

Contagious McKean–Vlasov problems with common noise: From smooth to singular feedback through hitting times^{*†}

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Abstract

We consider a family of McKean–Vlasov equations arising as the large particle limit of a system of interacting particles on the positive half-line with common noise and feedback. Such systems are motivated by structural models for systemic risk with contagion. This contagious interaction is such that when a particle hits zero, the impact is to move all the others toward the origin through a kernel which smooths the impact over time. We study a rescaling of the impact kernel under which it converges to the Dirac delta function so that the interaction happens instantaneously and the limiting singular McKean–Vlasov equation can exhibit jumps. Our approach provides a novel method to construct solutions to such singular problems that allows for more general drift and diffusion coefficients and we establish weak convergence to relaxed solutions in this setting. With more restrictions on the coefficients we can establish an almost sure version showing convergence to strong solutions. Under some regularity conditions on the contagion, we also show a rate of convergence up to the time the regularity of the contagion breaks down. Lastly, we perform some numerical experiments to investigate the sharpness of our bounds for the rate of convergence.

Keywords: McKean–Vlasov problem; weak convergence; common noise; contagion.

MSC2020 subject classifications: 60G57; 60H15; 60H30; 82C22; 34B16.

Submitted to EJP on August 10, 2023, final version accepted on April 22, 2025.

Supersedes arXiv:2307.10800.

^{*}Data availability statement has been added to the end of the article at the authors' request on 25 June 2025.

[†]This research has been supported by the EPSRC Centre for Doctoral Training in Mathematics of Random Systems: Analysis, Modelling and Simulation (EP/S023925/1).

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1 Introduction

In this paper, we study the limiting behaviour of the family of conditional McKean–Vlasov equations

$$\left\{ \begin{array}{l} dX_t^\varepsilon = b(t, X_t^\varepsilon, \nu_t^\varepsilon) dt + \sigma(t, X_t^\varepsilon) \sqrt{1 - \rho(t, \nu_t^\varepsilon)^2} dW_t + \sigma(t, X_t^\varepsilon) \rho(t, \nu_t^\varepsilon) dW_t^0 \\ \quad - \alpha(t) d\mathcal{L}_t^\varepsilon, \\ \tau^\varepsilon = \inf\{t > 0 : X_t^\varepsilon \leq 0\}, \\ \mathbf{P}^\varepsilon = \mathbb{P}[X^\varepsilon \in \cdot | W^0], \quad \nu_t^\varepsilon = \mathbb{P}[X_t^\varepsilon \in \cdot, \tau^\varepsilon > t | W^0], \\ L_t^\varepsilon = \mathbb{P}[\tau^\varepsilon \leq t | W^0], \quad \mathcal{L}_t^\varepsilon = \int_0^t \kappa^\varepsilon(t-s) L_s^\varepsilon ds, \end{array} \right. \quad (1.1)$$

as ε tends to zero. Here, W and W^0 are independent standard Brownian motions, and κ^ε is a rescaled mollifier which converges to the Dirac delta as ε goes to 0. Such equations arise as the limit of a large particle system, where W is usually referred to as the *idiosyncratic noise* (of a representative particle) and W^0 as the *common noise*. Also for the same reason, L^ε is referred to as the *loss process* and quantifies the amount of mass that has crossed the boundary at zero by time t . A solution to this system consists of the random probability measure \mathbf{P}^ε and the loss process L^ε , conditional on W^0 .

In addition to the more classical measure dependence of the coefficients that characterise McKean–Vlasov equations, there is a further feedback mechanism through the loss process L^ε : depending on the value of $\alpha(t) \geq 0$, L^ε pushes X^ε towards zero, causing the value of L^ε to increase, hence pushing X^ε even closer to 0. The integral kernel κ^ε , which is parameterised by some $\varepsilon > 0$, is a key element of the model and captures a latency in the transmission of L^ε to X^ε present in real-world systems. Precise conditions on the coefficient functions will be given later.

One motivation for this model arises in systemic risk, where X^ε represents the *distance-to-default* of a prototypical institution in a financial network with infinitely many entities, see for example [18]. In this setting, L_t^ε denotes the proportion of institutions that have defaulted by time t and is the cause of *endogenous contagion* through the feedback mechanism. In this model, we use the kernel κ^ε to capture feedback where, when a financial institution defaults and their positions are unwound, the counterparties experience a decrease in the value of their assets over time. A model for bank runs using such a smooth transmission of boundary losses was analysed in [3] in the presence of common noise. Moreover, [20] study a related mean-field model for neurons interacting gradually through threshold hitting times, albeit without common noise.

If the support of κ^ε is contained in the interval $[0, \gamma]$ with $\gamma \ll 1$, then the integral $\int_0^t \kappa^\varepsilon(t-s) L_s^\varepsilon ds$ is approximately equal to L_t^ε . In this article, we prove convergence in the following sense: if we fix a kernel κ and rescale it with a variable $\varepsilon > 0$ by $\kappa^\varepsilon(t) = \varepsilon^{-1} \kappa(\varepsilon^{-1}t)$, then we have convergence in the M_1 -topology of X^ε to X , where X is a (relaxed) solution to

$$\left\{ \begin{array}{l} dX_t = b(t, X_t, \nu_t) dt + \sigma(t, X_t) \sqrt{1 - \rho(t, \nu_t)^2} dW_t + \sigma(t, X_t) \rho(t, \nu_t) dW_t^0 \\ \quad - \alpha(t) dL_t, \\ \tau = \inf\{t > 0 : X_t \leq 0\}, \\ \mathbf{P} = \mathbb{P}[X \in \cdot | W^0, \mathbf{P}], \quad \nu_t = \mathbb{P}[X_t \in \cdot, \tau > t | W^0, \mathbf{P}], \\ L_t = \mathbb{P}[\tau \leq t | W^0, \mathbf{P}]. \end{array} \right. \quad (1.2)$$

Intuitively, (1.1) is a smoothed approximation to (1.2). Our motivation for taking the limit as $\varepsilon \rightarrow 0$ is to investigate the convergence to the system where the feedback is felt instantaneously, which captures the situation when the latency is small compared to the timescale of interest. It is well known that equations of the form (1.2) may develop jump discontinuities, as we will elaborate below.

Variants and special cases of (1.2) have been the subject of extensive research in the field. In the simplest scenario, where b and ρ are both zero, σ is equal to 1, and α is a positive constant, we obtain the probabilistic formulation of the *supercooled Stefan problem*. The Stefan problem, introduced in [32], describes the temperature and the phase boundary of a material undergoing a phase transition, typically from a solid to a liquid. The supercooled Stefan problem describes the freezing of a supercooled liquid (i.e. a liquid which is below its freezing point). Many authors constructed classical solutions [13, 14, 15, 24, 19] for time intervals where L_t is regular. In the PDE literature, it was first established in [31] that L_t may explode in finite time.

In the probabilistic formulation where the initial condition has finite mass, L_t is bounded but there can still exist a $t_* \in (0, \infty)$ such that $\lim_{s \uparrow t_*} L'_s = \infty$, referred to as a blow-up. Moreover, given suitable assumptions, for $\alpha(t)$ sufficiently large, a jump of $t \mapsto L_t$ must occur, as per [17, Theorem 1.1]. With added common noise, there is a set of paths of positive probability where a jump must happen, [25, Theorem 2.1].

The probabilistic reformulation provides a natural way to restart the system following a blow-up. From this perspective, for arbitrary initial conditions, see [17, Example 2.2], there may be infinitely many solutions: it may be possible for two solutions to be equal up to the first jump time t and then take jumps of different sizes. To address this ambiguity that arises at a jump time, a condition is typically imposed that selects the smallest possible jump sizes. This condition is known as the *physical jump condition*, defined as:

$$\Delta L_t = \inf\{x > 0 : \nu_{t-}([0, \alpha x]) < x\}, \quad (1.3)$$

where $\Delta L_t := L_t - L_{t-}$. The intuitive interpretation of (1.3) is that if we take the density of $X_{t-} \mathbb{1}_{\tau \geq t}$ and displace it by an αx amount towards 0, then the mass of the system below zero is exactly x . So it is the minimal amount by which we may displace our density such that the displacement and the mass below zero correspond. From a modelling perspective, the physical jump condition is the preferred choice of jump sizes due to its economic and physical interpretations.

Extensive research has been conducted to investigate various properties of *physical solutions* to the equation with $b = \rho = 0$, $\sigma = 1$, and α a positive constant [29, 28, 17, 26, 25, 27, 22, 9]. The paper [11] establishes that when X_{0-} possesses a bounded density that changes monotonicity finitely many times, then L is unique, and for any $t \geq 0$, L is continuously differentiable on $(t, t+h)$ for some $h > 0$. Additionally, in [17], it is demonstrated that for an initial condition with a bounded density that is Hölder continuous near the boundary, L is unique, continuous, and has a weak derivative until some explosion time. The work in [26] extends these results by showing that if the initial condition possesses an L^2 -density, then we have uniqueness for a short time after the explosion time. Moreover, irrespective of the initial condition, there exists a minimal loss process that will be dominated by any other loss process that solves the equation [9]. It has been established that such minimal solutions are physical [9, Theorem 6.5]. However, it remains unclear whether physical solutions are necessarily minimal due to the lack of uniqueness for general initial conditions.

Returning to (1.2), recent advances have been made in the study of general coefficients, specifically $t \mapsto b(t)$, $t \mapsto \sigma(t)$, and $t \mapsto \rho(t)$, in the presence of common noise. In Remark 2.5 from [28], a generalized Schauder fixed-point argument is presented to construct *strong solutions* in this setting. Strong solutions refer to the property $\mathbf{P} = \mathbf{P}(X \in \cdot \mid W^0)$, indicating that the random probability measure \mathbf{P} is adapted to the σ -algebra generated by the common noise. In [25], an underlying finite particle system was shown to converge to *relaxed* (or *weak*) *solutions* (see Definition 2.1), satisfying the aforementioned physical jump condition (with coefficients $(t, x) \mapsto b(t, x)$, $t \mapsto \sigma(t)$, and $t \mapsto \rho(t)$). Weak/relaxed solutions are characterised by having $\mathbf{P} = \mathbf{P}(X \in \cdot \mid W^0, \mathbf{P})$,

instead of $\mathbf{P} = \mathbb{P}(X \in \cdot \mid W^0)$, see Definition 2.1. As the empirical distributions of the finite particle systems converge weakly to \mathbf{P} , there is no guarantee that \mathbf{P} will be adapted to the σ -algebra generated by the common noise. The existence of such *strong* solutions in the sense just discussed, for the common noise problem satisfying the physical jump condition (1.3), has not yet been addressed in the literature.

The main contributions and structure of this paper are as follows:

- Firstly, in Section 2, we prove Theorem 2.5 and Corollary 2.6 showing the weak convergence of solutions of (1.1) to relaxed solutions of (1.2) as $\varepsilon \rightarrow 0$, i.e., as the smoothed feedback mechanism becomes instantaneous in the limit. As a by-product, this gives a novel method for establishing the existence of solutions to (1.2), avoiding time regularity assumptions on $\sigma\rho$ as needed in [25]. Furthermore, we derive an upper bound on the jump sizes, Theorem 2.5, and, under additional assumptions on the coefficients, Corollary 2.6, show that the loss process L satisfies the physical jump condition (1.3).
- Secondly, in Section 3, we show in Theorem 3.8 that, if the coefficients depend solely on time and α is a constant, then we may upgrade our mode of convergence from weak to almost sure. As a consequence of the method employed, we can guarantee that the limiting loss process will be W^0 -measurable and satisfy the physical jump condition. In addition, we have the existence of strong solutions in this setting.
- Lastly, in Section 4, for constant coefficients and without common noise, we provide in Proposition 4.1 an explicit rate of convergence of the smoothed approximations to the singular system prior to the first time the regularity of the loss function breaks down. We also give numerical tests of the convergence order in scenarios of different regularity, with and without common noise.

2 Weak convergence of smoothed feedback systems

Fix a finite time horizon $T > 0$, and let $\mathcal{P}(A)$ denote the set of probability measures on a measurable space (A, \mathcal{A}) . When A is a metric space, $\mathcal{B}(A)$ denotes the Borel σ -algebra. Let $\mathbf{M}_{\leq 1}(A)$ denote the space of sub-probability measures, which we shall endow with the topology of weak convergence. For any interval I and metric space X , let $C(I, X)$ denote the space of continuous functions from I to X . Similarly, $D(I, X)$ denotes the space of càdlàg functions from I to X . We shall employ the shorthand notation \mathcal{C}_X and D_X for $C(I, X)$ and $D(I, X)$, respectively, when the interval I is clear.

For every $\varepsilon > 0$, we fix a probability space $(\Omega^\varepsilon, \mathcal{F}^\varepsilon, \mathbb{P}^\varepsilon)$ that supports two independent Brownian motions. To simplify the notation, we will denote these Brownian motions by W and W^0 ; however, it is important to note that they may not be equal for different values of ε . Similarly, we adopt the simplified notations \mathbb{P} and \mathbb{E} to refer to \mathbb{P}^ε and the expectation under the measure \mathbb{P}^ε respectively. In this section, we characterise the weak limit of the system given by the following equation as ε tends to zero:

$$\left\{ \begin{array}{l} dX_t^\varepsilon = b(t, X_t^\varepsilon, \nu_t^\varepsilon) dt + \sigma(t, X_t^\varepsilon) \sqrt{1 - \rho(t, \nu_t^\varepsilon)^2} dW_t + \sigma(t, X_t^\varepsilon) \rho(t, \nu_t^\varepsilon) dW_t^0 \\ \quad - \alpha(t) d\mathfrak{L}_t^\varepsilon, \\ \tau^\varepsilon = \inf\{t > 0 : X_t^\varepsilon \leq 0\}, \\ \mathbf{P}^\varepsilon = \mathbb{P}[X^\varepsilon \in \cdot \mid W^0], \quad \nu_t^\varepsilon := \mathbb{P}[X_t^\varepsilon \in \cdot, \tau^\varepsilon > t \mid W^0], \\ L_t^\varepsilon = \mathbb{P}[\tau^\varepsilon \leq t \mid W^0], \quad \mathfrak{L}_t^\varepsilon = \int_0^t \kappa^\varepsilon(t-s) L_s^\varepsilon ds, \end{array} \right. \quad (2.1)$$

where $t \in [0, T]$. The coefficient b (σ , ρ , or α , respectively) is a measurable map from $[0, T] \times \mathbb{R} \times \mathbf{M}_{\leq 1}(\mathbb{R})$ ($[0, T] \times \mathbb{R}$, $[0, T] \times \mathbf{M}_{\leq 1}(\mathbb{R})$, or $[0, T]$, respectively) into \mathbb{R} . The

initial condition, denoted by X_{0-} , is assumed to be independent of the Brownian motions and positive almost surely. Finally, we define $\kappa^\varepsilon(t) := \varepsilon^{-1}\kappa(t\varepsilon^{-1})$.

