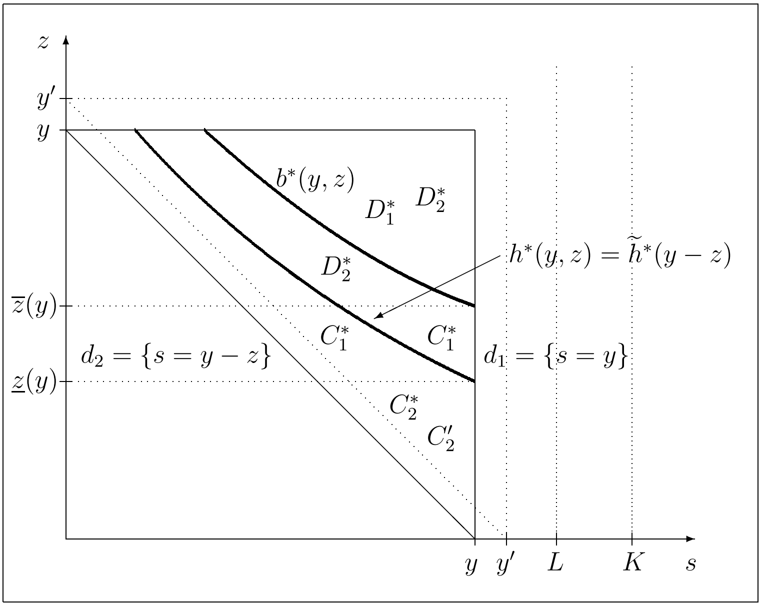

Fig. 2 A computer drawing of the optimal exercise boundaries $a^*(y, z)$, $b^*(y, z)$ and $h^*(y, z)$, for each $0 < y < L < K$ fixed

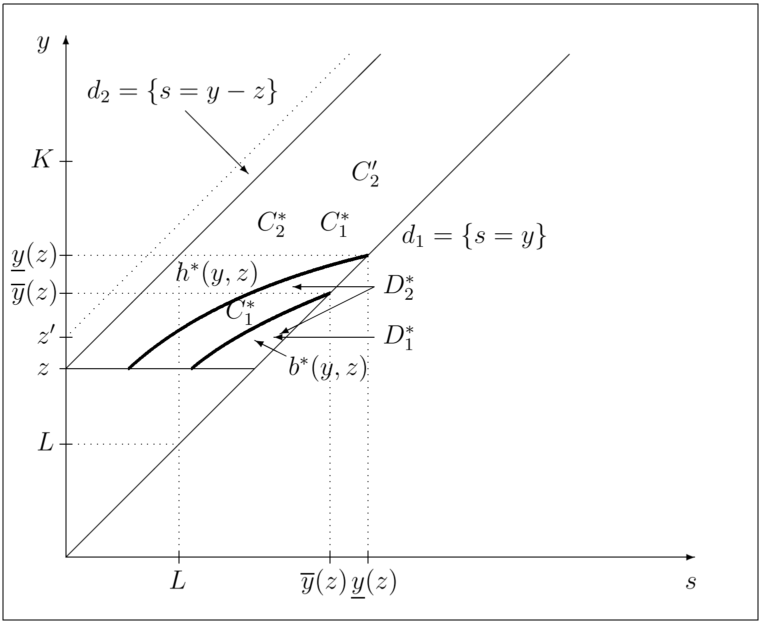

Fig. 3 A computer drawing of the optimal exercise boundaries $a^*(y, z)$, $b^*(y, z)$ and $h^*(y, z)$, for each $0 < z < L < K$ fixed

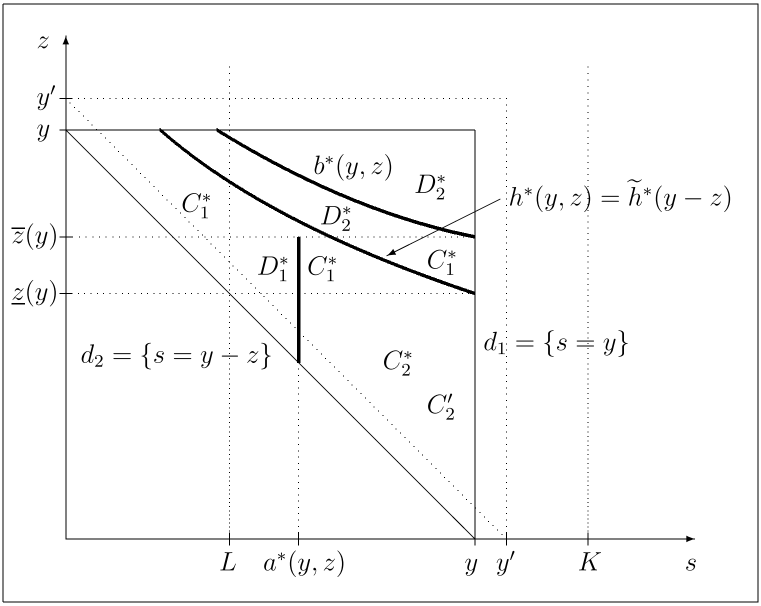

Fig. 4 A computer drawing of the optimal exercise boundaries $a^*(y, z)$, $b^*(y, z)$ and $h^*(y, z)$, for each $0 < L < y < K$ fixed

