Aequat. Math.

https://doi.org/10.1007/s00010-025-01209-x

Aequationes Mathematicae

Darboux and shift-compactness paradigms: Automatic continuity in the Gołąb-Schinzel and the Goldie equations

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On the 100th birth anniversary of János Aczél and 80th birth anniversary of Nick Bingham[1].

Abstract. Using shift-compactness, the continuity theorems of Baire and Luzin and a variant of Darboux's boundedness theorem, we deduce directly the continuity of positive solutions of the Gołąb-Schinzel equation and of the kernels of the closely related Goldie and generalized Goldie equation, from appropriate assumptions of the Baire property or (Lebesgue) measurability.

Mathematics Subject Classification. Primary 22A10, 43A05; Secondary 28C10.

Keywords. Darboux's boundedness theorem, Shift-compactness, Gołąb-Schinzel equation, Goldie equation, Group norm, Group logarithm.

1. Introduction. Measure case

We give short new proofs of the continuity of (Lebesgue) *measurable* and of *Baire* (i.e. with the Baire property) positive solutions of the Gołąb-Schinzel functional equation [31]

$$h(x + h(x)y) = h(x)h(y), \tag{GS}$$

for $x, y \in \mathbb{G}_h := \{t : h(t) > 0\}$, the latter assumed non-null/non-meagre. Aczél and Gołąb [1] studied this equation in order to identify one-parameter subsemigroups of the affine group without recourse to analytical-differential methods. Its close connection with Beurling's important extension of the celebrated Weiner Tauberian Theorem [5] and also with Beurling regular variation,

[1]Paper presented at the Bingham Colloquium at LSE celebrating on 19th March 2025 the anniversary of that day.

Published online: 15 September 2025

Birkhäuser

for which see [11,13,38], warrants as direct a proof as possible of continuity (and a read-out of its solution – for which see Section 6, although glimpses of the form of h abound in the proofs).

Positivity of h is natural in the regular variation context and below we assume (GS) is restricted to $\mathbb{G}_h$. Relaxing this restriction, one may consider non-negative h with a wider domain $\mathbb{R}$ similarly to [23] (where the domain is a vector space). In the general Goldie equation of Section 5 a wider domain seems necessary to support our analysis.

The continuity proof below may be viewed as the natural generalization of the analogous simple classical proof of continuity for measurable/Baire solutions of the Cauchy equation via Darboux's theorem (that a locally bounded additive function is continuous; cf. Section 2).

The theorem and proof for (GS) extend to measurable real-valued functions defined on Euclidean spaces (see the Concluding Remarks), but its inspiration is clearest (simplest) on the real line. The Baire category analogue could have been viewed as an instance of the Banach Continuous Homomorphism theorem [19, Ch. 11] (cf. [37]) in view of the group operation below, were it not for the fact that to apply Banach's theorem would require verification that the said group structure is topological under the Euclidean metric. But that is not a given here. We note the related Banach-Mehdi theorem, that for complete normed vector spaces an additive Baire map is continuous, has a similar but simpler proof, which inspired the present approach (see e.g. [19, Th. 12.1.5]). The Baire variant asserted above is contained in a result of Brzdęk [24], being implied by his hypothesis that $\{x : 0 < |h(x)| < a\}$ contains a non-meagre Baire subset for some $a \in (0, \infty)$. We use a somewhat similar hypothesis to deduce local boundedness of h.

As we shall confirm, non-zero continuous solutions of (GS) on $\mathbb{R}$ take one of the two forms

$$h(x) = 1 + \rho x, \qquad \max\{1 + \rho x, 0\},$$

for $\rho \in \mathbb{R}$: see [25] for a survey. Thus the second of these yields solutions that are positive for $\rho > 0$ on $(-1/\rho, \infty)$, and for $\rho < 0$ on $(-\infty, -1/\rho)$, the former suiting asymptotic analysis according to the received convention in regular variation. In consequence we admit only domains of solutions that are *unbounded on the right*. For other solutions, including 'trivial' ones with range $\{-1, 0, 1\}$, see [31] and [41].

Two key ideas allow transparent passage from the classical Cauchy result to the Gołąb-Schinzel case.

The first key in this approach is an adaptation of Darboux's Theorem through the use of the Popa binary operation [41]

$$s \circ_h t := s + h(s)t,$$

which we abridge to $s \circ t$ whenever context allows, by analogy to the circle operation of ring theory. When h satisfies (GS) this operation endows with a group structure the set $\mathbb{G}_h := \{x \in \mathbb{R} : h(x) > 0\}$, assumed non-null/non-meagre, on which h generates a group norm (below) in the measurable h context, and likewise in the Baire case, turning h into a homomorphism[1]. The latter feature would allow the Banach-Neeb Theorem to operate in the case when h is Baire rather than measurable (as $\mathbb{G}_h$ is a Baire space), were the group known to be topological, which initially is not the case here.

The second key is the shift-compactness property (which, as we will see, flows in the present context, from the Baire and Luzin Continuity Theorems for Baire/measurable functions, a 'third' key in effect):

Definition. In a metric group X an arbitrary ('target') subset $T \subseteq X$ is *shift-compact* if for any *null sequence* $z_m \to 1_X$ there is $t \in T$ such that i.o. (*infinitely often*) $tz_m \in T$, that is

$$\mathbb{M}_t := \{m \in \mathbb{N} : tz_m \in T\} \text{ is infinite.}$$

This concept broadens Parthasarathy's for whom the context was the *convolution semi-group* of (Borel) probabilities on a separable metric group X. See [17] for an appraisal of the connection.

For $X = (\mathbb{R}, +)$ the property was initially studied (specialized to co-finite sets $\mathbb{M}_t$) by Kestelman [34], motivated by work of Banach, and again later by Borwein and Ditor [22]. The target sets there were Baire non-meagre sets or measurable non-null sets and these are shift-compact [8], [19, §4.2]. Its close connection to Karamata's Uniform Convergence Theorem [5, §1.2] inspired much investigation of this area [6,8]. For the extensive usage of this property see [19]. We note that the concept of shift-compactness was independently later discovered by Banakh and Jabłońska in [4] who refer to the negated property as null-finiteness.

In Lemma 1, for any null sequence w_n and any non-null measurable set/non-meagre Baire set T, for some (indeed, for almost all) $t \in T$, a *subsequence* of $\{t \circ w_n\}_n$ embeds in T. This is obtained by a subtle modification of the usual proof of the Kestelman-Borwein-Ditor shift-compactness theorem [9].

We recall that a non-negative *group norm* satisfies three properties:

(i) positivity: $||x|| > 0$, unless x is the identity,
(ii) symmetry: $||x^{-1}|| = ||x||$,
(iii) subadditivity: $||xy|| \leq ||x|| + ||y||$.

According to the Birkhoff-Kakutani Theorem [19, Ch. 6], a group that is first-countable (i.e. having a countable local neighbourhood base at the identity) and with continuous right translation (a 'right-topological' group) is normable iff inversion and multiplication are continuous at the identity. A normed group thus need not be a topological group, but it is a metric group,

[1] Under $\circ_h$ the group $\mathbb{G}_h$ is a subgroup of $G^* := \{x : h(x) \neq 0\}$, which decomposes into two sets as in the (limiting) case of the punctured line $\mathbb{R}^*$ under multiplication, with $\mathbb{R}_+ := (0, \infty)$ its subgroup.

metrized for instance by the left-invariant metric $d_L(x,y) := ||x^{-1}y||$. For background see also [3].

The group identity of $(\mathbb{G}_h, \circ)$ is 0 and the inverse of any element x is $-x/h(x)$ (cf. [41]), denoted below by x_h^{-1}. Put

$$L(a) = L_h(a) := \int_0^a \frac{dx}{h(x)} \text{ and } ||a|| := |L(a)|.$$

(Absolute value is needed for elements of $\mathbb{G}_h$ to the left of 0.) We show in Lemma 3 below that $L(a)$ is finite and in Proposition 1 that $L(.)$ is a *group logarithm* so that $||.||$ is a group norm for $\mathbb{G}_h$. (This is consistent with known group logarithms and norms in the *Popa groups* $\mathbb{G}_\rho$ arising from $h(x) = 1+\rho x$ [15, Th. 1].) Thus we use Lebesgue measure on $\mathbb{R}$ to make the group $\mathbb{G}_h$ a topological measure space (cf. [30]), which one might briefly call a 'topological measure group'.

Our first result does not require h to satisfy (GS), i.e. (right-) 'shifting' via $\circ_h$ need not itself be a group operation – sufficient here is the embracing group structure of $(\mathbb{R}, +)$.