One way to view X^ε is as the mean-field limit of an interacting particle system where particles interact through their first hitting time of zero. The interactions among particles are smoothed out over time by convolving with the kernel κ^ε . As ε approaches zero, the effect of interactions occurs over increasingly smaller time intervals. As κ^ε is a mollifier, it is natural to expect, as ε tends to zero, $\mathfrak{L}_t^\varepsilon$ to converge to the instantaneous loss at time t . That is to say, along a suitable subsequence, the random tuple $\{(\mathbf{P}^\varepsilon, W^0, W)\}_{\varepsilon>0}$ would have a limit point (\mathbf{P}, W^0, W) where $\mathbf{P} = \mathbb{P}[X \in \cdot | W^0]$ and X solves

$$\begin{cases} dX_t = b(t, X_t, \nu_t) dt + \sigma(t, X_t) \sqrt{1 - \rho(t, \nu_t)^2} dW_t + \sigma(t, X_t) \rho(t, \nu_t) dW_t^0 \\ \quad - \alpha(t) dL_t, \\ \tau = \inf\{t > 0 : X_t \leq 0\}, \\ \mathbf{P} = \mathbb{P}[X \in \cdot | W^0], \quad \nu_t := \mathbb{P}[X_t \in \cdot, \tau > t | W^0], \\ L_t = \mathbb{P}[\tau \leq t | W^0], \end{cases} \quad (2.2)$$

with $X_0 = X_{0-} + \alpha(0)L_0$. In this system, the feedback is felt instantaneously and is characterised by the common noise W^0 . In what follows, for technical reasons, we construct an extension \tilde{X} of the process X . For an arbitrary stochastic process Z , we define its extended version as follows,

$$\tilde{Z}_t = \begin{cases} Z_{0-} & t \in [-1, 0), \\ Z_t & t \in [0, T], \\ Z_T + W_t - W_T & t \in (T, T+1]. \end{cases} \quad (2.3)$$

We artificially extend the processes to be constant on $[-1, 0)$ and by a pure Brownian noise term on $(T, T+1]$. Therefore, the extension to \mathbf{P}^ε is $\tilde{\mathbf{P}}^\varepsilon := \mathbb{P}(\tilde{X}^\varepsilon \in \cdot | W^0)$. Consequently, the random measure $\tilde{\mathbf{P}}^\varepsilon$ remains W^0 -measurable. We show that the collection of measures $\{\tilde{\mathbf{P}}^\varepsilon\}_{\varepsilon>0}$ is tight; hence there exists a subsequence $(\varepsilon_n)_{n \geq 1}$ that converges to zero such that $\tilde{\mathbf{P}}^{\varepsilon_n}$ converges weakly to the random measure \mathbf{P} . However, as the mode of convergence is weak, we cannot expect that the limit point \mathbf{P} is also measurable with respect to W^0 .

Hence, we relax our notion of solution to (2.2), which leads to the definition of *relaxed solutions* employed in the literature when studying the mean-field limit of particle systems with common noise [25] and also in the mean-field game literature with common source of noise [4].

Definition 2.1 (Relaxed solutions). *Let the coefficient functions b, σ, ρ , and α be given along with the initial condition X_{0-} at time $t = 0-$. We define a relaxed solution to (2.2) as a family (X, W, W^0, \mathbf{P}) on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\begin{cases} dX_t = b(t, X_t, \nu_t) dt + \sigma(t, X_t) \sqrt{1 - \rho(t, \nu_t)^2} dW_t + \sigma(t, X_t) \rho(t, \nu_t) dW_t^0 \\ \quad - \alpha(t) dL_t, \\ \tau = \inf\{t > 0 : X_t \leq 0\}, \\ \mathbf{P} = \mathbb{P}[X \in \cdot | W^0, \mathbf{P}], \quad \nu_t := \mathbb{P}[X_t \in \cdot, \tau > t | W^0, \mathbf{P}], \\ L_t = \mathbb{P}[\tau \leq t | W^0, \mathbf{P}], \end{cases} \quad (2.4)$$

with $X_0 = X_{0-} + \alpha(0)L_0$, $L_{0-} = 0$, $X_{0-} \perp (W, W^0, \mathbf{P})$, and $(W^0, \mathbf{P}) \perp W$, where (W, W^0) is a two-dimensional Brownian motion, X is a càdlàg process, and \mathbf{P} is a random probability measure on the space of càdlàg paths $D([-1, T+1], \mathbb{R})$.

As the drift and correlation function depend on a flow of measures, we still want them to satisfy some notion of linear growth and Lipschitzness in the measure component. We

will also require some spatial and temporal regularity such that (2.1) is well-posed. We will suppose that our coefficients b , σ , ρ , κ and α satisfy the following assumptions.

Assumption 2.2. (i) (Regularity of b) For all $t \in [0, T]$ and $\mu \in \mathbf{M}_{\leq 1}(\mathbb{R})$, the map $x \mapsto b(t, x, \mu)$ is $\mathcal{C}^2(\mathbb{R})$. Moreover, there exists a constant $C_b > 0$ such that

$$\begin{aligned} |b(t, x, \mu)| &\leq C_b (1 + |x| + \langle \mu, |\cdot| \rangle), \quad |\partial_x^{(n)} b(t, x, \mu)| \leq C_b, \quad n = 1, 2, \\ |b(t, x, \mu) - b(t, x, \tilde{\mu})| &\leq C_b (1 + |x| + \langle \mu, |\cdot| \rangle) d_0(\mu, \tilde{\mu}), \end{aligned}$$

where

$$d_0(\mu, \tilde{\mu}) = \sup \left\{ |\langle \mu - \tilde{\mu}, \psi \rangle| : \|\psi\|_{Lip} \leq 1, |\psi(0)| \leq 1 \right\}$$

for any $\mu, \tilde{\mu} \in \mathbf{M}_{\leq 1}(\mathbb{R})$.

(ii) (Space/time regularity of σ) The map $(t, x) \mapsto \sigma(t, x)$ is $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$. Moreover, there exists a constant $C_\sigma > 0$ such that

$$|\sigma(t, x)| \leq C_\sigma, \quad |\partial_t \sigma(t, x)| \leq C_\sigma, \quad \text{and} \quad |\partial_x^{(n)} \sigma(t, x)| \leq C_\sigma \quad \text{for } n = 1, 2.$$

(iii) (d_1 -Lipschitzness of ρ) For all $t \in [0, T]$, there exists a constant $C_\rho > 0$ such that

$$|\rho(t, \mu) - \rho(t, \tilde{\mu})| \leq C_\rho (1 + \langle \mu, |\cdot| \rangle) d_1(\mu, \tilde{\mu}),$$

where

$$d_1(\mu, \tilde{\mu}) = \sup \left\{ |\langle \mu - \tilde{\mu}, \psi \rangle| : \|\psi\|_{Lip} \leq 1, \|\psi\|_\infty \leq 1 \right\}$$

for any $\mu, \tilde{\mu} \in \mathbf{M}_{\leq 1}(\mathbb{R})$.

(iv) (Non-degeneracy) For all $t \in [0, T]$, $x \in \mathbb{R}$, and $\mu \in \mathbf{M}_{\leq 1}(\mathbb{R})$, the constants C_σ and C_ρ assumed above are such that $0 < C_\sigma^{-1} \leq \sigma(t, x)$ and $0 \leq \rho(t, \mu) \leq 1 - C_\rho^{-1}$.

(v) (Temporal regularity of α) The map $t \mapsto \alpha(t)$ is $\mathcal{C}^1([0, T])$ and increasing with $\alpha(0) \geq 0$.

(vi) (Sub-Gaussian initial law) The initial law, ν_{0-} is sub-Gaussian,

$$\exists \gamma > 0 \quad \text{s.t.} \quad \nu_{0-}(\lambda, \infty) = O(e^{-\gamma \lambda^2}) \quad \text{as } \lambda \rightarrow \infty,$$

and has a density $V_{0-} \in L^2(0, \infty)$ such that $\|x V_{0-}\|_{L^2}^2 = \int_0^\infty |x V_{0-}(x)|^2 dx < \infty$.

(vii) (Regularity of mollifier) The function $\kappa \in \mathcal{W}_0^{1,1}(\mathbb{R}_+)$, the Sobolev space with one weak derivative in L^1 and zero at 0, such that κ is non-negative, and $\|\kappa\|_1 = 1$.

Under Assumption 2.2, [18] showed existence and uniqueness of solutions to a stochastic partial differential equation (SPDE) that any solution to (2.1) will satisfy.

Theorem 2.3 ([18, Theorem 2.6]). There is a unique strong solution to the SPDE

$$\begin{aligned} d\langle \mu_t, \phi \rangle &= \langle \mu_t, b(t, \cdot, \mu_t) \partial_x \phi \rangle dt + \frac{1}{2} \langle \mu_t, \sigma(t, \cdot)^2 \partial_{xx} \phi \rangle dt \\ &\quad + \langle \mu_t, \sigma(t, \cdot) \rho(t, \mu_t) \partial_x \phi \rangle dW_t^0 - \langle \mu_t, \alpha(t) \partial_x \phi \rangle d\mathfrak{L}_t^\varepsilon, \\ \mathfrak{L}_t^\varepsilon &:= \int_0^t \kappa^\varepsilon(t-s) L_s ds, \\ L_t &:= 1 - \mu_t(0, \infty), \end{aligned} \tag{2.5}$$

where the coefficients b , σ , ρ , κ^ε , and α satisfy Assumption 2.2, and $\phi \in \mathcal{C}_0$, the set of Schwartz functions that are zero at 0.

From [18, Theorem 2.6], we can deduce the existence of solutions to (2.1).

Theorem 2.4 ([18, Theorem 2.7]). *Let (ν^ε, W^0) be the unique strong solution to the SPDE (2.5). Then, for any Brownian motion $W \perp (X_{0-}, W^0)$, we have*

$$\nu_t^\varepsilon = \mathbb{P} [X_t^\varepsilon \in \cdot, \tau^\varepsilon > t \mid W^0],$$

where X^ε is the solution to the conditional McKean–Vlasov diffusion

$$\begin{cases} dX_t^\varepsilon = b(t, X_t^\varepsilon, \nu_t^\varepsilon) dt + \sigma(t, X_t^\varepsilon) \sqrt{1 - \rho(t, \nu_t^\varepsilon)^2} dW_t + \sigma(t, X_t^\varepsilon) \rho(t, \nu_t^\varepsilon) dW_t^0 \\ \quad - \alpha(t) d\mathfrak{L}_t^\varepsilon, \\ \mathbf{P}^\varepsilon = \mathbb{P} [X^\varepsilon \in \cdot \mid W^0], \\ L_t^\varepsilon = \mathbb{P} [\tau^\varepsilon \leq t \mid W^0], \quad \mathfrak{L}_t^\varepsilon = \int_0^t \kappa^\varepsilon(t-s) L_s^\varepsilon ds, \\ \tau^\varepsilon = \inf\{t > 0 : X_t^\varepsilon \leq 0\}, \end{cases}$$

with initial condition $X_{0-} \sim \nu_{0-}$.

With [18, Theorem 2.6] and [18, Theorem 2.7], it can be inferred that solutions to (2.1) are unique.

The existence of solutions to (2.1) allows us to introduce the main result of this section, showing that solutions to (2.4) exist as limit points of the collection of smoothed equations.

Theorem 2.5 (Existence and convergence generalised). *Let \tilde{X}^ε be the extended version of X^ε in (2.1) and set $\tilde{\mathbf{P}}^\varepsilon = \text{Law}(\tilde{X}^\varepsilon \mid W^0)$. Then, the family of random tuples $\{(\tilde{\mathbf{P}}^\varepsilon, W^0, W)\}_{\varepsilon>0}$ is tight. Any subsequence $\{(\tilde{\mathbf{P}}^{\varepsilon_n}, W^0, W)\}_{n \geq 1}$, for a positive sequence $(\varepsilon_n)_{n \geq 1}$ which converges to zero, has a further subsequence which converges weakly to some (\mathbf{P}, W^0, W) . Here, W^0 and W are standard Brownian motions, \mathbf{P} is a random probability measure $\mathbf{P} : \Omega \rightarrow \mathcal{P}(D_{\mathbb{R}})$.*

Given this limit point, there is a background space which carries a stochastic process X such that (X, W^0, W, \mathbf{P}) is a relaxed solution to (2.4) in the sense of Definition 2.1. X_{0-} , W and (W^0, \mathbf{P}) are all mutually independent and

$$\Delta L_t = \mathbb{P} \left[\inf_{s < t} X_s > 0, X_t \leq 0 \mid W^0, \mathbf{P} \right] \leq \inf \{x \geq 0 : \nu_{t-}([0, \alpha(t)x]) < x\} \quad \text{a.s.} \quad (2.6)$$

for all $t \geq 0$.

The notation $\text{Law}(\tilde{X}^\varepsilon \mid W^0)$ stands for the conditional law of \tilde{X}^ε given W^0 , which indeed defines a random W^0 -measurable probability measure on $D([-1, T+1], \mathbb{R})$. Under stronger assumptions, namely b , σ , and ρ being of the form $(t, x) \mapsto b(t, x)$, $t \mapsto \sigma(t)$, $t \mapsto \rho(t)$ and α is a positive constant, there are established results in the literature for a lower bound on the jumps of the loss function. By Proposition 3.5 in [25], the jumps of the loss satisfy

$$\Delta L_t \geq \inf \{x \geq 0 : \nu_{t-}([0, \alpha x]) < x\} \quad \text{a.s.}$$

Due to the generality of the coefficients, we were not able to establish if (2.6) holds with equality. The primary reason is the lack of independence between the term driven by the idiosyncratic noise and the remainder of the terms that X is composed of. Hence, the technique employed in [25, Proposition 3.5] may not be readily applied or extended to our setting. Regardless, given these two results, under stronger assumptions, we have the following existence result.

Corollary 2.6 (Existence of physical solutions). *Let the coefficients b , σ , and ρ be of the form $(t, x) \mapsto b(t, x)$, $t \mapsto \sigma(t)$, $t \mapsto \rho(t)$ and satisfy Assumption 2.2. Then provided*

$\alpha(t) \equiv \alpha > 0$ and constant, there exists a relaxed solution to

$$\begin{cases} dX_t = b(t, X_t) dt + \sigma(t) \sqrt{1 - \rho(t)^2} dW_t + \sigma(t) \rho(t) dW_t^0 - \alpha dL_t, \\ \tau = \inf\{t > 0 : X_t \leq 0\}, \\ \mathbf{P} = \mathbb{P}[X \in \cdot | W^0, \mathbf{P}], \quad \nu_t := \mathbb{P}[X_t \in \cdot, \tau > t | W^0, \mathbf{P}], \\ L_t = \mathbb{P}[\tau \leq t | W^0, \mathbf{P}]. \end{cases} \quad (2.7)$$

Moreover, we have the minimal jump constraint

$$\Delta L_t = \inf\{x \geq 0 : \nu_{t-}([0, \alpha x]) < x\} \quad \text{a.s.}$$

for all $t \geq 0$. This determines the jump sizes of L .

This presents a minor generalisation of the results in [25]. In their work, the authors imposed the condition that $t \mapsto \sigma(t)\rho(t)$ must be Hölder continuous with an exponent strictly greater than $1/2$. This was necessary in [25, Lemma 3.16] to deduce the independence between the idiosyncratic noise and the common noise in the limiting system (2.7) due to their particle system approach to the problem. Here, such explicit assumptions on the regularity of ρ are not necessary, as we obtain the independence between the idiosyncratic noise and the common noise from the approximating system without additional conditions. Consequently, we can consider Corollary 2.6 to be an extension of Theorem 3.2 in [25].

2.1 Limit points of the smoothed system

In order to show the existence of a limit point of X^ε , we must first choose a suitable topology to establish convergence. By Theorem 2.4 in [18], L^ε is continuous for every $\varepsilon > 0$, but the loss of the limiting process may in fact jump. Skorokhod's M_1 -topology is sufficiently rich to facilitate the convergence of continuous functions to those with jumps.

Remark 2.7. The equation (1.1) has been posed in slightly more generality (with $\alpha(t, X_t^\varepsilon, \nu_t^\varepsilon)$) in [18]. As the convergence is strictly in the M_1 -topology, not in the J_1 -topology, we only consider α to be of the form $\alpha(t)$. If we considered it to also be a function of X_t and/or ν_t , then we cannot expect to obtain an equation of the form (1.2) in the limit as $\varepsilon \downarrow 0$, due to X , ν , and L having jumps at the same time.

The theory in [33] requires our càdlàg processes to be uniformly right-continuous at the initial time point and left-continuous at the terminal time point, when working with functions on compact time domains. As we are starting from an arbitrary initial condition X_{0-} which is positive almost surely, the limiting process may exhibit a jump immediately at time 0 given sufficient mass near the boundary. For this reason, we shall embed the process from $D([0, T], \mathbb{R})$ into $D([-1, \bar{T}], \mathbb{R})$, where $\bar{T} = T + 1$, using the extension defined in (2.3). Unless stated otherwise, for notational convenience we shall denote the latter space, $D([-1, \bar{T}], \mathbb{R})$, by $D_{\mathbb{R}}$. Recall, $\tilde{\mathbf{P}}^\varepsilon$ is defined to be the law of \tilde{X}^ε conditional on W^0 . That is $\tilde{\mathbf{P}}^\varepsilon := \text{Law}(\tilde{X}^\varepsilon | W^0)$.