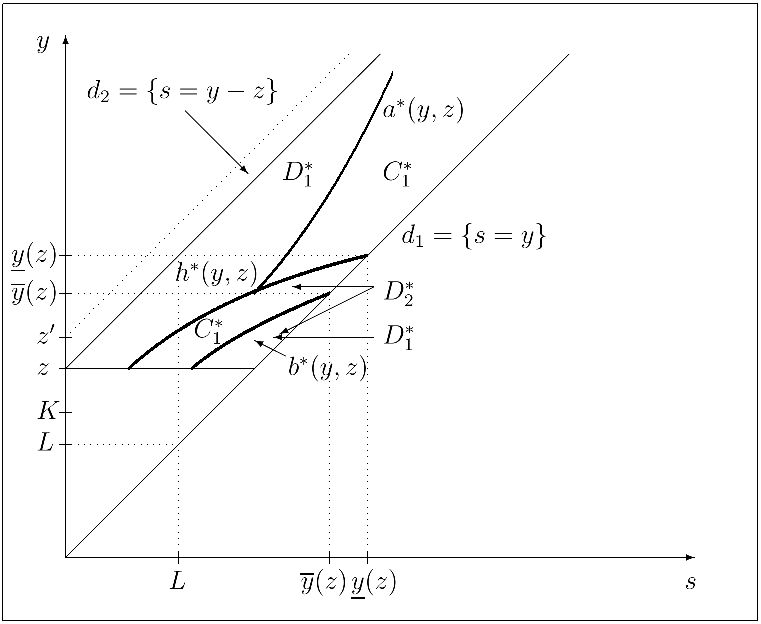

Fig. 5 A computer drawing of the optimal exercise boundaries $a^*(y, z)$, $b^*(y, z)$ and $h^*(y, z)$, for each $0 < z < L < K$ fixed

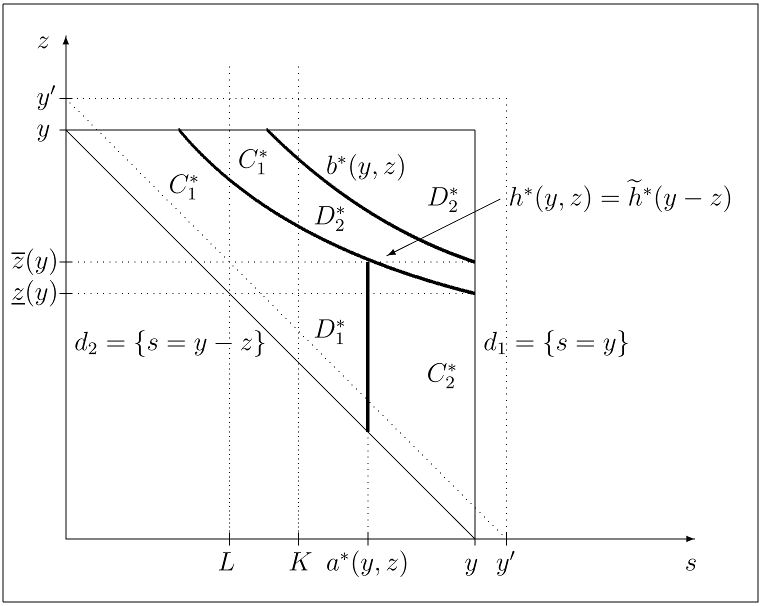

Fig. 6 A computer drawing of the optimal exercise boundaries $a^*(y,z)$, $b^*(y,z)$ and $h^*(y,z)$, for each $0 < L < K < y$ fixed

$$\partial_y a(y,z) = \frac{\gamma_2 (y/a(y,z))^{\gamma_1} - \gamma_1 (y/a(y,z))^{\gamma_2}}{\gamma_1 \gamma_2 (y-K)((y/a(y,z))^{\gamma_1} - (y/a(y,z))^{\gamma_2})} \tag{3.43}$$

for $0 < y - z \leq a(y,z) < h^*(y,z) \wedge y < b(y,z)$ and $y > K$. We also apply the condition of Eq. 3.23 to the functions $C_j(y,z;b(y,z))$, for $j = 1, 2$, in Eq. 3.28 to get that the candidate boundary $b(y,z)$ satisfies the ordinary differential equation (with a parameter)