Lemma 1. *For h measurable, $w_n \to 0$ a null sequence, and $T \subseteq \mathbb{R}$ non-null measurable, there is $t \in T$ for which $t \circ_h w_n \in T$ infinitely often, i.e. the following set is infinite:*

$$\{n \in \mathbb{N} : t + h(t)w_n \in T\}.$$

Proof. By Luzin's Continuity Theorem ([40, Th. 8.2], [33, Th. 17.12], [21, Th. 2.210 and 7.14(ix)], [32, p. 243]), reducing T by a subset of smaller measure, if needed, we assume w.l.o.g. (without loss of generality) h to be continuous on T and bounded thereon, by B say. Choose inductively, as follows, non-null descending compact subsets $T_n \subseteq T$ (hence with non-empty intersection), and increasing integers m_n with

$$t + h(t)w_{m_n} \in T_{n-1} \text{ for } t \in T_n.$$

Given the compact set T_n, let s_0 be a Lebesgue-density point of T_n. Pick an interval I_n around s_0 with $|I_n \cap T_n|/|I_n| > 3/4$. Also fix $m = m_{n+1} > m_n$ such that $B|w_n| \leq |I_n|/8$ for all $n \geq m$, so that with $t \in T_n$

$$|h(t)w_n| \leq |I_n|/8 \text{ for } n \geq m.$$

Let $J_n = I_n \backslash T_n$ and set

$$T_n' := \{t + h(t)w_m : t \in I_n \cap T_n\}.$$

Write $h_m(t) := h(t)w_m$. If $s = t + h_m(t) \in J_n$ for some $t \in I_n \cap T_n$, then $t = s - h_m(t) \in J_n - h_m(t)$ and $|h_m(t)| \leq |I_n|/8$. So

$$|J_n - \bigcup\{h_m(t) : t \in I_n \cap T_n \text{ with } t + h_m(t) \in J_n\}| \leq |I_n|/2.$$

Put

$$T_{n+1} := \{t \in I \cap T_n : t + h(t)w_m \in I_n \cap T_n\} \subseteq T_n.$$

Then $|T_{n+1}| \geq |I_n|/2$ and is closed, by continuity of h on T. Furthermore, $t + h(t)w_{m_{n+1}} \in T_n$ for $t \in T_{n+1}$. Consider

$$t \in \bigcap T_n \subseteq T.$$

For $n > 2$, since $t \in T_n$, we have $t \circ w_{m_n} = t + h(t)w_{m_n} \in T_{n-1} \subseteq T$. □

The above proof reduces to the standard proof of shift-compactness in the additive group $(\mathbb{R}, +)$ when $h(t) \equiv 1$. We can now see that $\mathbb{G}_h$ is an interval extending to $+\infty$ but its exact form comes at the end (in Section 6, Theorem 5(ii)).

Corollary 1. *For measurable h satisfying (GS), if $\mathbb{G}_h$ is non-null, then 0 is an interior point of $\mathbb{G}_h$, so that $\mathbb{G}_h$ is open in $\mathbb{R}$ (and so locally compact). Furthermore, if $\mathbb{G}_h$ is unbounded to the right, then $\mathbb{R}_+ \subseteq \mathbb{G}_h$.*

Proof. Suppose 0 is not an interior point of $\mathbb{R}$. Then, for $n = 1, 2, ...$, we may choose $z_n \in (-1/n, 1/n)\backslash\mathbb{G}_h$. As $\mathbb{G}_h$ is measurable non-null, for some $t \in \mathbb{G}_h$ one has $t \circ_h z_n \in \mathbb{G}_h$ i.o. by Lemma 1. For such n, referring to the group inverse t_h^{-1} for which $h(t_h^{-1}) = 1/h(t)$,

$$z_n = -t/h(t) + (t + h(t)z_n)/h(t) = t_h^{-1} \circ_h (t \circ_h z_n) \in \mathbb{G}_h,$$

a contradiction. So $[-\varepsilon, \varepsilon] \subseteq \mathbb{G}_h$ for some $\varepsilon > 0$. Then $[t - h(t)\varepsilon, t + h(t)\varepsilon] \subseteq \mathbb{G}_h$, for each $t \in \mathbb{G}_h$, as $h(t \pm \delta h(t)) = h(t \circ (\pm\delta)) = h(t)h(\pm\delta) > 0$ for $0 < \delta \leq \varepsilon$.

So 0 is an interior point and $\mathbb{G}_h$ is open.

To proceed further we exploit some ideas from [23, Th. 3] adapting to the present context, as function arguments here are restricted to $\mathbb{G}_h$. Two preliminary observations are needed (to make this account 'free standing').

Take $A := \{a : h(a) = 1\}$, which is additive, and $M = \{h(x) : x \in \mathbb{G}_h\}$, which is a multiplicative subgroup of $\mathbb{R}_+$.

Observation 1. For $x, y \in \mathbb{G}_h$, if $h(x) = h(y)$, then $x - y \in A$; indeed, $1 = h(x)h(y)^{-1} = h(x)h(-y/h(y)) = h(x - y)$.

Observation 2. It follows now that, for any $a, x \in \mathbb{G}_h$, if $h(a) = 1$, then by (*GS*)

$$h(x + h(x)a) = h(x),$$

so that $h(x)a \in A$, i.e. $A\mu = A$ for $\mu \in M$ and so also $MA = A$.

We now return to the proof of the final claim, which splits into three cases according as (i) M is finite, (ii) M is infinite and $A \neq \{0\}$, and (iii) $A = \{0\}$.

Case (i): M is finite. Then $M = \{1\}$, i.e. $h \equiv 1$, and so here $x + y = x + h(x)y \in \mathbb{G}_h$ for $x, y \in \mathbb{G}_h$. So since $[0, \varepsilon) \subseteq \mathbb{G}_h = A$, by additivity $\mathbb{R}_+ \subseteq \mathbb{G}_h$. (In fact $\mathbb{G}_h = \mathbb{R}$.)

Case (ii): M is infinite and $A \neq \{0\}$. Here M accumulates at 0. Indeed, if $\mu \in M\backslash\{1\}$, then $\mu^n \to 0$, assuming w.l.o.g. $\mu < 1$ (otherwise replace μ by $1/\mu$). Thus $A = MA$ is dense in $\mathbb{R}_+$. Here $a + x = a + h(a)x \in \mathbb{G}_h$, for $a \in A$ and $x \in \mathbb{G}_h$. But $[0, \varepsilon) \subseteq \mathbb{G}_h$, so $\mathbb{G}_h \supseteq A + [0, \varepsilon) \supseteq \mathbb{R}_+$, by the density of

A in $\mathbb{R}_+$. (Here again $\mathbb{G}_h = \mathbb{R}$.) The last step may be compared to classical additivity results: cf. [5, Cor. 1.1.4 and 1.1.5].

Case (iii): $A = \{0\}$. Here, for $x, y \in \mathbb{G}_h$, if $h(x) = h(y)$, then $x - y = 0$ by Observations 2, i.e. h is injective on $\mathbb{G}_h$. For any $x \in \mathbb{G}_h\backslash\{0\}$, as $h(x) \neq h(0) = 1$, put $z(x) := x/(1 - h(x))$. For any $x, y \in \mathbb{G}_h$,

$$h(x + h(x)y) = h(y + xh(y)),$$

both equal to $h(x)h(y)$ by (*GS*). Since $x \circ_h y$ and $y \circ_h x$ are in $\mathbb{G}_h$, by injectivity,

$$x + h(x)y = y + xh(y).$$

Equivalently $x - xh(y) = y - yh(x)$, and this is equivalent to

$$z(x) = x/(1 - h(x)) = y/(1 - h(y)) = z(y).$$

Thus $z(x)$ is constant on $\mathbb{G}_h$. Writing the constant value as $-1/\rho$ we obtain

$$x/(1 - h(x)) = -1/\rho, \text{ i.e. } h(x) = 1 + \rho x.$$

As declared at the outset, solutions bounded from above are disallowed. So if $\mathbb{G}_h$ is to be unbounded on the right, then $\rho > 0$ and $\mathbb{G}_h = (-1/\rho, \infty)$. For an alternative proof, see the Remark 4 below. □

Remarks 1. For continuous h the functional equation $h(x + h(x)y) = h(y + xh(y))$ is studied on $\mathbb{R}$ in [25, Lemma 1] where $\mathbb{G}_h$ is found to decompose into the union of a finite and an infinite interval.

2. In Case (iii) above, if $z(x) = x + h(x)z(x) \in \mathbb{G}_h$, then for $z = z(x)$

$$h(z) = h(x + h(x)z) = h(x)h(z),$$

so after cancelling by $h(z)$ – a contradiction. So $z \notin \mathbb{G}_h$, and indeed $z = z(x) = -1/\rho \notin \mathbb{G}_h$ by above.

3. Case (iii) uses the reverse of the injectivity argument of Theorem 5(i).

4. An alternative proof for Case (iii). Here h may also be Baire, to which we refer later. As we declared at the outset, we do not allow solutions bounded from above. If $\mathbb{G}_h$ is assumed bounded from below, consider $x \notin \mathbb{G}_h$ for some least $x > 0$. Choose x_n in $\mathbb{G}_h$ with $x_n \to x$. Then $h(x_n) \geq L$ for some $L > 0$, as otherwise $h(x_n) \to 0$ and so the group inverse $(x_n)_h^{-1} = -x_n/h(x_n) \in \mathbb{G}_h$ shows $\mathbb{G}_h$ to be unbounded from below. The positive sequence $z_n := (x - x_n)/h(x_n) \leq (x - x_n)/L$ is null, so for large enough n is in $\mathbb{G}_h$. Then $x = x_n + h(x_n)z_n \in \mathbb{G}_h$, a contradiction. Thus $[0, \infty) \subseteq \mathbb{G}_h$.

