Chapter 4
Lakatos and the Euclidean Programme

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Abstract Euclid's *Elements* inspired a number of foundationalist accounts of mathematics, which dominated the epistemology of the discipline for many centuries in the West. Yet surprisingly little has been written by recent philosophers about this conception of mathematical knowledge. The great exception is Imre Lakatos, whose characterisation of the Euclidean Programme in the philosophy of mathematics counts as one of his central contributions. In this essay, we examine Lakatos's account of the Euclidean Programme with a critical eye, and suggest an alternative picture that builds on his while differing from it in a number of important ways.

Keywords Euclideanism · Epistemological foundationalism · Lakatos · Mathematical knowledge · Axioms

In the *Elements*, Euclid begins by laying down definitions, postulates and common notions. From these, he methodically solves problems and proves theorems, deriving the geometry of his time step by step. Euclid himself offers no philosophical gloss on his method; as a mathematician rather than a philosopher, that is to be expected. Others, however, have not shied away from doing so. They have read into the *Elements* a methodological ideal to be emulated throughout mathematics and elsewhere. The *Elements* has inspired a foundationalist vision of mathematical knowledge and of knowledge in general.

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Although Euclidean foundationalism has been much praised over the centuries, it has been little analysed by recent philosophers. An important exception is Imre Lakatos, whom the present volume honours. Lakatos was no Euclidean; quite the contrary. But he believed in knowing his enemy, so was careful to describe the Euclidean picture in some detail. With Euclideanism as a foil, he developed his own 'quasi-empiricist' and fallibilist epistemology of mathematics.

Following in Lakatos's footsteps, we will take a closer look at Euclideanism. Our main motivation is that although the picture is commonly referred to, it is not entirely clear what it is. Contemporary philosophers are superficially familiar with 'Euclidean foundationalism' in the philosophy of mathematics; but dig down, and the details are fuzzy. Euclidean foundationalism is like a great-aunt who has always been around and seems very familiar, though you have never bothered to get to know her. When you finally have a long conversation with her, you realise quite how interesting she is, even if you don't necessarily agree with her. Lakatos would have concurred: as he put it, '[t]he fascinating story of the Euclidean programme and of its breakdown has not yet been written' (1962, p. 6). The first half of that story, before the breakdown, must start with what the programme actually is.

Our essay is devoted to drilling down into the details of Euclideanism, with Lakatos as our guide. It falls into two parts. The first and principal part (Sect. 4.1) outlines Lakatos's views about Euclidean foundationalism, which we follow him in calling the Euclidean Programme, or EP for short. Along the way, we analyse his account of it, noting where we part company with him. In Sect. 4.2, using Lakatos's discussion as a springboard but moving beyond it, we characterise the EP in our preferred way by means of seven principles. Our own assessment of where the EP stands today is too lengthy and unrelated to Lakatos for inclusion in this volume; it may be found in our recent book *The Euclidean Programme* (2024). In the present essay, we compare and contrast Lakatos's account of the EP with our own.

4.1 Lakatos on the EP

Lakatos wrote about the EP in several places. It crops up in his writings as something to be opposed, attacked and rejected, sometimes head-on, sometimes glancingly. An article in which the focus is squarely on the EP is the relatively early piece 'Infinite Regress and Foundations of Mathematics' (Lakatos, 1962). In this article, Lakatos considers several ways of organising knowledge in a deductive system. Here is how he describes the Euclidean mode of organisation:

> I call a deductive system a '*Euclidean theory*' if the propositions at the top (*axioms*) consist of perfectly well-known terms (*primitive terms*), and if there are *infallible truth-value-injections* at this top of the truth-value *True*, which flows downwards through the deductive channels of truth-transmission (*proofs*) and inundates the whole system. (If the truth-value at the top was *False*, there would of course be no current of truth-value in the system.) Since the Euclidean programme implies that all knowledge can be deduced from a finite set of trivially true propositions consisting only of terms with a trivial meaning-load, I

> shall call it also the *Programme of Trivialization of Knowledge*. Since a Euclidean theory contains only indubitably true propositions, it operates neither with conjectures nor with refutations. In a fully-fledged Euclidean theory meaning, like truth, is injected at the top and it flows down safely through meaning-preserving channels of nominal definitions from the primitive terms to the (abbreviatory and therefore theoretically superfluous) defined terms. A Euclidean theory is *eo ipso* consistent, for all the propositions occurring in it are true, and a set of true propositions is certainly consistent.[1] (1962, pp. 4–5)

In the rest of this section, we'll dissect this passage, and others, to extract some of the key features that Lakatos ascribes to the EP. We should clarify at the outset that we aren't concerned with Lakatos's criticisms of the EP, or with his criticism of dogmatic epistemology more generally or formalism more specifically. We will not, for example, consider how Lakatos's criticism of formalism stands up in today's age of computer proof. Our chief concern is his *description* of the EP, and our question is whether he got this target right. Of course, the EP is a rational reconstruction, not a historically attested manifesto, so there is some leeway in how to describe it. Nevertheless, given its history, there is something to get right here. And in our opinion, Lakatos gets some aspects of the EP right, but not others. This will be our concern in the rest of this section.

4.1.1 Truth

Both in the core passage above and elsewhere, Lakatos emphasises that the axioms of a Euclidean theory are true, or at least aspire to be true. He is clearly right that the axioms of a Euclidean theory are (supposed to be) true, and that this is an essential aspect of the EP.

Lakatos is not particularly clear about precisely *why* the truth of the axioms is an essential feature of the EP, and appears to take this for granted. An obvious point is that the EP is a foundationalist account of mathematical knowledge, and knowledge implies truth. It also chimes with how mathematicians down the ages have thought of, say, geometry, unhesitatingly taking its axioms to be correct.