$$\partial_z b(y,z) = \frac{\gamma_2 ((y-z)/b(y,z))^{\gamma_1} - \gamma_1 ((y-z)/b(y,z))^{\gamma_2}}{\gamma_1 \gamma_2 (L - K + z)(((y-z)/b(y,z))^{\gamma_1} - ((y-z)/b(y,z))^{\gamma_2})} \tag{3.44}$$

for $0 < a(y,z) < y - z < h^*(y,z) \leq b(y,z)$ and $K - L < z < y$. Note that the right-hand sides of the expressions in Eqs. 3.43 and 3.44 are (locally) continuous in $(y, z, a(y,z))$ and $(y, z, b(y,z))$ and (locally) Lipschitz in $a(y,z)$ and $b(y,z)$, for each $y > K$ and $K - L < z < y$ fixed, respectively. Thus, by means of the classical results on the existence and uniqueness of solutions for first-order nonlinear ordinary differential equations, the equations in Eqs. 3.43 and 3.44 admit (locally) unique solutions, which can be constructed by means of Picard's method of successive approximations (see Subsection 3.5 below for further constructions and references).

We also note that, since all the boundaries $b^*(y,z)$ and $h^*(y,z)$ are located in the regions $\{(s,y,z) \in E \,|\, K - L < z \leq y\}$ and $\{(s,y,z) \in E \,|\, 0 < y - z < L\}$, respectively, the inequalities $0 < L - y + z < z - K + L$ (and thus $1/(L - y + z) > 1/(L - K + z)$) are satisfied in their intersection, when $y > K$ holds, while the inequalities $0 < L - K + z < z - y + L$ (and thus $1/(L - y + z) < 1/(L - K + z)$) are satisfied there, when $y < K$ holds, for all

$0 < y - z \le L$. In this case, we may conclude by means of the classical comparison theorem for solutions of first-order (nonlinear) ordinary differential equations that, since the (negative) right-hand side of Eq. 3.44 is larger (not smaller) than the (negative) right-hand side of Eq. 3.16, we have that the inequalities $-\infty \le \partial_z h^*(y,z) \le \partial_z b^*(y,z) \le 0$ hold, when $y > K$, but the inequalities $-\infty \le \partial_z b^*(y,z) \le \partial_z h^*(y,z) \le 0$, when $y > K$, for all $0 < y - z \le L$.

3.5 The Maximal and Minimal Admissible Solutions $a^*(y,z)$ and $b^*(y,z)$

We finally consider the *maximal* and *minimal admissible* solutions to some first-order nonlinear ordinary differential equations as the largest and smallest possible solutions $a^*(y,z)$ and $b^*(y,z)$ to the equations in Eqs. 3.43 and 3.44, which satisfy the inequalities $0 < y - z \le a^*(y,z) < h^*(y,z) \wedge y$ and $0 < y - z < h^*(y,z) \le b^*(y,z)$, for all $y > K$ and $K - L < z < y$, respectively. By virtue of the classical results on the existence and uniqueness of solutions for first-order nonlinear ordinary differential equations, we may conclude that these equations admit (locally) unique solutions, because the facts that their right-hand sides represent (locally) continuous functions in $(y, z, a(y, z))$ and $(y, z, b(y, z))$ and (locally) Lipschitz functions in $a(y, z)$ and $b(y, z)$, for each $y > K$ and $K - L < z < y$ fixed, respectively (cf. also Peskir 1998, Subsection 3.9 for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations). Then, it is shown by means of technical arguments based on Picard's method of successive approximations that there exist unique solutions $a(y, z)$ and $b(y, z)$ to the equations in Eqs. 3.43 and 3.44 started at some points $(a(y_0,z_0), y_0, z_0)$ and $(b(y_0,z_0), y_0, z_0)$, for each $y_0 > K$ and $K - L < z_0 < y_0$ (cf. also Graversen and Peskir 1998, Subsection 3.2 and Peskir 1998, Example 4.4 for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations).