As a preliminary to Lemma 3 below we need:

Lemma 2. *If h satisfies (GS) and is continuous on $\mathbb{G}_h$, then $\mathbb{G}_h$ is connected. So, if $\mathbb{G}_h$ is unbounded to the right, then $\mathbb{R}_+ := (0, \infty) \subseteq \mathbb{G}_h$.*

Proof. By the assumed continuity of h, the group inversion map: $x \mapsto x_h^{-1} = -x/h(x)$ is continuous on $\mathbb{G}_h$. Write $\mathbb{G}_h^{\pm} = \mathbb{G}_h \cap \mathbb{R}_{\pm}$. By Corollary 1 $\mathbb{G}_h$ is open, so suppose that $\mathbb{G}_h^-$ has an unbounded connected component different from $\mathbb{R}_-$. Let I be a second, bounded component of $\mathbb{G}_h$ in $\mathbb{R}_-$, then the image J of I under inversion is a connected component in $\mathbb{R}_+$. This cannot be all of $\mathbb{G}_h^+$ otherwise $\mathbb{G}_h^-$ would be connected (by inversion). Let $x > 0$ be the infimum of J . Then there exist $x_n \in J \subseteq \mathbb{G}_h^+$ with $h(x_n) \to 0$ and limit x. So $(x_n)_h^{-1}$ is unbounded, yet lies in the bounded set I, a contradiction. Hence $\mathbb{G}_h^- = \mathbb{R}_-$ and is connected; hence also $\mathbb{G}_h^+$ is connected. So $\mathbb{G}_h$ is connected. $\square$

The next result needs to be interpreted with the hindsight of Theorem 5(ii), where we identify the form of h and so of $\mathbb{G}_h$. Specialized to $\kappa = h$ it is of immediate use in Proposition 1, but the Goldie equations of subsequent sections prompt an interest in the 'pexiderized' version of (GS) displayed below.

Lemma 3. *For h measurable satisfying (GS), any positive measurable function κ satisfying*

$$\kappa(x \circ_h y) = \kappa(x)\kappa(y) \qquad (x, y \in \mathbb{G}_h)$$

is locally bounded away from 0 and locally bounded above, both relative to non-null compact subsets of $\mathbb{G}_h$. Furthermore, κ is continuous on $\mathbb{G}_h$. In particular, these properties hold of h for h a positive measurable solution of (GS) and so h is continuous on $\mathbb{G}_h$, which is connected.

Proof. Consider any non-null compact $C \subseteq \mathbb{G}_h$. Suppose that $c_n \in C$ has $\kappa(c_n) \to 0$. W.l.og. c_n converges to $c \in C$. Put $z_n := c_n - c \to 0$, so that $w_n = z_n/h(c)$ is also null. But

$$c \circ_h w_n = c + z_n = c_n.$$

So $w_n \in \mathbb{G}_h$ and $\kappa(w_n) \to 0$, since $\kappa(c) > 0$ and

$$\kappa(c)\kappa(w_n) = \kappa(c \circ_h w_n) = \kappa(c_n) \to 0.$$

Passing to a non-null subset $C' \subseteq C$, we may assume, again by Luzin's Continuity Theorem, that κ is continuous on C' . By Lemma 1, for some $t \in C'$, $t \circ_h w_n \in C'$ i.o. and so passing to a subsequence w.l.o.g. for all n. Now $t + h(t)w_n \to t$, so by continuity on C' at t,

$$\kappa(t) = \lim_n \kappa(t \circ w_n) = \kappa(t) \lim_n \kappa(w_n) = 0,$$

contradicting that $\kappa(t) > 0$. So κ is bounded away from 0 on C.

If instead $\kappa(s_n) \to \infty$, the same argument yields the contradiction that $\kappa(t) = \infty$, showing that κ is bounded above on C.

Now consider any null sequence z_n. By Corollary 1, w.l.o.g. each z_n is in $\mathbb{G}_h$. By what has just been proved, w.l.o.g. we may suppose by passing to a subsequence that $\kappa(z_n)$ is convergent. Again by Lemma 1, for some $t \in C'$ the sequence $t \circ_h z_n \in C'$ i.o. and again by passing to a further subsequence we may suppose this holds for all n. Since κ is continuous on C',

$$\kappa(t)\kappa(z_n) = \kappa(t \circ_h z_n) \to \kappa(t),$$

and so $\kappa(z_n) \to 1 = \kappa(0)$, as κ is positive. That is, κ is continuous at 0.

Now for any $s \in \mathbb{G}_h$ and any null sequence w_n, which we may suppose is in $\mathbb{G}_h$ by Corollary 1, take $z_n := w_n/h(s) \to 0$, which again we may suppose is in $\mathbb{G}_h$; then

$$s \circ z_n = s + h(s)z_n = s + w_n \to s,$$

and, as $\kappa(z_n) \to 1$,

$$\kappa(s + w_n) = \kappa(s \circ_h z_n) = \kappa(s)\kappa(z_n) \to \kappa(s).$$

That is, κ is continuous on $\mathbb{G}_h$.

In the case $\kappa = h$, continuity of h implies $\mathbb{G}_h$ is connected, by Lemma 2. □

Proposition 1. *For h a measurable solution of (GS), the corresponding norm $||.||$ is a group norm on the Popa group $(\mathbb{G}_h, \circ_h)$ under which $\mathbb{G}_h$ is a topological group with a natural left-invariant metric equivalent to the Euclidean norm on $\mathbb{R}$.*

Proof. By Lemma 3, $\mathbb{G}_h$ is connected. For $a \neq 0$ we write $[0, a]$ for the (inclusive) interval between 0 and a irrespective of their order. Integrals below are implicitly calculated over such modified intervals. By Lemma 3, since $h(x)$ is bounded above and below on the compact interval from 0 to a, we have $0 < |L(a)| < \infty$ and so $||a||$ is well-defined and positive. Furthermore,

$$|a| \cdot \inf\{1/h(x) : x \in [0, a]\} \leq ||a|| \leq |a| \cdot \sup\{1/h(x) : x \in [0, a]\}.$$

The substitution

$$y = a \circ x = a + h(a)x$$

gives $dy = h(a)dx$ and so

$$\int_0^{-a/h(a)} \frac{dx}{h(x)} = \int_a^0 \frac{dy}{h(a)h(x)}.$$

By (GS), we conclude that

$$\int_0^{-a/h(a)} \frac{dx}{h(x)} = -\int_0^a \frac{dy}{h(a \circ x)} = -\int_0^a \frac{dy}{h(y)},$$

and $|L(-a/h(a))| = |N(a)|$. Again with $y = a \circ x$ and any b, c,

$$\int_{a\circ c}^{a\circ b} \frac{dx}{h(x)} = \int_{a\circ c}^{a\circ b} \frac{h(a)dx}{h(a)h(x)} = \int_c^b \frac{dy}{h(y)}, \tag{$*$}$$

corresponding, as earlier, to a notional metric $d_L(a \circ b, a \circ c) = ||b^{-1}a^{-1}ac|| = ||b^{-1}c||$. So by $(*)$

$$\begin{aligned} L(a) + L(b) &= \int_0^a \frac{dx}{h(x)} + \int_0^b \frac{dy}{h(y)} = \int_0^a \frac{dx}{h(x)} + \int_a^{a\circ b} \frac{dx}{h(x)} \\ &= \int_0^{a\circ b} \frac{dy}{h(y)} = L(a \circ b). \end{aligned}$$

Thus
$$||a \circ b|| = |L(a \circ b)| = |L(a) + L(b)| \leq ||a|| + ||b||,$$
and so all three properties of the norm are verified.

Since the norm is defined by integration, absolute continuity of the integral guarantees that inversion and product are continuous. □

We may now state and prove:

Theorem 1. *Any measurable solution h of (GS) with $\mathbb{G}_h = \{x : h(x) > 0\}$ non-null is continuous on $\mathbb{G}_h$. Furthermore, $\mathbb{R}_+ \subseteq \mathbb{G}_h$ and also $(-\infty, 0] \cap \mathbb{G}_h$ is an interval so that $\mathbb{G}_h$ is itself an interval.*

Proof. For $x \in \mathbb{G}$, put
$$f(x) = \log h(x),$$
which we view as a homomorphism from $(\mathbb{G}, \circ)$ to $(\mathbb{R}, +)$, since
$$f(a \circ b) = f(a) + f(b).$$
Interpreted in the context of a normed group, Darboux's Theorem [29], [19, Th. 9.4.1], [35], (cf. [10], or see below) asserts that if f is locally bounded at 0, then f is continuous at 0, and hence throughout the Popa group, f being a homomorphism.

By assumption, for some $B > 0$, the set
$$S := \{x : 0 < |f(x)| < B\}$$
here is (measurable and) non-null. If f is unbounded at the origin, we may choose $z_n \to 0$ in $\mathbb{G}$ with $|f(z_n)|$ unbounded. By Lemma 1 S is shift-compact, so we may choose $t \in S$ with $t \circ z_n \in S$ infinitely often. Then, as above with t_h^{-1} the group inverse of t,
$$|f(z_n)| = |f(t_h^{-1} \circ t \circ z_n)| = |f(t_h^{-1}) + f(t \circ z_n)| \leq |f(t_h^{-1})| + B,$$
for infinitely many n. But this contradicts unboundedness.

By Darboux's Theorem (see next section), f and so also h is continuous, also in the sense of the Euclidean norm, by Proposition 1.

As for the final claim, $(0, \infty) \subseteq \mathbb{G}_h$ by Lemma 2, so restricted to $[0, \infty)$ the inversion mapping $x \mapsto x_h^{-1} = -x/h(x) \in (-\infty, 0]$ is continuous and so its image is connected. □

2. Darboux Theorem

On a normed group $(G, \circ)$,
$$||x \circ y|| \leq ||x|| + ||y|| \leq 2 \max\{||x||, |y||\},$$

somewhat reminiscent of the p-adic ultrametric, cf. [28]. On $(\mathbb{G}_h, \circ_h)$, where $\mathbb{G}_h := \{x : h(x) > 0\}$ and $x \circ_h y := x + h(x)y$ with h positive on $\mathbb{R}$, satisfying (GS), and bounded by L on $|x| < \delta$, one has for x, y with $|x|, |y| < \delta$:

$$|x \circ_h y| \leq |x + h(x)y| \leq |x| + L|y| \leq \max\{|x|, |y|\}(1 + L),$$

so that the Euclidean norm restricted to $\mathbb{G}_h$ is a prenorm in the following sense.

