

Kristóf Bérczi

Eötvös Loránd University Budapest, Hungary kristof.berczi@ttk.elte.hu

László Lovász

HUN-REN Alfréd Rényi Institute of Mathematics Budapest, Hungary laszlo.lovasz@ttk.elte.hu

Boglárka Gehér

Eötvös Loránd University Budapest, Hungary bogigeher@gmail.com

Balázs Maga

HUN-REN Alfréd Rényi Institute of Mathematics Budapest, Hungary magab@renyi.hu

András Imolay

Eötvös Loránd University Budapest, Hungary andras.imolay@ttk.elte.hu

Tamás Schwarcz

London School of Economics and Political Science London, United Kingdom tamas.schwarcz@ttk.elte.hu

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1 Introduction

Coupling of probability measures, generally attributed to Doeblin [11], is a key concept in probability theory through which random variables can be compared with each other. For two probability measures, a *coupling* is a joint probability measure over the product of the underlying spaces, such that the marginals correspond to the given probability measures. Formally, if μ_1 and μ_2 are probability measures on spaces (S_1, \mathcal{B}_1) and (S_2, \mathcal{B}_2) , respectively, a coupling is a joint probability measure μ on the product space $(S_1 \times S_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ such that the marginal of μ on S_i is μ_i for i = 1, 2, ...i.e., $\mu(X_1 \times S_2) = \mu_1(X_1)$ for any $X_1 \in \mathcal{B}_1$ and $\mu(S_1 \times X_2) = \mu_2(X_2)$ for any $X_2 \in \mathcal{B}_2$. Usually, the goal is to choose a coupling that allows for comparisons between the probability measures, such as minimizing differences or distances between them. Coupling techniques have found applications in various areas. In optimal transport theory, the goal is to find the most efficient way to transform one distribution into another, with applications in economics, statistical physics, and machine learning [9]. In stochastic processes, coupling helps to describe the joint behavior of random variables and to analyze and bound convergence rates in chains with more complex dependencies [1, 24].

Interestingly, a similar notion has also appeared in a combinatorial line of research on matroids. Let $M_1 = (S_1, r_1)$ and $M_2 = (S_2, r_2)$ be finite matroids over ground sets S_1 and S_2 with rank functions r_1 and r_2 , respectively. A matroid $M = (S_1 \times S_2, r)$ is called a *quasi-product* of M_1 and M_2 if the restriction of M to $\{x\} \times S_2$ is isomorphic to M_2 by the natural bijection with S_2 for all non-loops $x \in S_1$ and is the zero matroid for loops x, and analogously with the two factors interchanged. If furthermore $X \times Y$ is a flat of M for all flats X of M_1 and Y of M_2 , then M is called a *product* of M_1 and M_2 . Finally, if $r(X \times Y) = r_1(X) \cdot r_2(Y)$ holds for every $X \subseteq S_1, Y \subseteq S_2$, then M is a *tensor product* of M_1 and M_2 . Note that at this point, it is not

Abstract

The study of matroid products traces back to the 1970s, when Lovász and Mason studied the existence of various types of matroid products with different strengths. Among these, the tensor product is arguably the most important, which can be considered as an extension of the tensor product from linear algebra. However, Las Vergnas showed that the tensor product of two matroids does not always exist. Over the following four decades, matroid products remained surprisingly underexplored, regaining attention only in recent years due to applications in tropical geometry, information theory, and the limit theory of matroids.

In this paper, inspired by the concept of coupling in probability theory, we introduce the notion of coupling for matroids – or, more generally, for submodular set functions. This operation can be viewed as a relaxation of the tensor product. Unlike the tensor product, however, we prove that a coupling always exists for any two submodular functions and can be chosen to be increasing if the original functions are increasing. As a corollary, we show that two matroids always admit a matroid coupling, leading to a novel operation on matroids. Our construction is algorithmic, providing an oracle for the coupling matroid through a polynomial number of oracle calls to the original matroids.

We apply this construction to derive new necessary conditions for matroid representability and establish connection between tensor products and Ingleton's inequality. In addition, we verify the existence of set functions that are universal with respect to a given property, meaning any set function over a finite domain with that property can be obtained as a quotient.

CCS Concepts

• Mathematics of computing \rightarrow Matroids and greedoids; • Theory of computation \rightarrow Submodular optimization and polymatroids; Discrete optimization.

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This work is licensed under a Creative Commons Attribution 4.0 International License. *STOC '25, Prague, Czechia* © 2025 Copyright held by the owner/author(s). ACM ISBN 979-8-4007-1510-5/25/06 https://doi.org/10.1145/3717823.3718152 clear whether any two matroids admit a tensor product, a product, or even a quasi-product. Mason [28] and Lovász [25] provided constructions yielding a product of rank $r_1(S_1) + r_2(S_2) - 1$ via Dilworth truncation. The existence of a tensor product, however, was asked as an open question in [25]. Note that if both M_1 and M_2 are linear matroids over the same field, then it is easy to construct a tensor product by taking the tensor product of the matrices representing them. Nevertheless, Las Vergnas [23] later showed that the situation is generally not so fortunate, as, for example, the Vámos matroid and the rank-2 uniform matroid on three elements do not admit a tensor product.

One can think of the tensor product of two matroids as a construction where the product of independent sets is itself independent. However, since a tensor product does not exist for every pair of matroids, Mason [29] raised the question of whether a freest product exists among the possible products. Las Vergnas [23] answered this question in the negative and showed that the matroid obtained via Dilworth truncation as in [25, 29] is the freest quasi-product among those satisfying $r(\{x, x'\} \times \{y, y'\}) \leq 3$ for all $x, x' \in S_1$ and $y, y' \in S_2$, where r denotes the rank function of the product matroid.

1.1 Our Results and Techniques

In this paper, we extend the concept of coupling to matroids and, more broadly, to submodular functions. We prove that, unlike tensor products, couplings always exist, leading to a novel operation on matroids. This result is particularly interesting because it shows that a finite number of matroids can be encoded into a single one, while also taking into account how these matroids interact with each other. As an application, we establish the existence of functions that are universal with respect to some property, meaning that any set function with that property can be obtained as a quotient. This result provides new insights into the developing limit theory of matroids and submodular functions.

Let $\varphi_1: 2^{S_1} \to \mathbb{R}$ and $\varphi_2: 2^{S_2} \to \mathbb{R}$ be set functions defined over finite ground sets S_1 and S_2 , respectively. The notion of a tensor product can be naturally extended to set functions by calling a function $\varphi: 2^{S_1 \times S_2} \to \mathbb{R}$ a tensor product of φ_1 and φ_2 if $\varphi(X_1 \times X_2) = \varphi_1(X_1) \cdot \varphi_2(X_2)$ holds for every $X_1 \subseteq S_1$ and $X_2 \subseteq S_2$. As a weaker concept, we call φ a *coupling* of φ_1 and φ_2 if $\varphi(X_1 \times S_2) = \varphi_1(X_1) \cdot \varphi_2(S_2)$ for every $X_1 \subseteq S_1$ and $\varphi(S_1 \times X_2) = \varphi_1(S_1) \cdot \varphi_2(X_2)$ for every $X_2 \subseteq S_2$. In other words, a coupling is a set function on the product set, where the projections onto each coordinate return the corresponding φ_i , up to a constant multiplier. It is worth emphasizing that neither a tensor product nor a coupling is uniquely determined from the factors φ_1 and φ_2 .

Motivated by the goal of defining the coupling of matroids, Section 3 focuses on submodular functions. First, we show that any two submodular functions admit a submodular coupling (Theorem 3.1). One remarkable feature of the proof is that it provides an explicit formula for the coupling function using two arbitrary modular functions. However, the proof itself does not imply that if both φ_1 and φ_2 are monotonically increasing, then the coupling function φ is also monotonically increasing. Since this property is essential for the coupling of two matroids to result in a matroid, next we show that a k_1 -polymatroid function and a k_2 -polymatroid function have

a $(k_1 \cdot k_2)$ -polymatroid coupling, which is also integer-valued if the original functions are (Theorem 3.2); see Section 2 for definitions. The proof follows a similar idea as that of the submodular case, but the modular functions in question need to be chosen to be elements in the base polyhedra of φ_1 and φ_2 , respectively. As a corollary, we get that any two matroids admit a coupling (Corollary 3.6) which in turn implies that there exists a finite matroid that contains every matroid of fixed rank over a ground set of fixed size (Corollary 3.7).

We provide several characterizations of two matroids having a coupling that is also a tensor product (Theorem 3.8), and establish a strong connection between tensor products and Ingleton's inequality, a fundamental tool in the theory of representable matroids (Theorem 3.11). As a corollary, we get a new proof for the fact that the uniform matroid $U_{2,3}$ and the Vámos matroid do not admit a tensor product.

We separately study a subclass of increasing submodular functions with strong structural properties, called "coverage functions" in combinatorial optimization. These functions were originally introduced by Choquet for analytical studies. He defined them in terms of a sequence of inequalities strengthening submodularity, and proved their equivalence with what is now the combinatorial definition. We prove that a coverage function and an increasing submodular function have an increasing submodular tensor product (Corollary 3.14), and that any two coverage functions have a tensor product that is a coverage function (Corollary 3.15). The proofs rely on the characterization of the extreme rays of the cone of coverage functions, as established by Choquet. Finally, we describe how to extend our results on submodular functions defined on finite ground sets to those with infinite domains (Theorems 3.16 and 3.17).

Let φ be a set function over some finite ground set *S*, and let $Q = (S_1, \ldots, S_q)$ be a partition of *S* into *q* possibly empty parts. The *quotient* of φ with respect to *Q* is a set function φ_Q over $S_Q = \{s_1, \ldots, s_q\}$ defined by $\varphi_Q(X) = \varphi(\bigcup_{s_i \in X} S_i)$ for $X \subseteq S_Q$. In particular, if $\varphi: 2^{S_1 \times S_2} \to \mathbb{R}$ is a coupling of $\varphi_1: 2^{S_1} \to \mathbb{R}$ and $\varphi_2: 2^{S_2} \to \mathbb{R}$ and Q_1 and Q_2 are the partitions of $S_1 \times S_2$ into fibers of the form $\{x\} \times S_2$ and $S_1 \times \{y\}$, respectively, then $\varphi_1 = \varphi_{Q_1}/\varphi_2(S_2)$ and $\varphi_2 = \varphi_{Q_2}/\varphi_1(S_1)$. The positive results of Section 3 motivate the following question: Given a property of set functions, is there a function that is universal with respect to that property, in the sense that every function exhibiting the property can be obtained as its quotient?

In Section 4, we examine this problem and show that, under some natural assumptions, the answer is positive (Theorem 4.2). Although the proof is analytical and may therefore be less appealing to those interested in combinatorial optimization, the result has far-reaching implications for functions defined on finite ground sets as well. For example, we obtain that there exists a submodular function $\varphi: 2^{\mathbb{Z}} \rightarrow [0, 1]$ such that any nonnegative normalized submodular function is a quotient of φ (Corollary 4.3), and we prove an analogous result for coverage functions as well (Corollary 4.4).