Hence, in order to construct the appropriate functions $a^*(y,z)$ and $b^*(y,z)$ which satisfy the nonlinear first-order ordinary differential equations in Eqs. 3.43 and 3.44 and stays strictly below and above the appropriate upper and lower planes d_1 and d_2, we can follow the arguments of (Peskir 2014, Subsection 3.5) and construct the sequences of the so-called *bad-good solutions* to the equations in Eqs. 3.43 and 3.44 which intersect the upper and lower planes d_1 and d_2, respectively. For this purpose, for any increasing sequence $(y'_l)_{l\in\mathbb{N}}$ such that $y'_l > K$ and $y'_l \uparrow \infty$ as $l \to \infty$, and any increasing sequence $(z'_l)_{l\in\mathbb{N}}$ such that $K - L < z'_l < y$ and $z'_l \uparrow y$ as $l \to \infty$, we can construct the sequences of solutions $a_l(y,z)$ and $b_l(y,z)$, for $l \in \mathbb{N}$, to the equations in Eqs. 3.43 and 3.44, such that $a_l(y'_l, z) = y'_l$ and $b_l(y, z'_l) = y - z'_l$ holds, for each $y > K$ and $K - L < z < y$ fixed, and every $l \in \mathbb{N}$, respectively. It follows from the structure of the equations in Eqs. 3.43 and 3.44 that the inequalities $\partial_y a_l(y'_l, z) < 1$ and $\partial_z b_l(y, z'_l) > -1$ should hold for the derivatives of the corresponding functions, for each $y > K$ and $K - L < z < y$ fixed, and every $l \in \mathbb{N}$, respectively (cf. also Pedersen 2000, pages 979-982 for the analysis of solutions to another first-order nonlinear differential equation). Observe that, by virtue of the uniqueness of solutions mentioned above, we know that each two curves $y \mapsto a_l(y,z)$ and $y \mapsto a_m(y,z)$ as well as $z \mapsto b_l(y,z)$ and $z \mapsto b_m(y,z)$ cannot intersect, for each $y > K$ and $K - L < z < y$, and $l, m \in \mathbb{N}$, such that $l \neq m$, and thus, we see that the sequence $(a_l(y,z))_{l\in\mathbb{N}}$ is decreasing and the sequence $(b_l(y,z))_{l\in\mathbb{N}}$ is increasing, so that the limits $a^*(y,z) = \lim_{l\to\infty} a_l(y,z)$ and $b^*(y,z) = \lim_{l\to\infty} b_l(y,z)$ exist, for each $y > K$ and

$K - L < z < y$, respectively. We may therefore conclude that $a^*(y, z)$ and $b^*(y, z)$ provide the maximal and minimal solutions to the equations in Eqs. 3.43 and 3.44 such that the inequalities $0 < a^*(y, z) < h^*(y, z) \wedge y$ and $b^*(y, z) \geq h^*(y, z) > y - z$ hold, for all $y > K$ and $K - L < z < y$, respectively. Note that the maximality and minimality of the solutions $a^*(y, z)$ and $b^*(y, z)$ to the equations in Eqs. 3.43 and 3.44 follows from the fact that the candidate value functions $V(s, y, z; a^*(y, z))$ and $V(s, y, z; b^*(y, z))$ associated with these boundaries should be *superharmonic* for the Markov process $(t, S_t, Y_t, Z_t)_{t \geq 0}$ (cf. Peskir 1998, Section 3, Formula (3.29)).

Moreover, since the right-hand sides of the first-order nonlinear ordinary differential equations in Eqs. 3.43 and 3.44 are (locally) Lipschitz in (y, z), one can deduce by means of Gronwall's inequality that the functions $a_l(y, z)$ and $b_l(y, z)$, for each $l \in \mathbb{N}$, are continuous, so that the functions $a^*(y, z)$ and $b^*(y, z)$ are continuous on $y > K$ and $K - L < z < y$. The appropriate *maximal admissible* solutions to first-order nonlinear ordinary differential equations and the associated maximality principle for solutions to optimal stopping problems which is equivalent to the superharmonic characterisation of the payoff functions were established in Peskir (1998) and further developed in Graversen and Peskir (1998), Pedersen (2000), Guo and Shepp (2001), Gapeev (2007), Guo and Zervos (2010), Peskir (2012)-Peskir (2014), Glover et al. (2013), Ott (2013), Kyprianou and Ott (2014), Gapeev and Rodosthenous (2014), Gapeev and Rodosthenous (2016a, 2016b), Rodosthenous and Zervos (2017), and Gapeev et al. (2021) among other subsequent papers (cf. also Peskir and Shiryaev 2006, Chapter I; Chapter V, Section 17 for other references).

4 Main Results and Proofs

In this section, being based on the facts proved above, we formulate and prove the main results of the paper concerning the problem of pricing of the perpetual American compound fixed-strike option on the maximum drawdown in Eq. 2.5 with Eqs. 2.6 and 2.27.

Theorem 4.1 *Let the process (S, Y, Z) be given by Eqs. 1.1-1.2 and 2.1 with some constants $r > 0$, $\delta > 0$ and $\sigma > 0$. Then, the value function of the inner optimal stopping problem in Eq. 2.6, for some $L > 0$ fixed, admits the representation*

$$U^*(s, y, z) = \begin{cases} U(s, y, z; h^*(y, z)), & if \quad 0 < y - z \leq s < h^*(y, z) \leq y, \\ U(s, y, z; \underline{y}(z), \underline{z}(y)), & if \quad 0 < y - z \leq s \leq y < h^*(y, z), \\ L - y + z, & if \quad 0 < y - z < h^*(y, z) \leq s \leq y, \end{cases} \tag{4.1}$$

while the optimal stopping time has the form

$$\eta^* = \inf \left\{ t \geq 0 \;\middle|\; S_t \geq h^*(Y_t, Z_t) \right\} \tag{4.2}$$

where the candidate value function and candidate stopping boundary are specified as follows:

(i) the function $U(s, y, z; h^*(y, z))$ is given by Eqs. 3.7-3.8 and the boundary $h^*(y, z)$ (increasing in y but decreasing in z) provides the minimal solution to the first-order nonlinear ordinary differential equation in Eq. 3.16 such that $y - z < h^*(y, z) \leq y$, for $0 < y - z \leq L$;

(ii) the function U(s, y, z; y(z), z(y)) is given by Eq. 3.9, where the coefficients $D_j(y, z; y(z), z(y))$, *for* $j = 1, 2$, *represent a solution to the two-dimensional system of first-order linear partial differential equations in Eqs. 3.10-3.11 satisfying the conditions of Eqs. 3.12-3.13 and 3.14-3.15.*

Theorem 4.2 *Let the process (S, Y, Z) be given by Eqs. 1.1-1.2 and 2.1 with some constants* $r > 0$, $\delta > 0$, *and* $\sigma > 0$. *Then, the value function of the outer optimal stopping problem in Eq. 2.27, for some* $K > L > 0$ *fixed,*

admits the representation

$$V^*(s, y, z) = \begin{cases} V(s, y, z; a^*(y, z)), & if \quad y - z \le a^*(y, z) < s \le y < b^*(y, z), \\ V(s, y, y; b^*(y, z)), & if \quad a^*(y, z) < y - z \le s < b^*(y, z) \le y, \\ V(s, y, z; y(z), z(y)), & if \quad a^*(y, z) \le y - z < s < y \le b^*(y, z), \\ 0, & if \quad y - z \le s \le a^*(y, z) < y \\ & or \quad y - z < b^*(y, z) \le s \le y, \end{cases} \tag{4.3}$$

while the outer optimal stopping time has the form

$$\tau^* = \inf \left\{ t \ge 0 \;\middle|\; S_t \notin \left(a^*(Y_t, Z_t), b^*(Y_t, Z_t) \right) \right\} \tag{4.4}$$

where the candidate value function and candidate stopping boundaries are specified as follows:

(i) the function $V(s, y, z; a^*(y, z))$ *is given by Eqs. 3.25-3.26, where* $a^*(y, z)$ *represents the maximal solution of the first-order nonlinear ordinary differential equation in Eq. 3.43 such that* $y - z \le a^*(y, z) < h^*(y, z) \wedge y$, *for* $y > K$, *where the boundary* $h^*(y, z)$ *is specified in Theorem 4.1 above;*

(ii) the function $V(s, y, z; b^*(y, z))$ *is given by Eqs. 3.27-3.28, where the boundary* $b^*(y, z)$ *represents the minimal solution of the first-order nonlinear ordinary differential equation in Eq. 3.44 such that* $y - z < h^*(y, z) \wedge y \le b^*(y, z) \wedge y$, *for* $K - L < z < y$, *where the boundary* $h^*(y, z)$ *is specified in Theorem 4.1 above;*

(iii) the function V(s, y, z; y(z), z(y)) is given by Eq. 3.31, where the coefficients $C_j(y, z; y(z), z(y))$, *for* $j = 1, 2$, *represent a solution of the two-dimensional system of first-order linear partial differential equations in Eqs. 3.32-3.33 satisfying the appropriate conditions of Eqs. 3.34-3.35 or Eqs. 3.36-3.37 or Eqs. 3.38-3.39 or Eqs. 3.40-3.41.*

Recall that we can put $y = s$ and $z = 0$ to obtain the value of the original perpetual American compound fixed-strike lookback maximum drawdown put option pricing problem of Eq. 1.3 from the value of the double optimal stopping problem of Eq. 2.2, which is decomposed into the sequence of single optimal stopping problems of Eqs. 2.5 and 2.6, where the problem of Eq. 2.5 is equivalent to the one in Eq. 2.27. Since the assertion of Theorem 4.1 (cf. also Gapeev and Rodosthenous 2016b, Theorem 4.1) is proved using the similar arguments as used in the proof of Theorem 4.2, we only give proof of the latter result below.

Proof of Theorem 4.2 In order to verify the assertion stated above, it remains for us to show that the function defined in Eq. 4.3 coincides with the value function in Eq. 2.27 and that the stopping time τ^* in Eq. 4.4 is optimal with the boundaries $a^*(y, z)$ and $b^*(y, z)$