Definition. Say that $||.||$ is a *pre-norm* on the group G if it is positive except at 1_G and for some $K > 1$ and $\delta_K > 0$

$$||x \circ y|| \leq K \max\{||x||, ||y||\}, \text{for } ||x||, ||y|| \leq \delta_K.$$

Variant Darboux Theorem. *For a pre-norm $||.||$ on a group G, if $f : (G, \circ) \to (\mathbb{R}, +)$ is a homomorphism,*

$$f(x \circ y) = f(x) + f(y),$$

bounded in some pre-norm neighbourhood of 1_G, then f is continuous. In particular this is so for the group norm of Section 1 on $\mathbb{G}_h$.

Proof. First note that for any $\delta < \delta_K$, for $||x|| < \delta/K^N N$

$$||x^N|| \leq K^N ||x||.$$

Indeed, this holds for $N = 1$. Proceed by induction. If $||x|| < \delta/K^{N+1}(N+1) < \delta/K^N N$, then $||x^N|| < K^N ||x|| < K^N \delta/K^N N = \delta/N < \delta_K$ and so, as $K > 1$,

$$||x \circ x^N|| \leq K \max\{||x^N||, ||x||\} \leq K \max\{K^N ||x||, ||x||\} \leq K^{N+1} ||x||.$$

Suppose that f has a local bound of L on $||t|| < \delta$. W.l.o.g. $\delta < \delta_K$. Now let $\varepsilon > 0$ be given. Choose an integer N with $N > L/\varepsilon$. Consider t with $||t|| \leq \delta/NK^N$. Then $||t|| < \delta$ and

$$||t^N|| \leq K^N ||t|| \leq K^N \delta/NK^N < \delta : \qquad |f(t^N)| \leq L.$$

By additivity,

$$||Nf(t)|| = ||f(t^N)|| \leq L : \qquad ||f(t)|| \leq L/N < \varepsilon.$$

Thus f is continuous at 1_G and so continuous.

We have already seen that for h a measurable solution of (GS) with $\mathbb{G}_h$ non-null there is a group norm equivalent to the Euclidean norm and that h is locally bounded. In the next section we study the case of h Baire.

3. Baire case

In this section h is a Baire solution to (GS), meaning that $h^{-1}(U)$ is *Baire* (has the Baire property) for U open. We will need a definition and a theorem.

Definition. (Category convergence, [7]) A sequence of Baire functions h_n satisfies the *category convergence condition* (cc) if, for any non-empty open set U, there is a non-empty open set $V \subseteq U$ such that, for each $k \in \omega\mathbb{N}$,

$$\bigcap\nolimits_{n \geq k} V \backslash h_n^{-1}(V) \text{ is meagre.} \tag{cc}$$

Equivalently, off a meagre set of t,

$$t \in V \Longrightarrow h_n(t) \in V \text{ i.o.}$$

This holds in the case of $h_n(t) = t + h(t)z_n$ for $z_n \to 0$ and V Euclidean open, since

$$t \in V \Longrightarrow t + h(t)z_n \in V \text{ i.o. (co-finitely!).}$$

However, for a density-open V the condition is essentially the result that V is shift-compact in the sense of shifts under $\circ_h$ as in Lemma 1 above.

Category Embedding Theorem. *([7, 19]; Th. 10.2.2). Let X be a topological space. If $h_n : X \to X$ are Baire functions satisfying (cc) with pre-images of meagre sets being meagre, then, for any Baire set $T \subseteq X$, for quasi-all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T, \quad \textit{i.e.} \quad h_n(t) \in T \textit{ i.o.}$$

Corollary 2. (Lemma 1–Baire version). *For $w_n \to 0$ a null sequence and $T \subseteq \mathbb{R}$ a non-meagre Baire set, there is $t \in T$ for which $t \circ w_n \in T$ i.o., i.e. the following set is infinite:*

$$\{n \in \mathbb{N} : t + h(t)w_n \in T\}.$$

Proof. This follows from the Category Embedding Theorem by applying the following result. □

Lemma 4. (Meagre pre-images). *For h Baire, $\{z_n\}_n$ null and M meagre, the following pre-image is meagre for each n*

$$h_n^{-1}(M) = \{s - h(s)z_n/h(z_n) : s \in M\}.$$

Proof. It is enough to consider a nowhere dense set M. Write $w_n = -z_n/h(z_n) = (z_n)_h^{-1}$ for the group inverse. Then the right w_n-shift is the inverse of the right z_n-shift:

$$s = h_n(t) := t + h(t)z_n = t \circ z_n,$$
$$t = h_n^{-1}(s) = s \circ (z_n)_h^{-1} = s + h(s)w_n.$$

As $h(x)$ is Baire, so is $h_n(t) := t + h(t)z_n$. So w.l.o.g. h_n is continuous off a mearge set $M' \supseteq M$, by Baire's Continuity Theorem [33, Th. 8.38], [40, Th.

8.1], [19, Th. 2.1.4]. We work relative to the complement X of M'. Choose V an arbitrary non-empty open set in X to exclude M. Now $h_n^{-1}(V)$ is open, by continuity as V omits M; so omits M'; now choose non-empty open $W \subseteq h_n^{-1}(V)$ to exclude M. Suppose if possible that W does not avoid $h_n^{-1}(M)$. Then for some $s \in M$

$$t = h_n^{-1}(s) = s + h(s)w_n \in W, \text{ so } h_n(t) \in h_n(W) \subset V,$$

so $s = h_n(t) \in V \cap M$, a contradiction. □

Theorem 2. *Any solution h of (GS) with the Baire property (that pre-images of open sets are Baire) and with $\{x : h(x) > 0\} = h^{-1}(0, \infty)$ non-meagre is continuous. Furthermore, $\mathbb{R}_+ \subseteq \mathbb{G}_h$ and also $(-\infty, 0] \cap \mathbb{G}_h$ is an interval so that $\mathbb{G}_h$ is itself an interval.*

Proof. This follows analogously to Theorem 1 by interpreting Corollary 1 and Lemma 3 in the Baire context, replacing Luzin's by Baire's Continuity Theorem. □

Alternative Integration Proof. One can follow the norm-based Darboux proof of §1 also in the Baire case. Begin by restricting the solution function h to a co-meagre $\mathcal{G}_\delta$ set $H := \mathbb{G}_h \backslash M$ (with M a meagre, centrally symmetric $\mathcal{F}_\sigma$ set) on which $h_H := h|H$ is *continuous* with domain *non-null.* This is possible here, as Corollary 1 (interpreted in the Baire context via Corollary 2) implies that $\mathbb{G}_h$ is open. (Also Remark 4 of Section 1 implies that if $\mathbb{G}_h$ is assumed bounded from below, then $[0, \infty) \subseteq \mathbb{G}_h$.) That yields a group-norm, albeit only on H, via Lemma 1, as H is non-null. Fortunately, w.l.o.g. H is a subgroup of $\mathbb{G}_h$. Indeed, as $h(0) = h(0)^2$ by (GS), by positivity $h(0) = 1$, and so $h(x)h(-x/h(x)) = h(0) = 1$, again by (GS). Thus $h(-x/h(x)) = 1/h(x)$ yields relative continuity at the group inverse x_h^{-1} from relative continuity at $x \in H$ (both relative to H). A norm can now be assigned to $x \in M$ via completion, which the density of the non-meagre subgroup H allows:

$$||x|| := \limsup_{\delta>0} \{||q|| : q \in (x - \delta, x + \delta) \cap H\}.$$

Hence $L(x)$, as defined previously, may be interpreted as an improper integral:

$$L(x) = L_H(x) := \int_0^x \frac{dt}{h_H(t)} = \lim_{\substack{q \to x \\ t \in H}} \int_0^q \frac{dt}{h_H(t)} \qquad (x \in \mathbb{G}_h).$$

As noted earlier, $\mathbb{G}_h$ is open and so locally compact in $\mathbb{R}$, implying the Subgroup Theorem applies here ([9, Ths 6.11, 6.13] or [36, Ch. VI. 13. XII], cf. [BinO2025, 9.3.1]), so that the co-meagre subgroup H is clopen. Suppose $H \neq \mathbb{G}_h$; then, being open, $\mathbb{G}_h \backslash H$ is non-meagre and yet contained in the meagre set M, a contradiction. Thus h is continuous on all of $\mathbb{G}_h$. □

4. The Goldie equation

The Goldie equation below, also linked to regular variation [5] and [12], is simpler than the Gołąb-Schinzel equation in that it is already defined on a Popa group $\mathbb{G}_h$ with $h(x) = h_\rho(x) = 1 + \rho x$. We show how continuity of solutions here is also provided by the Darboux paradigm.

We note that (the $\rho = 0$) versions of this equation were studied by Aczél [2] in connection with the characterization of geometric and power means and in the equivalence of certain utility representations.

Notation: In view of the needs in the subsequent section, it is convenient below to use $\circ_h$ for a general function h and to write $\circ_\rho$ when h is specialized to $h(x) = h_\rho(x)$.

Theorem 3. *Taking $x \circ_\rho y = x + h_\rho(x)y$, let K solve the Goldie equation*

$$K(x \circ_\rho y) = K(x) + g(x)K(y) \quad (x, y \in \mathbb{G}_\rho)$$

with $K(1) \neq 0$, $g(0) = 1$, and $\{x : |K(x)| > 0\}$ non-meagre/non-null. Then K is continuous at the origin, and so K and g are both continuous.