It is worth emphasizing that for functions over finite ground sets, our proofs are algorithmic in the sense that given value oracles for φ_1 and φ_2 , one can construct a value oracle for their coupling φ with the desired properties in polynomial time.

1.2 Related Work and Motivation

Understanding how matroids can be combined does not only help in studying properties of larger systems formed from simpler matroid components, but it is closely related to a range of problems and techniques in combinatorial optimization. In what follows, we give a brief overview of related topics.

Tropical Geometry. Tensor products of matroids have deep connections to tropical geometry. A key consequence of Las Vergnas's counterexample is that the tensor product of tropical linear spaces does not always yield a tropical linear space, see e.g. [18]. The notion of tensor products can be extended to more than two matroids. If all k matroids in the product are identical to a given matroid M = (S, r), the tensor product defines a matroid structure on the set of ordered *k*-tuples from *S*, known as the *k*-th power of *M*. For unordered k-tuples, we consider symmetric tensors, which remain invariant under any permutation of their variables. The concept of symmetric powers of matroids was first introduced by Lovász [25] and Mason [29], who noted that not all matroids admit higher symmetric powers. Draisma and Rincón [14] established a link between tropical ideals and matroid symmetric powers by showing that the Bergman fan of the direct sum of the Vámos matroid and $U_{2,3}$ is not a tropically realizable variety. In [2], Anderson proved an equivalence between valuated matroids with arbitrarily large symmetric powers and tropical linear spaces represented as varieties of tropical ideals. Brakensiek, Dhar, Gao, Gopi, and Larson [8] explored the connection between rigidity matroids of graphs and matroids arising from linear algebraic constructions like tensor products and symmetric products. Matroid tensor products in tropical geometry were also examined in [15, 20].

Linear Matroids. A matroid is called linear if its independent sets can be represented by vectors in a vector space over some field \mathbb{F} . Such matroids are particularly interesting because they allow for a rich connection between linear algebra and combinatorial structures. For deciding if a matroid is regular, i.e., representable over every field, Seymour's decomposition theorem [32] implies an algorithm, as such decompositions can be found efficiently [33]. However, Truemper [33] showed that many representability questions cannot be efficiently solved. This includes deciding representability over a specific field, over all fields with a given characteristic, and, most importantly, over any field. Since determining representability is oracle-hard in general, the focus of research has shifted toward finding necessary or sufficient conditions. Ingleton's inequality [19] is one of the most prominent examples, giving a necessary condition for a matroid to be linear. Additional necessary conditions, known as extension properties, have also been extensively studied; see, e.g., [3, 4]. Surprisingly, we show that the representability of a matroid is closely related to having a tensor product with $U_{2,3}$.

Limits of Matroids. The limit theory of graphs provides powerful tools for analyzing sequences of graphs and their structural similarities through analytic methods. By defining convergence in terms of distributions of small subgraphs (left-convergence) and homomorphisms into small graphs (right-convergence), it aids in understanding complex networks and their applications in various fields of mathematics and computer science; we refer the interested reader to [26] for a thorough introduction. In [5], a new form of right-convergence called *quotient-convergence* was introduced for set functions, which eventually led to a notion of convergence of matroids through their rank functions. The limit object of such a sequence is a submodular function [27]. One of the main research subjects is understanding the structure and properties of these limit objects, which heavily relies on examining their quotients and the coupling of the functions in the sequence.

1.3 Organization

The rest of the paper is organized as follows. In Section 2, we introduce basic definitions, notation, and relevant results on submodular functions, matroids, and polymatroids. Section 3 is devoted to verifying the existence of couplings for various functions, first focusing on submodular functions in Section 3.1 and then extending these results to polymatroid functions in Section 3.2. In Section 3.3, we show how these observations lead to one of the main results of the paper: the existence of matroidal couplings that are almost tensor products. Section 3.3.1 explores applications of this result. To understand the fine line between couplings and tensor products, we establish a necessary and sufficient condition for a coupling to be a tensor product in Section 3.3.2. Finally, in Section 3.3.3, we provide new necessary conditions for matroid representability. As an application, we verify the existence of functions that are universal with respect to certain properties in Section 4. The proof is analytical and hence is not constructive; we encourage first-time readers to skip the technical parts of Sections 4.1 and 4.2. As an application of the abstract existence theorem, we deduce that universal functions exist for several set function properties; these examples can be found in Section 4.3.

2 Preliminaries

Basic Notation. We denote the sets of *reals* and *integers* by \mathbb{R} and \mathbb{Z} , and add + or > 0 as a subscript when considering *nonnegatives* or *strictly positive* values only. The *cardinality of* \mathbb{R} is denoted by c. For a positive integer k, we use $[k] := \{1, \ldots, k\}$ while $[0] = \emptyset$ by convention. Given a ground set *S*, a subset $X \subseteq S$ and $y \in S$, the sets $X \setminus \{y\}$ and $X \cup \{y\}$ are abbreviated as X - y and X + y, respectively. Moreover, we often denote a single element set $\{x\}$ by x when this causes no confusion. The *complement* of X is denoted by $X^c = S \setminus X$. Given a function $\mu: S \to \mathbb{R}_+$, we use the notation $\mu(X) := \sum_{x \in X} \mu(x)$. If $S = S_1 \times S_2$ for some sets S_1 and S_2 , then for any $Z \subseteq S$, $x \in S_1$ and $y \in S_2$, we use $Z_x = Z \cap (\{x\} \times S_2)$ and $Z^y = Z \cap (S_1 \times \{y\})$ to denote the x- and y-fibers of Z, respectively. Furthermore, we use $\pi_i: S \to S_i$ for denoting the *coordinate maps*, that is, $\pi_1(Z) = \{x \in S_1 \mid Z_x \neq \emptyset\}$ and $\pi_2(Z) = \{y \in S_2 \mid Z^y \neq \emptyset\}$.

Matroids. For basic definitions on matroids, we refer the reader to [30]. A *matroid* M = (S, r) is defined by its finite *ground set* S and its *rank function* $r: 2^S \to \mathbb{Z}_+$ that satisfies the *rank axioms*: (R1) $r(\emptyset) = 0$, (R2) $X \subseteq Y \Rightarrow r(X) \leq r(Y)$, (R3) $r(X) \leq |X|$, and (R4) $r(X)+r(Y) \geq r(X \cap Y)+r(X \cup Y)$. Here, (R1) is a standard normalizing assumption, while (R2) requires the rank function to be increasing, (R3) its subcardinality, and (R4) its submodularity. The *rank* of the matroid is r(S), and by the *normalized rank function* of M we mean the set function r/r(S). A subset $X \subseteq S$ is called *independent* if |X| = r(X). An inclusionwise minimal non-independent set forms a *circuit*, while a *loop* is a circuit consisting of a single element. A *flat* is a set $F \subseteq S$ such that r(F+e) = r(F) + 1 holds for every $e \in S \setminus F$. A matroid is called *modular* if $r(F_1) + r(F_2) = r(F_1 \cap F_2) + r(F_1 \cup F_2)$ for any pair of flats F_1, F_2 . For sets $X, Y \subseteq S$, we say that X spans Y in M if $r(X \cup Y) = r(X)$.

Given a subset $S' \subseteq S$, the *restriction* of M to S' is a matroid M|S' = (S', r') whose rank function is the restriction of r to subsets of S'. The *contraction* of S' results in a matroid $M/S' = (S \setminus S', r')$ with rank function $r'(X) = r(X \cup S') - r(S')$ for each $X \subseteq S \setminus S'$. A matroid N that can be obtained from M by a sequence of restrictions and contractions is called a *minor* of M. The *direct sum* $M_1 \oplus M_2$ of matroids $M_1 = (S_1, r_1)$ and $M_2 = (S_2, r_2)$ on disjoint ground sets is the matroid $M = (S_1 \cup S_2, r)$ whose independent sets are the disjoint unions of an independent set of M_1 and an independent set of M_2 , that is, $r(X) = r_1(X \cap S_1) + r_2(X \cap S_2)$ for $X \subseteq S_1 \cup S_2$.

A matroid is *linear* or *representable* over some field \mathbb{F} if there exists a family of vectors from a vector space over \mathbb{F} whose linear independence relation is the same as the independence relation of the matroid. If the matroid is representable over any field, then it is called *regular*. A *uniform matroid* of rank *r* over a ground set of size *n* has rank function $r(X) = \min\{|X|, r\}$ and is denoted by $U_{r,n}$. Given pairwise disjoint sets $S_1 \cup \cdots \cup S_q \subseteq S$, we define the corresponding *partition matroid* M = (S, r) by setting $r(X) = |\{i \mid X \cap S_i \neq \emptyset\}|$ for all $X \subseteq S$. It is not difficult to see that both uniform and partition matroids are representable over the reals.

Submodular and Polymatroid Functions. By S, we always denote a finite ground set. A set function $\varphi: 2^S \to \mathbb{R}$ is normalized if $\varphi(S) = 1$. We say that φ is increasing if $X \subseteq Y$ implies $\varphi(X) \leq \varphi(Y)$, and decreasing if $X \subseteq Y$ implies $\varphi(X) \geq \varphi(Y)$. The function φ is submodular if

$$\varphi(X) + \varphi(Y) \ge \varphi(X \cap Y) + \varphi(X \cup Y)$$

for all *X*, *Y* \subseteq *S*, *supermodular* if $-\varphi$ is submodular, and *modular* if it is both sub- and supermodular.

A function $\varphi: 2^S \to \mathbb{R}_+$ is called a *polymatroid function* if it is an increasing, submodular function with $\varphi(\emptyset) = 0$. Furthermore, if $\varphi(X) \leq k \cdot |X|$ holds for some $k \in \mathbb{R}_+$ and for every subset $X \subseteq S$, then φ is a *k*-polymatroid function. Observe that polymatroid functions retain three of the four basic properties of matroid rank functions: (R1), (R2), and (R4), omitting only the subcardinality and integrality requirement. Thus, matroid rank functions form a subclass of polymatroid functions.

For a nonnegative submodular function $\varphi: 2^S \to \mathbb{R}_+$, its *base polyhedron* is defined as

$$B(\varphi) := \{ x \in \mathbb{R}^{S}_{+} \mid x(S) = \varphi(S), \ x(Z) \le \varphi(Z) \text{ for every } Z \subseteq S \}$$

The following is a fundamental result in polyhedral combinatorics, see e.g. [17, Corollary 14.2.3].

PROPOSITION 2.1. If φ is a nonnegative submodular function, then $B(\varphi)$ is non-empty. Furthermore, if φ is integer-valued, then the vertices of $B(\varphi)$ are integer-valued.