being the solution of the system in Eqs. 3.19-3.24 specified in either Eqs. 3.25-3.26 with Eq. 3.43 or Eqs. 3.27-3.28 with Eq. 3.44. For this purpose, let us denote by $V(s, y, z)$ the right-hand side of the expression in Eq. 4.3 associated with $a^*(y,z)$ and $b^*(y,z)$. Then, it is shown by means of straightforward calculations from the previous section that the function $V(s, y, z)$ solves the system of Eqs. 2.36-2.42. Recall that the function $V(s, y, z)$ is $C^{2,1,1}$ on the closure $\overline{C}_1^*$ of the set C_1^* and is equal to 0 on the set D_1^*, which are defined in Eqs. 2.32 and 2.33, respectively. Hence, taking into account the assumption that the boundaries $a^*(y,z)$ and $b^*(y,z)$ are (at least piecewise) continuously differentiable, for all $0 < z < y$, by applying the change-of-variable formula from Peskir (2007, Theorem 3.1) to the process $(e^{-rt}V(S_t, Y_t, Z_t))_{t \ge 0}$ (cf. also Peskir and Shiryaev 2006, Chapter II, Section 3.5 for a summary of the related results and further references), we obtain the expression

$$
\begin{aligned}
&e^{-rt}\, V(S_t, Y_t, Z_t) = V(s,y,z) + M_t \\
&+ \int_0^t e^{-ru}\, (\mathbb{L}V - rV)(S_u, Y_u, Z_u)\, I\big((Y_u - Z_u) \vee a^*(Y_u, Z_u) < S_u < b^*(Y_u, Z_u) \wedge Y_u\big)\, du \\
&+ \int_0^t e^{-ru}\, \partial_y V(S_u, Y_u, Z_u)\, I(S_u = Y_u)\, dY_u + \int_0^t e^{-ru}\, \partial_z V(S_u, Y_u, Z_u)\, I(S_u = Y_u - Z_u)\, dZ_u
\end{aligned}
\tag{4.5}
$$

for all $t \ge 0$. Here, the process $M = (M_t)_{t \ge 0}$ defined by

$$
M_t = \int_0^t e^{-ru}\, \partial_s V(S_u, Y_u, Z_u)\, I\big(Y_u - Z_u < S_u < Y_u\big)\, \sigma\, S_u\, dB_u \tag{4.6}
$$

is a continuous local martingale with respect to the probability measure $\mathbb{Q}_{s,y,z}$. Note that, since the time spent by the process (S, Y, Z) at the parts $\{(s,y,z) \in E \mid s = a^*(y,z)\}$ and $\{(s,y,z) \in E \mid s = b^*(y,z)\}$ of the boundary surface ∂C_1 as well as at the diagonals $d_1 = \{(s,y,z) \in E \mid 0 < y - z < s = y\}$ and $d_2 = \{(s,y,z) \in E \mid 0 < s = y - z < y\}$ is of the Lebesgue measure zero (cf., e.g. Borodin and Salminen 2002, Chapter II, Section 1), the indicators in the second line of the formula in Eq. 4.5 as well as in the expression of Eq. 4.6 can be ignored. Moreover, since the component Y increases only when the process (S, Y, Z) is located on the upper diagonal d_1, while the component Z increases only when the process (S, Y, Z) is located on the lower diagonal d_2, the indicators appearing in the third line of Eq. 4.5 can also be set equal to one.

It follows from straightforward calculations and the arguments of the previous section that the function $V(s, y, z)$ satisfies the left-hand second-order ordinary differential equation in Eq. 2.36, which together with the left-hand conditions of Eqs. 2.37-2.38 and 2.40 as well as the fact that the left-hand inequality in Eq. 2.42 holds imply that the inequality $(\mathbb{L}V - rV)(s,y,z) \le -H(s,y,z)$ is satisfied, for all $(s,y,z) \in E$ such that $0 < y - z < s < y$ with $s \neq a^*(y,z)$ and $s \neq b^*(y,z)$. Moreover, we observe directly from the expressions in either Eqs. 3.25-3.26 or Eqs. 3.27-3.28 with Eqs. 3.29-3.30 that the value function $V(s, y, z)$ is convex, because its first-order partial derivative $\partial_s V(s,y,z)$ is increasing, while its second-order partial derivative $\partial_{ss} V(s,y,z)$ is positive, on the interval $(y-z) \vee a^*(y,z) < s < b^*(y,z) \wedge y$. Thus, we may conclude that the inequality in Eq. 2.41 holds, which together with the conditions of Eqs. 2.37-2.38 and 2.40 imply that the inequality $V(s,y,z) \ge 0$ is satisfied, for all $(s,y,z) \in E$. Let $(\varkappa_n)_{n \in \mathbb{N}}$ be the localising sequence of stopping times for the process M from Eq. 4.6 such that $\varkappa_n = \inf\{t \ge 0 \,|\, |M_t| \ge n\}$, for each $n \in \mathbb{N}$. It therefore follows from the expression in Eq. 4.5 that the inequalities

$$\begin{aligned}
&\int_0^{\tau\wedge\varkappa_n} e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^{\tau\wedge\varkappa_n} e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u \\
&\le e^{-r(\tau\wedge\varkappa_n)}\, V(S_{\tau\wedge\varkappa_n}, Y_{\tau\wedge\varkappa_n}, Z_{\tau\wedge\varkappa_n}) \\
&\qquad + \int_0^{\tau\wedge\varkappa_n} e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^{\tau\wedge\varkappa_n} e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u \\
&\le V(s, y, z) + M_{\tau\wedge\varkappa_n}
\end{aligned} \tag{4.7}$$