To show tightness and convergence of the collection of random measures $\{\tilde{\mathbf{P}}^\varepsilon\}_{\varepsilon > 0}$, we shall follow the ideas in [10] and [25]. To begin, we first derive a Grönwall-type estimate of the smoothed system uniformly in ε . These estimates are necessary to show the tightness of $\{\tilde{\mathbf{P}}^\varepsilon\}_{\varepsilon > 0}$ and the existence of a limiting random measure. In the following Proposition and its sequels, C will denote a constant that may change from line to line, and we will denote the dependencies of the value of C in its subscript. To further simplify notation, we use Y_t^ε , $Y_t^{0,\varepsilon}$, and $\mathcal{Y}_t^\varepsilon$ to denote

$$\int_0^t \sigma(u, X_u^\varepsilon) \sqrt{1 - \rho^2(u, \nu_u^\varepsilon)} dW_u, \quad \int_0^t \sigma(u, X_u^\varepsilon) \rho(u, \nu_u^\varepsilon) dW_u^0, \quad \text{and} \quad Y_t^\varepsilon + Y_t^{0,\varepsilon}$$

respectively. We shall use \tilde{Y}_t^ε , $\tilde{Y}_t^{0,\varepsilon}$, and $\tilde{\mathcal{Y}}_t^\varepsilon$ to denote their corresponding extensions as defined in (2.3).

Proposition 2.8 (Grönwall upper bound). *For any $p \geq 1$ and $t \leq T$, there exists a constant $C_{\alpha,b,p,T,\sigma} > 0$ independent of $\varepsilon > 0$ such that*

$$\mathbb{E} \left[\sup_{s \leq T} |X_s^\varepsilon|^p \right] \leq C_{\alpha,b,p,T,\sigma}. \quad (2.8)$$

Proof. By the linear growth condition on b and the triangle inequality, we have

$$|X_t^\varepsilon| \leq |X_{0-}| + C_b \int_0^t 1 + \sup_{u \leq s} |X_u^\varepsilon| + \mathbb{E} [|X_{s \wedge \tau^\varepsilon}^\varepsilon| |W^0|] ds + \sup_{s \leq T} |\mathcal{Y}_s^\varepsilon| + \|\alpha\|_\infty.$$

By [18, Lemma A.3], $\int_0^T \mathbb{E} |X_{s \wedge \tau^\varepsilon}^\varepsilon|^p ds < \infty$ for any $p \geq 1$. Therefore, a simple application of Grönwall's inequality shows that

$$\sup_{s \leq t} |X_s^\varepsilon| \leq C_{T,b,\alpha} \left(|X_{0-}| + \int_0^t \mathbb{E} [|X_{s \wedge \tau^\varepsilon}^\varepsilon| |W^0|] ds + \sup_{s \leq T} |\mathcal{Y}_s^\varepsilon| + 1 \right).$$

By (vi) in Assumption 2.2, X_{0-} has finite L^p -moments for every $p > 0$. Furthermore, by employing the Burkholder-Davis-Gundy inequality to control $\mathbb{E}[\sup_{s \leq T} |\mathcal{Y}_s^\varepsilon|]$, we may deduce that $\mathbb{E}[\sup_{s \leq t} |X_s^\varepsilon|^p] < \infty$ for all $t \geq 0$ and $p \geq 1$. Now, observing that $|X_{t \wedge \tau^\varepsilon}^\varepsilon| \leq \sup_{s \leq t} |X_s^\varepsilon|$, we have by the monotonicity of expectation and Jensen's inequality that

$$\begin{aligned} \sup_{s \leq t} |X_s^\varepsilon|^p &\leq C_{T,b,\alpha}^p \left(|X_{0-}| + \int_0^t \mathbb{E} [|X_{s \wedge \tau^\varepsilon}^\varepsilon| |W^0|] ds + \sup_{s \leq T} |\mathcal{Y}_s^\varepsilon| + 1 \right)^p \\ &\leq C_{T,b,\alpha,p} \left(|X_{0-}|^p + \int_0^t \mathbb{E} \left[\sup_{u \leq s} |X_u^\varepsilon|^p \mid W^0 \right] ds + \sup_{s \leq T} |\mathcal{Y}_s^\varepsilon|^p + 1 \right). \end{aligned}$$

Taking expectations and applying Fubini's theorem followed by Grönwall's inequality, we obtain

$$\mathbb{E} \left[\sup_{s \leq T} |X_s^\varepsilon|^p \right] \leq C_{T,b,\alpha,p} \mathbb{E} \left[|X_{0-}|^p + \sup_{s \leq T} |\mathcal{Y}_s^\varepsilon|^p + 1 \right]. \quad (2.9)$$

Lastly, by the Burkholder-Davis-Gundy inequality and (ii) from Assumption 2.2, we may bound (2.9) independently of ε . This completes the proof. \square

The collection $\{\tilde{\mathbf{P}}^\varepsilon\}_{\varepsilon>0}$ consists of $\mathcal{P}(D_{\mathbb{R}})$ -valued random measures. To show that this collection is tight, we need to find, for any $\gamma > 0$, a compact set K_γ in $\mathcal{P}(D_{\mathbb{R}})$ so that $\text{Law}(\tilde{\mathbf{P}}^\varepsilon)(K_\gamma) = \mathbb{P}(\tilde{\mathbf{P}}^\varepsilon \in K_\gamma) \leq \gamma$ for all $\varepsilon > 0$. In other words, we need the probability measures $\{\text{Law}(\tilde{\mathbf{P}}^\varepsilon)\}_{\varepsilon>0}$ to be tight in $\mathcal{P}(\mathcal{P}(D_{\mathbb{R}}))$. Rather than tackling this directly, we follow an indirect approach of first showing that the measures $\{\text{Law}(\tilde{X}^\varepsilon)\}_{\varepsilon>0}$ are tight in $\mathcal{P}(D_{\mathbb{R}})$ and then we construct K_γ from there. Due to how the processes are defined, the latter follows easily from properties of the M_1 -topology and [1, Theorem 1].

Proposition 2.9 (Tightness of smoothed random measures). *Let $\mathfrak{T}_{M_1}^{\text{wk}}$ denote the topology of weak convergence on $\mathcal{P}(D_{\mathbb{R}})$ induced by the M_1 -topology on $D_{\mathbb{R}}$. Then, the collection $\{\tilde{\mathbf{P}}^\varepsilon\}_{\varepsilon>0} = \{\text{Law}(\tilde{X}^\varepsilon \mid W^0)\}_{\varepsilon>0}$ is tight on $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$ under Assumption 2.2.*

Proof. Define $\tilde{P}^\varepsilon := \text{Law}(\tilde{X}^\varepsilon)$. By [33, Theorem 12.12.3], we need to verify two conditions to show the tightness of the measures on $D_{\mathbb{R}}$ endowed with the M_1 -topology:

- (i) $\lim_{\lambda \rightarrow \infty} \sup_{\varepsilon>0} \tilde{P}^\varepsilon(\{x \in D_{\mathbb{R}} : \|x\|_\infty > \lambda\}) = 0$.
- (ii) For any $\eta > 0$, we have $\lim_{\delta \rightarrow 0} \sup_{\varepsilon>0} \tilde{P}^\varepsilon(\{x \in D_{\mathbb{R}} : w_{M_1}(x, \delta) \geq \eta\}) = 0$, where w_{M_1} is the oscillatory function of the M_1 -topology, defined as in [33, Section 12, Equation 12.2].

To show the first condition, we observe that by the definition of the extension of our process, we have

$$\sup_{t \leq \bar{T}} |\tilde{X}_t^\varepsilon| \leq \sup_{t \leq T} |X_t^\varepsilon| + \sup_{t \leq 1} |W_{T+t} - W_T| + \|\alpha\|_\infty.$$

Then, it is clear by Markov's inequality and Proposition 2.8 that for any $\lambda > 0$,

$$\mathbb{P} \left[\sup_{t \leq \bar{T}} |\tilde{X}_t^\varepsilon| > \lambda \right] = O(\lambda^{-1})$$

uniformly in ε . Therefore, by taking the supremum over ε and then the \limsup over λ , the first condition holds. We shall not show the second condition directly. By [1, Theorem 1], the second condition is equivalent to showing:

- (I) There exists some $C > 0$, uniformly in ε , such that $\mathbb{P} \left[\mathcal{H}_R(\tilde{X}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon, \tilde{X}_{t_3}^\varepsilon) \geq \eta \right] \leq C\eta^{-4}|t_3 - t_1|^2$ for all $\eta > 0$ and $-1 \leq t_1 \leq t_2 \leq t_3 \leq \bar{T}$ where $\mathcal{H}_R(x_1, x_2, x_3) = \inf_{\lambda \in [0,1]} |x_2 - (1-\lambda)x_1 - \lambda x_3|$.
- (II) $\lim_{\delta \rightarrow 0} \sup_{\varepsilon > 0} \mathbb{P} \left[\sup_{t \in (-1, -1+\delta)} |\tilde{X}_t^\varepsilon - \tilde{X}_{-1}| + \sup_{t \in (\bar{T}-\delta, \bar{T})} |\tilde{X}_t^\varepsilon - \tilde{X}_{\bar{T}}| \geq \eta \right] = 0$ for all $\eta > 0$.

Note that by Assumption 2.2, we have α is non-decreasing and non-negative. Therefore, by the properties of Lebesgue-Stieltjes integration, $t \mapsto \int_0^t \alpha(s) d\mathfrak{L}_s^\varepsilon$ is non-decreasing. As monotone functions are immaterial to the M_1 modulus of continuity, we have

$$\mathcal{H}_R(\tilde{X}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon, \tilde{X}_{t_3}^\varepsilon) \leq |Z_{t_1} - Z_{t_2}| + |Z_{t_2} - Z_{t_3}|,$$

where Z is given by

$$Z_t = X_{0-} + \int_0^{t \wedge T} b(u, X_u^\varepsilon, \nu_u^\varepsilon) du + \tilde{\mathcal{Y}}_t$$

for $t \geq 0$ and $Z_t = X_{0-}$ for $t < 0$. Hence, to show (I), it is sufficient to bound the increments of Z . Note that when $s < t < -1$, Z is constant. Therefore, trivially we have $\mathbb{E} \left[|Z_t - Z_s|^4 \right] \leq C(t-s)^2$ for any $C > 0$. When $0 \leq s < t$, by the formula above for Z we have that

$$Z_t - Z_s = \int_{s \wedge T}^{t \wedge T} b(u, X_u^\varepsilon, \nu_u^\varepsilon) du + \tilde{\mathcal{Y}}_t - \tilde{\mathcal{Y}}_s. \quad (2.10)$$

Employing the linear growth condition on b and Proposition 2.8,

$$\mathbb{E} \left[\left| \int_{s \wedge T}^{t \wedge T} b(u, X_u^\varepsilon, \nu_u^\varepsilon) du \right|^4 \right] \leq C(t-s)^4 \left(1 + \mathbb{E} \left[\sup_{u \leq T} |X_u^\varepsilon|^4 \right] \right) = O((t-s)^2) \quad (2.11)$$

uniformly in ε . By the Burkholder-Davis-Gundy inequality and the upper bound on σ , it is clear that

$$\mathbb{E} \left[|\tilde{\mathcal{Y}}_t - \tilde{\mathcal{Y}}_s|^4 \right] = O((t-s)^2) \quad (2.12)$$

uniformly in ε . Therefore, by Markov's inequality,

$$\begin{aligned} \mathbb{P} \left[\mathcal{H}_R(\tilde{X}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon, \tilde{X}_{t_3}^\varepsilon) \geq \eta \right] &\leq \eta^{-4} \mathbb{E} \left[\mathcal{H}_R(\tilde{X}_{t_1}^\varepsilon, \tilde{X}_{t_2}^\varepsilon, \tilde{X}_{t_3}^\varepsilon)^4 \right] \\ &\leq C\eta^{-4} \mathbb{E} \left[|Z_{t_1} - Z_{t_2}|^4 + |Z_{t_2} - Z_{t_3}|^4 \right] \\ &\leq C\eta^{-4} ((t_2 - t_1)^2 + (t_3 - t_2)^2) \\ &\leq C\eta^{-4} (t_3 - t_1)^2, \end{aligned}$$

where all the constants hold uniformly in ε . To verify the second condition, we observe that for any $\eta > 0$ and $\delta < 1$,

$$\begin{aligned} \mathbb{P} \left[\sup_{t \in (-1, -1+\delta)} |\tilde{X}_t^\varepsilon - \tilde{X}_{-1}^\varepsilon| \geq \frac{\eta}{2} \right] &= \mathbb{P} \left[\sup_{t \in (-1, -1+\delta)} |X_{0-} - X_{0-}| \geq \frac{\eta}{2} \right] = 0, \text{ and} \\ \mathbb{P} \left[\sup_{t \in (\bar{T}-\delta, \bar{T})} |\tilde{X}_t^\varepsilon - \tilde{X}_{\bar{T}}^\varepsilon| \geq \frac{\eta}{2} \right] &= \mathbb{P} \left[\sup_{t \in (\bar{T}-\delta, \bar{T})} |W_t - W_{\bar{T}}| \geq \frac{\eta}{2} \right] = O(\delta^2), \end{aligned}$$

uniformly in ε . Hence, we have shown that

$$\mathbb{P} \left[\sup_{t \in (-1, -1+\delta)} |\tilde{X}_t^\varepsilon - \tilde{X}_{-1}^\varepsilon| + \sup_{t \in (\bar{T}-\delta, \bar{T})} |\tilde{X}_t^\varepsilon - \tilde{X}_{\bar{T}}^\varepsilon| \geq \eta \right] = O(\delta^2) \quad \text{for all } \eta > 0, \delta < 1.$$

Therefore, together, conditions (I) and (II) show

$$\sup_{\varepsilon > 0} \tilde{P}^\varepsilon(\{x \in D_{\mathbb{R}} : w_{M_1}(x, \delta) \geq \eta\}) = O(\delta^2),$$

for all $\delta < 1$ uniformly in ε . This shows condition (ii).

Based on the above, we can now employ Markov's inequality and Prokhorov's theorem to deduce that $\{\text{Law}(\tilde{\mathbf{P}}^\varepsilon)\}_{\varepsilon > 0}$ is tight in $\mathcal{P}(\mathcal{P}(D_{\mathbb{R}}))$. To begin, fix a $\gamma > 0$. For any $l, k \in \mathbb{N}$, we may find a $\lambda_l, \delta_{k,l} > 0$ such that

$$\begin{aligned} \tilde{P}^\varepsilon(A_{k,l}^{\mathbb{G}}) &< \gamma 2^{-(k+2l+1)} \quad \forall k \in \mathbb{N} \quad \text{uniformly in } \varepsilon, \\ \text{where} \quad A_{0,l} &= \{x \in D_{\mathbb{R}} : \|x\| \leq \lambda_l\}, \\ A_{k,l} &= \left\{ x \in D_{\mathbb{R}} : w_{M_1}(x, \delta_{k,l}) < \frac{1}{k+2l} \right\}. \end{aligned}$$

We define $A_l = \cap_{k \geq 0} A_{k,l}$. By [33, Theorem 12.12.2], A_l has compact closure in the M_1 -topology. The closure of A_l is denoted by \bar{A}_l . Furthermore, by construction $\tilde{P}^\varepsilon(A_l^{\mathbb{G}}) \leq \sum_{k \geq 0} \tilde{P}^\varepsilon(A_{k,l}^{\mathbb{G}}) \leq \gamma 4^{-l}$. By the subadditivity of measures and Markov's inequality,

$$\mathbb{P} \left[\bigcup_{l=1}^{\infty} \left\{ \tilde{\mathbf{P}}^\varepsilon(A_l^{\mathbb{G}}) > 2^{-l} \right\} \right] \leq \sum_{l \geq 1} \mathbb{P} \left[\tilde{\mathbf{P}}^\varepsilon(A_l^{\mathbb{G}}) > 2^{-l} \right] \leq \sum_{l \geq 1} 2^l \mathbb{E} \left[\tilde{\mathbf{P}}^\varepsilon(A_l^{\mathbb{G}}) \right] \leq \sum_{l \geq 1} \frac{\gamma 2^l}{4^l} = \gamma. \quad (2.13)$$

Finally, let $K := \{\mu \in \mathcal{P}(D_{\mathbb{R}}) : \mu(\bar{A}_l^{\mathbb{G}}) \leq 2^{-l} \forall l \in \mathbb{N}\}$. As $D_{\mathbb{R}}$ endowed with the M_1 -topology is a Polish space, Prokhorov's theorem may be applied, and it suffices to show that the set of measures K is tight; hence, K will then have compact closure in $\mathcal{P}(D_{\mathbb{R}})$ by Prokhorov's theorem. It is clear by construction that the set of measures K is tight as the sets \bar{A}_l are compact in $D_{\mathbb{R}}$ endowed with the M_1 -topology. By (2.13), we have $\text{Law}(\tilde{\mathbf{P}}^\varepsilon)(\bar{K}^{\mathbb{G}}) \leq \gamma$, uniformly in ε . As γ was arbitrary, this completes the proof. \square

2.2 Continuity of hitting times

Note that $(D_{\mathbb{R}}, M_1)$ is a Polish space by [33, Theorem 12.8.1] and its Borel σ -algebra is generated by the marginal projections, [33, Theorem 11.5.2]. Hence, the topological space $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$ is also a Polish space. Therefore, by invoking Prokhorov's Theorem, [2, Theorem 5.1], tightness is equivalent to being sequentially pre-compact. So, we may choose a weakly convergent subsequence $\{\tilde{\mathbf{P}}^{\varepsilon_n}\}_{n \geq 1}$ for a positive sequence $(\varepsilon_n)_{n \geq 1}$ which converges to zero. Let \mathbf{P}^* denote the limit point of this sequence. Using this limit point, we will construct a probability space and a stochastic process that will be a solution to (2.4).