A further point is that the major figures in the history of mathematics and its philosophy that one would want to identify as Lakatos's targets are all explicit that mathematics is a body of truths, starting from true axioms or first principles. For

[1] There is a footnote accompanying a sentence in this passage (ending with the words '*Programme of Trivialization of Knowledge*'). The footnote refers to Pascal's *De l'esprit géométrique* ('On the Geometrical Mind'), which Lakatos calls the EP's *locus classicus*. Never published in Pascal's lifetime, the *Esprit* is a short work that influenced the *Port-Royal Logic*. In his one-sentence footnote, Lakatos refers to it as 'Pascal [1657–8]', but more recent scholarship has tended to settle on 1655 as the date of its composition, following Jean Mesnard, the editor of Pascal's works. Lakatos's reference to it is interesting in that the work is largely unknown to English-speaking philosophers; indeed, the philosophical literature on the *Esprit* in the analytic tradition is almost non-existent, even today. For discussion of the *Esprit* by us, see Sect. 5.2 of *The Euclidean Programme*.

example, Aristotle gives an account of how *episteme* ('understanding' or 'scientific knowledge', which he identifies as the highest epistemic state) can be gained from demonstrations, in terms that resemble Lakatos's characterisation of proof in the EP. And Aristotelian demonstrations all start from true principles (*Posterior Analytics* I.2 71b17–25).[2] Centuries later, Descartes gives a Euclidean account of *scientia* (the epistemic ideal of the scholastic and early modern periods) where first principles are truths which are understood so clearly and distinctly as to be rationally indubitable (1637, pp. 16–17; AT 6, p. 19).[3] And Pascal, who Lakatos paints as the arch-Euclidean (see footnote 1, above) unambiguously describes the axioms of geometry as 'vérités' ('truths').

So on the truth of the axioms, we are in complete agreement with Lakatos. No commitment is thereby made to any particular analysis or philosophical account of truth. As we shall see below, however, we do not entirely share Lakatos's view about the role that truth plays in a Euclidean theory.

4.1.2 Flow

One of the most interesting features of the passage above is Lakatos's metaphor of the *flow* of truth in a Euclidean system, downward from axioms at the 'top' of the theory to theorems at the 'bottom'. To illustrate the significance of this point, Lakatos contrasts the Euclidean Programme with the 'Empiricist Programme':

> The Euclidean programme proposes to build up Euclidean theories with foundations in meaning and truth-value at the top, lit by the *natural light of Reason*, specifically by arithmetical, geometrical, metaphysical, moral, etc. intuition. The Empiricist programme proposes to build up Empiricist theories with foundations in meaning and truth-value at the bottom, lit by the *natural light of Experience*. Both programmes however rely on Reason (specifically on logical intuition) for the safe transmission of meaning and truth-value. (1962, p. 5)

It is important to appreciate that empiricist theories do not need to be strictly empirical. In particular, one could have an empiricist account of mathematics, in Lakatos's sense. The salient epistemological point is that in an empiricist theory, the relevant flow is not *downward*, from axioms to theorems, but rather *upward* from 'basic statements' (perhaps observations, or elementary arithmetical sentences) to higher-level statements (perhaps theoretical scientific principles or mathematical axioms).

[2] Although Aristotle predates Euclid by a few decades, one way to read him is as an early exponent of the EP. There are several important commonalities between Aristotle's account of method and the EP as described by Lakatos. And naturally, Aristotle's account of geometric method was influenced by the geometry of his time, likely very similar to that of Euclid's time. For more details, see Sect. 4 of *The Euclidean Programme*.

[3] The first citation is to Olscamp's English translation of the *Discourse on Method* (2001). The second is to Adam and Tannery's *Oeuvres de Descartes* (volume, page).

Lakatos returns to Euclideanism in a later work, 'A Renaissance of Empiricism in the Recent Philosophy of Mathematics'.[4] In this article, he clarifies that empiricist theories are *quasi-empirical*. The relevant direction of flow is upward, and what is transmitted is typically falsity, rather than truth. If a theory, empirical or otherwise, implies a false basic statement, this 'inundates' the system, which is refuted. To think that truth, in addition to falsity, can be transmitted upward is to indulge in what Lakatos refers to in a Popperian vein as the *inductivist delusion* (1976a, p. 41).

We do not wish to dwell on this point, since we are not concerned here with quasi-empirical theories. We simply note, against Lakatos, that the inductivist idea of basic statements retransmitting truth to the axioms which imply them is not obviously a delusion, as he characterises it. It is common enough, of course, to take a scientific theory to be confirmed to some degree when its observational predictions are correct. And in mathematics as well, many philosophers have thought that axioms are given some degree of confirmation when they imply elementary truths which we already take ourselves to know (such as that $2 + 2 = 4$, for instance).[5] But Lakatos is clearly correct about the direction of flow in the Euclidean account of mathematics. Indeed, so deeply embedded is this idea that it has seemed obvious to many that the direction of flow is top-down in mathematics, an idea just as obvious as that the direction of flow in the empirical sciences is bottom-up.

But Lakatos does more than just identify this commitment of the EP. In the above passages, and in others to be quoted later, he consistently talks of truth-value, and of meaning, as flowing through the channels of the system, a point to which we return with a more critical eye later in this section. His focus on meaning and truth notwithstanding, he also offers what we see as an absolutely crucial insight. This is the observation that the EP is less about *what* flows from axioms to theorems and more about *how* it flows. As Lakatos puts it:

> We can get a long way merely by discussing *how* anything flows in a deductive system without discussing the problem of *what in fact flows* there, infallible truth or only, say, Russellian 'psychologically incorrigible' truth, Braithwaitian 'logically incorrigible' truth, Wittgensteinian 'linguistically incorrigible' truth or Popperian corrigible falsity and 'verisimilitude', Carnapian probability. (1962, p. 6)

What makes the EP distinctive as a methodological account of mathematics, therefore, is its emphasis on mathematicians' prior access to the axioms from which they establish theorems by means of proof. Different Euclideans might mean different things by the terms 'access' and 'establish'. With his key observation that what flows is of lesser interest than how it flows, Lakatos is a locksmith who has opened the way to a proper understanding of the EP.