hold with any stopping time τ of the process S, for each $n \in \mathbb{N}$ fixed. Then, taking the expectation with respect to $\mathbb{Q}_{s,y,z}$ in Eq. 4.7, by means of Doob's optional sampling theorem, we get

$$\begin{aligned}
&\mathbb{E}_{s,y,z}\bigg[\int_0^{\tau\wedge\varkappa_n} e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^{\tau\wedge\varkappa_n} e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u\bigg] \\
&\le \mathbb{E}_{s,y,z}\bigg[e^{-r(\tau\wedge\varkappa_n)}\, V(S_{\tau\wedge\varkappa_n}, Y_{\tau\wedge\varkappa_n}, Z_{\tau\wedge\varkappa_n}) \\
&\qquad + \int_0^{\tau\wedge\varkappa_n} e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^{\tau\wedge\varkappa_n} e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u\bigg] \\
&\le V(s, y, z) + \mathbb{E}_{s,y,z}\big[M_{\tau\wedge\varkappa_n}\big] = V(s, y, z)
\end{aligned} \tag{4.8}$$

for all $(s, y, z) \in E$ and each $n \in \mathbb{N}$. Hence, letting n go to infinity and using Fatou's lemma, we obtain from the expressions in Eq. 4.8 that the inequalities

$$\begin{aligned}
&\mathbb{E}_{s,y,z}\bigg[\int_0^{\tau} e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^{\tau} e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u\bigg] \\
&\le \mathbb{E}_{s,y,z}\bigg[e^{-r\tau}\, V(S_\tau, Y_\tau, Z_\tau) \\
&\qquad + \int_0^{\tau} e^{-ru}\, H(S_u, Y_u, Z_u)\, du + \int_0^{\tau} e^{-ru}\, I\big(Y_u < h^*(Y_u, Z_u)\big)\, dY_u\bigg] \\
&\le V(s, y, z)
\end{aligned} \tag{4.9}$$

are satisfied with any stopping time τ, for all $(s, y, z) \in E$ and each $n \in \mathbb{N}$.

We now prove the fact that the couple of boundaries $a^*(y, z)$ and $b^*(y, z)$ specified above is optimal. By virtue of the fact that the function $V(s, y, z)$ from the right-hand side of the expression in Eq. 4.3 associated with the boundaries $a^*(y, z)$ and $b^*(y, z)$ satisfies the equation of Eq. 2.36 and the conditions of Eq. 2.37, and taking into account the structure of τ^* in Eq. 4.4, it follows from the expression in Eq. 4.5 that the equalities

$$
\begin{aligned}
&\mathbb{E}_{s,y,z}\Big[\int_0^{\tau^*\wedge\varkappa_n} e^{-ru}\,H(S_u,Y_u,Z_u)\,du+\int_0^{\tau^*\wedge\varkappa_n} e^{-ru}\,I\big(Y_u<h^*(Y_u,Z_u)\big)\,dY_u\Big]\\
&=\mathbb{E}_{s,y,z}\Big[e^{-r(\tau^*\wedge\varkappa_n)}\,V(S_{\tau^*\wedge\varkappa_n},Y_{\tau^*\wedge\varkappa_n},Z_{\tau^*\wedge\varkappa_n})\\
&\qquad+\int_0^{\tau^*\wedge\varkappa_n} e^{-ru}\,H(S_u,Y_u,Z_u)\,du+\int_0^{\tau^*\wedge\varkappa_n} e^{-ru}\,I\big(Y_u<h^*(Y_u,Z_u)\big)\,dY_u\Big]\\
&=V(s,y,z)+\mathbb{E}_{s,y,z}\big[M_{\tau^*\wedge\varkappa_n}\big]=V(s,y,z)
\end{aligned}
\tag{4.10}
$$

hold, for all $(s,y,z)\in E$ and each $n\in\mathbb{N}$. Observe that, taking into account the arguments from (Shepp and Shiryaev 1993, pages 635-636) as well as the assumption $K>L>0$, it follows from the structure of the function $G(s,y,z)$ in Eq. 2.5 and the stopping time τ^* in Eq. 4.4 that the property

$$
\begin{aligned}
&\mathbb{E}_{s,y,z}\Big[\sup_{t\ge 0} e^{-r(\tau^*\wedge t)}\,G(S_{\tau^*\wedge t},Y_{\tau^*\wedge t},Z_{\tau^*\wedge t})\Big]\\
&\le \mathbb{E}_{s,y,z}\Big[\sup_{t\ge 0} e^{-r(\tau^*\wedge t)}\,(Y_{\tau^*\wedge t}-K+L)\Big]\le \mathbb{E}_{s,y,z}\Big[\sup_{t\ge 0} e^{-r(\tau^*\wedge t)}\,Y_{\tau^*\wedge t}\Big]<\infty
\end{aligned}
\tag{4.11}
$$