Before proceeding, we seek to show that for a co-countable set of times t , $L_t^{\varepsilon_n} = \tilde{\mathbf{P}}^{\varepsilon_n}(\tau_0(\eta) \leq t)$ converges weakly to $L_t^* := \mathbf{P}^*(\tau_0(\eta) \leq t)$, where τ_0 is a function on $D_{\mathbb{R}}$ whose value is the first hitting time of 0. To be explicit, we define

$$\mathbb{T} := \{t \in [-1, \bar{T}] : \mathbb{E}[\mathbf{P}^*(\eta_t = \eta_{t-})] = 1, \mathbb{E}[\mathbf{P}^*(\tau_0(\eta) = t)] = 0\},$$

and

$$\tau_0(\eta) := \inf\{t \geq -1 : \eta_t \leq 0\} \quad (2.14)$$

with the convention that $\inf\{\emptyset\} = \bar{T}$. Our first result is that for $\text{Law}(\mathbf{P}^*)$ -almost every measure μ , μ -almost every path $\eta \in D_{\mathbb{R}}$ is constant on the interval $[-1, 0)$.

Lemma 2.10. *For $\text{Law}(\mathbf{P}^*)$ -almost every measure μ , we have that $\sup_{s < 0} |\eta_s - \eta_{-1}| = 0$ for μ -almost every path η .*

Proof. As $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$ is a Polish space, we may apply Skorokhod's Representation Theorem [2, Theorem 6.7]. Hence, there exists a common probability space, and $\mathcal{P}(D_{\mathbb{R}})$ -valued random variables $(\mathbf{Q}^n)_{n \geq 1}$ and \mathbf{Q}^* such that

$$\text{Law}(\mathbf{Q}^n) = \text{Law}(\tilde{\mathbf{P}}^{\varepsilon_n}), \quad \text{Law}(\mathbf{Q}^*) = \text{Law}(\mathbf{P}^*), \quad \text{and} \quad \mathbf{Q}^n \rightarrow \mathbf{Q}^* \quad \text{a.s.}$$

It is straightforward to see-by [33, Theorem 13.4.1]-the following maps from $D_{\mathbb{R}}$ into itself

$$\eta \mapsto \left(t \mapsto \inf_{s \leq t} \{\eta_s - \eta_{-1}\} \right) \quad \eta \mapsto \left(t \mapsto \sup_{s \leq t} \{\eta_s - \eta_{-1}\} \right)$$

are continuous. Now, for a $t \in \mathbb{T} \cap (-1, 0)$, the maps c_t and \tilde{c}_t from $D_{\mathbb{R}}$ onto \mathbb{R} such that

$$c_t(\eta) = \inf_{s \leq t} \{\eta_s - \eta_{-1}\} \quad \tilde{c}_t(\eta) = \sup_{s \leq t} \{\eta_s - \eta_{-1}\}$$

are continuous. Therefore, by the Continuous Mapping Theorem, $c_t^{\#} \mathbf{Q}^n \rightarrow c_t^{\#} \mathbf{Q}^*$ and $\tilde{c}_t^{\#} \mathbf{Q}^n \rightarrow \tilde{c}_t^{\#} \mathbf{Q}^*$ almost surely in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$.

Fix a $\gamma > 0$. Then by the Portmanteau Theorem and Fatou's lemma, we have

$$\begin{aligned} \mathbb{E} \left[\mathbf{P}^* \left(\inf_{s \leq t} \{\eta_s - \eta_{-1}\} < -\gamma \right) \right] &= \mathbb{E} \left[\mathbf{Q}^* \left(\inf_{s \leq t} \{\eta_s - \eta_{-1}\} < -\gamma \right) \right] \\ &\leq \mathbb{E} \left[\liminf_{n \rightarrow \infty} \mathbf{Q}^n \left(\inf_{s \leq t} \{\eta_s - \eta_{-1}\} < -\gamma \right) \right] \\ &\leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[\mathbf{Q}^n \left(\inf_{s \leq t} \{\eta_s - \eta_{-1}\} < -\gamma \right) \right] \\ &= \liminf_{n \rightarrow \infty} \mathbb{E} \left[\tilde{\mathbf{P}}^{\varepsilon_n} \left(\inf_{s \leq t} \{\eta_s - \eta_{-1}\} < -\gamma \right) \right] \\ &= \liminf_{n \rightarrow \infty} \tilde{P}^{\varepsilon_n} \left(\inf_{s \leq t} \{\eta_s - \eta_{-1}\} < -\gamma \right) = 0, \end{aligned}$$

where the last equality follows from the embedding of X^{ε_n} from $D([0, T], \mathbb{R})$ into $D([-1, \bar{T}], \mathbb{R})$. So, by continuity of measure and the Monotone Convergence Theorem, as γ was arbitrary, we have

$$\mathbb{E} \left[\mathbf{P}^* \left(\inf_{s \leq t} \{\eta_s - \eta_{-1}\} < 0 \right) \right] = 0.$$

Similarly, $\mathbb{E} [\mathbf{P}^* (\sup_{s \leq t} \{\eta_s - \eta_{-1}\} > 0)] = 0$. \square

As \tilde{X}^ε is fundamentally a time-changed Brownian motion with drift, it is not hard to show that, with probability one, \tilde{X}^ε will take a negative value on any open neighbourhood of its first hitting time of zero. This property is preserved by weak convergence for almost every realisation of \mathbf{P}^* . Furthermore, as the Lebesgue-Stieltjes integral $\int_0^t \alpha(s) d\mathcal{L}_s^\varepsilon$ takes non-negative values, by weak convergence we expect that for $\text{Law}(\mathbf{P}^*)$ -almost every measure μ , μ -almost every path $\eta \in D_{\mathbb{R}}$ will have only downward jumps.

Lemma 2.11 (Strong crossing property). *For any $h > 0$, we have*

$$\mathbb{E} \left[\mathbf{P}^* \left(\inf_{s \in (\tau_0(\eta), (\tau_0(\eta)+h) \wedge \bar{T})} \{ \eta_s - \eta_{\tau_0(\eta)} \} \geq 0, \tau_0(\eta) < \bar{T} \right) \right] = 0, \quad (2.15)$$

$$\mathbb{E} [\mathbf{P}^* (\eta : \Delta \eta_t \leq 0 \quad \forall t \leq \bar{T})] = 1. \quad (2.16)$$

Proof. As with the space of càdlàg functions, we shall employ the shorthand notation $\mathcal{C}_{\mathbb{R}}$ for this proof to denote $\mathcal{C}([-1, \bar{T}], \mathbb{R})$. Now, as σ is non-degenerate and bounded by assumption, by the Kolmogorov-Chentsov Tightness Criterion, [23] and [7], we have that $(\tilde{\mathcal{Y}}^\varepsilon)_{\varepsilon>0}$ is tight. Additionally, we define the random variable $\tilde{Z}^\varepsilon := \langle \tilde{\mathbf{P}}^\varepsilon, \sup_{u \leq \bar{T}} |\eta_u| \rangle$. By definition of $\tilde{\mathbf{P}}^\varepsilon$, $\mathbb{E}[\tilde{Z}^\varepsilon] = \mathbb{E}[\sup_{u \leq \bar{T}} |\tilde{X}_u^\varepsilon|]$. Therefore, $\mathbb{E}[\tilde{Z}^\varepsilon]$ is uniformly bounded by Proposition 2.8 and hence $\{\tilde{Z}^\varepsilon\}_{\varepsilon>0}$ is tight on \mathbb{R} .

As marginal tightness implies joint tightness, we have $\mathbb{P}_{x,y,z}^\varepsilon := \text{Law}(\tilde{X}^\varepsilon, \tilde{\mathcal{Y}}^\varepsilon, \tilde{Z}^\varepsilon)$ is tight in $\mathcal{P}(D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \times \mathbb{R})$ by Proposition 2.9. Given a suitable subsequence, also denoted by $(\varepsilon_n)_{n \geq 1}$ for simplicity, we have $\tilde{\mathbf{P}}^{\varepsilon_n} \Rightarrow \mathbf{P}^*$ and $\mathbb{P}_{x,y,z}^{\varepsilon_n} \Rightarrow \mathbb{P}_{x,y,z}^*$. Here \mathbb{P}_x^* and \mathbb{P}_y^* are used to denote the first and second marginal respectively.

Intuitively, $\mathbb{E}[\mathbf{P}^*(\cdot)]$ and \mathbb{P}_x^* should have the same law as we are averaging over the stochasticity inherited by the common noise. By definition of \mathbf{P}^ε and $\mathbb{P}_{x,y,z}^\varepsilon$, for any continuous bounded function $f : D_{\mathbb{R}} \rightarrow \mathbb{R}$, we have that

$$\mathbb{E} [\langle \mathbf{P}^{\varepsilon_n}, f \rangle] = \langle \mathbb{P}_{x,y,z}^{\varepsilon_n}, f \rangle.$$

As $D_{\mathbb{R}}$ is a Polish space, by a Monotone Class Theorem argument and Dynkin's Lemma, we have

$$\mathbb{E} [\mathbf{P}^*(A)] = \mathbb{P}_x^*(A) = \mathbb{P}_{x,y,z}^*(A \times \mathcal{C}_{\mathbb{R}} \times \mathbb{R}) \quad \forall A \in \mathcal{B}(D_{\mathbb{R}}). \quad (2.17)$$

Define the canonical processes X^* , Y^* and Z^* on $(D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty) \times (\mathbb{R}, |\cdot|)$, where for $(\eta, \omega, z) \in D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \times \mathbb{R}$, $X^*(\eta, \omega, z) = \eta$, $Y^*(\eta, \omega, z) = \omega$ and $Z^*(\eta, \omega, z) = z$. By considering the parametric representations, the map $\eta \mapsto \sup_{u \leq \bar{T}} |\eta_u|$ is M_1 -continuous for any $\eta \in D_{\mathbb{R}}$. Hence, by the linear growth condition on b , the Continuous Mapping Theorem, and the Portmanteau Theorem, for any $s, t \in \mathbb{T}$ with $s < t$ and $\gamma > 0$,

$$\begin{aligned} & \mathbb{P}_{x,y,z}^* \left(X_t^* - X_s^* \leq Y_t^* - Y_s^* + C_b(t-s)(1 + \sup_{u \leq \bar{T}} |X_u^*| + Z^*) + \gamma \right) \\ & \geq \limsup_{n \rightarrow \infty} \mathbb{P} \left[\tilde{X}_t^{\varepsilon_n} - \tilde{X}_s^{\varepsilon_n} \leq \tilde{\mathcal{Y}}_t^{\varepsilon_n} - \tilde{\mathcal{Y}}_s^{\varepsilon_n} + C_b(t-s)(1 + \sup_{u \leq \bar{T}} |\tilde{X}_u^{\varepsilon_n}| + \tilde{Z}^{\varepsilon_n}) + \gamma \right] \\ & = 1. \end{aligned} \quad (2.18)$$

The last equality follows from the fact that for any ε_n

$$\begin{aligned} \tilde{X}_t^{\varepsilon_n} - \tilde{X}_s^{\varepsilon_n} &= \tilde{\mathcal{Y}}_t^{\varepsilon_n} - \tilde{\mathcal{Y}}_s^{\varepsilon_n} + \int_{(s \vee 0) \wedge T}^{(t \vee 0) \wedge T} b(u, X_u^{\varepsilon_n}, \nu_u^{\varepsilon_n}) \mathbf{1}_{[0, T]}(u) du - \int_{(s \vee 0) \wedge T}^{(t \vee 0) \wedge T} \alpha(u) d\mathcal{L}_u^{\varepsilon_n} \\ &\leq \tilde{\mathcal{Y}}_t^{\varepsilon_n} - \tilde{\mathcal{Y}}_s^{\varepsilon_n} + C_b(t-s) \left(1 + \sup_{u \leq \bar{T}} |\tilde{X}_u^{\varepsilon_n}| + \tilde{Z}^{\varepsilon_n} \right). \end{aligned}$$

Sending $\gamma \rightarrow 0$ countably and employing the right continuity of X^* and Y^* , we deduce

$$X_t^* - X_s^* \leq Y_t^* - Y_s^* + C_b(t-s)(1 + \sup_{u \leq \bar{T}} |X_u^*| + Z^*) \quad \forall s < t \quad \mathbb{P}_{x,y,z}^* \text{-a.s.}$$

Furthermore, $\Delta X_t^* \leq 0$ for all t $\mathbb{P}_{x,y,z}^*$ -almost surely. By Lemma A.1, Y^* is a continuous local martingale with respect to the filtration generated by (X^*, Y^*) . It is clear that $\tau_0(X^*)$ is a stopping time with respect to the filtration generated by (X^*, Y^*) . So the claim follows from Lemma A.2 if $\tau_0(X^*) \geq 0$ and $\mathbb{E}[\sup_{u \leq \bar{T}} |X_u^*| + Z^*] < \infty$. For the former condition, it is sufficient to show

$$\mathbb{P}_x^* \left(\inf_{s < 0} \eta_s \leq 0 \right) = 0.$$

As $\mathbb{P}_x^{\varepsilon_n} \Rightarrow \mathbb{P}_x^*$, then by Skorokhod's Representation Theorem, there exist (Z^n) and Z on a common probability space such that $\text{Law}(Z^n) = \mathbb{P}_x^{\varepsilon_n}$, $\text{Law}(Z) = \mathbb{P}_x^*$, and $Z^n \rightarrow Z$ almost surely in $(D_{\mathbb{R}}, M_1)$. By the Portmanteau Theorem, for any $\gamma > 0$

$$\mathbb{P}[Z_{-1} < \gamma] \leq \liminf_{n \rightarrow \infty} \mathbb{P}[Z_{-1}^n < \gamma] = \mathbb{P}[X_{0-} < \gamma] = O(\gamma^{1/2}),$$

as X_{0-} has an L^2 -density by Assumption 2.2 (vi). So $\mathbb{P}_x^*(\eta_{-1} \leq 0) = \mathbb{P}[Z_{-1} \leq 0] = 0$. By Lemma 2.10 and (2.17), we have

$$1 = \mathbb{E} \left[\mathbb{P}^* \left(\sup_{s < 0} |\eta_s - \eta_{-1}| = 0 \right) \right] = \mathbb{P}_x^* \left(\sup_{s < 0} |\eta_s - \eta_{-1}| = 0 \right).$$

Therefore, $X_s^* > 0$ for every $s \in [-1, 0)$ $\mathbb{P}_{x,y,z}^*$ -almost surely. Hence, $\tau_0(X^*) \geq 0$ almost surely. Furthermore, as $\eta \mapsto \sup_{u \leq \bar{T}} |\eta_u|$ is an M_1 -continuous map, $\mathbb{E}[\sup_{u \leq \bar{T}} |X_u^*| + Z^*] < \infty$ follows from a simple application of the Continuous Mapping Theorem and Proposition 2.8. Therefore, we deduce,

$$\begin{aligned} & \mathbb{E} \left[\mathbb{P}^* \left(\inf_{s \in (\tau_0(\eta), (\tau_0(\eta) + h) \wedge \bar{T})} \{\eta_s - \eta_{\tau_0(\eta)}\} \geq 0, \tau_0(\eta) < \bar{T} \right) \right] \\ & \leq \mathbb{P}_{x,y,z}^* \left(\inf_{s \in (\tau_0^*, (\tau_0^* + h) \wedge \bar{T})} \left\{ Y_s^* - Y_{\tau_0^*}^* + C_b(s - \tau_0^*)(1 + \sup_{u \leq \bar{T}} |X_u^*| + Z^*) \right\} \geq 0, \tau_0^* < \bar{T} \right) \\ & = 0, \end{aligned}$$

where $\tau_0^* = \tau_0(X^*)$ and the final equality is due to Lemma A.2. \square

Now we have all the ingredients to show that τ_0 is an M_1 -continuous map.