Years later, in the 'Renaissance' article, Lakatos re-iterated the flow idea: truth is injected at the top and flows down to the bottom. Indeed, he draws the very

[4] This essay, posthumously published as Lakatos (1976a), is an expanded version of an earlier 1967 paper.

[5] For an influential expression of this idea, see Russell (1907).

distinction between Euclidean and quasi-empirical theories in these terms: as he says, '[i]t is the *how* of the flow that is decisive' (1976a, p. 29).

The insight we extract from Lakatos is, as we put it elsewhere, that the EP is all about Euclidean hydraulics (2024, p. 4). For comparison, consider the Phillips Machine, an analogue computer developed in 1949. In this machine, coloured water moves through a series of clear pipes in order to model the flow of money in the economy. By analogy, in a Euclidean theory, some theoretical good corresponds to the coloured water. This for Lakatos is truth, but for us (see Sect. 4.2) it will be something epistemic, such as (rational) certainty, knowledge, or justification. Much as the colour of the water is inessential to the modelling process in the Phillips Machine, the choice of a particular theoretical good is immaterial to the Euclidean explanation of mathematics, at least in broad outline. What matters is that the theoretical good is injected with the axioms, and flows downward through the logical structure of the theory to the theorems.

4.1.3 What Is Injected?

Although we agree with Lakatos that the axioms of a Euclidean theory are supposed to be true, we part ways with him in a crucial respect on the point of truth. In the passage cited above from the 'Foundations' paper, Lakatos speaks of an injection of truth and meaning. This idea that truth is injected into the theory via the axioms persists into the 'Renaissance' paper, where he writes:

> Classical epistemology has for two thousand years modelled its ideal of a theory, whether scientific or mathematical, on its conception of Euclidean geometry. The ideal theory is a deductive system with an indubitable truth-injection at the top (a finite conjunction of axioms) – so that truth, flowing down from the top through the safe truth-preserving channels of valid inferences, inundates the whole system. (1976a, p. 28)

We find this talk at best misleading, at worst confused. What would it even mean for truth itself to flow from axioms to theorems? In mathematics at least, truth is not tensed: mathematical propositions are either eternally true or eternally false. The theorems of geometry are all eternally true, and there is no literal sense in which the truth of one proposition is transmitted to another. Of course, logicians like to speak of rules being 'truth-preserving', but that image is more easily literalised than the flow or transmission idea: it simply means that if the rule's premises are true then so is the conclusion. It's possible, of course, that Lakatos meant no more than this. A similar point applies to meaning: the meanings of the theorems do not depend on the meanings of the axioms. Although perhaps that view is more tenable than the analogous one about truth, especially if the axioms are consciously stipulated at the start of the practice rather than extracted from it.

We highlighted above that the crucial point about flow in a Euclidean theory is its direction. We think the flow metaphor is best construed as transmission of an *epistemic* good of some sort. What this good is exactly will vary from one Euclidean theorist to another. But the EP as we see it represents an epistemological conception. It is best thought of as a form of epistemic foundationalism, in which the axioms enjoy an initial justification (for instance), which flows to the theorems when these are proved.[6] The channel along which the epistemic good flows, from axioms to theorems, is an epistemic path the mathematician themself follows, or could follow.[7]

As mentioned, it's quite possible that Lakatos appreciated this point but wrote misleadingly. (Or to be fairer to him, that he wrote in a way that two philosophers in the 2020s taking him very literally find misleading.) He seems to recognise as much in passages such as the following:

> Whether a deductive system is Euclidean or quasi-empirical is decided by the pattern of truth value flow in the system. The system is Euclidean if the characteristic flow is the transmission of truth from the set of axioms 'downwards' to the rest of the system – logic here is an *organon of proof*; it is quasi-empirical if the characteristic flow is retransmission of falsity from the false basic statements 'upwards' towards the 'hypothesis' – logic here is an *organon of criticism*. (1976a, p. 29)

The focus on the role of proof in Euclidean theories and criticism in quasi-empirical ones is most welcome. But despite that, Lakatos seems to think epistemic facts enable—in the best case, *guarantee*—truth-injection, rather than constitute the injection itself. This is plain in the talk of truth value being transmitted from axioms to theorems. And as he put it much earlier, empiricists 'criticized the guarantee of the intuitive Euclidean truth-injection: self-evidence' (1962, p. 9). We shall clarify our way of putting things in the next section. For now, we simply note that our characterisation of the EP as involving the transmission of an epistemic good from axioms to theorems is sufficiently general to encompass the main historical figures that one would wish to characterise as Euclideans.

4.1.4 Finitude

In the long quotation from 'Foundations' at the start of Sect. 4.1, Lakatos describes the axioms of a Euclidean theory as a 'finite set of trivially true propositions'. In the 'Renaissance' passage cited at the start of Sect. 4.1.3, he characterises the 'top' of a Euclidean theory as 'a finite conjunction of axioms'. No clear justification for this is given by Lakatos; he claims only that the finitude of the axioms is implied by the EP (1962, p. 4).

That sets of axioms should be finite is a broadly, but not entirely, correct historical observation. Axiomatic theories prior to the twentieth century, including

[6] However, some epistemic goods possessed by the axioms will not flow from axioms to theorems. For instance, *self*-evidence will not be transferred via deduction.

[7] Since deduction is truth-preserving, and the axioms of a Euclidean theory are true, so are the theorems. This point stands even if we think of *Flow* as a primarily epistemic, rather than alethic or semantic, principle.

Euclid's geometry, are finite. Even today, most axiomatic theories of mainstream mathematical interest are finitary, in an important sense. This makes Lakatos's inclusion of this point defensible, though it needs to be made more precise. In particular, we must take due notice of axiom schemata. First-order Peano Arithmetic (PA), for example, cannot be finitely axiomatised, and hence cannot be presented as a finite conjunction. But PA *can* be finitely formulated as long as schemata are allowed, the usual way of doing so being to adopt a schematic form of the induction axiom. Euclideans should allow this sort of latitude.