holds, for all $(s,y,z)\in E$. We also note that the variable $e^{-r\tau^*}V(S_{\tau^*},Y_{\tau^*},Z_{\tau^*})$ is finite on the event $\{\tau^*=\infty\}$ as well as recall from the arguments of Beibel and Lerche (1997) and Pedersen (2000, Theorem 2.5) that the property $\mathbb{Q}_{s,y,z}(\tau^*<\infty)=1$ holds, for all $(s,y,z)\in E$. Hence, letting n go to infinity and using the conditions of Eq. 2.37, we can apply the Lebesgue dominated convergence theorem to the expression of Eq. 4.10 to obtain the equality

$$
\mathbb{E}_{s,y,z}\Big[\int_0^{\tau^*} e^{-ru}\,H(S_u,Y_u,Z_u)\,du+\int_0^{\tau^*} e^{-ru}\,I\big(Y_u<h^*(Y_u,Z_u)\big)\,dY_u\Big]=V(s,y,z) \tag{4.12}
$$

for all $(s,y,z)\in E$, which together with the inequalities in Eq. 4.9 directly implies the desired assertion. We finally recall from the results of Subsection 2.5 above implied by standard comparison arguments applied to the value functions of the appropriate optimal stopping problems that the inequalities $a^*(y,z)\le h^*(y,z)\wedge y$ and $b^*(y,z)\ge h^*(y,z)>y-z$, for $y>K$ and $K-L<z<y$, respectively, should hold for the optimal stopping boundaries with the boundary $h^*(y,z)$ which is specified in Theorem 4.1 above, so that the verification is complete. □

Corollary 4.3 *The optimal method of sequentially exercising the perpetual American compound fixed-strike lookback option with the outer value $P^*(s,y,z)$ in Eq. 2.5, which is equivalent to the one $V^*(s,y,z)=P^*(s,y,z)-G(s,y,z)$ of Eq. 2.27, and the inner value $U^*(s,y,z)$ of Eq. 2.6, acts as follows. After the outer option with the equivalent value function from Eq. 2.27 is exercised at the first exit time τ^* from Eq. 4.4 with either the boundaries $a^*(y,z)[<h^*(y,z)]$ and $b^*(y,z)[\ge h^*(y,z)]$ specified in Theorem 4.2 above, the inner option should be exercised at the first hitting time*

$$
\zeta^*=\inf\big\{t\ge\tau^*\;\big|\;S_t\ge h^*(Y_t,Z_t)\big\} \tag{4.13}
$$

with the boundary $h^*(y, z)$ *which is specified in Theorem 4.1 above. In other words, an investor should enter the market when the price of the underlying asset S falls down to the stochastic boundary* $a^*(Y, Z)[< h^*(Y, Z)]$ *and then exit the market when the price rises up to the stochastic boundary* $h^*(Y, Z)$*. Furthermore, the investor should enter and exit the market, simultaneously, when the price S rises up to the stochastic boundary* $b^*(Y, Z)[\geq h^*(Y, Z)]$*. The rational (no-arbitrage) costs of the related exercise strategy is then given by the value* $P^*(s, y, z) = G(s, y, z) + V^*(s, y, z)$ *in Eq. 2.5 with* $V^*(s, y, z)$ *in Eq. 2.27.*

Remark 4.4 Note that in the cases in which one starts from the stretch, that is, when $y = s$ and $z = 0$ holds, the subsequent exercise of the *outer* and *inner* perpetual American fixed-strike lookback options with the value functions in Eqs. 2.5 and 2.6 may actually follow the subsequent exercise of the *standard* perpetual American lookback put and call options with the value functions in Gapeev (2022) and Gapeev et al. (2022). More precisely, after the process S starts at some $s = y$ and $z = 0$, the *outer* option *should be exercised* at the time at which the process S reaches either a lower or an upper boundary $a^*(Y, Z)[< h^*(Y, Z)]$ or $b^*(Y, Z)[\geq h^*(Y, Z)]$, respectively. On the one hand, in the case in which the process S reaches the lower boundary $a^*(Y, Z)$ first, the *inner* option *should then be exercised* at the time at which the underlying asset price process reaches the upper boundary $h^*(Y, Z)$. On the other hand, in the case in which the process S reaches the upper boundary $b^*(Y, Z)$ first, the *inner* option *should then be exercised* instantly, because the equality $h^*(Y, Z) \leq b^*(Y, Z)$ holds. Roughly speaking, if we consider such a strategy from the point of view of *buy low and sell high*, the boundary $a^*(Y, Z)$ [with $h^*(Y, Z)$] is then the *stop loss* one, while the boundary $b^*(Y, Z)$ [with $h^*(Y, Z)$] is then the *take profit* one.

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Declarations

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