Corollary 2.12 (Hitting time continuity). *For $\text{Law}(\mathbb{P}^*)$ -almost every measure μ , we have that the hitting time map $\tau_0 : D_{\mathbb{R}} \rightarrow \mathbb{R}$ is continuous in the M_1 -topology for μ -almost every $\eta \in D_{\mathbb{R}}$.*

Proof. By Lemma 2.11, for $\text{Law}(\mathbb{P}^*)$ -almost every measure μ , μ -almost every path $\eta \in D_{\mathbb{R}}$ will have only downward jumps and one of the following conditions holds:

1. $\tau_0(\eta) < \bar{T}$ and η takes a negative value on any neighbourhood of $\tau_0(\eta)$,
2. $\tau_0(\eta) = \bar{T}$ and $\inf_{s \leq \bar{T}} \eta_s > 0$,
3. $\tau_0(\eta) = \bar{T}$ and $\eta_{\bar{T}} = 0$.

If 1 holds, then by Lemma A.3, τ_0 is M_1 -continuous at η . If 2 holds— $\tau_0(\eta) = \bar{T}$ and $\inf_{s \leq \bar{T}} \eta_s > 0$ —then for any approximating sequence $(\eta^n)_{n \geq 1} \subset D_{\mathbb{R}}$ in the M_1 -topology, we must have $\inf_{s \leq \bar{T}} \eta_s^n > 0$ eventually as the parametric representations get arbitrarily close in the uniform topology. Therefore, as $\inf_{s \leq \bar{T}} \eta_s^n > 0$ eventually, by definition $\tau_0(\eta^n) = \bar{T}$

eventually. Therefore, $\tau_0(\eta)$ is M_1 -continuous at η . If 3 holds, when $\tau_0(\eta) = \bar{T}$ and $\eta_{\bar{T}} = 0$, then for any $\gamma > 0$, with $\bar{T} - \gamma$ being a continuity point, we must have $\inf_{s \leq \bar{T} - \gamma} \eta_s > 0$ because η only jumps downwards. So for any approximating sequence $(\eta^n)_{n \geq 1} \subset D_{\mathbb{R}}$ in the M_1 -topology, eventually $\inf_{s \leq \bar{T} - \gamma} \eta_s^n > 0$. Hence, eventually $\tau_0(\eta^n) > \bar{T} - \gamma$. As $\gamma > 0$ can be made arbitrarily close to zero, by definition, $\lim_{n \rightarrow \infty} \tau_0(\eta^n) = \tau_0(\eta) = \bar{T}$. Therefore, τ_0 is M_1 -continuous at η . \square

With the result stating that the hitting time is an M_1 -continuous map, weak convergence of the loss function follows immediately.

Lemma 2.13 (Continuity of conditional feedback). *For $\text{Law}(\mathbf{P}^*)$ -almost every measure $\mu \in \mathcal{P}(D_{\mathbb{R}})$, the map $\mu \mapsto \mu(\tau_0(\eta) \leq t)$ is continuous with respect to $\mathfrak{T}_{M_1}^{wk}$ for all $t \in \mathbb{T}^\mu \cap [0, \bar{T})$. \mathbb{T}^μ is the set of continuity points of $t \mapsto \mu(\tau_0(\eta) \leq t)$.*

Proof. By Corollary 2.12, for $\text{Law}(\mathbf{P}^*)$ -almost every measure μ , τ_0 is M_1 -continuous for μ -almost every η . We fix such a μ and consider a sequence $(\mu^n)_{n \geq 1}$ converging to μ in $\mathcal{P}(D_{\mathbb{R}})$. By Skorokhod's Representation Theorem,

$$\mu^n(\tau_0(\eta) \leq t) = \mathbb{E} [\mathbb{1}_{\tau_0(Z^n) \leq t}] \quad \text{and} \quad \mu(\tau_0(\eta) \leq t) = \mathbb{E} [\mathbb{1}_{\tau_0(Z) \leq t}],$$

where τ_0 is continuous for almost all paths Z and $Z^n \rightarrow Z$ almost surely in $(D_{\mathbb{R}}, M_1)$. Now, for any $t \in \mathbb{T}^\mu := \{t \in [-1, \bar{T}] : \mu(\tau_0(\eta) = t) = 0\}$, by the Monotone Convergence Theorem,

$$\mathbb{P}[\tau_0(Z) = t] = \mu(\tau_0(\eta) \leq t) - \lim_{s \uparrow t} \mu(\tau_0(\eta) \leq s) = 0. \quad (2.19)$$

Therefore, employing the continuity of τ_0 and (2.19), we have

$$\mathbb{E} [\mathbb{1}_{\tau_0(Z^n) \leq t}] \rightarrow \mathbb{E} [\mathbb{1}_{\tau_0(Z) \leq t}],$$

by the Dominated Convergence Theorem. So, we conclude that

$$\mu^n(\tau_0(\eta) \leq t) \rightarrow \mu(\tau_0(\eta) \leq t) \quad \forall t \in \mathbb{T}^\mu. \quad \square$$

Furthermore, we have weak convergence of the mollified loss to the singular loss.

Corollary 2.14 (Convergence of delayed loss). *For $\text{Law}(\mathbf{P}^*)$ -almost every measure μ , $\int_0^t \kappa^{\varepsilon_n}(t-s) \mu^n(\tau_0(\eta) \leq s) ds$ converges to $\mu(\tau_0(\eta) \leq t)$ for any $t \in \mathbb{T}^\mu$ and any sequence $(\mu^n)_{n \geq 1}$ that converges to μ in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$.*

Proof. By Lemma 2.13, $\mu^n(\tau_0 \leq t)$ converges to $\mu(\tau_0 \leq t)$ for any $t \in \mathbb{T}^\mu$ when τ_0 is an M_1 -continuous map μ -almost surely. Such measures μ have full $\text{Law}(\mathbf{P}^*)$ -measure by Corollary 2.12. Furthermore, for every such μ ,

$$(s \mapsto \mu^n(\tau_0(\eta) \leq s) \mathbb{1}_{[0, t]}(s)) \xrightarrow{n \rightarrow \infty} (s \mapsto \mu(\tau_0(\eta) \leq s) \mathbb{1}_{[0, t]}(s)) \quad (2.20)$$

in the M_1 -topology as functions from $[-1, t] \rightarrow \mathbb{R}$, as the functions are non-decreasing, [33, Corollary 12.5.1]. Now, for any $t \in \mathbb{T}^\mu \cap (0, \bar{T}]$,

$$\begin{aligned} \left| \int_0^t \kappa^{\varepsilon_n}(s) \mu^n(\tau_0(\eta) \leq t-s) ds - \mu(\tau_0(\eta) \leq t) \right| &\leq \left| \int_0^t \kappa^{\varepsilon_n}(s) (\mu^n(\tau_0(\eta) \leq t-s) - \mu(\tau_0(\eta) \leq t-s)) ds \right| \\ &\quad + \left| \int_0^t \kappa^{\varepsilon_n}(s) (\mu(\tau_0(\eta) \leq t-s) - \mu(\tau_0(\eta) \leq t)) ds \right| \\ &\quad + \left| \int_0^t \kappa^{\varepsilon_n}(s) ds - 1 \right| \mu(\tau_0(\eta) \leq t) \\ &= I + II + III. \end{aligned}$$

For any $\delta > 0$, we observe

$$\begin{aligned} I &\leq \sup_{t-\delta \leq s \leq t} |\mu^n(\tau_0(\eta) \leq s) - \mu(\tau_0(\eta) \leq s)| \int_0^\delta \kappa^{\varepsilon_n}(s) \, ds + \int_\delta^\infty \kappa^{\varepsilon_n}(s) \, ds, \\ II &\leq \sup_{t-\delta \leq s \leq t} |\mu(\tau_0(\eta) \leq s) - \mu(\tau_0(\eta) \leq t)| \int_0^\delta \kappa^{\varepsilon_n}(s) \, ds + \int_\delta^\infty \kappa^{\varepsilon_n}(s) \, ds, \\ III &\leq \int_t^\infty \kappa^{\varepsilon_n}(s) \, ds. \end{aligned}$$

As M_1 -convergence implies local uniform convergence at continuity points, [33, Theorem 12.5.1], and t is a continuity point, by setting $\delta = \varepsilon_n^{1/2}$ and sending $n \rightarrow \infty$, we have that I , II , and III all go to zero. \square

2.3 Martingale arguments and convergence

As marginal tightness implies joint tightness, $\{(\tilde{\mathbf{P}}^\varepsilon, W^0, W)\}_{\varepsilon>0}$ is tight in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}}) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$ where $(\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$ is shorthand notation for $(\mathcal{C}([0, T], \mathbb{R}), \|\cdot\|_\infty)$, the space of continuous functions from $[0, T]$ to \mathbb{R} endowed with the topology of uniform convergence. From now on, we fix a weak limit point (\mathbf{P}^*, W^0, W) along a subsequence $(\varepsilon_n)_{n \geq 1}$ for which ε_n converges to zero. Although we have fixed a limit point, all the following results will hold for any limit point.

Let $\mathbb{P}^n := \text{Law}(\tilde{\mathbf{P}}^{\varepsilon_n}, W^0, W)$ and $\mathbb{P}_{\mu, \omega^0, \omega}^* := \text{Law}(\mathbf{P}^*, W^0, W)$. So $\mathbb{P}^n \Rightarrow \mathbb{P}_{\mu, \omega^0, \omega}^*$. For completeness, we will define the probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}_{\mu, \omega^0, \omega}^*)$ where $\Omega^* = \mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$ and \mathcal{F}^* is the corresponding Borel σ -algebra. Define the random variables \mathbf{P}^* , W^0 , and W on Ω^* such that for any tuple (μ, ω^0, ω) ,

$$\mathbf{P}^*(\mu, \omega^0, \omega) = \mu, \quad W^0(\mu, \omega^0, \omega) = \omega^0, \quad \text{and} \quad W(\mu, \omega^0, \omega) = \omega.$$

Hence, the joint law of (\mathbf{P}^*, W^0, W) is $\mathbb{P}_{\mu, \omega^0, \omega}^*$ and $\mathcal{F}^* = \sigma(\mathbf{P}^*, W^0, W)$. We also define the limiting loss function $L^* := \mathbf{P}^*(\tau_0(\eta) \leq \cdot)$ and the co-countable set of times

$$\mathcal{T} := \{t \in [-1, \bar{T}] : \mathbb{P}_{\mu, \omega^0, \omega}^*(\eta_t = \eta_{t-}) = 1, \mathbb{P}_{\mu, \omega^0, \omega}^*(L_t^* = L_{t-}^*) = 1\}. \quad (2.21)$$

Looking at the approximating system, we know $(\tilde{\mathbf{P}}^\varepsilon, W^0) \perp W$ for any $\varepsilon > 0$. Even though \mathbf{P}^* is the weak limit of W^0 -measurable random variables, weak convergence does not allow us to guarantee that limit points will be W^0 -measurable. Regardless, we may exploit the independence from the approximating system to deduce the independence of (\mathbf{P}^*, W^0) and W in the limit. To fix the notation, let $\mathbb{P}_{\mu, \omega^0}^*$ denote the projection of the measure $\mathbb{P}_{\mu, \omega^0, \omega}^*$ onto its first two coordinates and \mathbb{P}_ω^* denote the projection onto its final coordinate. Then we intuitively expect $\mathbb{P}_{\mu, \omega^0, \omega}^* = \mathbb{P}_{\mu, \omega^0}^* \otimes \mathbb{P}_\omega^*$.

Lemma 2.15 (Independence from idiosyncratic noise). *Let \mathbf{P}^* , W^0 , and W be the canonical random variables on the probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}_{\mu, \omega^0, \omega}^*)$ defined above. Then, (\mathbf{P}^*, W^0) is independent of W .*

Proof. As $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$ and $(\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$ are Polish spaces, it is sufficient to show that for any $f \in \mathcal{C}_b(\mathcal{P}(D_{\mathbb{R}}))$ and $g, h \in \mathcal{C}_b(\mathcal{C}_{\mathbb{R}})$,

$$\langle \mathbb{P}_{\mu, \omega^0, \omega}^*, f \otimes g \otimes h \rangle = \langle \mathbb{P}_{\mu, \omega^0}^*, f \otimes g \rangle \langle \mathbb{P}_\omega^*, h \rangle. \quad (2.22)$$

The result follows from the Dominated Convergence Theorem and Dynkin's Lemma. Now, (2.22) follows readily by weak convergence and the Portmanteau Theorem as

$$\begin{aligned} \langle \mathbb{P}_{\mu, \omega^0, \omega}^*, f \otimes g \otimes h \rangle &= \lim_{n \rightarrow \infty} \langle \mathbb{P}^n, f \otimes g \otimes h \rangle \\ &= \lim_{n \rightarrow \infty} \langle \mathbb{P}^n, f \otimes g \rangle \langle \mathbb{P}^n, h \rangle \\ &= \langle \mathbb{P}_{\mu, \omega^0}^*, f \otimes g \rangle \langle \mathbb{P}_\omega^*, h \rangle. \end{aligned}$$

The equality in the second line follows from the independence of $(\tilde{\mathbf{P}}^\varepsilon, W^0)$ from W . \square

We shall use $\mathbb{P}_{\mu, \omega^0, \omega}^*$ to construct a probability space where we can define a process that will solve (2.2) in the sense of Definition 2.1. Prior to that, we need to define the map employed in the martingale arguments that follow. This allows us to deduce that the process we construct will be of the correct form. For any $\varepsilon > 0$, we define the following functionals $\mathcal{M}, \mathcal{M}^\varepsilon : \mathcal{P}(D_{\mathbb{R}}) \times D_{\mathbb{R}} \rightarrow D_{\mathbb{R}}$:

$$\mathcal{M}^\varepsilon(\mu, \eta) = \eta - \eta_{-1} - \int_0^\cdot b(s, \eta_s, \nu_s^\mu) ds + \int_0^\cdot \alpha(s) d\mathfrak{L}_s^{\mu, \varepsilon}, \quad (2.23)$$

$$\mathcal{M}(\mu, \eta) = \eta - \eta_{-1} - \int_0^\cdot b(s, \eta_s, \nu_s^\mu) ds + \int_{[0, \cdot]} \alpha(s) dL_s^\mu, \quad (2.24)$$

where for any $\mu \in \mathcal{P}(D_{\mathbb{R}})$,

$$\nu_t^\mu := \mu(\eta_t \in \cdot, \tau_0(\eta) > t), \quad L_t^\mu := \mu(\tau_0(\eta) \leq t), \quad \mathfrak{L}_t^{\mu, \varepsilon} = \int_0^t \kappa^\varepsilon(t-s) L_s^\mu ds,$$

and b satisfies Assumption 2.2. For any $s_0, t_0 \in \mathbb{T}^\mu \cap [0, T)$ with $s_0 < t_0$ and $\{s_i\}_{i=1}^k \subset [0, s_0] \cap \mathbb{T}$, we define the function

$$F : D_{\mathbb{R}} \rightarrow \mathbb{R}, \quad \eta \mapsto (\eta_{t_0} - \eta_{s_0}) \prod_{i=1}^k f_i(\eta_{s_i}), \quad (2.25)$$

for arbitrary $f_i \in \mathcal{C}_b(\mathbb{R})$. We define the functionals:

$$\begin{cases} \Psi^\varepsilon(\mu) = \langle \mu, \eta \mapsto F(\mathcal{M}^\varepsilon(\mu, \eta)) \rangle, \\ \Upsilon^\varepsilon(\mu) = \langle \mu, \eta \mapsto F((\mathcal{M}^\varepsilon(\mu, \eta))^2 - \int_0^\cdot \sigma(s, \eta_s)^2 ds) \rangle, \\ \Theta^\varepsilon(\mu, \omega) = \left\langle \mu, \eta \mapsto F\left(\mathcal{M}^\varepsilon(\mu, \eta) \times \omega - \int_0^\cdot \sigma(s, \eta_s) \sqrt{1 - \rho(s, \nu_s^\mu)^2} ds\right) \right\rangle, \\ \Theta^{0, \varepsilon}(\mu, \omega^0) = \langle \mu, \eta \mapsto F(\mathcal{M}^\varepsilon(\mu, \eta) \times \omega^0 - \int_0^\cdot \sigma(s, \eta_s) \rho(s, \nu_s^\mu) ds) \rangle. \end{cases} \quad (2.26)$$

Lastly, the corresponding functionals without the mollification, denoted by $\Psi(\mu)$, $\Upsilon(\mu)$, $\Theta(\mu, \omega)$ and $\Theta^0(\mu, \omega^0)$, are defined in exactly the same way as $\Psi^\varepsilon(\mu)$, $\Upsilon^\varepsilon(\mu)$, $\Theta^\varepsilon(\mu, \omega)$, and $\Theta^{0, \varepsilon}(\mu, \omega^0)$ with \mathcal{M}^ε replaced by \mathcal{M} .