Moreover, there are important Euclidean thinkers who contradict this point, or remain silent on it. For example, Aristotle, who can be identified as a forerunner of the EP, is explicit in the *Posterior Analytics* that science as a whole requires an infinity of axioms (I.32 88b6). Prominent advocates of the EP in the seventeenth century do not share this commitment as far as we know, but nor are we aware of an active commitment to the finitude of the axioms in these writers. No clear endorsement is discernible in the relevant works of Descartes (*The Discourse on Method,* including *The Geometry*) or Pascal (*On the Geometric Mind*), for instance.

So on this point, we broadly agree with Lakatos, but insist that more care be taken over its formulation, and that the principle is not central to the EP. We return to it in the next section.

4.1.5 Triviality

As we saw in the quotation from 'Foundations', Lakatos takes the axioms of a Euclidean theory to be 'trivially true' and says they bear a 'trivial meaning-load' (1962, pp. 4–5). What does Lakatos mean by 'trivial'? We confess we're not entirely sure.

One understanding of 'trivial' is logical. But this cannot be the sense of triviality that Lakatos has in mind. If the axioms were logical truths they would be redundant, as they would be deducible from anything. And no Euclidean theory could go beyond logic.

Another way to understand triviality is along logical empiricist lines: mathematical statements are void of content, in virtue of being analytically true. If this is what Lakatos intends, we disagree in the strongest terms. The central philosophical commitments of the EP, as both a general epistemology of mathematics and as a historical phenomenon, are entirely consistent with the axioms being substantive truths about a 'third realm' of abstract objects, or as being apprehended by a non-linguistic faculty of mathematical intuition. But given Lakatos's general disparagement of logical empiricism, and his explicit mention of intuition in 'Foundations', it is unlikely that this is his intended sense.

A third possibility is that triviality in 'Foundations' is related to non-explanatoriness in the 'Renaissance' article, where the term 'trivial' does not appear. The earlier Lakatos describes proof as *giving way* to explanation (1962, p. 14), as

Euclidean theories are replaced by their Empiricist successors. And the later Lakatos draws the contrast in the following way:

> [I]n a Euclidean theory the true basic statements at the 'top' of the deductive system (usually called 'axioms') *prove*, as it were, the rest of the system; in a quasi-empirical theory the (true) basic statements are *explained* by the rest of the system. (1976a, p. 29)

So, perhaps the intended sense of *trivial* in the early paper is simply that the axioms are non-explanatory. They represent fossilised truisms rather than theoretically hard-working explanatory principles that have a role to play in making bold theoretical conjectures.

Lakatos believes that in quasi-empirical theories of the type he favours, basic statements are explained by the rest of the system, but (by implicit contrast) that this is not the case in a Euclidean theory. We can agree on at least one point: that axioms are explanatory of theorems need not be built into a Euclidean theory. But we do not see why explanatoriness has to be ruled out either. Perhaps a Euclidean could think of the axioms as explaining the theorems. Indeed, this seems to be Aristotle's position (*Posterior Analytics* I.2 71b17–25), and his thought bears a strong resemblance to the EP, according to at least one major interpretative school.[8] So the thought that axioms explain theorems is not *per se* un-Euclidean or un-foundational. Perhaps there is good reason to think that the axioms cannot *in fact* explain the theorems in a Euclidean theory. But in so far as we are undertaking a rational reconstruction of the EP, we see no reason to include non-explanatoriness as one of its features.

There is a fourth way to interpret the word 'trivial'. This is the idea that axioms are self-evident; that their discovery, as opposed to their content, is trivial. This reading is suggested by Lakatos' insistence that the injection of truth value in a Euclidean theory is supposed to be infallible. In the development of his quasi-empirical account of mathematics, Lakatos sets himself fiercely against the axioms' alleged self-evidence and the indubitability of mathematics, so he clearly meant to bake this idea into the EP. If this is the intended sense of 'trivial', then we agree. Indeed, we take it to be central to the EP that axioms are thought of as self-evident (more on this in Sect. 4.2).

4.1.6 Primitive Terms

Related to the issue of triviality, Lakatos requires that the primitive terms of a Euclidean theory be *perfectly* well-known (1962, p. 4). What is significantly less clear to us, however, is why he insists on this.

In the context of the 1962 paper, the reason seems to be scepticism. Here, Lakatos is concerned with two regressive sceptical arguments that aim to show that

[8] See chapter 4 of *The Euclidean Programme*.

meaning and truth cannot be conclusively established (1962, p. 3). Meaning cannot be established because when defining an expression *E*, for instance, one must use at least one expression, call it *F*. Presumably *F* itself requires a definition, and if circularity is to be avoided, this definition will use expressions that are not defined in terms of *E* or *F*. Rather, a new term, *G*, must be introduced. *G* apparently needs its own non-circular definition, and so the regress goes on *ad infinitum.* There is also the more familiar regress in terms of proof and knowledge. If one claims to know some mathematical theorem by giving a proof, that proof will have premises, which in turn require their own proofs, and so on *ad infinitum* again.

Lakatos paints the EP as a response to *both* problems. The regress in proof is blocked by the axioms; these truths are known indubitably and are not in need of proof at all. If the EP is to block the semantic regress also, it is natural to think of the Euclidean theorist as making a similar pronouncement on the primitive terms of the theory: they are understood *perfectly*, and so are not in need of a definition or elucidation, and the regress is blocked. In short, Lakatos sees the EP as both an epistemological and a semantic manifestation of foundationalism.