Coverage Functions. A coverage function $\varphi: 2^S \to \mathbb{R}_+$ is defined by a bipartite graph G = (S, T; E) and weights $w \in \mathbb{R}_+^T$ by setting $\varphi(X) = \sum_{t \in N(X)} w_t$ for $X \subseteq S$, where N(X) denotes the set of neighbours of X in G. It is less known that coverage functions also have an alternative characterization involving inequalities [7]. Specifically, φ is a coverage function if and only if $\varphi(\emptyset) = 0$ and, for any $k \in \mathbb{Z}_{>0}$,

$$\sum_{K \subseteq [k]} (-1)^{|K|} \varphi \Big(A_0 \cup \bigcup_{i \in K} A_i \Big) \le 0 \qquad (k-\text{Alt})$$

holds for any choice of subsets $A_0, A_1, \ldots, A_k \subseteq S$. It is worth noting that for k = 1 and k = 2, inequality (*k*-ALT) is equivalent with the property that φ is increasing and increasing submodular, respectively. In general, for $k \geq 3$, the inequality describes a strengthened form of increasing submodularity. For any non-empty $A \subseteq S$, let $\varphi_A : 2^S \to \mathbb{R}$ be defined as

$$\varphi_A(X) = \begin{cases} 0 & \text{if } X \cap A = \emptyset, \\ 1 & \text{if } X \cap A \neq \emptyset. \end{cases}$$
(1)

It is not difficult to see that φ_A is a normalized coverage function. Choquet [10, Section 43] showed that these set functions correspond to the extreme rays of the convex cone of coverage functions; see also [27] for a different proof.

PROPOSITION 2.2 (CHOQUET). The set of normalized extremal elements of the convex cone of coverage functions over a finite set S is $\{\varphi_A \mid \emptyset \neq A \subseteq S\}.$

For further details on coverage functions, we recommend [7, 10, 27].

From Finite to Infinite Domain. The above definitions extend naturally to set functions over an infinite domain. For clarity, we denote the ground set by J rather than S whenever it is infinite. In this context, let (I, \mathcal{B}) be a set algebra, i.e., \mathcal{B} is a family of subsets of *J* with $\emptyset \in \mathcal{B}$ and closed under taking complements and finite unions. Note that these imply that \mathcal{B} is also closed under taking finite intersections. A set function φ assigns a real value to each member of \mathcal{B} . By a slight abuse of terminology, we call the members of \mathcal{B} measurable. For set algebras (J_1, \mathcal{B}_1) and (J_2, \mathcal{B}_2) , a function $f: J_1 \to J_2$ is measurable if $f^{-1}(B) \in \mathcal{B}_1$ for all $B \in \mathcal{B}_2$. For a set algebra (J, \mathcal{B}) , we call a nonempty set $B \in \mathcal{B}$ an *atom* of the algebra if $A \subseteq B, A \in \mathcal{B}$ implies $A = \emptyset$ or A = B. We refer to a finitely additive measure, which is not necessarily nonnegative, as a charge. Properties of set functions, such as being normalized, increasing, decreasing, submodular, supermodular, modular, and being a polymatroid can be generalized to functions over an infinite domain in a straightforward manner. Moreover, using the inequality-based definition, this is also doable for coverage functions. We call a set function finite if its domain is finite, and a set function is bounded if the range of the set function is bounded.

Let (J_1, \mathcal{B}_1) and (J_2, \mathcal{B}_2) be measurable spaces, and let $J = J_1 \times J_2$. The product algebra \mathcal{B} on J is defined as the algebra generated by $\{X_1 \times J_2 \mid X_1 \in \mathcal{B}_1\} \cup \{J_1 \times X_2 \mid X_2 \in \mathcal{B}_2\}$, and it is denoted by $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$, see e.g. [16]. We refer to the measurable space (J, \mathcal{B}) thus obtained as the product of measurable spaces (J_1, \mathcal{B}_1) and (J_2, \mathcal{B}_2) . Let $\varphi_1 : \mathcal{B}_1 \to \mathbb{R}$ and $\varphi_2 : \mathcal{B}_2 \to \mathbb{R}$ be set functions. Then, $\varphi : \mathcal{B} \to \mathbb{R}$ is a coupling of φ_1 and φ_2 if $\varphi(X_1 \times J_2) = \varphi_1(X_1) \cdot \varphi_2(J_2)$ for every $X_1 \in \mathcal{B}_1$ and $\varphi(J_1 \times X_2) = \varphi_1(J_1) \cdot \varphi_2(X_2)$ for every $X_2 \in \mathcal{B}_2$. Similarly to the finite case, for $x \in J_1, y \in J_2$ and $Z \in \mathcal{B}$, we use Z_x and Z^y for denoting the x- and y-fibers of Z, and π_1, π_2 denotes the coordinate maps. By $\mathbb{1}_Z$ we denote the indicator function of a set Z.

3 Coupling Submodular Functions

In this section we show that, unlike the tensor product, two matroids always admit a coupling. We first prove this statement for nonnegative submodular functions in Section 3.1 and then extend the discussion to polymatroid functions in Section 3.2. Matroids and the relation between coupling and tensor products are addressed in Section 3.3. We study k-alternating and coverage functions in Section 3.4. Finally, in Section 3.5 we explain how all these results can be extended to functions over an infinite ground set.

3.1 Submodular Functions

As a first step, we verify that couplings exist for submodular functions. Let $\varphi_1: 2^{S_1} \to \mathbb{R}_+$ and $\varphi_2: 2^{S_2} \to \mathbb{R}_+$ be submodular functions over ground sets S_1 and S_2 , and let $S = S_1 \times S_2$. Recall that for any $Z \subseteq S$, $x \in S_1$ and $y \in S_2$, we use $Z_x = Z \cap (\{x\} \times S_2)$ and $Z^y = Z \cap (S_1 \times \{y\})$ to denote the *x*- and *y*-fibers of *Z*, respectively. For any $\mu_1: S_1 \to \mathbb{R}_+$ and $\mu_2: S_2 \to \mathbb{R}_+$, define the set function *b* on *S* by

$$b(Z) \coloneqq \sum_{e_1 \in S_1} \mu_1(e_1) \cdot \varphi_2(\pi_2(Z_{e_1})) + \sum_{e_2 \in S_2} \mu_2(e_2) \cdot \varphi_1(\pi_1(Z^{e_2})) - \sum_{(e_1, e_2) \in Z} \mu_1(e_1) \cdot \mu_2(e_2).$$
(2)

Note that, by the nonnegativity of φ_1 and φ_2 , μ_1 and μ_2 can always be chosen such that $\mu_1(S_1) = \varphi_1(S_1)$ and $\mu_2(S_2) = \varphi_2(S_2)$. Therefore, the next theorem settles the existence of a coupling of two nonnegative submodular functions whenever $\varphi_1(\emptyset) = \varphi_2(\emptyset) = 0$.

THEOREM 3.1. Let $\varphi_1: 2^{S_1} \to \mathbb{R}_+$ and $\varphi_2: 2^{S_2} \to \mathbb{R}_+$ be submodular functions over ground sets S_1 and S_2 , respectively, satisfying $\varphi_1(\emptyset) = 0$ and $\varphi_2(\emptyset) = 0$. Furthermore, let $\mu_1: S_1 \to \mathbb{R}_+$ and $\mu_2: S_2 \to \mathbb{R}_+$ be such that $\mu_1(S_1) = \varphi_1(S_1)$ and $\mu_2(S_2) = \varphi_2(S_2)$. Then, the function b defined in (2) is a submodular coupling of φ_1 and φ_2 .

PROOF. Take arbitrary $X, Y \subseteq S_1 \times S_2$, $e_1 \in S_1$ and $e_2 \in S_2$. Then we have

$$\pi_2((X \cap Y)_{e_1}) = \pi_2(X_{e_1}) \cap \pi_2(Y_{e_1}),$$

$$\pi_2((X \cup Y)_{e_1}) = \pi_2(X_{e_1}) \cup \pi_2(Y_{e_1}),$$

$$\pi_1((X \cap Y)^{e_2}) = \pi_1(X^{e_2}) \cap \pi_1(Y^{e_2}),$$

$$\pi_1((X \cup Y)^{e_2}) = \pi_1(X^{e_2}) \cup \pi_1(Y^{e_2}).$$

Together with the submodularity of φ_1 and φ_2 and the nonnegativity of μ_1 and μ_2 , this implies that *b* is a nonnegative combination of submodular functions minus a modular function, and is therefore submodular.

It remains to show that *b* is a coupling. For each $Y_1 \subseteq S_1$ and $Y_2 \subseteq S_2$, we get

$$b(Y_1 \times Y_2) = \mu_1(Y_1) \cdot \varphi_2(Y_2) + \mu_2(Y_2) \cdot \varphi_1(Y_1) - \mu_1(Y_1) \cdot \mu_2(Y_2)$$
(3)
$$= \varphi_1(Y_1) \cdot \varphi_2(Y_2) - (\varphi_1(Y_1) - \mu_1(Y_1)) \cdot (\varphi_2(Y_2) - \mu_2(Y_2)).$$
(4)

Thus, by $\mu_1(S_1) = \varphi_1(S_1)$ and $\mu_2(S_2) = \varphi_2(S_2)$, we have $b(Y_1 \times Y_2) = \varphi_1(Y_1) \cdot \varphi_2(Y_2)$ whenever $Y_1 = S_1$ or $Y_2 = S_2$, concluding the proof of the theorem.

It is worth emphasizing that if φ_1 and φ_2 are integer-valued, they admit an integer-valued submodular coupling. This follows by choosing μ_1 and μ_2 to be integer-valued in the proof.

3.2 Polymatroid Functions

A noteworthy feature of the proof of Theorem 3.1 is that it relies on two arbitrarily chosen nonnegative modular functions satisfying $\mu_1(S_1) = \varphi_1(S_1)$ and $\mu_2(S_2) = \varphi_2(S_2)$. However, the resulting function *b* may not be increasing, even if φ_1 and φ_2 are so. Therefore, to show that polymatroid functions admit a polymatroid coupling, μ_1 and μ_2 must be chosen carefully.

THEOREM 3.2. Let $\varphi_1: 2^{S_1} \to \mathbb{R}_+$ be a k_1 -polymatroid function and $\varphi_2: 2^{S_2} \to \mathbb{R}_+$ be a k_2 -polymatroid function over ground sets S_1 and S_2 , respectively. Then, φ_1 and φ_2 have a $(k_1 \cdot k_2)$ -polymatroid coupling which is integer-valued if φ_1 and φ_2 are integer-valued.

PROOF. Let $\mu_1 \in B(\varphi_1)$ and $\mu_2 \in B(\varphi_2)$ be arbitrary elements of the base polyhedrons of φ_1 and φ_2 , respectively, which are integers if φ_1 and φ_2 are integers. Note that such elements exist by Proposition 2.1. Let *b* denote the submodular coupling of φ_1 and φ_2 defined as in (2) using μ_1 and μ_2 .

CLAIM 3.3. We have $b(\emptyset) = 0$ and $b(Y_1 \times Y_2) \le \varphi_1(Y_1) \cdot \varphi_2(Y_2)$ for each $Y_1 \subseteq S_1$ and $Y_2 \subseteq S_2$, where equality holds if $\varphi_1(Y_1) = \mu_1(Y_1)$ or $\varphi_2(Y_2) = \mu_2(Y_2)$.