Remark 2.16 (Measurability of measure flows). In (2.23) and (2.24), we are taking a fixed measure, μ , and computing the integral with respect to the measure flow $t \mapsto \nu_t^\mu$. The measurability of the functions b and σ is sufficient for this integral to be well-defined.

Using Corollary 2.14, we have the following proposition.

Proposition 2.17 (Functional Continuity I Generalised). *For $\mathbb{P}_{\mu, \omega^0, \omega}^*$ -almost every measure μ , we have that $\Psi^{\varepsilon_n}(\mu^n)$, $\Upsilon^{\varepsilon_n}(\mu^n)$, $\Theta^{\varepsilon_n}(\mu^n, \omega^n)$, and $\Theta^{0, \varepsilon_n}(\mu^n, \omega^{0, n})$ converge to $\Psi(\mu)$, $\Upsilon(\mu)$, $\Theta(\mu, \omega)$, and $\Theta^0(\mu, \omega^0)$ respectively, whenever $(\mu^n, \omega^{0, n}, \omega^n) \rightarrow (\mu, \omega^0, \omega)$ in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk}) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$, along a sequence for which $\sup_{n \geq 1} \langle \mu^n, \sup_{s \leq T} |\tilde{\eta}_s|^p \rangle$ is bounded for some $p > 2$ and ε_n that converges to zero.*

Proof. By Lemma 2.11 and the definition of \mathbb{T} , we have a set of μ' 's that have full $\mathbb{P}_{\mu, \omega^0, \omega}^*$ measure, such that

$$\mu \left(\inf_{s \in (\tau_0(\eta), (\tau_0(\eta) + h) \wedge \bar{T})} \{\eta_s - \eta_{\tau_0(\eta)}\} \geq 0, \tau_0(\eta) < \bar{T} \right) = 0$$

for any $h > 0$, $\mu(\eta_{s_i} = \eta_{s_i-}) = 1$, $\mu(\eta_{t_0} = \eta_{t_0-}) = 1$, and $\mu(\tau_0(\eta) = t_0) = 0$. First, we shall show that $\Psi^{\varepsilon_n}(\mu^n)$ converges to $\Psi(\mu)$. By Corollary 2.14, $\mathfrak{L}_t^{\mu^n, \varepsilon_n}$ converges to L_t^μ . It is

well-known that for any Borel measurable functions f and g of finite variation, we have for any $t > 0$

$$f_t g_t = f(0)g(0) + \int_{(0,t]} f_{s-} dg_s + \int_{(0,t]} g_{s-} df_s + \sum_{s \leq t} \Delta f_s \Delta g_s.$$

This, together with the continuous differentiability of α implies

$$\int_0^t \alpha(s) d\mathfrak{L}_s^{\mu, \varepsilon_n} = \alpha(t) \mathfrak{L}_t^{\mu, \varepsilon_n} - \int_0^t \mathfrak{L}_s^{\mu, \varepsilon_n} \alpha'(s) ds \rightarrow \alpha(t) L_t^\mu - \int_0^t L_s^\mu \alpha'(s) ds = \int_{[0,t]} \alpha(s) dL_s^\mu.$$

As $\mu^n \Rightarrow \mu$, by Skorokhod's Representation Theorem, there exists a $(Z^n)_{n \geq 1}$ and Z defined on a common probability space such that $\text{Law}(Z^n) = \mu^n$, $\text{Law}(Z) = \mu$ and $Z^n \rightarrow Z$ almost surely in $(D_{\mathbb{R}}, M_1)$. Hence,

$$\Psi^{\varepsilon_n}(\mu^n) = \mathbb{E}[F(\mathcal{M}^{\varepsilon_n}(\mu^n, Z^n))] \quad \text{and} \quad \Psi(\mu) = \mathbb{E}[F(\mathcal{M}(\mu, Z))].$$

By Lemma A.6,

$$\int_0^t b(s, Z_s^n, \nu_s^{\mu^n}) ds \rightarrow \int_0^t b(s, Z_s, \nu_s^\mu) ds \quad (2.27)$$

almost surely for any $t \geq 0$. Since \mathbb{T}^μ contains all the almost sure continuity points of Z , by the properties of M_1 -convergence and (2.27), we have

$$Z_t^n - Z_{-1}^n - \int_0^t b(s, Z_s^n, \nu_s^{\mu^n}) ds \rightarrow Z_t - Z_{-1} - \int_0^t b(s, Z_s, \nu_s^\mu) ds$$

almost surely for any $t \in \{t_0, s_0, \dots, s_k\}$. Hence, we deduce that $F(\mathcal{M}^{\varepsilon_n}(\mu^n, Z^n))$ converges almost surely to $F(\mathcal{M}(\mu, Z))$ in \mathbb{R} . Lastly, we observe

$$\langle \mu^n, |\mathcal{M}_t^{\varepsilon_n}(\mu^n, \cdot)|^p \rangle \leq C \left(\left\langle \mu^n, \sup_{s \leq T} |\tilde{\eta}_s|^p \right\rangle + 1 \right), \quad (2.28)$$

for some constant C that depends on p and b only but is uniform in n . Therefore, $F(\mathcal{M}^{\varepsilon_n}(\mu^n, Z^n))$ is uniformly L^p -bounded as

$$|F(\mathcal{M}^{\varepsilon_n}(\mu^n, Z^n))|^p \leq C (|\mathcal{M}_{t_0}^{\varepsilon_n}(\mu^n, Z^n)|^p + |\mathcal{M}_{s_0}^{\varepsilon_n}(\mu^n, Z^n)|^p),$$

and $\mathbb{E}[|\mathcal{M}_{t_0}^{\varepsilon_n}(\mu^n, Z^n)|^p] = \langle \mu^n, |\mathcal{M}_{t_0}^{\varepsilon_n}(\mu^n, \cdot)|^p \rangle$ where the latter is uniformly bounded in n for some $p > 2$ by (2.28) and assumption. Therefore, by Vitali's Convergence Theorem, it follows that $\Psi^{\varepsilon_n}(\mu^n)$ converges to $\Psi(\mu)$.

The convergence of $\Upsilon^{\varepsilon_n}(\mu^n)$, $\Theta^{\varepsilon_n}(\mu^n, \omega^n)$, and $\Theta^{0, \varepsilon_n}(\mu^n, \omega^{0, n})$, to $\Upsilon(\mu)$, $\Theta(\mu, \omega)$ and $\Theta^0(\mu, \omega^0)$ respectively follows from similar arguments. As σ and ρ are totally bounded by Assumption 2.2 (ii) and (iv), $\Upsilon^{\varepsilon_n}(\mu^n)$, $\Theta^{\varepsilon_n}(\mu^n, \omega^n)$, and $\Theta^{0, \varepsilon_n}(\mu^n, \omega^{0, n})$ are L^p -bounded uniformly in n . The continuity of σ and the almost sure convergence of Z^n to Z in the M_1 -topology ensures that

$$\int_0^t \sigma(s, Z_s^n)^2 ds \rightarrow \int_0^t \sigma(s, Z_s)^2 ds$$

almost surely for all $t \geq 0$. Lastly, by the boundedness of σ and ρ , a straightforward computation shows that

$$\begin{aligned} |\sigma(t, x)\rho(t, \mu) - \sigma(t, x)\rho(t, \tilde{\mu})| &\leq C(1 + \langle \mu, |\cdot| \rangle) d_1(\mu, \tilde{\mu}) \leq C(1 + \langle \mu, |\cdot| \rangle) d_0(\mu, \tilde{\mu}), \\ \left| \sigma(t, x)\sqrt{1 - \rho(t, \mu)^2} - \sigma(t, x)\sqrt{1 - \rho(t, \tilde{\mu})^2} \right| &\leq C(1 + \langle \mu, |\cdot| \rangle) d_1(\mu, \tilde{\mu}) \leq C(1 + \langle \mu, |\cdot| \rangle) d_0(\mu, \tilde{\mu}). \end{aligned}$$

Therefore, the functions $(t, x, \mu) \mapsto \sigma(t, x)\rho(t, \mu)$ and $(t, x, \mu) \mapsto \sigma(t, x)\sqrt{1 - \rho(t, \mu)^2}$ satisfy Assumption 2.2 (i). Now we may apply Lemma A.6 and conclude

$$\begin{aligned} \int_0^t \sigma(s, Z_s^n) \sqrt{1 - \rho(s, \nu_s^{\mu^n})^2} ds &\rightarrow \int_0^t \sigma(s, Z_s) \sqrt{1 - \rho(s, \nu_s^\mu)^2} ds, \\ \int_0^t \sigma(s, Z_s^n) \rho(s, \nu_s^{\mu^n}) ds &\rightarrow \int_0^t \sigma(s, Z_s) \rho(s, \nu_s^\mu) ds, \end{aligned}$$

almost surely for all $t \geq 0$. \square

The remainder of this section aims to show that the conditional law of $\{\tilde{X}^{\varepsilon_n}\}$ converges weakly to a random variable X which will have the dynamics defined in (2.7). This is achieved in the following two steps:

1. First, we construct a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that $\mathcal{M}, \mathcal{M}^2 - \int_0^\cdot \sigma(s, \eta_s)^2 ds, \mathcal{M} \times W - \int_0^\cdot \sigma(s, \eta_s) \sqrt{1 - \rho(s, \nu_s^\mu)^2} ds$, and $\mathcal{M} \times W^0 - \int_0^\cdot \sigma(s, \eta_s) \rho(s, \nu_s^\mu) ds$, defined as in (2.24), are continuous martingales.
2. Secondly, we construct a stochastic process X on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that the tuple $(X, W, W^0, \mathbf{P}^*)$ is a solution to (2.4) in the sense of Definition 2.1.

To this end, we now proceed to show the above two claims. We begin by defining the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ where $\bar{\Omega} = \Omega^* \times D_{\mathbb{R}} = \mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \times D_{\mathbb{R}}$ and $\bar{\mathcal{F}}$ is the Borel σ -algebra $\mathcal{B}(\bar{\Omega})$. We define the probability measure

$$\bar{\mathbb{P}}(A) := \int_{\mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}} \mu(\{\eta : (\mu, \omega^0, \omega, \eta) \in A\}) d\mathbb{P}_{\mu, \omega^0, \omega}^*(\mu, \omega^0, \omega), \quad (2.29)$$

for any $A \in \mathcal{B}(\bar{\Omega})$. Observe by construction, for any $A \in \mathcal{B}(\bar{\Omega})$,

$$\bar{\mathbb{P}}(A) = \mathbb{E}^*[\langle \mathbf{P}^*, \mathbb{1}_A(\mathbf{P}^*, W^0, W, \cdot) \rangle].$$

Furthermore, under $\bar{\mathbb{P}}$, W^0 and W are still Brownian motions, and (\mathbf{P}^*, W^0) is independent of W . This is immediate, as for any $A \in \mathcal{B}(\mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}})$ and $B \in \mathcal{B}(\mathcal{C}_{\mathbb{R}})$,

$$\bar{\mathbb{P}}[(\mathbf{P}^*, W^0) \in A, W \in B] = \bar{\mathbb{P}}(A \times B \times D_{\mathbb{R}}) = \mathbb{P}_{\mu, \omega^0, \omega}^*(A \times B).$$

Given these ingredients, we may now show our first claim.

Proposition 2.18. *Let \mathcal{M} be given as in (2.24). Then $\mathcal{M}, \mathcal{M}^2 - \int_0^\cdot \sigma(s, \pi_s(\cdot))^2 ds, \mathcal{M} \times W - \int_0^\cdot \sigma(s, \pi_s(\cdot)) \sqrt{1 - \rho(s, \nu_s^\mu)^2} ds$, and $\mathcal{M} \times W^0 - \int_0^\cdot \sigma(s, \pi_s(\cdot)) \rho(s, \nu_s^\mu) ds$ are all continuous martingales on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, where*

$$\pi_s : D_{\mathbb{R}} \rightarrow \mathbb{R}, \pi_s(\eta) = \eta_s \quad \text{and} \quad \nu_s^\cdot : \mathcal{P}(D_{\mathbb{R}}) \rightarrow \mathbf{M}_{\leq 1}(\mathbb{R}), \nu_s^\cdot(\mu) = \nu_s^\mu.$$

Proof. If \mathcal{M} is continuous, then the continuity of the other processes follows from the continuity of \mathcal{M} and the continuity of integration. For simplicity, we shall use \mathcal{N} to denote any one of $\mathcal{M}, \mathcal{M} \times W - \int_0^\cdot \sigma(s, \pi_s(\cdot)) \sqrt{1 - \rho(s, \nu_s^\mu)^2} ds, \mathcal{M} \times W^0 - \int_0^\cdot \sigma(s, \pi_s(\cdot)) \rho(s, \nu_s^\mu) ds$ or $\mathcal{M}^2 - \int_0^\cdot \sigma(s, \pi_s(\cdot))^2 ds$. Hence to show that \mathcal{N} is a martingale, it is sufficient by a Monotone Class argument that

$$\bar{\mathbb{E}}[F(\mathcal{N})] = 0, \quad (2.30)$$

To begin, recall that $\mathbb{P}^{\varepsilon_n} \implies \mathbb{P}^*$, where $\mathbb{P}^{\varepsilon_n} = \text{Law}(\tilde{\mathbf{P}}^{\varepsilon_n}, W^0, W)$ and $\mathbb{P}^* = \text{Law}(\mathbf{P}^*, W^0, W)$. By Skorokhod's Representation Theorem, we may find $\{(\mathbf{Q}^n, B^n, \tilde{B}^n)\}_{n \geq 1}$ and $(\mathbf{Q}^*, B^*, \tilde{B}^*)$ defined on a common probability space such that $\mathbb{P}^{\varepsilon_n} = \text{Law}(\mathbf{Q}^n, B^n, \tilde{B}^n)$,

$\mathbb{P}^* = \text{Law}(\mathbf{Q}^*, B^*, \tilde{B}^*)$ and $(\mathbf{Q}^n, B^n, \tilde{B}^n) \rightarrow (\mathbf{Q}^*, B^*, \tilde{B}^*)$ almost surely in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}}) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_{\infty}) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_{\infty})$. By the definition of $\tilde{\mathbf{P}}^{\varepsilon_n}$, for any $p > 1$, we have:

$$\mathbb{E} \left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n}, \sup_{s \leq \bar{T}} |\eta_s|^p \right\rangle \right] = \mathbb{E} \left[\sup_{s \leq \bar{T}} |\tilde{X}^{\varepsilon_n}|^p \right] \leq C, \quad (2.31)$$

where the constant C is derived from Proposition 2.8. As the map $\eta \mapsto \sup_{s \leq \bar{T}} |\eta_s|$ is M_1 -continuous, the Portmanteau Theorem yields:

$$\mathbb{E} \left[\left\langle \mathbf{P}^*, \sup_{s \leq \bar{T}} |\eta_s|^p \right\rangle \right] \leq C. \quad (2.32)$$