It is questionable, however, whether Lakatos's requirement is justifiable in historical terms. It is not clear (to us, at any rate) that the history of the EP is so closely connected to semantic scepticism. In the *Posterior Analytics*, Aristotle discusses a relevant issue. He claims that '[a]ll teaching and all learning of an intellectual kind proceed from pre-existent knowledge', clarifying that '[t]here are two ways in which we must already have knowledge: of some things we must already believe that they are, of others we must grasp what the items spoken about are (and of some things both).' The requirement that we *must grasp what the items spoken about are* is something like a requirement that the terms of a theory be previously understood, and Aristotle gives the example 'of the triangle, that it means *this*' as something we need to know in order to learn about triangles (I.1 71a1–16).[9] Now of course one must know the meaning of the term 'triangle' *in some sense* to have knowledge of triangles. But there is nothing here to suggest that such understanding must be perfect. Rather, the use of the demonstrative seems to suggest that Aristotle requires simply that one be able to identify triangles when confronted with them, which on its own seems to fall far short of understanding 'triangle' perfectly. And, at least at this juncture, Aristotle is not even responding to the sceptical regress about meaning that Lakatos had in mind. Rather, he is responding to 'the puzzle in the *Meno*' (I.1 71a29–31), that one cannot enquire after what one is ignorant of, since one will not know what to enquire after, nor will one recognize the correct answer to the query if one comes across it.

Reading the great Euclideans of the seventeenth century also casts Lakatos's claims in a dubious light. Descartes, of course, is extremely concerned to respond to scepticism. But he is most naturally read as responding to *epistemological* scepticism about the possibility of knowledge (*scientia*) rather than to semantic scepticism about the meaning of terms in mathematics or elsewhere.[10]

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[9] The translations here are from Barnes's edition (Aristotle 1993).

[10] See *The Meditations* (AT 7) for instance.

The situation is even worse when we turn to Pascal, Lakatos's paradigmatic Euclidean. Pascal is clearly alive to both of Lakatos's regresses. He claims that, ideally at least, we would like to define *all* the geometric terminology, and give a proof of *all* the propositions of geometry. But neither is possible in practice, thanks to the threat of infinite regress. We must, therefore, use primitive terms that are so clear that we cannot explain them in clearer terms, and unproved principles that are so obvious as to admit of no proof from principles more obvious still. But he goes on to assert that trying to further explain the geometrical primitives would cause more confusion than it would resolve.[11] This makes it clear that Pascal also requires primitive terms to be understood, but suggests that the understanding may be imperfect, otherwise there would be no confusion to even try to resolve. Pascal is also clear that geometric knowledge is in good standing when it is obtained by the method he outlines. Hence failing to live up to our initial ideals is not to the detriment of geometry, contrary to what Lakatos asserts about the Euclidean position on semantics.

In short, we see the primitive terms requirement as a convenience that Lakatos adds to his characterisation of the EP in order to present it as a broader form of foundationalism than the textual evidence allows for. Lakatos's thoughts on primitive terms and meaning in mathematics are certainly interesting. Yet we do not find in practice that historical Euclideans address these semantic issues in the way Lakatos describes, if indeed they address them at all.

4.1.7 Formality

In the earlier 'Foundations', Lakatos also comments parenthetically that deductions in a Euclidean system need not be formal. Changing the clause's italicisation to emphasise this aspect of it:

> The basic definitional characteristic of a (*not necessarily formal*) deductive system is the principle of retransmission of falsity from the 'bottom' to the 'top' . . . (1962, p. 4)

Lakatos is of course famous for denigrating 'formalist' philosophies of mathematics, or at least insisting that there is a lot more to the philosophy of mathematics than 'formalist' approaches. As he explains in the introduction to *Proofs and Refutations* (1976b), formalists identify mathematics with its formalised axiomatic version; Carnap, Church, Peano, Russell and Whitehead are examples of formalists in this sense. As far as the EP goes, however, Lakatos builds no requirement of formality into it. Formalism, in the hands of some of its advocates, is a late nineteenth/early twentieth-century incarnation of Euclideanism. Yet the two should not be identified

[11] Pascal (1655, p. 396).

more generally, on pain of being blind to all pre-nineteenth-century forms of Euclideanism, which were non-formal.

In this, Lakatos is surely correct. Logicians today conceive of a theory as a set of sentences in a formal language that is closed under the deductive apparatus of some formal logic. Yet Euclidean theories can be formulated in natural languages, and until the late nineteenth century that is how they were formulated. So it would be entirely anachronistic to insist that a Euclidean theory must be formal.[12] To do so would rule out huge swathes of mathematics, and even reconstructions of mathematics, by definition.

Lakatos stresses a related point. Not only can entailment in a Euclidean theory be informal, it can also be non-logical (1962, p. 13). Subject-specific inferential rules and construction techniques may be taken to be perfectly legitimate components of informal deductions if certain conditions are met; for instance, if they are licensed by spatial intuition in geometry. If such modes of inference are ineliminably used in a proof, the conclusion is implied by the premises but does not follow from them in the deductive sense of modern formal logic. In order to cast a suitably wide historical net, this sort of implication should count as well on our reconstruction of the EP.

Just as significant is the fact that which entailments we count as logical depends on which background logic we use. However, it would be anachronistic to insist that all Euclidean theories employ the same background logic, since different Euclideans may disagree on the legitimacy of particular inference rules. For example, a more modern Euclidean would likely contest Aristotle's rule that from *All As are Bs* one can infer *Some A is B*. In short, we must not insist that a Euclidean theory is a set of *formal* sentences, nor that it is closed under a *formal* deduction relation.

4.1.8 Lakatos and the EP

Although we will not dwell on Lakatos's own assessment of the Euclidean picture, it is clear which side he is on. He thinks that '[f]rom the seventeenth to the twentieth century Euclideanism has been on a great retreat', and that rearguard attempts to 'break through beyond the hypotheses, towards the peaks of *first principles* ' have all failed. The upshot: '[t]he fallible sophistication of the empiricist programme has won, the infallible triviality of Euclideans has lost'. That said, the 'four hundred years of retreat seems to have by-passed mathematics', and Lakatos clearly sees his own role as being to wield the axe in this subject too.[13] The point of his most famous work in the philosophy of mathematics, *Proofs and Refutations*, is to show 'that informal, quasi-empirical, mathematics does not grow through a monotonous

[12] As Barnes highlights, ancient logicians did not even have the concept of a formal language (2005, p. 512).