PROOF. The equality $b(\emptyset) = 0$ follows from $\varphi_1(\emptyset) = \varphi_2(\emptyset) = 0$. For each $Y_1 \subseteq S_1$ and $Y_2 \subseteq S_2$, using (4), $\mu_1(Y_1) \leq \varphi_1(Y_1)$, and $\mu_2(Y_2) \leq \varphi_2(Y_2)$, we get

$$b(Y_1 \times Y_2) = \varphi_1(Y_1) \cdot \varphi_2(Y_2) - (\varphi_1(Y_1) - \mu_1(Y_1)) \cdot (\varphi_2(Y_2) - \mu_2(Y_2)) \leq \varphi_1(Y_1) \cdot \varphi_2(Y_2).$$

Here equality holds if $\varphi_1(Y_1) = \mu_1(Y_1)$ or $\varphi_2(Y_2) = \mu_2(Y_2)$.

The function *b* thus obtained is still not necessarily increasing. To get an increasing set function, we define φ on *S* by setting

$$\varphi(Z) \coloneqq \min\{b(Z') \mid Z' \supseteq Z\}$$

The following technical claim is folklore, see e.g. [17]. We include a proof to make the paper self-contained.

CLAIM 3.4. φ is an increasing submodular function over $S_1 \times S_2$.

PROOF. Clearly, φ is increasing by definition. To prove submodularity, let $Z_1, Z_2 \subseteq S_1 \times S_2$. Then, for i = 1, 2, there exists $Z'_i \supseteq Z_i$ such that $\varphi(Z_i) = b(Z'_i)$. Then, using the submodularity of b and the definition of φ , we get

$$\begin{split} \varphi(Z_1) + \varphi(Z_2) &= b(Z'_1) + b(Z'_2) \\ &\geq b(Z'_1 \cap Z'_2) + b(Z'_1 \cup Z'_2) \\ &\geq \varphi(Z_1 \cap Z_2) + \varphi(Z_1 \cup Z_2) \end{split}$$

proving the claim.

Finally, we show that φ and *b* coincide on product sets.

CLAIM 3.5. $\varphi(Y_1 \times Y_2) = b(Y_1 \times Y_2)$ holds for each $Y_1 \subseteq S_1$ and $Y_2 \subseteq S_2$.

PROOF. We need to show that $b(Z) \ge b(Y_1 \times Y_2)$ holds if $Y_1 \times Y_2 \subseteq Z \subseteq S_1 \times S_2$. Using that φ_2 is increasing and $\mu_2(X) \le \varphi_2(X)$ holds for any $X \subseteq S_2$, we get

$$\sum_{e_{1} \in S_{1}} \mu_{1}(e_{1}) \cdot \varphi_{2}(\pi_{2}(Z_{e_{1}}))$$

$$\geq \sum_{e_{1} \in Y_{1}} \mu_{1}(e_{1}) \cdot \varphi_{2}(Y_{2}) + \sum_{e_{1} \in S_{1} \setminus Y_{1}} \mu_{1}(e_{1}) \cdot \varphi_{2}(\pi_{2}(Z_{e_{1}}))$$

$$\geq \sum_{e_{1} \in Y_{1}} \mu_{1}(e_{1}) \cdot \varphi_{2}(Y_{2}) + \sum_{e_{1} \in S_{1} \setminus Y_{1}} \mu_{1}(e_{1}) \cdot \mu_{2}(\pi_{2}(Z_{e_{1}}))$$

$$= \mu_{1}(Y_{1}) \cdot \varphi_{2}(Y_{2}) + \sum_{(e_{1},e_{2}) \in Z \setminus (Y_{1} \times S_{2})} \mu_{1}(e_{1}) \cdot \mu_{2}(e_{2}). \quad (5)$$

Similarly, using that φ_1 is increasing and $\mu_1(X) \le \varphi_1(X)$ holds for any $X \subseteq S_1$, we get

$$\begin{split} &\sum_{e_2 \in S_2} \mu_2(e_2) \cdot \varphi_1(\pi_1(Z^{e_2})) \\ &\geq \sum_{e_2 \in Y_2} \mu_2(e_2) \cdot \varphi_1(Y_1) + \sum_{e_2 \in S_2 \setminus Y_2} \mu_2(e_2) \cdot \varphi_1(\pi_1(Z^{e_2})) \\ &\geq \sum_{e_2 \in Y_2} \mu_2(e_2) \cdot \varphi_1(Y_1) + \sum_{e_2 \in S_2 \setminus Y_2} \mu_2(e_2) \cdot \mu_1(\pi_1(Z^{e_2})) \\ &= \mu_2(Y_2) \cdot \varphi_1(Y_1) + \sum_{(e_1, e_2) \in Z \setminus (S_1 \times Y_2)} \mu_1(e_1) \cdot \mu_2(e_2). \end{split}$$
(6)

Using the definition (2) of b, the equality (3), and the sum of the inequalities (5) and (6), we get

$$\begin{split} b(Z) &\geq \mu_1(Y_1) \cdot \varphi_2(Y_2) + \sum_{(e_1,e_2) \in Z \setminus (Y_1 \times S_2)} \mu_1(e_1) \cdot \mu_2(e_2) \\ &+ \mu_2(Y_2) \cdot \varphi_1(Y_1) + \sum_{(e_1,e_2) \in Z \setminus (S_1 \times Y_2)} \mu_1(e_1) \cdot \mu_2(e_2) \\ &- \sum_{(e_1,e_2) \in Z} \mu_1(e_1) \cdot \mu_2(e_2) \\ &= \mu_1(Y_1) \cdot \varphi_2(Y_2) + \mu_2(Y_2) \cdot \varphi_1(Y_1) \\ &+ \sum_{\substack{(e_1,e_2) \in Z \\ e_1 \notin Y_1, e_2 \notin Y_2}} \mu_1(e_1) \cdot \mu_2(e_2) - \sum_{(e_1,e_2) \in Y_1 \times Y_2} \mu_1(e_1) \cdot \mu_2(e_2) \\ &\geq \mu_1(Y_1) \cdot \varphi_2(Y_2) + \mu_2(Y_2) \cdot \varphi_1(Y_1) - \mu_1(Y_1) \cdot \mu_2(Y_2) \\ &= b(Y_1 \times Y_2). \end{split}$$

This concludes the proof of the claim.

By Claim 3.4, φ is increasing submodular. Claim 3.5 implies that $\varphi(\emptyset) = b(\emptyset) = 0$, thus $\varphi \ge 0$. Using $\varphi_1(S_1) = \mu_1(S_1)$, $\varphi_2(S_2) = \mu_2(S_2)$, and Claims 3.3 and 3.5, we get that $\varphi(Y_1 \times S_2) = b(Y_1 \times S_2) = \varphi_1(Y_1) \cdot \varphi_2(S_2)$ and $\varphi(S_1 \times Y_2) = b(S_1 \times Y_2) = \varphi_1(S_1) \times \varphi_2(Y_2)$ holds for $Y_1 \subseteq S_1$ and $Y_2 \subseteq S_2$, that is, φ is a coupling of φ_1 and φ_2 . For $(e_1, e_2) \in S_1 \times S_2$, by Claims 3.3 and 3.5, we have

$$\varphi(\{(e_1, e_2)\}) = b(\{(e_1, e_2)\}) \le \varphi_1(\{e_1\}) \cdot \varphi_2(\{e_2\}) \le k_1 \cdot k_2,$$

that is, φ is a $(k_1 \cdot k_2)$ -polymatroid function. Finally, if φ_1 and φ_2 are integer-valued, then φ is integer-valued as well by construction. This finishes the proof of the theorem.

3.3 Matroids and Tensor Products

One of the main motivations for our work was to extend the notion of coupling to matroids. For ease of discussion, we say that $M = (S_1 \times S_2, r)$ is a coupling of matroids $M_1 = (S_1, r_1)$ and $M_2 = (S_2, r_2)$ if r is a coupling of r_1 and r_2 . In Section 3.3.1, we show that matroids always admit a coupling that nearly satisfies the requirements of being a tensor product. Then, in Section 3.3.2, we characterize couplings that are tensor products as well. Based on our observations, in Section 3.3.3, we highlight a surprising phenomenon: the linearity of a matroid is closely related to whether it admits a tensor product with the uniform matroid $U_{2,3}$. This result could serve as a useful tool in studying linear matroids.

3.3.1 Matroid Couplings. Since the rank function of any matroid is an integer-valued 1-polymatroid function, Theorem 3.2 can be applied. Actually, the proof implies an even stronger result.

COROLLARY 3.6. Let $M_1 = (S_1, r_1)$ and $M_2 = (S_2, r_2)$ be matroids, let B_1 be a basis of M_1 and B_2 be a basis of M_2 , and set $S := S_1 \times S_2$. Then, M_1 and M_2 have a coupling M = (S, r) such that $r(Y_1 \times Y_2) =$ $r_1(Y_1) \cdot r_2(Y_2)$ whenever $Y_1 \subseteq B_1$ or $Y_2 \subseteq B_2$.

PROOF. Let $\mu_1 := \chi_{B_1}$ and $\mu_2 := \chi_{B_2}$ be the characteristic vectors of B_1 and B_2 , respectively. Clearly, $\chi_i \in B(r_i)$ holds for i = 1, 2. By Theorem 3.2, M_1 and M_2 admit a coupling M = (S, r), where r(Z) is

$$\min_{W \supseteq Z} \Big\{ \sum_{e_1 \in B_1} r_2(\pi_2(W_{e_1})) + \sum_{e_2 \in B_2} r_1(\pi_1(W^{e_2})) - |W \cap (B_1 \times B_2)| \Big\}.$$

Let $Y_1 \subseteq S_1$ and $Y_2 \subseteq S_2$. By Claim 3.5, the minimum is attained on W = Z for $Z = Y_1 \times Y_2$. Furthermore, if $Y_1 \subseteq B_1$ or $Y_2 \subseteq B_2$, then $\varphi_1(Y_1) = \mu_1(Y_1)$ or $\varphi_2(Y_2) = \mu_2(Y_2)$, and thus $r(Y_1 \times Y_2) = r_1(Y_1) \cdot r_2(Y_2)$ by Claim 3.3. This completes the proof of the corollary. \Box

Corollary 3.6 has a somewhat surprising implication: for a fixed rank *r* and ground set size *n*, there exists a function defined on a finite ground set that contains every rank-*r* matroid in some sense. For $r, n \in \mathbb{Z}_+$, let $N_{r,n}$ denote the number of rank-*r* matroids on *n* elements up to isomorphism.

COROLLARY 3.7. There exists a matroid M over $[n]^{N_{r,n}}$ of rank $r^{N_{r,n}}$ such that any rank-r matroid on [n] is a projection of M to one of its coordinates up to a constant factor.

PROOF. The result follows by applying the coupling of matroid pairs iteratively for every rank-r matroid over a ground set of size n.

Recall that two matroids do not necessarily admit a tensor product. However, the main message of Corollary 3.6 is that there always exists a coupling that locally satisfies the properties of the tensor product in the sense that $r(Y_1 \times Y_2) = r_1(Y_1) \cdot r_2(Y_2)$ holds, not universally for all Y_1, Y_2 , but specifically when at least one of $Y_1 \subseteq B_1$ or $Y_2 \subseteq B_2$ is satisfied. In particular, this implies that $M|(S_1 \times \{y\})$ is isomorphic to M_1 for each $y \in B_2$ by the natural bijection to S_1 , and $M|(\{x\} \times S_2)$ is isomorphic to M_2 for each $x \in B_1$ by the natural bijection to S_2 .