In this functional equation context we will refer to K above as a *kernel.*

We offer several proofs. In the first proof, we provide a *group logarithm* on the range $\mathcal{R}(K)$, which it emerges is a Popa group of the form $\mathbb{G}_\sigma$ for some σ. This demonstrates similarity to the Gołąb-Schinzel case.

Darboux Proof. We may exclude the case $K \equiv 0$. Then the range $\mathcal{R}(K)$ is a Popa group, by [18, Theorem 7.1A] (applicable here as $\mathcal{N}(\rho) = \{t : \rho t = 0\} = \{0\}$ and $\mathcal{R}(K) \neq K(\mathcal{N}(\rho))$). Now

$$K(x) + g(x)K(y) = K(x \circ y) = K(x) \circ_\sigma K(y) = K(x) + (1 + \sigma K(x))K(y)$$

implies $g(x) - 1 = \sigma K(x)$. The group logarithm of the Popa group $\mathbb{G}_\sigma$ is as in Section 1:

$$L_\sigma(a) = \int_0^a \frac{dt}{h_\sigma(t)} = \int_0^a \frac{dt}{1 + \sigma t} = \frac{1}{\sigma}\log(1 + \sigma a).$$

Replacing K by K/σ in the Goldie equation, if necessary, we may assume $\sigma = 1$. Applying this logarithm, take

$$f(x) := \log(1 + K(x)).$$

Then f is a homomorphism into $(\mathbb{R}, +)$:

$$\begin{aligned} f(x \circ_\rho y) &= \log(1 + K(x \circ_\rho y)) = \log[1 + K(x) + (1 + K(x))K(y)] \\ &= \log[(1 + K(x))(1 + K(y))] = f(x) + f(y). \end{aligned}$$

Take $h = h_\rho$ in Lemma 1; then as in Theorem 1, being Baire/measurable, f is locally bounded. By Darboux's theorem, f is continuous and hence so also is K. (first proof) $\square$

Second Proof. Note that $K(0) = 0$, since $g(0) = 1$ and

$$K(0) = K(0 \circ_\rho 0) = K(0) + g(0)K(0).$$

We first show that K is continuous at 0. Suppose otherwise. Then $K(z_n)$ does not converge to 0 for some null z_n (which is in $\mathbb{G}_\rho$). W.l.o.g. K is continuous on $T := \{x : |K(x)| > 0\}$, again by Luzin's Continuity Theorem. By Lemma 1, $t \circ_\rho z_n \in T$ i.o. for some $t \in T$. If $g(t) = 0$, then taking $s = t_\rho^{-1} \circ_\rho x$ for x arbitrary implies that

$$K(x) = K(t \circ_\rho s) = K(t) + g(t)K(s) = K(t),$$

i.e. that K is constant, a contradiction since $K(0) = 0 \neq K(1)$. So $g(t) \neq 0$ and then

$$K(t) = \lim K(t \circ_\rho z_n) = K(t) + g(t) \lim K(z_n).$$

But this implies $\lim K(z_n) = 0$, contrary to assumption. So K is continuous at $t = 0$. We complete the proof by proving the following lemma (needed again below and in the next section), which does not require the exact form of h.

Lemma 5. *Assume the kernel K is continuous at* 0. *Then*

(i) *K is continuous;*
(ii) *if h is continuous, then also g is continuous.*

Proof. (i) For z_n null, continuity of K at 0 gives $K(z_n) \to 0$; furthermore,

$$K(t + h(t)z_n) = K(t \circ_h z_n) = K(t) + g(t)K(z_n) \to K(t).$$

So K is continuous at any $t \in \mathbb{G}_h$. (For any t and null sequence w_n again take $z_n := w_n/h(t)$, which is null, and then $t + w_n = t \circ_h z_n$.)

(ii) For any $t_n \to t \in \mathbb{G}_h$, by continuity of h, $t_n \circ_h 1 = t_n + h(t_n) \to t + h(t) = t \circ_h 1$,

$$K(t_n) + g(t_n)K(1) = K(t_n \circ_h 1) \to K(t \circ_h 1) = K(t) + g(t)K(1),$$

yielding $g(t_n) \to g(t)$ (by (i) and as $K(1) \neq 0$), so that g is continuous.

In particular, this holds for the auxiliary in the Goldie equation.(second proof) □

Towards a third Darboux proof, we need the following to motivate our assumptions below. Here again we need h to be Baire/measurable.

Lemma 6. *A Baire/measurable solution K of the Goldie equation with $\{x : |K(x)| > 0\}$ non–meagre/non-null is locally bounded.*

Proof. For some bound $B > 0$ the set $T := \{x : 0 < |K(x)| < B\}$ is non-null. Suppose that $|K(z_n)|$ is unbounded above for some null z_n. By Lemma 1, for some t the sequence $t \circ_\rho z_n \in T$ i.o. Then

$$|g(t)K(z_n)| = |K(t \circ_\rho z_n) - K(t)| \leq |K(t \circ_\rho z_n)| + |K(t)| < B + |K(t)|,$$

contradicting unboundedness unless $g(t)=0$. But, if this were the case, taking $s=t_\rho^{-1}\circ_\rho x$ for x arbitrary implies as before that

$$K(x)=K(t\circ_\rho s)=K(t),$$

i.e. that K is constant, a contradiction since $K(0)=0\neq K(1)$. $\square$

Remark. In Lemma 6, suppose that $|g(z_n)|$ is unbounded above for some null z_n. By Lemma 6 we may suppose that $K(z_n)$ is bounded. For some $t\in T$ with $K(t)>0$ the sequence $t\circ_\rho z_n\in T$ i.o. Then by Lemma 1

$$K(t\circ_\rho z_n)=K(z_n\circ_\rho t)=K(z_n)+g(z_n)K(t).$$

So

$$|g(z_n)K(t)|=|K(t\circ_\rho z_n)-K(z_n)|\leq B+|K(z_n)|,$$

contradicting unboundedness, since $t\in T$. $\square$

The following result shows off another Darboux paradigm. However, the same conclusion also follows from the identity $K(x)=g(x)-1$ noted above, and also more directly from commutativity of $\circ_\rho$, as follows.

$$K(x)+g(x)K(y)=K(y)+g(y)K(x),\ \text{so}\ K(x)[1-g(y)]=K(y)[1-g(x)],$$
$$\frac{K(x)}{1-g(x)}=\frac{K(y)}{1-g(y)}=c,\ \text{so}\ K(x)=c(g(x)-1).$$

This last needs a single y with $g(y)\neq 1$ and $K\neq 0$ to define c (see [12, Th. 1]).

Proposition 2. *In the Goldie equation, if the kernel K is locally bounded and the auxiliary g is continuous at* 0, *then K is continuous.*

Proof. We assume K is bounded by L in some neighbourhood of 0. Iterating powers from the left under $\circ_\rho$ so that $t=t\circ_\rho t^{n-1}=t+h_\rho(t)t^{n-1}$ yields

$$\begin{aligned}K(t\circ_\rho t^{n-1}) &= K(t)+g(t)K(t^{n-1})=K(t)+g(t)(K(t)+g(t)K(t^{n-2}))\\ &= ...=K(t)(1+g(t)+...+g(t)^{n-1}))=K(t)\cdot\begin{cases} n, & \text{if } g(t)=1,\\ \frac{1-g(t)^{n-1}}{1-g(t)}, & \text{if } g(t)\neq 1.\end{cases}\end{aligned}$$

For any N, as $g(0)=1$, provided t lies in a neighbourhood of 0 with $|g(t)-1|<1/N$, we have for appropriate $n=n(N)$ that

$$|NK(t)|\leq|K(t^n)|\leq L,\ \text{i.e.}\ |K(t)|<L/N.$$

Then, as in the Darboux variant, $|K(t)|<\varepsilon$ for $N>L/\varepsilon$ and so K is continuous at 0. Consequently K is continuous by Lemma 5(i). (third proof) $\square$

5. The General Goldie equation

Here we use shift-compactness to deduce continuity aspects of the general Goldie equation, previously studied in [18] albeit assuming continuity, namely

$$K(x+h(x)y)=K(x)+g(x)K(y)\quad (x\in\mathbb{G}_h, y\in\mathbb{R})$$

with $K(1)\neq 0$, $g(0)=h(0)=1$, where $\mathbb{G}_h:=\{t:h(t)>0\}$ with $1\in\mathbb{G}_h$ is Baire non-meagre, respectively measurable non-null. In the functional equation context here, we will again refer to K as a *kernel*.

Above we have relaxed the condition $y\in\mathbb{G}_h$ of earlier sections, thus extending the domain of K to $\mathbb{R}$, as $K(y)=K(0+h(0)y)$. This *wider domain assumption* is for two reasons. Firstly, the present domain choice aligns with that of [18]. Secondly, until h is identified, one cannot assume that $\mathbb{G}_h$ is a group, which occasionally impedes the analysis. However, for $a\in\mathbb{G}_h$, the left a-shift $t\mapsto s=a\circ_h t=a+h(a)t$, defined on $\mathbb{R}$, posseses a well-defined *anti-shift* (inverse on $\mathbb{R}$), $s\mapsto t=(s-a)/h(a)$; this allows the interpretation $K(s)=K(a\circ_h t)$. To aid matters we assume also that $\mathbb{G}_h$ is *inversion invariant* in that $-a/h(a)\in\mathbb{G}_h$ for $a\in\mathbb{G}_h$ (a 'reflection' in 0). Nonetheless, we record at the critical junctures whenever these stronger assumptions are invoked (Propositions 4 and 6, Corollary 3, and in Corollary 4, where we use the inversion-invariance assumption).