[13] The quotations in this paragraph so far are from his (1962, p. 10). The theme of Euclidean theories' decline, especially outside mathematics, is repeated in his (1976a, p. 30).

increase of the number of indubitably established theorems [the Euclidean picture] but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations' (1976b, p. 5). In the words of Pi, one of that great book's Greek-alphabet-named characters: '[h]euristic is concerned with language-dynamics, while logic is concerned with language-statics' (1976b, p. 93). The latter could equally well apply to the Euclidean picture, which is static rather than dynamic. To present mathematical knowledge in static fashion, as an unchanging pyramidal-shaped system of immutable truths, is to belie it. Towards the end of *Proofs and Refutations*, its author comments:

> In deductivist style, all propositions are true and all inferences are valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. Counterexamples, refutations, criticism cannot possibly enter. An authoritarian air is secured for the subject by beginning with disguised monster-barring and proof-generated definitions and with the fully-fledged theorem, and by suppressing the primitive conjecture, the refutations, and the criticism of the proof. Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility. (1976b, p. 142)

Like Lakatos, we are also critical of the Euclidean Programme, though we do not have the space to discuss our criticisms of it here. These may be found in *The Euclidean Programme*.

We conclude this section by raising a more general sort of worry. One may criticise Lakatos's ambition to even discuss the EP in the relatively ahistorical way that he does. Perhaps it is historically insensitive to throw a single critical blanket over a great swath of the past.[14] Perhaps one should not try to capture the essence of Euclideanism in the way Lakatos tried to. Perhaps there is no such thing as Euclideanism, only a series of authors inspired by *The Elements* in different ways.

We have some sympathy with this complaint, but only up to a point. Clearly, different authors tempted by Euclideanism have stressed different points and added their individual imprint to its expression. Indeed, the body of work attributed to Euclid has varied across time and place, so that different Euclideans may have even drawn their inspiration from varied sources. That said, we believe there is an identifiable body of doctrine reasonably called 'The Euclidean Programme' that runs through the ages, even if it is not precisely defined and differs from writer to writer. As philosophers, we see our role as trying to identify these doctrines and, once identified, to assess them. Euclideanism is not at bottom different from the many other 'isms' philosophers blithely engage with in fairly ahistorical fashion—just within epistemology, think of coherentism, foundationalism, internalism, externalism, etc. If there is room for discussion of these 'isms' in a relatively abstract way, so should there be for Euclideanism.

A more sensitive approach might be to compare a rational reconstruction that tries to capture the centre of gravity of a body of thought—Euclideanism—within

[14] We owe this phrase to Brendan Larvor.

individual historical writers' conceptions. Although there is no space for the latter here, we have attempted it in our book, which compares the EP as an abstract methodological ideal with some historical authors: Aristotle, Euclid, Descartes, Pascal and other more recent ones. This leads on to the next section: having seen what Lakatos thinks the EP is, it is high time we say what we take it to be.

4.2 The Euclidean Programme in Seven Principles

Lakatos's discussion of the EP was instructive. Let's try to capture the general picture and the lessons learnt in a more structured way, and set aside the historical scruples just mentioned.

In *The Euclidean Programme*, we argue that the EP is characterised by seven principles. Three of these are core principles, which we take to be present in any historical manifestation of the Euclidean Programme worthy of the name (2024, p. 8). The other four principles are peripheral to the programme, manifesting themselves in many, but not all, occurrences of the EP throughout history (2024, p. 9).

4.2.1 Core Principles

The first core principle is that the axioms of a Euclidean theory are supposed to be true. We agree with Lakatos that this is essential to the EP, and see its inclusion as mandatory. All the major figures in the Euclidean tradition subscribe to some version of it: Aristotle (on one common interpretation), Descartes, and Pascal, to mention just a few.[15]

The second core principle of the EP in our reconstruction is that the truth of the axioms should be *self-evident*. While this is not something Lakatos focused on,[16] all the major Euclidean thinkers subscribe to this principle, or something very similar. As a distinctly foundationalist epistemology, the EP requires knowledge of the axioms to be completely secure and unmediated by inference. Given that this aspect of the programme is so historically well-attested, we shall not dwell on it here. Suffice it to say that we are sympathetic with Lakatos' assessment that our assurance of the truth of the axioms should be infallible in the context of a Euclidean theory.

[15] A difficult question is how Euclidean Euclid himself was. As we see it, the EP is a programme inspired by the methodology of the *Elements*, whether or not its author was what we would now call a Euclidean foundationalist. For a little more detail, see chapter 3 of *The Euclidean Programme*.

[16] Unless self-evidence is how we are supposed to understand his idea of 'triviality'.

Given our interpretation of the Euclidean Programme as a foundationalist epistemology of mathematical propositions, rather than an account of mathematical terms, we have no parallel to Lakatos' requirement that the primitive terms of a Euclidean theory be perfectly well-understood. At best, some requirement on the understanding of the terms can be inferred from the requirement that the axioms be self-evident. For if we cannot even understand what proposition is expressed by an axiom, then the truth of the proposition expressed can hardly be evident to us. But *perfect* understanding of the terms is a far stronger requirement than merely understanding them well enough to grasp the self-evidence of the axioms in which they appear.

The easiest way to appreciate this point is to think of simple logical propositions involving imperfectly understood terms. For example, it should be as obvious as can be that 'all democracies are democracies' is true. It is equally obvious that if 'all horses are ungulates' and 'all ungulates are mammals' are both true, then 'all horses are mammals' is true. One can appreciate this even if the terms 'democracies' and 'ungulates' (or even 'horses' and 'mammals') are not understood with perfect clarity. But we can go beyond logical truths. It is self-evident, for example, that a gallon of water is less than a gallon and a fluid ounce of water, even to those with only the haziest understanding of American units of measurement.

So on our picture, the mathematician must have an understanding of the primitive terms which is developed enough to allow them to understand the axioms. But their grasp of the terms may be less than perfect. If, consequently, their understanding of the axioms is less than perfect, this is permissible so long as the mathematician is still able to appreciate their self-evidence. To return to our hydraulic metaphor, there is no requirement that the water in the Phillips Machine be chemically pure.