Now, for any $K > 0$, we can write

$$\begin{aligned} \mathbb{E} [|\langle \mathbf{Q}^n, \sup_{s \leq \bar{T}} |\eta_s|^p \rangle - \langle \mathbf{Q}^*, \sup_{s \leq \bar{T}} |\eta_s|^p \rangle|] &\leq \mathbb{E} [|\langle \mathbf{Q}^n, \sup_{s \leq \bar{T}} |\eta_s|^p \wedge K \rangle - \langle \mathbf{Q}^*, \sup_{s \leq \bar{T}} |\eta_s|^p \wedge K \rangle|] \\ &\quad + \mathbb{E} [|\langle \mathbf{Q}^n, (\sup_{s \leq \bar{T}} |\eta_s|^p - K) \mathbb{1}_{\sup_{s \leq \bar{T}} |\eta_s|^p > K} \rangle|] \\ &\quad + \mathbb{E} [|\langle \mathbf{Q}^*, (\sup_{s \leq \bar{T}} |\eta_s|^p - K) \mathbb{1}_{\sup_{s \leq \bar{T}} |\eta_s|^p > K} \rangle|]. \end{aligned}$$

By equation (2.31), and a combination of Hölder's and Markov's inequalities, the second term on the right-hand side above is $o(1)$ as $K \rightarrow \infty$ uniformly in n . Similarly, by (2.32), the third term on the right-hand side above is $o(1)$ as $K \rightarrow \infty$. For the first term, we recall that $\mathbf{Q}^n \rightarrow \mathbf{Q}^*$ almost surely in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$. Hence, as $\eta \mapsto \sup_{s \leq \bar{T}} |\eta_s|^p \wedge K$ is an M_1 -continuous and bounded function, we have by the almost sure convergence and the Portmanteau Theorem that $\langle \mathbf{Q}^n, \sup_{s \leq \bar{T}} |\eta_s|^p \wedge K \rangle$ converges almost surely towards $\langle \mathbf{Q}^*, \sup_{s \leq \bar{T}} |\eta_s|^p \wedge K \rangle$. Therefore, by the Dominated Convergence Theorem, the first term on the right-hand side above is $o(1)$ as $n \rightarrow \infty$. Therefore, we can make the right-hand side arbitrarily small by first fixing K sufficiently large, then choosing n sufficiently large. Hence, we have

$$\mathbb{E} [|\langle \mathbf{Q}^n, \sup_{s \leq \bar{T}} |\eta_s|^p \rangle - \langle \mathbf{Q}^*, \sup_{s \leq \bar{T}} |\eta_s|^p \rangle|] \rightarrow 0.$$

Now, fixing a $p > 2$, as we have L^1 -convergence, we can find a subsequence (also denoted by n for simplicity) such that $\langle \mathbf{Q}^n, \sup_{s \leq \bar{T}} |\eta_s|^p \rangle$ converges almost surely to $\langle \mathbf{Q}^*, \sup_{s \leq \bar{T}} |\eta_s|^p \rangle$. By (2.32), $\langle \mathbf{Q}^*, \sup_{s \leq \bar{T}} |\eta_s|^p \rangle < +\infty$ almost surely. Therefore, restricting ourselves to this subsequence going forward, we have

$$\sup_{n \geq 1} \left\langle \mathbf{Q}^n, \sup_{s \leq \bar{T}} |\eta_s|^p \right\rangle < \infty$$

almost surely.

By definition of \mathcal{M} , for the same $p > 2$ fixed above

$$\mathbb{E} \left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n}, \left| \mathcal{M}_t(\tilde{\mathbf{P}}^{\varepsilon_n}, \cdot) \right|^p \right\rangle \right] = \mathbb{E} \left[\left| \int_0^t \sigma(s) \sqrt{1 - \rho(s)^2} dW_s + \int_0^t \sigma(s) \rho(s) dW_s^0 \right|^p \right] \leq Ct^p, \quad (2.33)$$

where the constant C depends on the constant from applying Burkholder–Davis–Gundy, p , and the bounds on σ but is independent of ε . Hence, $\mathbb{E}[\langle \mathbf{Q}^n, |\mathcal{M}_t(\mathbf{Q}^n, \cdot)|^p \rangle] < \infty$ uniformly in n .

Employing Proposition 2.18 and Vitali's Convergence Theorem, we have

$$\bar{\mathbb{E}}[F(\mathcal{N})] = \mathbb{E}^* [\langle \mathbf{P}^*, F(\mathcal{N}(\mathbf{P}^*, W^0, W, \cdot)) \rangle] = \lim_{n \rightarrow \infty} \mathbb{E} [\langle \mathbf{P}^{\varepsilon_n}, F(\mathcal{N}^{\varepsilon_n}(\mathbf{P}^{\varepsilon_n}, W^0, W, \cdot)) \rangle],$$

where $\mathcal{N}^{\varepsilon_n}$ represents one of $(\mathcal{M}^{\varepsilon_n})^2 - \int_0^\cdot \sigma(s, \pi_s(\cdot))^2 ds$, $\mathcal{M}^{\varepsilon_n} \times W^0 - \int_0^\cdot \sigma(s, \pi_s(\cdot)) \rho(s, \nu_s^\varepsilon) ds$, $\mathcal{M}^{\varepsilon_n} \times W - \int_0^\cdot \sigma(s, \pi_s(\cdot)) \sqrt{1 - \rho(s, \nu_s^\varepsilon)^2} ds$, or $\mathcal{M}^{\varepsilon_n}$ depending on \mathcal{N} . Recall for arbitrary $f_i \in \mathcal{C}_b(\mathbb{R})$, $F(\eta) = (\eta_{t_0} - \eta_{s_0}) \prod_{i=1}^k f_i(\eta_{s_i})$. So

$$\mathbb{E} [\langle \mathbf{P}^{\varepsilon_n}, F(\mathcal{N}^{\varepsilon_n}(\mathbf{P}^{\varepsilon_n}, W^0, W, \cdot)) \rangle] = \mathbb{E} \left[\left(\tilde{\mathcal{N}}_{t_0}^{\varepsilon_n} - \tilde{\mathcal{N}}_{s_0}^{\varepsilon_n} \right) \prod_{i=1}^k f_i(\tilde{\mathcal{N}}_{s_i}^{\varepsilon_n}) \right], \quad (2.34)$$

where $\tilde{\mathcal{N}}^{\varepsilon_n}$ is one of $\tilde{\mathcal{Y}}^{\varepsilon_n}$, $(\tilde{\mathcal{Y}}^{\varepsilon_n})^2 - \int_0^\cdot \sigma(s, \tilde{X}^{\varepsilon_n})^2 ds$, $\tilde{\mathcal{Y}}^{\varepsilon_n} \times W - \int_0^\cdot \sigma(s, \tilde{X}^{\varepsilon_n}) \sqrt{1 - \rho(s, \nu_s^{\varepsilon_n})^2} ds$, or $\tilde{\mathcal{Y}}^{\varepsilon_n} \times W^0 - \int_0^\cdot \sigma(s, \tilde{X}^{\varepsilon_n}) \rho(s, \nu_s^{\varepsilon_n}) ds$ depending on the choice of \mathcal{N} . By the boundness assumption on σ , Assumption 2.2 (ii), $\tilde{\mathcal{N}}^{\varepsilon_n}$ is a martingale. As $s_1 \leq \dots \leq s_k \leq s_0 < t_0$, we have (2.34) equals zero by the tower property. Hence, we have shown (2.30).

Lastly, to see the continuity of \mathcal{M} , define the function

$$\tilde{F} : D_{\mathbb{R}} \rightarrow \mathbb{R}, \quad \eta \mapsto |\eta_t - \eta_s|^4,$$

for $s, t \in \mathbb{T} \cap [0, T)$. As before, define the functionals

$$\tilde{\Psi}^\varepsilon(\mu) = \left\langle \mu, \tilde{F}(\mathcal{M}^\varepsilon(\mu, \cdot)) \right\rangle, \quad \tilde{\Psi}(\mu) = \left\langle \mu, \tilde{F}(\mathcal{M}(\mu, \cdot)) \right\rangle.$$

Following the same proof as in Proposition 2.17, we have that for $\mathbb{P}_{\mu, \omega^0, \omega}^*$ -almost every measure μ , $\tilde{\Psi}^{\varepsilon_n}(\mu^n)$ converges to $\tilde{\Psi}(\mu)$ whenever $\mu^n \rightarrow \mu$ in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$ along a sequence for which

$$\sup_{n \geq 1} \langle \mu^n, \sup_{s \leq T} |\eta_s|^p \rangle < \infty$$

for some $p > 4$. We have finite moments for any $p > 1$, by (2.31). Therefore, by functional continuity and Vitali's convergence theorem, for any $s, t \in \mathbb{T} \cap [0, T)$ we have

$$\begin{aligned} \mathbb{E} |\mathcal{M}_t - \mathcal{M}_s|^4 &= \mathbb{E}^* \left[\left\langle \mathbf{P}^*, |\mathcal{M}_t(\mathbf{P}^*, \cdot) - \mathcal{M}_s(\mathbf{P}^*, \cdot)|^4 \right\rangle \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n}, |\mathcal{M}_t^{\varepsilon_n}(\tilde{\mathbf{P}}^{\varepsilon_n}, \cdot) - \mathcal{M}_s^{\varepsilon_n}(\tilde{\mathbf{P}}^{\varepsilon_n}, \cdot)|^4 \right\rangle \right]. \end{aligned}$$

By the definition of $\tilde{\mathbf{P}}^{\varepsilon_n}$ and the Burkholder–Davis–Gundy inequality,

$$\mathbb{E} \left[\left\langle \tilde{\mathbf{P}}^{\varepsilon_n}, |\mathcal{M}_t^{\varepsilon_n}(\tilde{\mathbf{P}}^{\varepsilon_n}, \cdot) - \mathcal{M}_s^{\varepsilon_n}(\tilde{\mathbf{P}}^{\varepsilon_n}, \cdot)|^4 \right\rangle \right] = \mathbb{E} \left| \tilde{\mathcal{Y}}_t^{\varepsilon_n} - \tilde{\mathcal{Y}}_s^{\varepsilon_n} \right|^4 \leq C |t - s|^2,$$

where the constant C is uniform in n . As \mathbb{T} is dense, by Kolmogorov's Criterion, there exists a continuous process that is a modification of \mathcal{M} . Since \mathcal{M} is right continuous and \mathbb{T} is dense, these two processes are indistinguishable. Hence, \mathcal{M} has a continuous version. \square

Now, we have all the ingredients to prove Theorem 2.5.

Proof of Theorem 2.5. By Proposition 2.9, $\{(\tilde{\mathbf{P}}^\varepsilon, W^0, W)\}_{\varepsilon > 0}$ is tight. By Prokhorov's Theorem, tightness on Polish spaces is equivalent to being sequentially precompact. Therefore, for any subsequence $\{(\mathbf{P}^{\varepsilon_n}, W^0, W)\}_{n \geq 1}$, where $(\varepsilon_n)_{n \geq 1}$ is a positive sequence that converges to zero, we have a convergent sub-subsequence. Fix a limit point (\mathbf{P}^*, W^0, W) of this subsequence. Having fixed (\mathbf{P}^*, W^0, W) , we define the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ exactly as in (2.29). Now, define the càdlàg process X by

$$X : \bar{\Omega} \rightarrow D_{\mathbb{R}}, \quad (\mu, \omega^0, \omega, \eta) \mapsto \eta.$$

Then, by the construction of $\bar{\mathbb{P}}$ and the fact that $\mathbb{P}_{\mu, \omega^0, \omega}^* = \mathbb{P}_{\mu, \omega^0}^* \times \mathbb{P}_{\omega}^*$ by Lemma 2.15, for all $A \in \mathcal{B}(D_{\mathbb{R}})$, $S \in \mathcal{B}(\mathcal{P}(D_{\mathbb{R}}) \times \mathcal{C}_{\mathbb{R}})$, we have

$$\bar{\mathbb{P}}[X \in A, (\mathbf{P}^*, W^0) \in S] = \int_S \mu(A) d\mathbb{P}_{\mu, \omega^0}^*.$$

Here, $\mathbb{P}_{\mu, \omega^0, \omega}^* = \text{Law}(\mathbf{P}^*, W^0, W)$, $\mathbb{P}_{\mu, \omega^0}^* = \text{Law}(\mathbf{P}^*, W^0)$, and $\mathbb{P}_{\omega}^* = \text{Law}(W)$. Consequently,

$$\bar{\mathbb{P}}[X \in A | \mathbf{P}^*, W^0] = \mathbf{P}^*(A) \quad \forall A \in \mathcal{B}(D_{\mathbb{R}}).$$

By Proposition 2.18,

$$\mathcal{M}_t = X_t - X_{-1} - \int_0^t b(s, X_s, \nu_s^*) ds - \int_{[0, t]} \alpha(s) d\mathbf{P}^*(\tau_0(X) \leq s)$$

is a continuous local martingale with

$$\begin{aligned} \langle \mathcal{M} \rangle_t &= \int_0^t \sigma(s, X_s)^2 ds, \quad \langle \mathcal{M}, W \rangle_t = \int_0^t \sigma(s, X_s) \sqrt{1 - \rho(s, \nu_s^*)^2} ds, \\ \langle \mathcal{M}, W^0 \rangle_t &= \int_0^t \sigma(s, X_s) \rho(s, \nu_s^*) ds, \end{aligned}$$

where $\nu_s^* := \mathbf{P}^*(X_s \in \cdot, \tau_0(X) > s)$. As W^0 and W are standard independent Brownian motions, by Lévy's Characterisation Theorem we have that

$$\mathcal{M}_t = \int_0^t \sigma(s, X_s) \left(\sqrt{1 - \rho(s, \nu_s^*)^2} dW_s + \rho(s, \nu_s^*) dW_s^0 \right).$$

Now, as $-1 \in \mathbb{T}$, the map $\eta \mapsto \eta_{-1}$ is μ -almost surely continuous for $\mathbb{P}_{\mu, \omega^0, \omega}$ -almost every measure μ . A simple application of the Portmanteau Theorem shows that $X_{-1} \sim \nu_{0-}$. By Lemma 2.10, we deduce that $X_{0-} \sim \nu_{0-}$. The independence between (\mathbf{P}^*, W^0) and W follows from Lemma 2.15. A similar argument as employed in Lemma 2.15 shows that $X_{0-} \perp (\mathbf{P}^*, W^0, W)$. Lastly, by Lemma A.9,

$$\Delta L_t^* \leq \inf\{x \geq 0 : \nu_{t-}^*([0, \alpha(t)x]) < x\} \quad \text{a.s.}$$

for all $t \geq 0$. □

3 Stronger mode of convergence

One of the limitations of the method in Section 2 is that it fails to yield a strong solution. That is, \mathbf{P} is not equal to $\text{Law}(X | W^0)$. This is due to the mode of convergence employed being weak. To the best of our knowledge, there are no results in the existing literature relating to the existence of strong physical solutions in the setting with common noise. By Remark 2.5 from [28], the existence of strong solutions in the setting when b , σ and ρ are functions of time only is shown; however, it remains unclear whether these solutions are physical or not.

The work introduced in [9] provided an alternative framework to construct solutions to systems with simplified dynamics and without common noise. This is done by a fixed-point approach. Notably, the constructed solutions possess a minimality property, meaning that any alternative solution to the system will dominate the solution obtained in [9]. By utilising the mean-field limit of a perturbed finite particle system approximation, the authors deduce that minimal solutions are in fact physical.