For further alignment with the Goldie equation, we assume that h *preserves positivity*, i.e. that if $a,b\in\mathbb{G}_h$, then $h(a+h(a)b)>0$, so that $a\circ_h b\in\mathbb{G}_h$ – a 'quasi' semi-group, lacking associativity (and likewise for g); see [26, Th.1] why adding the converse implication results in h satisfying (GS); [2] eventually in Proposition 5 below we recover that here too h satisfies (GS). An *additional hypothesis* is helpful: we will assume that $-(\varepsilon,\varepsilon)\subseteq\mathbb{G}_h$ for some $\varepsilon>0$, so that $\mathbb{G}_h$ is open. We recall from [18] that $\mathbb{G}_h=\mathbb{G}_g:=\{t:g(t)>0\}$: the first of many parallel behaviours between the auxiliaries (see below). As a frequent ingredient in proofs, we mention another instance. Given continuity of K, and of h at 0, for any null z_n,

$$\begin{aligned}K(1)&=\lim K(z_n+1)=\lim K(z_n+h(z_n)/h(z_n))\\&=\lim[K(z_n)+g(z_n)K(1/h(z_n))]=K(1)\lim g(z_n),\end{aligned}$$

i.e. $\lim g(z_n)=1=g(0)$, and we thus see g is also continuous at 0.

The 'unitary' case of $g\equiv 1$ embraces (logarithmically) the Gołąb-Schinzel equation, allowing the solution read-out in Section 6 Theorem 5 (by an appeal to monotonicity/injectivity, cf. Corollary 3).

When $K(x)\equiv cx$, the equation reduces for $c\neq 0$ to

$$h(x)=g(x)\quad (x\in\mathbb{G}_h),$$

[2] For $h\geq 0$, this guarantees that both the left-shift $t\mapsto a\circ_h t$ and its anti-shift inverse preserve positivity, leading to $h(a)h(b)/h(a\circ b)=1$ when 0 is an interior point of $\mathbb{G}_h$.

which says nothing about the nature of the auxiliaries. (For the converse direction of $h = g$ implying linearity, see the Remarks at the end of Section 6.) Thus we will at least need the n assumption that g and h are like K: respectively Baire, or measurable.

Theorem 4. *With the assumptions just given that g and h are like K, i.e. all Baire or respectively all measurable, then K is continuous. If further h is continuous at 0, then the auxiliaries g, h are continuous.*

The proof will emerge from the sequence of results below. The form of the solution triplet is then known (with $h = h_\rho$ and $g(t) = (h_\rho(t))^{\theta/\rho}$, including $g(t) = e^{\theta t}$ when $\rho = 0$), see e.g. [39].

Lemma 7. *For non-trivial K, if K is differentiable anywhere on $\mathbb{G}_h$, then it is differentiable everywhere on $\mathbb{G}_h$, and then*

$$K'(t) = \frac{g(t)}{h(t)} K'(0).$$

Proof. Since $K(0) = 0$ for $s \neq 0$ with $s, t \in \mathbb{G}_h$, the formula follows from:

$$\frac{K(t + h(t)s) - K(t)}{sh(t)} = \frac{g(t)}{h(t)} \frac{K(s) - K(0)}{s}. \qquad \square$$

Lemma 8. *Assume that both K and the auxiliary h are Baire/measurable and $\mathbb{G}_h := \{t : h(t) > 0\}$ is correspondingly Baire non-meagre/measurable non-null. Then K is continuous.*

Proof. As in Section 4 (second proof), consider any null sequence z_n (in $\mathbb{G}_h$). Restricting attention to some subset of $\mathbb{G}_h$ on which by Luzin's Continuity Theorem K is continuous, by Lemma 1 for some $t \in \mathbb{G}_h$ a subsequence $t+h(t)z_n$ converges to a continuity point t of K. So

$$K(t) = \lim K(t + h(t)z_n) = K(t) + g(t) \lim K(z_n),$$

implying continuity of K at 0, as $K(0) = 0$ (since this last implies $K(z_n) \to 0$). Note that here again $g(t) \neq 0$, since $h(t) > 0$ implies constancy near t : indeed, otherwise, taking $s = x/h(t) \in \mathbb{G}_h$ for x near 0, gives for $g(t) = 0$

$$K(t + x) = K(t \circ_h s) = K(t) + g(t)K(s) = K(t).$$

In turn, this implies $K'(t) = 0$ and by the formula in Lemma 7 constancy on $\mathbb{G}_h$, a contradiction.

Furthermore, as in Lemma 5(i), given continuity of K at 0, for arbitrary $s \in \mathbb{G}_h$ and with $s_n \to s$ also in $\mathbb{G}_h$, take the 'anti-shifts' $z_n := (s_n - s)/h(s) \to 0$ (in $\mathbb{G}_h$ for large n). Then, since $s_n = s + h(s)z_n$,

$$K(s) = K(s) + g(s) \lim K(z_n) = \lim K(s + h(s)z_n) = \lim K(s_n).$$

That is, K is continuous at any $s \in \mathbb{G}_h$.

Proposition 3. *For K a continuous kernel, the zeros of K form a discrete set: neither the origin nor any other point is an accumulation of zeros of K.*

Proof. Suppose that a sequence of zeros ζ_n (in $\mathbb{G}_h$) of K converges to 0 from the right. W.l.o.g. $K(1) > 0$, and there is a maximal open interval (c, d) around 1, where K is positive. Here $K(c) = 0$ by continuity (this invokes our wider domain assumption). However, the shifts $c_n := c + h(c)\zeta_n$ are zeros of K converging from the right to c, as

$$K(c + h(c)\zeta_n) = K(c) + g(c)K(\zeta_n) = 0,$$

but these lie ultimately in (c, d), contradicting positivity of K on (c, d). A similar argument applies at d if there is a sequence of zeros converging from the left at 0.

Now suppose that $a \in \mathbb{G}_h$ accumulates zeros a_n to the right, so that by continuity also $K(a) = 0$. Then the anti-shifts $\zeta_n := (a_n - a)/h(a) > 0$ (in $\mathbb{G}_h$ for large n) are zeros of K accumulating to the right of 0, since

$$0 = K(a_n) - K(a) = K(a + h(a)\zeta_n) - K(a) = g(a)K(\zeta_n),$$

and $g(a) > 0$ as $a \in \mathbb{G}_h = \mathbb{G}_g$. □

Corollary 3. *If the kernel K is continuous, then it is strictly monotone on $\mathbb{G}_h$ locally to each zero; furthermore, if the origin is the unique zero of K, then K is strictly monotone on $\mathbb{G}_h$.*

Proof. Suppose first that the origin is the unique zero of K and that w.l.o.g. $K(t) > 0$ to the right of 0. Given $0 < y < x$ with $y \in \mathbb{G}_h$ take the anti-shift $z = (x - y)/h(y)$, so that $x = y + h(y)z$ with $z > 0$. Then, as $K(z) > 0$,

$$K(x) = K(y + h(y)z) = K(y) + g(y)K(z) > K(y).$$

Here again we have used the wider domain assumption (for z).

Now let k be the first zero of K to the right of 0. If $0 < y < x < y + h(y)k$, then $K(y) < K(x)$.[3] □

Corollary 4. *If K is a continuous kernel with a least positive zero at $k \in \mathbb{G}_h$, then $h(k) = 1$, and so $\{nk : n \in \mathbb{Z}\}$ is the set of all zeros of K.*

Proof. For any zero $\zeta \in \mathbb{G}_h \backslash \{0\}$ with $0 < h(\zeta) < 1$, note that the inductive definition $\zeta_h^{n+1} := \zeta \circ_h \zeta_h^n$ gives $\zeta_h^{n+1} \in \mathbb{G}_h$ and $K(\zeta_h^{n+1}) = K(\zeta \circ_h \zeta_h^n) = K(\zeta) + g(\zeta)K(\zeta_h^n) = 0$. But

$$\zeta_h^{n+1} = \zeta + h(\zeta)\zeta_h^n = \zeta(1 + ... + h(\zeta)^n) \to \omega := \frac{\zeta}{1 - h(\zeta)},$$

so these zeros accumulate (with ω a zero), a contradiction. So $h(\zeta) \geq 1$ and in particular $h(k) \geq 1$. If $h(k) > 1$, then $\kappa := -k/h(k) \in \mathbb{G}_h$ is also a negative

[3] For a more detailed account, see the Appendix.

zero of K (appealing to the inversion-invariance assumption), since $0 = K(0) = K(k \circ_h \kappa) = K(k) + g(k)K(\kappa)$ as $g(k) > 0$. So $h(\kappa) \geq 1$. Then $k_1 = -\kappa/h(\kappa) \leq -\kappa < k$ and $k_1 \in \mathbb{G}_h$ is a smaller positive zero of K, as now $K(0) = K(\kappa \circ k_1) = K(\kappa) + g(\kappa)K(k_1)$ and $g(\kappa) > 0$ (as $h(\kappa) > 0$), a contradiction. So $h(k) = 1$. Likewise $h(\kappa) = 1$ and so $\kappa = -k$. By induction, as $h(k) = 1$, both $(n+1)k = k \circ nk \in \mathbb{G}_h$ and $K((n+1)k) = K(k) + g(k)K(nk) = 0$. Likewise, $K((n+1)\kappa) = K(\kappa) + K(n\kappa) = 0$.