This brings us to the third core Euclidean principle. In one respect, we take it to be Lakatos' greatest contribution to the study of the EP that he identifies the flow idea as an essential and defining principle of it, a principle understood in distinct ways by distinct philosophers and mathematicians working in the tradition. However, we part company with Lakatos in one key respect. In his reconstruction of the EP, the flow from axioms to theorems is of semantic content, such as truth and meaning. We think this is an unfaithful representation of the historical Euclidean ideal, where the emphasis has been squarely on epistemological issues.

In our reconstruction, the direction of flow is indeed downwards, from axioms to theorems. But what is inherited is an epistemological good, which one exactly varying from one manifestation of the EP to the next. In addition to an account of the relevant epistemic good, a particular manifestation of the EP must include a principle governing the flow or transmission of said good. In a strong version, the epistemic good (such as justification) is perfectly preserved from premises to conclusion; in a weaker version, it is more or less preserved. Since different Euclidean thinkers have had different ideas about the relation that the mathematician bears to the axioms, and to the theorems, of a theory, our reconstruction simply uses the placeholder relation E to represent a mathematician's having this epistemic good with respect to a proposition. We allow for degrees as well, as this is relevant to the epistemic good in some cases. So for example, $E(x, p, d)$ might mean that x has a justified belief to degree d in p.[17]

Our account of the core principles of the EP is therefore as follows:

EP-Truth All axioms and theorems are true.

EP-Self-Evidence All axioms are self-evident. If a subject clearly grasps a self-evident proposition then she bears E to it to the maximal degree.

EP-Flow If a conclusion is deducible from some premises, and the subject clearly grasps this, and bears E to these premises to a high degree, she thereby bears E to the conclusion to the same, or a similarly high, degree.

The point of this reconstruction is to enable the systematic comparison of a range of historical views, both to the abstract prototype of the Euclidean view, and to one another. This has two important implications. The first is that the actual views of historical figures will inevitably diverge to some extent from the reconstructed ideal we have presented. That ideal is supposed to represent the views of multiple philosophers and mathematicians, which of course differ from one another. Consequently, in any historical manifestation of the EP, there will be extra details that do not appear in our account, details that will be fleshed out in slightly different ways, and numerous other small discrepancies. But we hope to have avoided the potential charge of anachronism or caricature, which, as Lakatos warns us, is often levelled at these kinds of projects by '[r]espectable historians' (1962, p. 4).

The second implication is that, since the reconstructed ideal of the EP is indeed supposed to represent these actual historical views (albeit imperfectly), we do expect our rational reconstruction to bear significant similarities with the specific historical manifestations of the programme. The reason we have identified the above principles as core is that, despite the disagreements on detail between the Euclidean thinkers of the past, all of them subscribe to some version of these three. We take them to be an accurate, if not exhaustive, characterisation of any species of the EP worth its salt.

4.2.2 Peripheral Principles

In order to try and account for popular, though less significant, currents in the history of the EP, we also give the following four peripheral principles:

EP-Finite The axioms are finitely many.

[17] Chapters 3–6 of *The Euclidean Programme* discuss several more specific accounts of this relation and how it might be taken to flow in the accounts of particular thinkers.

EP-General	All axioms are general propositions.
EP-Independence	Each axiom is independent of the others.
EP-Completeness	All truths of a certain kind can be deduced from the axioms.

Most Euclideans have historically subscribed to at least some of these, although we do not take them to characterise an essential aspect of Euclidean foundationalism. We now explain these peripheral principles, some of which are familiar from Lakatos, some of which are not.

We are happy to follow Lakatos in including *EP-Finite* as part of the EP, but only as a peripheral principle and with the caveat mentioned in the previous section. We do not take the relevant sense of finitude quite so literally as Lakatos, since schematic theories can have a finite presentation, despite having an infinite number of axioms. When we are dealing with historical theories that were formulated prior to the development of modern formal logic, it is anachronistic to ask whether the theory is 'really' a first-order theory with axiom schemata, or a second order theory, where a principle which would be schematic in the first-order context is formulated as a single axiom (e.g. the induction principle in arithmetic). But it seems to us that, for the purposes of Euclidean epistemology, a theory with a finite number of schemata is on a par with a truly finite theory. So we understand this principle as requiring only that a theory have a finite presentation. And the principle is only peripheral, since important Euclideans do not endorse it (see Sect. 4.1.4).

EP-General is also a popular principle amongst historical Euclideans. Although it is not a feature of the Euclidean Programme as Lakatos reconstructs it, we include it in our characterisation due to its prevalence in the history of the EP. It passes muster only as a peripheral principle, however, due to the existence of prevalent Euclideans who appear to deny it.[18]

We confess that it is not immediately clear what the requirement of generality actually amounts to. The logical form of a statement is not a particularly helpful guide here, since a (logically) singular statement, such as 'London is a city' can be made general in a narrow sense simply by being prefixed with a redundant universal quantifier. Nonetheless, the axioms of standard mathematical theories are general in a recognizable sense. For example, the standard axioms of arithmetic, such as that the natural numbers are closed under the successor relation, and that distinct numbers have distinct successors, clearly do not concern particular individuals. And even when an individual is mentioned, for instance 0, the singular term can be eliminated in favour of a definition using only general vocabulary (in this case, "the only natural number x such that $x+x=x$"). So we understand *EP-General* as requiring that the axioms include only general vocabulary, and expressions that are definable in terms of it.[19] This certainly accords with axiomatic theories as

[18] For example, see pp. 29–30 of *The Euclidean Programme* for our argument that Descartes does not subscribe to this principle.

[19] Of course, we have not actually defined the notion of general vocabulary. But that is a job for particular Euclideans, not for us.

they are usually found. A Euclidean-style presentation of geometry in which one of the axioms is about the properties of a triangle of sides 3, 4 and 5 units is not inconceivable; but it would be eccentric, and quite different from the usual historical practice.