It turns out that our construction is closely related to the notion of *amalgams* of matroids; see the full version [6] of the paper for further details.

3.3.2 Tensor Products. From the definitions alone, it is clear that coupling and the tensor product are closely related concepts. The following theorem characterizes which couplings satisfy the stronger properties of the tensor product.

THEOREM 3.8. Let $M_1 = (S_1, r_1)$, $M_2 = (S_2, r_2)$ and M = (S, r) be matroids such that $S = S_1 \times S_2$. Then, the following are equivalent:

- (i) $r(Y_1 \times Y_2) = r_1(Y_1) \cdot r_2(Y_2)$ for each $Y_1 \subseteq S_1$ and $Y_2 \subseteq S_2$,
- (ii) $r(\{e_1\} \times Y_2)) = r_1(e_1) \cdot r_2(Y_2)$ for each $Y_2 \subseteq S_2$ and $e_1 \in S_1$, $r(Y_1 \times \{e_2\})) = r_1(Y_1) \cdot r_2(e_2)$ for each $Y_1 \subseteq S_1$ and $e_2 \in S_2$, and $r(S) = r_1(S_1) \cdot r_2(S_2)$,
- (iii) *M* is a coupling of M_1 and M_2 and $r((e_1, e_2)) = r_1(e_1) \cdot r_2(e_2)$ for each $(e_1, e_2) \in S$.

PROOF. Las Vergnas [23] showed the equivalence of (i) and (ii), and it is also clear that (i) implies (iii). Assume now that (iii) holds, we prove that it implies (ii). For the proof, we will use the following simple observation.

CLAIM 3.9. If $J_1 \subseteq S_1$ is independent in M_1 and $J_2 \subseteq S_2$ is independent in M_2 , then

$$M|(J_1 \times S_2) = \bigoplus_{e_1 \in J_1} M|(\{e_1\} \times S_2)$$

and

$$M|(S_1 \times J_2) = \bigoplus_{e_2 \in J_2} M|(S_1 \times \{e_2\})$$

PROOF. Using that *M* is a coupling, we get that

$$\sum_{e_1 \in J_1} r(\{e_1\} \times S_2) = \sum_{e_1 \in J_1} r_1(e_1) \cdot r_2(S_2)$$
$$= |J_1| \cdot r_2(S_2)$$
$$= r_1(J_1) \cdot r_2(S_2)$$
$$= r(J_1 \times S_2),$$

implying $M|(J_1 \times S_2) = \bigoplus_{e_1 \in J_1} M|(\{e_1\} \times S_2)$. The second statement follows similarly. \Box

First we show that $r(Y_1 \times Y_2) \ge r_1(Y_1) \cdot r_2(Y_2)$ holds for each $Y_1 \subseteq S_1$ and $Y_2 \subseteq S_2$. Let $I_1 \subseteq Y_1$ and $I_2 \subseteq Y_2$ be maximal independent sets of M_1 and M_2 contained in Y_1 and Y_2 , respectively. Claim 3.9 implies that each circuit contained in $I_1 \times S_2$ intersects at most one of the sets of the form $\{e_1\} \times S_2$ for $e_1 \in I_1$. Similarly, each circuit contained in $S_1 \times I_2$ intersects at most one of the sets of the form $S_1 \times \{e_2\}$ for $e_2 \in I_2$. These show that each circuit in $I_1 \times I_2$ is a loop. Since $r((e_1, e_2)) = r_1(e_1) \cdot r_2(e_2) = 1$ holds for $(e_1, e_2) \in I_1 \times I_2$ by assumption, the set $I_1 \times I_2$ does not contain any loops. This proves that $I_1 \times I_2$ is independent in M, thus $r(Y_1 \times Y_2) \ge r_1(Y_1) \cdot r_2(Y_2)$.

Next we show that $r(\{e_1\} \times Y_2) = r_1(e_1) \cdot r_2(Y_2)$ holds for each $e_1 \in S_1$ and $Y_2 \subseteq S_2$. If e_1 is a loop in M_1 , then $r(\{e_1\} \times S_2) = r_1(e_1) \cdot r_2(S_2) = 0$, thus $r(\{e_1\} \times Y_2) = 0$. Otherwise, let B_1 be a basis of M_1 containing e_1 . By the previous paragraph, we have $r(\{b_1\} \times Y_2) \ge r_2(Y_2)$ for $b_1 \in B_1$. Furthermore, $M|(B_1 \times Y_2) =$

 $\bigoplus_{b_1 \in B_1} M | (\{b_1\} \times Y_2)$ by Claim 3.9. Thus,

$$r(S_{1} \times Y_{2}) \ge r(B_{1} \times Y_{2})$$

= $\sum_{b_{1} \in B_{1}} r(\{b_{1}\} \times Y_{2})$
 $\ge \sum_{b_{1} \in B_{1}} r_{2}(Y_{2})$
= $r_{1}(S_{1}) \cdot r_{2}(Y_{2})$
= $r(S_{1} \times Y_{2}),$

where the last equality holds by the assumption that *M* is a coupling. Therefore, equality holds throughout, implying that $r(\{b_1\} \times Y_2) = r_2(Y_2)$ for each $b_1 \in B_1$. In particular, this shows that $r(\{e_1\} \times Y_2) = r_1(e_1) \cdot r_2(Y_2)$. Similarly, we get that $r(Y_1 \times \{e_2\}) = r_1(Y_1) \cdot r_2(e_2)$ for each $Y_1 \subseteq S_1$ and $e_2 \in S_2$. Finally, $r(S) = r_1(S_1) \cdot r_2(S_2)$ follows from the assumption that *M* is a coupling.

If e_1 or e_2 is a loop in the corresponding matroid, then (e_1, e_2) is a loop in any coupling by definition. Hence if (e_1, e_2) is not a loop in a coupling, then $r((e_1, e_2)) = r_1(e_1) \cdot r_2(e_2)$ holds. Therefore, an interesting consequence of Theorem 3.8 is the following.

COROLLARY 3.10. Any loopless coupling of two matroids is also a tensor product.

3.3.3 Linear Matroids and Ingleton's Inequality. Tools for determining the linear representability of a matroid include the use of excluded minors and Ingleton's inequality [19], who showed that the rank function of any linear matroid M = (S, r) satisfies

$$r(A \cup B) + r(A \cup C) + r(A \cup D) + r(B \cup C) + r(B \cup D) \ge$$

$$r(A) + r(B) + r(A \cup B \cup C) + r(A \cup B \cup D) + r(C \cup D)$$
(7)

for every $A, B, C, D \subseteq S$. In this sense, the inequality describes a necessary condition for linearity. Since the uniform matroid $U_{2,3}$ is regular, it has a matroid tensor product with any linear matroid. In other words, for a matroid, the existence of a matroid tensor product with $U_{2,3}$ is also a necessary condition for linearity. Our next result shows that this property does not only imply Ingleton's inequality for polymatroid functions but also yields a slightly weaker form for general submodular functions; the proof is deferred to the full version of the paper [6].

THEOREM 3.11. Let $\varphi_1: 2^{S_1} \to \mathbb{R}$ be a submodular function and let $\varphi_2: 2^{S_2} \to \mathbb{R}_+$ denote the rank function of the uniform matroid $U_{2,3}$.

- (a) If φ_1 and φ_2 have a submodular tensor product, then φ_1 satisfies Ingleton's inequality (7) for all pairwise disjoint sets $A, B, C, D \subseteq S_1$.
- (b) If φ_1 and φ_2 have a tensor product which is a polymatroid function, then φ_1 satisfies Ingleton's inequality (7) for all sets $A, B, C, D \subseteq S_1$.

It was shown by Las Vergnas [23] that the Vámos matroid and the uniform matroid $U_{2,3}$ do not have a matroid tensor product. The Vámos matroid is the matroid on ground set $\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$ in which each subset of size 4 is a basis except for $A \cup B$, $A \cup C$, $A \cup D$, $B \cup C$ and $B \cup D$, where $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$ and $D = \{d_1, d_2\}$. As the rank function of the Vámos matroid does not satisfy Ingleton's inequality (7) for the pairwise disjoint sets *A*, *B*,

C, and *D*, Theorem 3.11 implies the following strengthening of the result of Las Vergnas [23].

COROLLARY 3.12. The rank function of the Vámos matroid does not have a submodular tensor product with the rank function of the uniform matroid $U_{2,3}$.

For any fixed ground set S_1 and fixed matroid $N = (S_2, r_2)$, the set of submodular (or polymatroid) functions $\varphi_1 \colon 2^{S_1} \to \mathbb{R}$ which have a tensor product with r_2 forms a polyhedron in $\mathbb{R}^{2^{S_1}}$, that is, they can be described as functions $\varphi_1: 2^{S_1} \to \mathbb{R}$ satisfying certain linear inequalities. If we choose N to be a regular matroid, then each of these inequalities is a linear rank inequality, that is, a linear inequality satisfied by the rank function of any representable matroid. Ingleton's inequality (7) was the first known linear rank inequality, and since then infinite families of independent linear rank inequalities have been found [12, 21]. Interestingly, all currently known linear rank inequalities follow from the so-called common informa*tion property* of the rank function of representable matroids [3, 12], though other linear inequalities have been found that are satisfied by rank functions of matroids representable over fields of certain characteristics but not by all representable matroids [13]. For defining the common information property, let us call every polymatroid function 0-*CI*-compliant, and call a polymatroid function $f: 2^S \to \mathbb{R}$ *k*-*CI*-compliant if for any subsets $A, B \subseteq S$, there exists a (k - 1)compliant polymatroid function $f': 2^{S'} \to \mathbb{R}$ and a subset $Z \subseteq S'$ such that $S \subseteq S'$, f'(X) = f(X) for all $X \subseteq S$, $f'(A \cup Z) = f(A)$, $f'(B \cup Z) = f(B)$, and $f'(Z) = f(A) + f(B) - f(A \cup B)$. Then, a polymatroid satisfies the common information property if it is k-CI*compliant* for each integer $k \ge 0$. Note that the common information property is not sufficient for the representability of a matroid, e.g. all rank-3 matroids satisfy it [3] while e.g. the non-Desargues matroid is not representable [19]. It would be interesting to see the relations between the class of polymatroid functions having tensor products with $U_{2,3}$, the class of polymatroid functions having tensor product with all regular matroids, and the class of polymatroid functions satisfying the common information property.

3.4 Coverage and *k*-Alternating Functions

Coverage functions naturally arise in applications like set cover, influence maximization, sensor placement, feature selection, and data summarization. The goal of this section is to show that if one of the functions is a coverage function, then there exist couplings with stronger properties. For some $k \in \mathbb{Z}_{>0}$, let us call a function $\varphi: 2^S \to \mathbb{R}_+$ *k*-alternating if $\varphi(\emptyset) = 0$ and it satisfies (*k*-ALT) for the given *k*.