Furthermore, for $0 < \delta < k$, if $nk + \delta \in \mathbb{G}_h$ then $-k + (nk + \delta) \in \mathbb{G}_h$ and by induction $\delta \in \mathbb{G}_\delta$. As $K(-k + nk + \delta) = K(nk + \delta)$ for each k, by reverse induction if $K(nk + \delta) = 0$, then $K(\delta) = 0$, contradicting the definition of k. Hence there are no other zeros on $[0, \infty)$. The same argument applies with κ for k. Hence the set of zeros is $\{nk : n \in \mathbb{Z}\}$. $\square$

Corollary 5. *For non-trivial K, if K is differentiable at some point in $\mathbb{G}_h$, then K has only one zero and is strictly monotonic by Corollary 3.*

Proof. By Lemma 7, for K somewhere differentiable in $\mathbb{G}_h$ and non-trivial, $K'(0) \neq 0$ and w.l.o.g. $K'(0) > 0$. So, as g, h are positive, strict monotonicity follows again from Lemma 7. $\square$

Remark. For g, h continuous and $K(z) > 0$ for small $z > 0$, [12, Th. 8] shows that $K(x)$ is differentiable and of the form $c \int_0^x (g(t)/h(t))\, \mathrm{d}t$. The proof uses Riemann telescoping sums generated this time by iterating powers from the right: $u^{n+1} = u^n \circ_h u = u^n + h(u^n)u$, giving 'Beck sequence' partitions of the range of integration.

In view of Proposition 5 below, it seems unlikely that K has more than one zero, unless h is 'ill-behaved': see Theorem 5 below. Nevertheless, the following observation is, if not noteworthy, then curious. 'Boundedness away from zero' was noted in Lemma 3.

Proposition 4. *For K a continuous kernel, if K has more than one zero and g and h are bounded and also bounded away from 0 in the neighbourhood of the origin, then g and h are continuous at 0.*

Proof. Let k be the first zero of K to the right of the origin. Let z_n be null with well-defined finite limits $\eta := \lim h(z_n) > 0$ and $\lim g(z_n) > 0$ (by boundedness away from 0). We check that $k\eta$ and k/η are zeros of K (appealing again to the wider domain assumption). As $\eta > 0$, we conclude that $k\eta \geq k$ and $k/\eta \geq k$ so that $\eta = 1 = h(0)$. Indeed

$$K(\eta k) = K(\lim(z_n + h(z_n)k)) = \lim g(z_n)K(k) = 0,$$
$$K(k) = \lim K(z_n + k) = \lim g(z_n)K(k/h(z_n)) = \lim g(z_n) \lim K(k/h(z_n)).$$

Given this deduction of continuity of h at 0, for any null z_n,

$$\begin{aligned} K(1) = \lim K(z_n + 1) &= \lim K(z_n + h(z_n)/h(z_n)) \\ &= \lim g(z_n)K(1/h(z_n)) = K(1) \lim g(z_n), \end{aligned}$$

by continuity of K, as noted earlier. So g is also continuous at 0. $\square$

Parallel behaviours of auxiliaries: Before proceeding towards our final result, we need to clarify that if h is continuous at 0, then assuming only continuity of K at 0 implies the same for g. This repeats the ideas in Lemma 5 but with a weaker hypothesis on h: first, exactly as in Lemma 5(i), for z_n null (in $\mathbb{G}_h$), continuity of K at 0 gives $K(z_n) \to 0$ and so

$$K(1 + h(1)z_n) = K(1 \circ_h z_n) = K(1) + g(1)K(z_n) \to K(1),$$

i.e. continuity of K at 1. Hence, for z_n null, continuity of h at 0 gives

$$K(1) = \lim K(z_n + h(z_n)) = K(1) \lim g(z_n),$$

yielding continuity of g at 0, as $K(1) \neq 0$. We resist deriving the many other parallels (from f to g, save to mention: boundedness, unboundedness, convergence to zero) and 'reverse parallels' (from g to f, which can require the monotonicity corollary).

Proposition 5. *Assume continuity at* 0 *of* g *and* h, *and that* K *is continuous but not linear. If* h *is Baire/measurable, then* h *is continuous and satisfies (GS). Likewise,* g *is continuous and satisfies a pexiderized variant of (GS) if* g *is correspondingly Baire/measurable.*

Proof. It is shown in [18], albeit in a normed vector space context, that assuming K, h, g continuous, for any $u > 0$,

$$K(su) = \lambda_u(s)K(u) \qquad (s \geq 0),$$

where $\lambda_u(s)$ is the *linking function* defined there – with suitable parameters. In fact, only the limits $\delta^g(u) := \lim_n(g(u/n) - 1)$ and $\delta^h(u) := \lim_n(h(u/n) - 1)$ being zero (equivalent to g and h continuous at 0) and the continuity of K are used.

The case when $\lambda_u(t) = \mathrm{id}(t) \equiv t$ yields K linear and $g = h$, as above, excluded by our assumptions. Otherwise, as in [BinO24a, Prop. 8.1], the relation

$$\lambda_u \left(\frac{h(a + h(a)b)}{h(a)h(b)} \right) = \frac{g(a + h(a)b)}{g(a)g(b)} \text{ for } a, b \in \mathbb{G}_h$$

holds for any two choices $u > 0$, so since all the graphs $\lambda_u(t)$ other than id cross only at $t = 1$ [BinO24a, Lemma 8.2], it follows that

$$\frac{h(a + h(a)b)}{h(a)h(b)} = 1 \quad \text{and hence} \quad g(a + h(a)b) = g(a)g(b),$$

as $\lambda_u(1) = 1$. Thus h satisfies the equation (GS) on $\mathbb{G}_h$; consequently, g satisfies the 'pexiderized' variant seen in Lemma 3.

Thus, if we assume h is Baire/measurable, then in view of the assumptions on $\mathbb{G}_h$, we conclude that h is continuous by Theorem 1 and, by known results (confirmed in Theorem 5 below) $h(t) = 1 + \rho t$ for some $\rho \in [0, \infty)$, i.e. that $\mathbb{G}_h$ is a Popa group. That in turn shows that if also g is Baire/measurable, then by the variant Darboux theorem g is continuous, in view of the assumptions on $\mathbb{G}_g = \mathbb{G}_h$. $\square$

Remarks. Evidently h will have continuity points provided $T_K := \{t \in \mathbb{G}_h : K(t) > 0\}$ is non-meagre/non-null. But it is unclear how this information might imply continuity at 0.

The results in this section may easily be transferred to a vector space context; the form of the auxiliaries is then as in [BinO24a, Theorem 8.1] and of the kernel as in [20, Th. 4a/4b].

6. The form of h

We derive the form of h directly, using two separate approaches. In the first we invoke a strong form of the property of h considered in Lemma 3.

Proposition 6. *For a continuous kernel K, if h is locally bounded away from 0 on $[0,\infty)$, then $[0,\infty) \subseteq \mathbb{G}_h$ and K is strictly monotonic.*

Proof. We show that 0 is the unique zero of K, so that Corollary 3 applies.

By appeal to compactness, one readily proves that h *locally bounded away from* 0 implies that h is bounded away from zero on compact subintervals of $[0,\infty)$, so that $[0,\infty) \subseteq \mathbb{G}_h$. Towards a contradiction, we now suppose that K has a least positive zero at k, that for some $L > 0$ and all $t \in [0,k]$, $h(t) \geq L$, and finally, as K is continuous and non-zero in $(0,k)$, that K is positive on $(0,k)$ – otherwise replace K by $-K$.

Now for $y \in (0,k)$ as $0 < k-y < k$ we have $K(k-y) > 0$ and $h(k-y) \geq L$. However, for any choice of $y \in (0,Lk) \cap (0,k)$, we reach the contradiction that both $y, k-y \in \mathbb{G}_h$ (as $h(k-y) > 0$) and

$$0 = K(k) = K(k-y+y) = K(k-y) + g(k-y)K(y/h(k-y)) > 0,$$

as $0 < y/h(k-y) < Lk/L = k$ and $g(k-y) > 0$. Thus, there is no least positive zero of K and so by Corollary 3, K is strictly monotonic. □

Theorem 5. (i) *For h measurable, $g(t) \equiv 1$ and K a continuous kernel (e.g. when K is measurable), equivalently for $\kappa(t) = \exp K(t)$ a positive solution of the pexiderized (GS) equation*

$$\kappa(x \circ_h y) = \kappa(x)\kappa(y),$$

there exists ρ such that in a neighbourhood of 0 the inner auxiliary satisfies

$$h(t) = 1 + \rho t.$$

If, further, h is locally bounded away from 0 , then K is strictly monotonic,

$$h(t) = 1 + \rho t \text{ for all } t \in \mathbb{G}_h,$$

and $\mathbb{G}_h := \{t : t > -1/\rho\}$, with the convention $-1/0 = -\infty$.

(ii) *With the assumptions on h of Theorem 4, the positive solutions of the Gołąb-Schinzel equation have the form*

$$h(t) = 1 + \rho t \text{ for } t > -1/\rho \text{ with } \rho \geq 0,$$

and again $\mathbb{G}_h := \{t : t > -1/\rho\}$.

Proof. (i) By Corollary 3, K is strictly monotonic in some neighbourhood of 0. Then, as $g(t) \equiv 1$,

$$K(s + h(s)t) = K(s) + K(t) = K(t + h(t)s).$$

So, for $s, t > 0$ restricted to that neighbourhood of 0,

$$s + h(s)t = t + h(t)s : \qquad (h(s) - 1)/s = (h(t) - 1)/t = \text{const}.$$

Setting the constant as ρ yields $h(t) = 1 + \rho t$ in that neighbourhood of 0.