EP-Independence requires that each axiom of the theory is not logically redundant. (In other words, no axiom is derivable from the rest of the axioms.) Like *EP-General*, this principle is not one that Lakatos builds into his reconstruction of the EP, so it is worth saying something to justify its inclusion. The most historically significant episode relating to independence concerns the status of Euclid's Parallel Postulate.[20] From ancient times until modern, a number of mathematicians attempted to prove this postulate from the other four. An interesting feature of these attempts is that the mathematicians were *already* convinced that the postulate was true. Moreover, they do not, in the main, seem to have thought that the Parallel Postulate wasn't evident or obvious. The issue was simply that they thought that a proof *could* be, and so *should* be given. As Proclus (one of Euclid's prominent commentators) puts it, '[the Parallel Postulate's] obvious character does not appear independently of demonstration but is turned by proof into a matter of knowledge' (1970, p. 151). There is evidence, therefore, of a historical current in thinking about mathematics which requires that no provable proposition is included as an axiom.[21] Much as with Lakatos's finitude requirement, independence is not discussed or endorsed by all the Euclidean theorists we consider. And independence problems have been of significant interest outside the Euclidean tradition too, for example in Hilbert's work, simply for their purely mathematical (as opposed to epistemological) significance. Thus we take *EP-Independence* to be a merely peripheral component of the programme.

EP-Completeness says that *all* truths in some important class can be deduced from the axioms. Although the issue is not prominent in Lakatos's characterisation of the EP, he is clearly aware of its presence in Euclidean thought generally; for example, he highlights that the Euclidean believes '*all knowledge* can be deduced from a finite set of trivially true propositions' (1962, p. 4, our emphasis). This characterisation is ambiguous (as we'll see below) and Lakatos does not return to it in the later 'Renaissance' article. There he writes only that the axioms prove 'the rest of the system' (1976a, p. 29), though he does discuss completeness in (what he sees as) some specific manifestations of the EP, such as logicist foundations for mathematics and Hilbert's finitist programme.

Completeness is properly seen as a schematic requirement. It can be understood in various ways corresponding to different understandings of which class of truths the axioms must be complete with respect to. A weak, though still mathematically

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20 In Heath's translation, the postulate is: 'That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles' (1968, p. 155).

21 We discuss the case in of the Parallel Postulate in more detail on pp. 10–12 of *The Euclidean Programme* (2024).

significant, understanding of this requirement is that the axioms should be complete with respect to the known truths in the relevant area of mathematics. Completeness in this sense facilitates the achievement of what Russell described as 'an organization of our knowledge, making it more manageable and more interesting' (1907, p. 580). A stronger principle demands completeness with respect to the knowable truths of the relevant mathematical discipline (although the notion of *knowability* here stands in need of clarification). The strongest completeness requirement is that every truth in the relevant mathematical domain is derivable from the axioms.

We include the (schematic) completeness principle in our reconstruction of the EP because of its historical prominence, particularly amongst canonical Euclideans such as Descartes.[22] Despite this, not all significant Euclideans subscribe to this principle, and indeed it is not a distinctively Euclidean principle at all. Even its strongest version is subscribed to by figures whose classification as Euclideans is best resisted, for example Hilbert and Kant.[23] Hence we include it only as a subsidiary principle.[24]

Our reconstruction of the EP does not include a principle stating that theorems are *dependent* on the axioms from which they are derived. It isn't clear to us whether Lakatos intended this principle to be included in his reconstruction of the EP, though something like it is perhaps suggested by his talk of truth flowing from the axioms to the theorems. And the idea is of course a historically prominent one. Frege famously claimed that the aim of proof was 'to afford us insight into the dependence of truths upon one another' (1884, Sect. 2). This relationship of dependency (*Abhängigkeit*) is supposed to be an objective matter, and similar views can be found in earlier writers as far back as Aristotle. But this relationship of dependency is clearly metaphysical,[25] concerning as it does a relation that holds between truths independently of our epistemological stance toward them. However, the EP is an epistemology, rather a metaphysics, of mathematics. Just as we resist Lakatos' attempt to substitute semantic notions for those that, in the EP, should be epistemological, so too we resist the importation of anything metaphysical into our picture, to whatever extent that is possible. While the dependency of theorems on axioms may have appealed to a number of Euclidean thinkers in the past, we do not reflect this with the inclusion of a relevant EP-principle, not even a peripheral one.

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[22] See p. 29 of *The Euclidean Programme* (2024).

[23] See, for example, (1787, A480/B508) for remarks by Kant, and (1902, p. 445) for remarks by Hilbert.

[24] Evaluating the plausibility of *EP-Completeness* would take us too far afield here, but it is tackled in Sect. 8.4 of *The Euclidean Programme* (2024).

[25] As emphasised by Shapiro (2009, p. 183), for instance.

4.3 Conclusion

With Lakatos's help, we reconstructed the EP. His idea that what matters is *how* some theoretical good flows from axioms to theorems, not *what* flows, was key to this reconstruction. In other ways, we parted company with Lakatos, for the reasons given. The next thing to do would be to assess the EP in light of developments in contemporary epistemology and contemporary mathematics. We take that next step in *The Euclidean Programme*, and also compare and contrast the ahistorical EP with some flesh-and-blood authors.

Lakatos, as we have had occasion to mention, was strongly opposed to the EP. But he was clear-sighted enough to recognise it as a formidable opponent. Well-versed in Popper's philosophy, he knew how hard existential claims are to refute. We will let him have the last word:

> A Euclidean never *has* to admit defeat: his programme is irrefutable. One can never refute the pure existential statement that there exists a set of trivial first principles from which all truth follows. Thus science may be haunted for ever by the Euclidean programme as a regulative principle, 'influential metaphysics'. A Euclidean can always deny that the Euclidean programme as a whole has broken down when a particular candidate for a Euclidean theory is tottering. (1962, pp. 6–7)

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