THEOREM 3.13. Let $\varphi_1: 2^{S_1} \to \mathbb{R}_+$ be a coverage function and $\varphi_2: 2^{S_2} \to \mathbb{R}_+$ be a k-alternating function over ground sets S_1 and S_2 , respectively. Then, φ_1 and φ_2 have a k-alternating tensor product which is integer-valued if φ_1 and φ_2 are integer-valued.

PROOF. Assume first that $\varphi_1 = \varphi_A$ for some $\emptyset \neq A \subseteq S_1$ as defined in (1). We define the tensor product of φ_A and φ_2 to be

$$(\varphi_A \otimes \varphi_2)(X) \coloneqq \varphi_2(\pi_2((A \times S_2) \cap X))$$

for any $X \subseteq S_1 \times S_2$. For each pair of sets $Y_1 \subseteq S_1, Y_2 \subseteq S_2$, we have $(\varphi_A \otimes \varphi_2)(Y_1 \times Y_2) = \varphi_2(Y_2)$ if $A \cap Y_1 \neq \emptyset$ and $(\varphi_A \otimes \varphi_2)(Y_1 \times Y_2) = 0$ if $A \cap Y_1 = \emptyset$, thus $\varphi_A \otimes \varphi_2$ is indeed a tensor product. We claim that

it is *k*-alternating as well. To see this, let $X_0, X_1, \ldots, X_k \subseteq S_1 \times S_2$. Then, the *k*-alternating condition (*k*-ALT) requires that

$$\begin{split} &\sum_{K\subseteq[k]} (-1)^{|K|} (\varphi_A \otimes \varphi_2) \Big(X_0 \cup \bigcup_{i \in K} X_i \Big) \\ &= \sum_{K\subseteq[k]} (-1)^{|K|} \varphi_2 \Big(\pi_2 \Big((A \times S_2) \cap (X_0 \cup \bigcup_{i \in K} X_i) \Big) \Big) \\ &= \sum_{K\subseteq[k]} (-1)^{|K|} \varphi_2 \Big(\pi_2 \Big((A \times S_2) \cap X_0 \Big) \cup \bigcup_{i \in K} \pi_2 \Big((A \times S_2) \cap X_i) \Big) \Big) \\ &= \sum_{K\subseteq[k]} (-1)^{|K|} \varphi_2 \Big(Y_0 \cup \bigcup_{i \in K} Y_i \Big) \\ &\leq 0. \end{split}$$

This follows from the *k*-alternating condition (*k*-ALT) for φ_2 and sets $Y_i = \pi_2((A \times S_2) \cap X_i)$.

In general, if φ_1 is a coverage function, then by Proposition 2.2, $\varphi_1 = \sum_{\emptyset \neq A \subseteq S_1} c_A \cdot \varphi_A$ for some nonnegative coefficients $c_A \in \mathbb{R}_+$ for all nonempty $A \subseteq S_1$. Define the tensor product of φ_1 and φ_2 to be

$$\varphi_1 \otimes \varphi_2 \coloneqq \sum_{\emptyset \neq A \subseteq S_1} c_A \cdot (\varphi_A \otimes \varphi_2).$$

Then, the resulting function over $S_1 \times S_2$ is a tensor product of φ_1 and φ_2 . Indeed, $(\varphi_1 \otimes \varphi_2)(Y_1 \times Y_2) = \sum_{\emptyset \neq A \subseteq S_1} c_A \cdot (\varphi_A \otimes \varphi_2)(Y_1 \times Y_2) = (\sum_{A \cap Y_1 \neq \emptyset} c_A) \cdot \varphi_2(Y_2) = \varphi_1(Y_1) \cdot \varphi_2(Y_2)$. Finally, observe that $\varphi_1 \otimes \varphi_2$ is *k*-alternating as it is a nonnegative combination of *k*-alternating functions.

As noted earlier, (*k*-ALT) for k = 1 corresponds to the increasing property, while for k = 2 it is equivalent to the increasing submodularity of the function. Thus, by applying Theorem 3.13 to the case when k = 2, we get the following.

COROLLARY 3.14. Any coverage function φ_1 and increasing submodular function φ_2 have an increasing submodular tensor product.

Another interesting special case is when both φ_1 and φ_2 are coverage functions.

COROLLARY 3.15. Any two coverage functions have a coverage tensor product.

3.5 Extension to the Infinite Case

The main results of Section 3.1 and Section 3.2 can be extended to pairs of bounded submodular set functions defined on infinite set algebras. Essentially the same ideas work as in the finite cases, i.e., in the proofs of Theorem 3.1 and Theorem 3.2, but the reasoning is slightly more technical, and we have to integrate instead of taking sums. Moreover, we will need to integrate with respect to nonnegative charges. We follow the definitions of [27]. Let (J, \mathcal{B}) be a set algebra, $\alpha : \mathcal{B} \to \mathbb{R}_+$ be a nonnegative charge, and $f : J \to \mathbb{R}_+$ be a nonnegative bounded measurable function. Define

$$\hat{\alpha}(f) \coloneqq \int_0^\infty \alpha(\{f \ge t\}) \, \mathrm{d}t.$$

Note that $\hat{\alpha}$ is *linear* in the sense that $\hat{\alpha}(c_1f_1 + c_2f_2) = c_1\hat{\alpha}(f_1) + c_2\hat{\alpha}(f_2)$ for all nonnegative bounded measurable functions f_1, f_2 , and $c_1, c_2 \in \mathbb{R}_+$. Furthermore, $\hat{\alpha}$ is *positive*, meaning that $\hat{\alpha}(g) \geq \hat{\alpha}(f)$ if $g(x) \geq f(x)$ for all $x \in J$.

Let (J_1, \mathcal{B}_1) and (J_2, \mathcal{B}_2) be set algebras and denote their product space by (J, \mathcal{B}) . Let $\varphi_1 \colon \mathcal{B}_1 \to \mathbb{R}_+, \varphi_2 \colon \mathcal{B}_2 \to \mathbb{R}_+$ be bounded submodular functions satisfying $\varphi_1(\emptyset) = \varphi_2(\emptyset) = 0$, and let $\alpha_1 \colon \mathcal{B}_1 \to \mathcal{B}_1$ $\mathbb{R}_+, \alpha_2 \colon \mathcal{B}_2 \to \mathbb{R}_+$ be charges. To make it easier to draw parallels with the finite case, by a slight abuse of notation, we denote by $\varphi_2(\pi_2(Z_x))$ the function $x \mapsto \varphi_2(\pi_2(Z_x))$ and by $\varphi_1(\pi_1(Z^y))$ the function $y \mapsto \varphi_1(\pi_1(Z^y))$ for any set $Z \in \mathcal{B}$. Any set Z of the product algebra has the form $Z = \bigcup_{i=1}^{n} A_i \times B_i$, where $A_i \in \mathcal{B}_1$ and $B_i \in \mathcal{B}_2$ for $i \in [n]$. Consider the finite algebras $\mathcal{A}_1, \mathcal{A}_2$ generated by the sets $\{A_i\}_{i=1}^n$ and the sets $\{B_i\}_{i=1}^n$, respectively. It is not difficult to see that $\varphi_2(\pi_2(Z_x))$ is constant on the atoms of \mathcal{A}_1 and $\varphi_1(\pi_1(Z^y))$ is constant on the atoms of \mathcal{A}_2 . Consequently, both functions take only finitely many values, and the preimages of these values belong to \mathcal{A}_1 and \mathcal{A}_2 , respectively. As $\mathcal{A}_i \subseteq \mathcal{B}_i$ for $i \in [2]$, we conclude that both functions are measurable. Similarly to (2), we can define

$$b(Z) \coloneqq \hat{\alpha}_1(\varphi_2(\pi_2(Z_x))) + \hat{\alpha}_2(\varphi_1(\pi_1(Z^y))) - \hat{\alpha}_1(\alpha_2(\pi_2(Z_x)))$$
(8)

for all $Z \in \mathcal{B}$. While this definition does not look symmetric at first, note that $\hat{\alpha}_1(\alpha_2(\pi_2(Z_x)))$ and $\hat{\alpha}_2(\alpha_1(\pi_1(Z^y)))$ agree on the semialgebra of product sets and thus agree on the generated algebra, see e.g. [31, Theorem 3.5.1 (ii)]. Observe that if $Z = Y_1 \times Y_2$ for some $Y_1 \in \mathcal{B}_1$ and $Y_2 \in \mathcal{B}_2$, then

$$\begin{aligned} \hat{\alpha}_1(\varphi_2(\pi_2(Z_X))) &= \alpha_1(Y_1)\varphi_2(Y_2), \\ \hat{\alpha}_2(\varphi_1(\pi_1(Z^y))) &= \varphi_1(Y_1)\alpha_2(Y_2), \\ \hat{\alpha}_1(\alpha_2(\pi_2(Z_X))) &= \alpha_1(Y_1)\alpha_2(Y_2). \end{aligned}$$

First we extend the statement of Theorem 3.1.

THEOREM 3.16. Let φ_1, φ_2 be nonnegative submodular functions over the set algebras (J_1, \mathcal{B}_1) and (J_2, \mathcal{B}_2) , respectively, satisfying $\varphi_1(\emptyset) = 0$ and $\varphi_2(\emptyset) = 0$. Furthermore, let $\alpha_1 \colon \mathcal{B}_1 \to \mathbb{R}_+$ and $\alpha_2 \colon \mathcal{B}_2 \to \mathbb{R}_+$ be charges such that $\alpha_1(S_1) = \varphi_1(S_1)$ and $\alpha_2(S_2) =$ $\varphi_2(S_2)$. Then, the function b defined in (8) is a submodular coupling of φ_1 and φ_2 .

PROOF. By the definition of b, we have

$$b(Y_1 \times Y_2) = \alpha_1(Y_1) \cdot \varphi_2(Y_2) + \alpha_2(Y_2) \cdot \varphi_1(Y_1) - \alpha_1(Y_1) \cdot \alpha_2(Y_2) = \varphi_1(Y_1) \cdot \varphi_2(Y_2) - (\varphi_1(Y_1) - \alpha_1(Y_1)) \cdot (\varphi_2(Y_2) - \alpha_2(Y_2)) = \varphi_1(Y_1) \cdot \varphi_2(Y_2),$$

if $Y_1 = J_1$ or $Y_2 = J_2$, hence *b* is indeed a coupling. To prove submodularity, note that for fix $x \in J_1$ and $y \in J_2$, the functions $\varphi_2(\pi_2(Z_x))$ and $\varphi_1(\pi_1(Z^y))$ are submodular while $\alpha_2(\pi_2(Z_x))$ is modular, hence *b* is submodular by the linearity and nonnegativity of α_1 and α_2 .