If further h is locally bounded away from 0, then by Proposition 6, K is strictly monotonic and so the previous argument is valid without restricting s or t to a neighbourhood of 0.

ii) For h a solution of (GS), take $K = \log h$, so that K solves the general Goldie equation with $g(t) \equiv 1$ implying K is continuous. So $h = \exp K$ is continuous and then h is locally bounded away from 0 near the origin by Corollary 1. So by the local result in part (i),

$$\frac{K(s) - K(0)}{s} = \frac{\log(1 + \rho s)}{s} \to \rho \text{ as } s \to 0.$$

So $K'(0) = \rho$. By Lemma 7, with $g(t) \equiv 1$ and $K = \log h$,

$$K'(t) = \frac{1}{h(t)}\rho \Longrightarrow \frac{h'(t)}{h(t)} = \frac{\rho}{h(t)} \Longrightarrow h(t) = \rho t + 1,$$

as $h(0) = 1$. Positivity for large $t > 0$ yields $\rho \geq 0$. □

Remarks. Our result complements the case $h = g$ of the general Goldie equation studied in [12, Th. 9] where for h locally bounded above and away from 0, the solution is found to be linear: $K(x) = cx$, provided $K(0) = 0$ (implied in the present context by the assumption $h(0) = g(0) = 1$). By recasting (GS) as a Goldie equation a further deduction is made there that the solution of (GS) for h locally bounded above and away from 0 is given by $h(x) = 1 + \rho x$. The assumptions there justify the use of 'Beck sequence' partitions for Riemann sums with the integrand being the ratio $g/h \equiv 1$, as might be suggested by Lemma 7.

7. Concluding remarks

1. Lemma 1 and its consequences exploit the embedding of $(\mathbb{G}_h, \circ)$ in $(\mathbb{R}, +)$. This is reminiscent of [12] where function behaviour on a (smaller) dense subgroup $(\mathbb{A}, +)$, embedded in $(\mathbb{R}, +)$, is deduced from that of a related function on $\mathbb{R}$. There the tool is the integral assuming continuous solutions. Here the tool in Theorem 5(ii) is the derivative.

2. *Alternative proof of Lemma 1–B.* Repeat the argument in the measure case of Lemma 1 verbatim (save for invoking Baire's Continuity Theorem) using the category density topology $\mathcal{D}_{\mathcal{B}}$ on $\mathbb{R}$ [14, Theorem 1], in which sets V are open if each point $v \in V$ has a Euclidean neighbourhood U with $U \backslash V$ meagre. $\square$

3. For *measurable* $h : \mathbb{R}^d \to \mathbb{R}_+$ satisfying (GS) in the form

$$h(x + h(x)y) = h(x)h(y) \qquad (x, y \in \mathbb{R}^d),$$

as above take $a \circ b = a + h(a)b$, and define a Borel measure by setting

$$\eta(B) = \int_B \frac{dx}{h(x)^d} \qquad \text{for Borel } B,$$

where dx refers to Lebesgue measure in $\mathbb{R}^d$ (cf. [27, 7.6.3]). As in Lemma 3, $\eta(B)$ is finite. Again as on the line, using the substitution $y = a \circ x = a + h(a)x$, for which the Jacobian may be computed to be $\partial y / \partial x = h(a)^d$, shows the measure to be left-invariant:

$$\eta(a \circ B) = \int_{a \circ B} \frac{dy}{h(y)^d} = \int_B \frac{h(a)^d dx}{h(a)^d h(x)^d} = \int_B \frac{dx}{h(x)^d} = \eta(B).$$

So η may be interpreted as Haar measure on $(\mathbb{R}^d, \circ)$.

Denoting by B_n the $1/n$ closed balls centered at 0, and writing $\triangle$ for symmetric difference, put

$$d_L(x, y) := \sup\nolimits_n \eta(x \circ B_n \triangle y \circ B_n).$$

This is a left-invariant Weil metric [16] (as used also in the proof of Struble's theorem [DieS, Th. 8.1]), which gives rise to the group norm

$$||x||_\eta := d_L(x, 0).$$

As in Proposition 1, $||x||_\eta$ is equivalent to the Euclidean norm $||x||_2$.

For *Baire* h satisfying (GS), apply the alternative integration approach of §3 relative to a non-null non-meagre subgroup H, which is locally compact (by Corollary 1, interpreted via Corollary 2).

4. The proof above for Theorem 2 holds also in $\mathbb{R}^d$. One may either refer to the CET-based proof of Lemma 1 – Baire, which applies quite generally, or repeat the alternative proof of Lemma 1– Baire in Remark 1 but with $\mathcal{D}_{\mathcal{B}}$ now the category density topology (as defined in [14, Th. 1]) derived in the group $(\mathbb{R}^d, +)$. The proof of Theorem 2 then continues to hold with $|x|$ interpreted as the Euclidean norm $||x||$ in $\mathbb{R}^d$.

Acknowledgements

It is a pleasure to thank both Nick Bingham and Charles Goldie for very helpful discussions.

Author contributions There is only one author and the contributions are his work.

Data Availability Statement No datasets were generated or analysed during the current study.

Declaration

Competing interests The authors declare no competing interests.

8. Appendix

Detailed Proof of Cor. 3. We check in greater detail the proof of Corollary 3, and how this leads to Theorem 5(i).

Let k be the first zero of K to the right of 0. W.l.o.g. we may suppose that $K(z) > 0$ for $0 < z < k$ (otherwise, replace K by $-K$). Let $\varepsilon > 0$ be given with $(-\varepsilon, \varepsilon) \subseteq \mathbb{G}_h$ and $\varepsilon < 1/3$ so that

$$\frac{1-\varepsilon}{1+\varepsilon} > \frac{1}{2}.$$

By continuity of h at 0, choose $\delta > 0$ so that $1 - \varepsilon < h(u) < 1 + \varepsilon$ for $|u| < \delta < \min\{k/4, \varepsilon\}$. Take $\bar{k} := (k-\delta)/(1+\varepsilon) < k$. Given $u < v < u + h(u)\bar{k}$, we shall have $z := (v - u)/h(u) < \bar{k} < k$ and so $K(z) > 0$.

Consider any u with $|u| < \delta$. By choice of δ, we have both

$$u + h(u)\bar{k} < \delta + (1+\varepsilon)\bar{k} = k,$$

and also

$$h(u)\bar{k} > (1-\varepsilon)(k-\delta)/(1+\varepsilon) > \frac{3k}{4}(1-\varepsilon)/(1+\varepsilon) > \frac{k}{2}\frac{1}{2} > \delta.$$

That is, the interval $(u, u + h(u)\bar{k})$ contains δ as $0 < \delta < u + h(u)\bar{k}$. So any v with $u < v < \delta$ satisfies $u < v < u + h(u)\bar{k}$. So with z as above, as $u \in \mathbb{G}_h$,

$$K(u) < K(u) + g(u)K(z) = K(u \circ z) = K(v).$$

Put $I_0 := (-\delta, \delta)$, then K is strictly increasing on I_0. $\square$

Alternative Proof of Prop. 6. By Proposition 3 we may assume for some $\delta > 0$ that $K(t) > 0$ for $t \in (0, \delta)$. We will show that K is (strictly) increasing on $[0, \delta]$. For now, suppose that, for some $a \geq 0$, $K(t)$ is continuous, positive and (strictly) increasing on $[0, a]$ but not increasing on $(0, b)$ for $b > a$. This last implies that for each $b > a$ there are witness points $u = u(b) < v = v(b)$ with $0 < u < v < b$ and $K(u) \geq K(v)$. Clearly, u, v are not both in $[0, a]$.

So first consider the case that for each $b > a$ the witness points have $a \leq u < v < b$ with $K(v) \leq K(u)$ and $K(a) \leq K(u)$. Since h is bounded from below we have $L := \min\{h(x) : a \leq x \leq a + \varepsilon h(a)\} > 0$ for some $\varepsilon > 0$ with $0 < \varepsilon < \delta$. Take $b = a + \varepsilon L/2$ and consider the corresponding witnesses u, v $\in [a, a + \varepsilon L/2] \subseteq [a, a + h(a)\varepsilon]$, as $h(a) \geq L$. So $h(u) \geq L$ and $h(v) \geq L$ with $v - u = (v - a) - (u - a) < \varepsilon L$. As $h(u) \geq L$, we may write $v = u + h(u)z$ with $z := (v - u)/h(u) < \varepsilon L/L < \delta$, so that $K(z) > K(0) = 0$. Then, as $h(u) > L > 0$ so that $g(u) > 0$ and $u \in \mathbb{G}_h$,

$$K(v) = K(u \circ_h z) = K(u) + g(u)K(z) > K(u),$$

a contradiction.

In particular, with $a = 0$ the above argument precludes witnesses $u < v$ in $[0, \delta)$ with $K(v) \leq K(u)$, implying that $K(u) < K(v)$ for all pairs $u < v$ from $(0, \delta)$. So we may now assume $a > 0$ (in fact we may now assume $a \geq \delta$).

Now consider the alternative. Here for each $b > a$ there is $y = y(b) \in (a, b)$ with $K(y) < K(a)$; indeed, the witnesses now satisfy $u < a < v < b$, so that $K(v) \leq K(u) < K(a)$ and we may take $y = v$. Taking $y_n \in (a, a + 1/n)$ with $K(y_n) < K(a)$, we may write $z_n := (y_n - a)/h(a) > 0$, as h is bounded away from 0. Then since $z_n \to 0$, for large enough n, $z_n < a$ and so, for such n, as $K(z_n) > 0$, so since $a \in \mathbb{G}_h$,

$$K(a) > K(y_n) = K(a \circ_h z_n) = K(a) + g(a)K(z_n) > K(a),$$

again a contradiction. □

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Received: May 30, 2025
Revised: May 30, 2025
Accepted: August 1, 2025