To extend the result of Theorem 3.2, we first have to define the measurable analogue of a base polytope. Let (J, \mathcal{B}) be a set algebra and $\varphi \colon \mathcal{B} \to \mathbb{R}_+$ be a polymatroid function. We say that a nonnegative charge $\alpha \colon \mathcal{B} \to \mathbb{R}_+$ is a *minorizing charge* if $\alpha(A) \leq \varphi(A)$ for all $A \in \mathcal{B}$. A minorizing charge is called *basic* if $\alpha(J) = \varphi(J)$. The set of all basic minorizing charges is denoted by bmm(φ). It was proved in [27, Corollary 5.3.] that bmm(φ) is nonempty. THEOREM 3.17. Let φ_1 , φ_2 be polymatroid functions on measurable spaces (J_1, \mathcal{B}_1) and (J_2, \mathcal{B}_2) , respectively. Then, there exists a polymatroid coupling φ of φ_1 and φ_2 on the product space (J, \mathcal{B}) .

PROOF. Choose $\alpha_1 \in \text{bmm}(\varphi_1)$ and $\alpha_2 \in \text{bmm}(\varphi_2)$, and define *b* as in (8).

CLAIM 3.18. We have $b(\emptyset) = 0$ and $b(Y_1 \times Y_2) \le \varphi_1(Y_1) \cdot \varphi_2(Y_2)$ for each $Y_1 \in \mathcal{B}_1$ and $Y_2 \in \mathcal{B}_2$, where equality holds if $\varphi_1(Y_1) = \alpha_1(Y_1)$ or $\varphi_2(Y_2) = \alpha_2(Y_2)$.

PROOF. The equality $b(\emptyset) = 0$ follows from $\varphi_1(\emptyset) = \varphi_2(\emptyset) = 0$. For each $Y_1 \in \mathcal{B}_1$ and $Y_2 \in \mathcal{B}_2$, $\alpha_1(Y_1) \leq \varphi_1(Y_1)$ and $\alpha_2(Y_2) \leq \varphi_2(Y_2)$, thus we get

$$\begin{split} b(Y_1 \times Y_2) \\ &= \varphi_1(Y_1) \cdot \varphi_2(Y_2) - (\varphi_1(Y_1) - \alpha_1(Y_1)) \cdot (\varphi_2(Y_2) - \alpha_2(Y_2)) \\ &\leq \varphi_1(Y_1) \cdot \varphi_2(Y_2). \end{split}$$

Here equality holds if $\varphi_1(Y_1) = \alpha_1(Y_1)$ or $\varphi_2(Y_2) = \alpha_2(Y_2)$.

The function *b* is not necessarily monotone. To overcome this, let $\varphi(Z) \coloneqq \inf_{Z' \supset Z} b(Z')$ for all $Z \in \mathcal{B}$.

CLAIM 3.19. φ is increasing and submodular.

PROOF. Clearly, φ is increasing by definition. To prove submodularity, let $Z_1, Z_2 \in \mathcal{B}$. Then, for any $\varepsilon > 0$ and for i = 1, 2, there exists $Z'_i \supseteq Z_i$ such that $\varphi(Z_i) + \varepsilon \ge b(Z'_i)$. We get

$$\begin{aligned} \varphi(Z_1) + \varphi(Z_2) &\geq b(Z'_1) + b(Z'_2) + 2\varepsilon \\ &\geq b(Z'_1 \cap Z'_2) + b(Z'_1 \cup Z'_2) + 2\varepsilon \\ &\geq \varphi(Z_1 \cap Z_2) + \varphi(Z_1 \cup Z_2) + 2\varepsilon. \end{aligned}$$

The claim follows by taking $\varepsilon \to 0$.

CLAIM 3.20. $\varphi(Y_1 \times Y_2) = b(Y_1 \times Y_2)$ holds for each $Y_1 \in \mathcal{B}_1$ and $Y_2 \in \mathcal{B}_2$.

PROOF. We need to show that $b(Z) \ge b(Y_1 \times Y_2)$ holds if $Y_1 \times Y_2 \subseteq Z \in \mathcal{B}$. Using that φ_2 is increasing and $\alpha_2(X) \le \varphi_2(X)$ holds for any $X \in \mathcal{B}_2$, we get

$$\begin{aligned} \hat{x}_1(\varphi_2(\pi_2(Z_X))) \\ &= \hat{\alpha}_1(\mathbb{1}_{Y_1} \cdot \varphi_2(\pi_2(Z_X))) + \hat{\alpha}_1(\mathbb{1}_{J_1 \setminus Y_1} \cdot \varphi_2(\pi_2(Z_X))) \\ &\ge \alpha_1(Y_1) \cdot \varphi_2(Y_2) + \hat{\alpha}_1(\alpha_2(\pi_2((Z \setminus (Y_1 \times J_2))_X))). \end{aligned}$$

Similarly,

$$\begin{aligned} \hat{\alpha}_{2}(\varphi_{1}(\pi_{1}(Z^{y}))) \\ &= \hat{\alpha}_{2}(\mathbb{1}_{Y_{2}} \cdot \varphi_{1}(\pi_{1}(Z^{y}))) + \hat{\alpha}_{2}(\mathbb{1}_{J_{2} \setminus Y_{2}} \cdot \varphi_{1}(\pi_{1}(Z^{y}))) \\ &\geq \varphi_{1}(Y_{1}) \cdot \alpha_{2}(Y_{2}) + \hat{\alpha}_{2}(\mathbb{1}_{J_{2} \setminus Y_{2}} \cdot \alpha_{1}(\pi_{1}(Z^{y}))) \\ &= \varphi_{1}(Y_{1}) \cdot \alpha_{2}(Y_{2}) + \hat{\alpha}_{1}(\alpha_{2}(\pi_{2}((Z \setminus (J_{1} \times Y_{2}))_{x}))). \end{aligned}$$

Finally,

$$\begin{aligned} \hat{\alpha}_{1}(\alpha_{2}(\pi_{2}(Z_{x}))) \\ &= \alpha_{1}(Y_{1}) \cdot \alpha_{2}(Y_{2}) + \hat{\alpha}_{1}(\alpha_{2}(\pi_{2}((Z \setminus (Y_{1} \times Y_{2}))_{x}))) \\ &\leq \alpha_{1}(Y_{1}) \cdot \alpha_{2}(Y_{2}) + \hat{\alpha}_{1}(\alpha_{2}(\pi_{2}((Z \setminus (Y_{1} \times J_{2}))_{x}))) \\ &+ \hat{\alpha}_{1}(\alpha_{2}(\pi_{2}((Z \setminus (J_{1} \times Y_{2}))_{x}))). \end{aligned}$$

Combining these three inequalities, the claim follows.

Combining Claims 3.18, 3.19 and 3.20, we deduce that φ is a polymatroid coupling of φ_1 and φ_2 . This concludes the proof of the theorem.

4 Universal Functions

In this section we show that, under certain conditions, if there exists a coupling of any two functions with some property P, then there exists a universal function with property P, i.e., a function that contains any finite function with property P as a quotient.

4.1 Preparations

If $Q = \{Q_i\}_{i=1}^n$ is a finite measurable partition of J, i.e., $\{Q_i\}_{i=1}^n$ is a partition and $Q_i \in \mathcal{B}$ for $i \in [n]$, then the *quotient* $\varphi/Q: 2^{[n]} \to \mathbb{R}$ denotes the function defined by $(\varphi/Q)(X) = \varphi(\bigcup_{i \in X} Q_i)$ for $X \subseteq [n]$. The notion of coupling can be naturally extended to more than two functions. Let $(J_1, \mathcal{B}_1), \ldots, (J_n, \mathcal{B}_n)$ be set algebras and denote their product space by (J, \mathcal{B}) . Furthermore, let $\varphi_i: \mathcal{B}_i \to \mathbb{R}$ be set functions for $i \in [n]$. We say that a function $\varphi: \mathcal{B} \to \mathbb{R}$ is a *coupling* of $\varphi_1, \ldots, \varphi_n$ if $\varphi(S_1 \times \cdots \times S_{i-1} \times X_i \times S_{i+1} \times \cdots \times S_n) = \varphi_i(X_i) \cdot \prod_{\ell \in [n] \setminus \{i\}} \varphi_j(J_\ell)$ for every $X_i \in \mathcal{B}_i$, $i \in [n]$.

Let (J_1, \mathcal{B}_1) and (J_2, \mathcal{B}_2) be two set algebras. The functions $\varphi_1: \mathcal{B}_1 \to \mathbb{R}$ and $\varphi_2: \mathcal{B}_2 \to \mathbb{R}$ are *isomorphic* if there exists a bijection $f: J_1 \to J_2$ such that f and f^{-1} are measurable and $\varphi_2(f(X)) = \varphi_1(X)$ for all $X \in \mathcal{B}_1$. By a *property* of set functions, we mean an isomorphism invariant class of set functions. We call a property *P* of set functions *finitary* if a function $\varphi \colon \mathcal{B} \to \mathbb{R}$ is in *P* if and only if φ/Q is in *P* for any finite measurable partition Q of J. A finitary property P is *closed* if for any measurable space (J, \mathcal{B}) and any finite measurable partition of J, the set of functions $\{f: \mathcal{B} \to \mathbb{R} \mid f/Q \in P\}$ is closed. We call a finitary property P *extendable* if for any $\varphi: 2^{[n]} \to \mathbb{R}$ with property *P* and any finite, pairwise disjoint sets A_1, \ldots, A_n with union A, there exists a function $\psi: 2^A \to \mathbb{R}$ with property *P* that extends φ in the sense that that $\varphi(S) = \psi(\bigcup_{i \in S} A_i)$ for all $S \subseteq [n]$. We call a finitary property *P liftable* if any two finite functions with property *P* admit a coupling with property P on the product of their ground sets. Note that liftability implies that any finite number of finite functions with property *P* admit a coupling with property *P* on the product of their ground sets. We say that property *P* is *b*-bounded for some $b \in \mathbb{R}_+$ if the range of any function with property *P* is contained in [-b, b]. Finally, a normalized function with property P is called *universal* for property P, if it contains all finite, normalized functions with property *P* as a quotient.

We need the following well-known lemma (which follows from e.g. [22, Problem 4.43]) and include a proof for completeness.

LEMMA 4.1. There exists a family of subsets $\mathcal{H} \subseteq 2^{\mathbb{Z}}$ with $|\mathcal{H}| = \mathfrak{c}$ such that for any disjoint finite subsets $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}$, we have

$$\bigcap_{H\in\mathcal{H}_1}H\cap\bigcap_{H\in\mathcal{H}_2}H^c\neq\emptyset$$

PROOF. Let

$$\mathcal{R} \coloneqq \left\{ A \subseteq \mathbb{R} \mid A = \bigcup_{i=1}^{n} [p_i, q_i] \text{ for some } n \in \mathbb{Z}_+ \text{ and} \\ \text{rational numbers } p_i, q_i \text{ for } i \in [n] \right\}.$$

Note that \mathcal{R} is countable, hence it is enough to find an appropriate system with $\mathcal{H} \subseteq 2^{\mathcal{R}}$. For any $x \in \mathbb{R}$, let $\mathcal{H}_x := \{A \in \mathcal{R} \mid x \in A\}$.

For any disjoint finite subsets $X_1 \subseteq \mathbb{R}$ and $X_2 \subseteq \mathbb{R}$ we can clearly find $A \in \mathcal{R}$ with $X_1 \subseteq A$ and $X_2 \cap A = \emptyset$. Hence the set system $\{\mathcal{H}_x\}_{x \in \mathbb{R}}$ satisfies the condition of the lemma. \Box

The proof of Theorem 4.2 relies on a compactness argument. For a set algebra (J, \mathcal{B}) , the space of set functions $\{f : \mathcal{B} \to \mathbb{R}\}$ can be identified with the product topological space $\prod_{A \in \mathcal{B}} \mathbb{R}$. For any $r : \mathcal{B} \to \mathbb{R}_+$, the space of functions

$$\left\{f\colon \mathcal{B}\to \mathbb{R}\ \Big|\ |f(A)|\leq r(A) \text{ for all } A\in \mathcal{B}\right\}=\prod_{A\in \mathcal{B}}\left[-r(A),r(A)\right]$$

is compact, as the product of compact sets is compact by Tychonoff's theorem [34]. We will use that a topological space *X* is compact if and only if $\bigcap_{C \in C} C \neq \emptyset$ for any collection *C* of closed subsets of *X* such that $\bigcap_{C \in C'} C \neq \emptyset$ for all finite subcollections $C' \subseteq C$.

4.2 **Proof of Existence**

Now we are ready to state the main result of this section.

THEOREM 4.2. For any finitary, closed, extendable, liftable and bbounded property P for some $b \ge 1$, there exists a universal function on the set algebra $(\mathbb{Z}, 2^{\mathbb{Z}})$.

PROOF. Let $\mathcal{H} \subseteq 2^{\mathbb{Z}}$ be a set given by Lemma 4.1. Let Φ be a set of representatives of functions with property P with finite domain whose size is a power of two, i.e., for all such isomorphism class Φ contains exactly one function. For a function $\varphi \in \Phi$, denote by S_{φ} the domain of φ . We may assume that for any two distinct $\varphi, \psi \in \Phi$, their domains are disjoint, that is, $S_{\varphi} \cap S_{\psi} = \emptyset$. Since P is extendable, it is enough to find a function $F: 2^{\mathbb{Z}} \to \mathbb{R}$ with property P that contains every member of Φ as a quotient.

Up to isomorphism, there are continuum real-valued functions on finite domain, hence $|\Phi| \leq \mathfrak{c}$. Consequently, we can choose $H_1^{\varphi}, \ldots, H_n^{\varphi} \in \mathcal{H}$ for all $\varphi \in \Phi$, where $n = \log_2(|S_{\varphi}|)$, such that $H_i^{\varphi_1} = H_j^{\varphi_2}$ only if $\varphi_1 = \varphi_2$ and i = j. Furthermore, for all $\varphi \in \Phi$ and all $a \in S_{\varphi}$, define $I_a \subseteq [n]$ and

$$T_a := \bigcap_{i \in I_a} H_i^{\varphi} \cap \bigcap_{i \in [n] \setminus I_a} \left(H_i^{\varphi} \right)^c,$$

where $n = \log_2(|S_{\varphi}|)$, such that $I_a \neq I_b$ for distinct $a, b \in S_{\varphi}$. It is enough to prove the existence of a function $F: 2^{\mathbb{Z}} \to \mathbb{R}$ such that Fhas property P (which is equivalent to every finite factor having property P) and

$$F\Bigl(\bigcup_{a\in A}T_a\Bigr)=\varphi(A)$$

for all $\varphi \in \Phi$ and $A \subseteq S_{\varphi}$, as the latter condition guarantees that it contains every finite function with property *P* as a quotient. Note that, by the property of the family \mathcal{H} given by Lemma 4.1, $\bigcup_{a \in A} T_a \neq \bigcup_{b \in B} T_b$ for any $\varphi, \psi \in \Phi$ and $A \subseteq S_{\varphi}, B \subseteq S_{\psi}$.

The function space $\{f: \mathcal{B} \to \mathbb{R} \mid |f(A)| \le b \text{ for all } A \in 2^{\mathbb{Z}}\}$ is compact. The condition that F/Q has property P is closed for every finite measurable partition Q of \mathbb{Z} . Furthermore, conditions of the form $F(\bigcup_{a \in A} T_a) = \varphi(A)$ are also closed. Hence, it is enough to prove that for any finite number of these conditions, we can find a function satisfying all of them. Let Q_1, \ldots, Q_n be finite partitions of \mathbb{Z} , and let $\bigcup_{a \in A_i} T_a$ be sets for some A_1, \ldots, A_m , where there exist distinct $\varphi_1, \ldots, \varphi_k \in \Phi$ such that $A_i \subseteq S_{\varphi_{f(i)}}$ for all $1 \le i \le m$ and

some function $f: [m] \to [k]$. We need to show that there exists a function $F': 2^{\mathbb{Z}} \to \mathbb{R}$ such that $F'(\bigcup_{a \in A_i} T_a) = \varphi_{f(i)}(A_i)$ for $1 \le i \le m$, and F'/Q_i has property *P* for $1 \le i \le n$.

Let φ be a coupling of $\varphi_1, \ldots, \varphi_k$ on ground set $S := S_{\varphi_1} \times \ldots \times S_{\varphi_k}$. Denote by \mathcal{A}_1 the algebra generated by the sets $\{H_j^{\varphi_i} \mid 1 \le i \le k, 1 \le j \le \log_2(|S_{\varphi_i}|)$. The atoms of \mathcal{A}_1 are of the form $\bigcap_{i=1}^k T_{a_i}$ for some $(a_1, \ldots, a_k) \in S$. Note that none of these atoms are empty by Lemma 4.1. The set algebra of S and \mathcal{A}_1 are isomorphic, as to an atom $(a_1, \ldots, a_k) \in S$ we can assign $\bigcap_{i=1}^k T_{a_i}$ of \mathcal{A}_1 , which induces the isomorphism $\eta: 2^S \to \mathcal{A}_1$,

$$\eta(A) = \bigcup_{(a_i)_{i=1}^k \in A} \left(\bigcap_{i=1}^k T_{a_i}\right)$$

for all $A \in 2^S$. Now we can define the function $F_1: \mathcal{A}_1 \to \mathbb{R}$ by $F_1(A) = \varphi(\eta^{-1}(A))$ for all $A \in \mathcal{A}_1$. By definition, this satisfies all equalities of the form $F_1(\bigcup_{a \in A_i} T_a) = \varphi_{f(i)}(A_i)$, and F_1 has property P as φ has property P. Let \mathcal{A}_2 be the algebra generated by the sets of \mathcal{A}_1 and the sets appearing in one of the partitions Q_i for $1 \le i \le n$. This is a refinement of \mathcal{A}_1 still having a finite number of atoms, hence, by extendability, there exists a function $F_2: \mathcal{A}_2 \to \mathbb{R}$ with property P that extends F_1 , i.e., $F_2|_{\mathcal{A}_1} = F_1$. It follows that F_2/Q_i has property P for $1 \le i \le n$. Finally, an arbitrary extension F' of F_2 to the domain $2^{\mathbb{Z}}$ concludes the proof of the theorem. \Box

4.3 Applications

We show two applications of Theorem 4.2. First, we show that there exists a universal increasing submodular set function, using Theorem 3.2. Any increasing, submodular set function which has value 0 on the empty set (apart from the constant 0) can be normalized, hence such a universal function contains all increasing, submodular set functions with value 0 on the empty set up to a constant multiplier.

COROLLARY 4.3. There exists a universal increasing submodular set function.

PROOF. Let *P* be the property of being normalized, increasing, and submodular. To apply Theorem 4.2, we have to show that *P* is a finitary, closed, extendable, liftable, 1-bounded property. For a set function $\varphi: \mathcal{B} \to \mathbb{R}$ on (J, \mathcal{B}) , having property *P* means

$$\varphi(A) \le \varphi(B)$$
 and $\varphi(C \cap D) + \varphi(C \cup D) \le \varphi(C) + \varphi(D)$ (9)

for all $A, B, C, D \in \mathcal{B}$ with $A \subseteq B$, and being normalized. Hence, it is clear that P is finitary, and 1-boundedness also holds by monotonicity and $\varphi(J) = 1$. Liftability follows from Theorem 3.2. Note that every inequality in (9) is satisfied by a closed subset of $\{f : \mathcal{B} \to \mathbb{R}\}$, and the functions with normalized property also form a closed subset, hence the set of functions having property P in $\{f : \mathcal{B} \to \mathbb{R}\}$ is closed, as it is the intersection of closed subsets. Consequently, $\{f : \mathcal{B} \to \mathbb{R} \mid f/Q \in P\}$ is closed for all finite measurable partitions, as it is a projection of $\{f : \mathcal{B} \to \mathbb{R}\}$ to the algebra generated by Q, proving that P is closed. Finally, if $\varphi : 2^{[n]} \to \mathbb{R}$ has property P, and A_1, A_2, \ldots, A_n are finite, pairwise disjoint sets with union A, then the set function $\psi : 2^A \to \mathbb{R}$ defined by $\psi(X) := \varphi(\nu(X))$ for all $X \subseteq A$, where $\nu(X) := \{i \in [n] \mid X \cap A_i \neq \emptyset\}$ for all $X \subseteq A$, is an extension with property *P*. Indeed, ψ is clearly an increasing extension, and for any *X*, *Y* \subseteq *A*,

$$\begin{split} \psi(X) + \psi(Y) &= \varphi(\nu(X)) + \varphi(\nu(Y)) \\ &\geq \varphi(\nu(X) \cap \nu(Y)) + \varphi(\nu(X) \cup \nu(Y)) \\ &\geq \varphi(\nu(X \cap Y)) + \varphi(\nu(X \cup Y)) \\ &= \psi(X \cap Y) + \psi(X \cup Y). \end{split}$$

Hence property *P* is extendable as well, so we can apply Theorem 4.2 to finish the proof of the corollary. \Box

In particular, Corollary 4.3 implies that there exists a function $F: 2^{\mathbb{Z}} \to \mathbb{R}$ that contains all normalized matroid rank functions as a quotient.

As a second application, we show that there exists a universal function for coverage functions using Corollary 3.15.

COROLLARY 4.4. There exists a universal coverage function.

PROOF. The proof is similar to that of Corollary 4.3. The coverage property is given by the inequalities of (*k*-ALT), hence this property is finitary, closed, and by monotonicity it is 1-bounded. Corollary 3.15 gives liftability. It remains to prove that if $\varphi: 2^{[n]} \to \mathbb{R}$ is a coverage function and A_1, A_2, \ldots, A_n are finite, pairwise disjoint sets with union *A*, then we can extend φ on *A*. By Proposition 2.2, we can write

$$\varphi = \sum_{\emptyset \neq X \subseteq [n]} c_X \varphi_X,$$

where φ_X is defined as in (1) and $c_X \ge 0$ for all $\emptyset \ne X \subseteq [n]$. Let $u(X) := \bigcup_{i \in X} A_i$ for all $X \subseteq [n]$ and define

$$\psi \coloneqq \sum_{\emptyset \neq X \subseteq [n]} c_X \varphi_{u(X)}.$$

Then ψ is a coverage function by Proposition 2.2, and it clearly extends φ , proving extendability. Hence the corollary follows by Theorem 4.2.

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Data Availability

No data are associated with this article. Data sharing is not applicable to this article.

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