This section extends this work to the case with common noise. Provided more restrictive assumptions on the coefficients than those introduced in Assumption 2.2,

we provide an algorithm to construct minimal W^0 -measurable solutions to the singular and smoothed system. Furthermore, we get almost sure convergence of the smoothed minimal system towards the singular minimal system. As a consequence, we are able to conclude that the minimal W^0 -measurable solution is, in fact, physical. This provides an alternative method to show minimal solutions are physical in the setting of [9].

We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ that satisfies the usual conditions and supports two independent Brownian motions. This differs from Section 2 as the filtered probability space may change as we change ε . The mode of convergence was weak in Section 2, therefore the smoothed systems needed not be defined on the same probability space. In this section, to be able to show a stronger mode of convergence, we require that our probability space and our Brownian motions are fixed because our methods employ a comparison principle approach.

We would like the loss process to be adapted and measurable with respect to the common noise. Hence, for measurability reasons, we define \mathcal{F}^{W^0} as the σ -algebra generated by W^0 and augmented to contain all \mathbb{P} -null sets. We define $\mathcal{F}_t^{W^0}$ to be the right continuous filtration generated by W^0 that contains all the information up to time t and augmented to contain all \mathbb{P} -null sets. To be precise, that is

$$\mathcal{F}_t^{W^0} = \left(\bigcap_{s > t} \sigma(\{W_u^0 : u \leq s\}) \right) \vee \sigma(\{N \in \mathcal{F} : \mathbb{P}(N) = 0\}).$$

As Brownian motion is continuous and has independent increments, W^0 is still a standard Brownian motion under the filtration $(\mathcal{F}_t^{W^0})_{t \geq 0}$.

We now propose our alternative method of solution construction. We will be considering the equation

$$\begin{cases} dX_t = b(t) dt + \sigma(t) \sqrt{1 - \rho(t)^2} dW_t + \sigma(t) \rho(t) dW_t^0 - \alpha dL_t, \\ \tau = \inf\{t > 0 : X_t \leq 0\}, \\ \mathbf{P} = \mathbb{P} \left[X \in \cdot \mid \mathcal{F}^{W^0} \right], \quad \nu_t := \mathbb{P} \left[X_t \in \cdot, \tau > t \mid \mathcal{F}_t^{W^0} \right], \\ L_t = \mathbb{P} \left[\tau \leq t \mid \mathcal{F}_t^{W^0} \right], \end{cases} \quad (3.1)$$

where $\alpha > 0$ is a constant. The coefficients b , σ , and ρ are a measurable maps from \mathbb{R} into \mathbb{R} satisfying Assumption 2.2. The system starts at time $0-$ with initial condition X_{0-} which is almost surely positive. We require no further assumptions on the initial condition.

Given any solution (X, L) to (3.1), we may view the paths of L living in the space

$$M := \{ \ell : \bar{\mathbb{R}} \rightarrow [0, 1] : \ell_{0-} = 0, \ell_\infty = 1, \ell \text{ increasing and càdlàg} \}.$$

M is the space of cumulative density functions on the extended real line. We endow M with the topology induced by the Lévy-metric

$$d_L(\ell^1, \ell^2) := \inf \{ \varepsilon > 0 : \ell_{t+\varepsilon}^1 + \varepsilon \geq \ell_t^2 \geq \ell_{t-\varepsilon}^1 - \varepsilon, \forall t \geq 0 \}.$$

The Lévy-metric metricizes weak convergence, hence we are endowing M with the topology of weak convergence as we can associate each ℓ with a distribution $\mu_\ell \in \mathcal{P}([0, \infty])$. Hence as M is endowed with the topology of weak convergence, then we observe that $\ell^n \rightarrow \ell$ in M if and only if $\ell_t^n \rightarrow \ell_t$ for all $t \in \mathbb{T} := \{t \geq 0 : \ell_{t-} = \ell_t\}$. With this topology, M is a compact Polish space. As in the previous section, we will let $D_{\mathbb{R}}$ denote the space of càdlàg functions from $[-1, \infty)$ to \mathbb{R} and we endow $D_{\mathbb{R}}$ with the M_1 -topology. As elements in M are increasing, then convergence in M is equivalent to convergence in $D_{\mathbb{R}}$.

3.1 Properties of Γ and existence of strong solutions

For any W^0 -measurable process ℓ with values in M , we define the operator Γ as

$$\begin{cases} dX_t^\ell = b(t) dt + \sigma(t)\sqrt{1 - \rho(t)^2} dW_t + \sigma(t)\rho(t) dW_t^0 - \alpha d\ell_t, \\ \tau^\ell = \inf\{t > 0 : X_t^\ell \leq 0\}, \\ \Gamma[\ell]_t = \mathbf{P}[\tau^\ell \leq t | \mathcal{F}_t^{W^0}]. \end{cases}$$

By the independence of increments of Brownian motion, $\mathbf{P}[\tau^\ell \leq t | \mathcal{F}_t^{W^0}] = \mathbf{P}[\tau^\ell \leq t | \mathcal{F}_t^{W^0}]$. Therefore, we may always choose a version of $\mathbf{P}[\tau^\ell \leq t | \mathcal{F}_t^{W^0}]$ such that $\Gamma[\ell]$ is a W^0 -measurable process with càdlàg paths. By artificially setting $\Gamma[\ell]_\infty = 1$, $\Gamma[\ell]$ has paths in M . First, we observe that Γ is a continuous operator.

Proposition 3.1 (Continuity of Γ). *Let ℓ^n and ℓ be a sequence of adapted W^0 -measurable processes that take values in M such that $\ell^n \rightarrow \ell$ almost surely in M . Then $\Gamma[\ell^n] \rightarrow \Gamma[\ell]$ almost surely in M .*

Proof. For simplicity, we shall denote X^{ℓ^n} by X^n and X^ℓ by X . As done previously, we may artificially extend X^n and X to be càdlàg processes on $[-1, \infty)$ by

$$\tilde{X}^n := \begin{cases} X_{0-} & t \in [-1, 0), \\ X_t^n & t \geq 0, \end{cases} \quad \tilde{X} := \begin{cases} X_{0-} & t \in [-1, 0), \\ X_t & t \geq 0, \end{cases}$$

By the coupling, $\tilde{X}^n + \alpha\ell^n = \tilde{X} + \alpha\ell$ for every n . Hence trivially $\tilde{X}^n + \alpha\ell^n \rightarrow \tilde{X} + \alpha\ell$ in $D_{\mathbb{R}}$. As convergence in M is equivalent to convergence in the M_1 -topology, $\ell^n \rightarrow \ell$ almost surely in $D_{\mathbb{R}}$. Addition is a M_1 -continuous map for functions that have jumps of common sign, [33, Theorem 12.7.3], therefore $\tilde{X}^n \rightarrow \tilde{X}$ almost surely in $D_{\mathbb{R}}$. It is clear that $\Delta\tilde{X}_t \leq 0$ for any $t \geq 0$ and

$$\mathbf{P}\left[\inf_{s \in (\tau_0(\tilde{X}), \tau_0(\tilde{X})+h)} \left\{ \tilde{X}_s - \tilde{X}_{\tau_0(\tilde{X})} \right\} \geq 0\right] = 0$$

for any $h > 0$ by Lemma A.2. Hence, τ_0 is an M_1 -continuous map at almost every path of \tilde{X} by Lemma A.3. By the Conditional Dominated Convergence Theorem, for any $t \in \mathbb{T}^{\Gamma[\ell]} := \{t \geq 0 : \mathbf{P}[\Gamma[\ell]_t = \Gamma[\ell]_{t-}] = 1\}$ we have

$$\Gamma[\ell^n]_t = \mathbb{E}\left[\mathbb{1}_{\{\tau_0(\tilde{X}^n) \leq t\}} \middle| \mathcal{F}^{W^0}\right] \longrightarrow \mathbb{E}\left[\mathbb{1}_{\{\tau_0(\tilde{X}) \leq t\}} \middle| \mathcal{F}^{W^0}\right] = \Gamma[\ell]_t \quad (3.2)$$

almost surely. Now, we fix a $\mathbb{D} \subset \mathbb{T}^{\Gamma[\ell]}$ such that \mathbb{D} is countable and dense in \mathbb{R}_+ . By (3.2), we may find a $\Omega_0 \in \mathcal{F}^{W^0}$ of full measure such that if we fix $\omega \in \Omega_0$ then (3.2) holds at ω for all $t \in \mathbb{D}$. Now we fix a $\gamma > 0$, $\omega \in \Omega_0$ and $t > 0$ such that $\Gamma[\ell]_t(\omega) = \Gamma[\ell]_{t-}(\omega)$. By continuity, there is a $s_1, s_2 \in \mathbb{D}$ such that $s_1 < t < s_2$ and

$$|\Gamma[\ell]_t(\omega) - \Gamma[\ell]_{s_1}(\omega)| + |\Gamma[\ell]_t(\omega) - \Gamma[\ell]_{s_2}(\omega)| < \gamma \quad (3.3)$$

Therefore for by monotonicity of $\Gamma[\ell^n]$ and the above we have

$$\begin{aligned} |\Gamma[\ell]_t(\omega) - \Gamma[\ell^n]_t(\omega)| &\leq |\Gamma[\ell]_{s_2}(\omega) - \Gamma[\ell]_t(\omega)| + |\Gamma[\ell]_{s_2}(\omega) - \Gamma[\ell^n]_{s_2}(\omega)| \\ &\quad + |\Gamma[\ell^n]_{s_1}(\omega) - \Gamma[\ell^n]_{s_2}(\omega)| = O(\gamma) \end{aligned}$$

for all n large. In the case when $t = 0$ is a continuity point, we set $s_1 = -1$. Hence we have convergence of $\Gamma[\ell^n](\omega)$ to $\Gamma[\ell](\omega)$ at the continuity points of $\Gamma[\ell](\omega)$. Therefore, by definition, $\Gamma[\ell^n](\omega)$ converges to $\Gamma[\ell](\omega)$ in M . As Ω_0 is a set of full measure, the result follows. \square

We observe that the map Γ also preserves almost sure monotonicity of the input processes.

Lemma 3.2 (Monotonicity of Γ). *Let ℓ^1 and ℓ^2 be W^0 -measurable processes with paths in M such that $\ell^1 \leq \ell^2$ almost surely, then $\Gamma[\ell^1] \leq \Gamma[\ell^2]$ almost surely.*

Proof. As $\ell^1 \leq \ell^2$ almost surely, then we have $X^{\ell^1} \geq X^{\ell^2}$ almost surely. It follows that $\tau^{\ell^1} \leq \tau^{\ell^2}$ almost surely. By monotonicity of conditional expectation,

$$\Gamma[\ell^1]_t = \mathbb{P} \left[\tau^{\ell^1} \leq t \mid \mathcal{F}_t^{W^0} \right] \leq \mathbb{P} \left[\tau^{\ell^2} \leq t \mid \mathcal{F}_t^{W^0} \right] = \Gamma[\ell^2]_t$$

almost surely for any $t \geq 0$. As $\Gamma[\ell^1]$ and $\Gamma[\ell^2]$ are càdlàg, we deduce $\Gamma[\ell^1]_t \leq \Gamma[\ell^2]_t$ for any $t \geq 0$ almost surely. \square

With these two results in hand, we have all the ingredients to construct W^0 -measurable solutions to (3.1).

Proposition 3.3. *There exists a càdlàg W^0 -measurable process \underline{L} which solves (3.1) and for any other càdlàg W^0 -measurable process L which satisfies (3.1), we have $\underline{L} \leq L$ almost surely.*

Proof. For any $n \geq 1$, we define inductively

$$\begin{cases} dX_t^n = b(t) dt + \sigma(t) \sqrt{1 - \rho(t)^2} dW_t + \sigma(t) \rho(t) dW_t^0 - \alpha d\Gamma^{n-1}[\mathbb{1}_{\{\infty\}}], \\ \tau^n = \inf\{t > 0 : X_t^n \leq 0\}, \\ \Gamma^n[\mathbb{1}_{\{\infty\}}]_t = \mathbb{P} \left[\tau^n \leq t \mid \mathcal{F}_t^{W^0} \right], \end{cases}$$

with $\Gamma^0[\mathbb{1}_{\{\infty\}}] = \mathbb{1}_{\{\infty\}}$ and $\Gamma^n[\mathbb{1}_{\{\infty\}}]$ is the application of Γ n -times to the function $\mathbb{1}_{\{\infty\}} \in M$. By Lemma 3.2, $\Gamma^{n+1}[\mathbb{1}_{\{\infty\}}] \geq \Gamma^n[\mathbb{1}_{\{\infty\}}]$ almost surely for any $n \in \mathbb{N}$. As these processes are càdlàg, we deduce $\Gamma^{n+1}[\mathbb{1}_{\{\infty\}}] \geq \Gamma^n[\mathbb{1}_{\{\infty\}}]$ for any $n \in \mathbb{N}$ almost surely. Let $\Omega_0 \in \mathcal{F}^{W^0}$ denote the set of full measure where the monotonicity holds for every n and we fix a $\mathbb{D} \subset \mathbb{R}_+$ that is countable and dense. As $\Gamma^n[\mathbb{1}_{\{\infty\}}]$ is increasing and bounded above, let

$$\ell_t := \lim_{n \rightarrow \infty} \Gamma^n[\mathbb{1}_{\{\infty\}}]_t \mathbb{1}_{\Omega_0} \quad \forall t \in \mathbb{D}.$$

It is clear for any $t \in \mathbb{D}$, ℓ_t is $\mathcal{F}_t^{W^0}$ -measurable. Therefore we define

$$\underline{L}_t := \lim_{s \downarrow t, s \in \mathbb{D}} \ell_s \quad \forall t \geq 0.$$

By construction, \underline{L}_t is a càdlàg W^0 -measurable process with paths in M . A similar proof as that used in the end of Proposition 3.1, shows that $\Gamma^n[\mathbb{1}_{\{\infty\}}] \rightarrow \underline{L}$ almost surely in M . Hence by Proposition 3.1, $\Gamma^{n+1}[\mathbb{1}_{\{\infty\}}] \rightarrow \Gamma[\underline{L}]$. As $\Gamma[\underline{L}]$ and \underline{L} are càdlàg W^0 -measurable processes that are limits of $\Gamma^n[\mathbb{1}_{\{\infty\}}]$, we may conclude that $\Gamma[\underline{L}] = \underline{L}$ almost surely. Lastly, if L is any càdlàg W^0 -measurable process that solves (3.1), then by Lemma 3.2 we have $\Gamma^n[\mathbb{1}_{\{\infty\}}] \leq L$ for all $n \in \mathbb{N}$ almost surely. Taking limit, we deduce $\underline{L} \leq L$ almost surely. \square

We now turn our attention to the smoothed version of (3.1). We will work on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ as in (3.1) that satisfies the usual conditions and supports two independent Brownian motions. For an $\varepsilon > 0$, we consider the McKean–Vlasov problem

$$\begin{cases} dX_t^\varepsilon = b(t) dt + \sigma(t) \sqrt{1 - \rho(t)^2} dW_t + \sigma(t) \rho(t) dW_t^0 - \alpha d\mathfrak{L}_t^\varepsilon, \\ \tau^\varepsilon = \inf\{t > 0 : X_t^\varepsilon \leq 0\}, \\ \mathbf{P}^\varepsilon = \mathbb{P} \left[X^\varepsilon \in \cdot \mid \mathcal{F}^{W^0} \right], \quad \nu_t^\varepsilon := \mathbb{P} \left[X_t^\varepsilon \in \cdot, \tau^\varepsilon > t \mid \mathcal{F}_t^{W^0} \right], \\ L_t^\varepsilon = \mathbb{P}^\varepsilon \left[\tau^\varepsilon \leq t \mid \mathcal{F}_t^{W^0} \right], \quad \mathfrak{L}_t^\varepsilon = \int_0^t \kappa^\varepsilon(t-s) L_s^\varepsilon ds, \end{cases} \quad (3.4)$$

where $\alpha > 0$ is a constant. The coefficients b , σ , ρ and κ are a measurable maps from \mathbb{R} into \mathbb{R} satisfying Assumption 2.2. The system starts at time $0-$ with the same initial condition, X_{0-} , as in (3.1). As the assumptions on X_{0-} is more general than those imposed in Section 2, we may not apply Theorem 2.4 to guarantee existence of solutions to (3.1). So, we propose an alternative proof to show existence of solutions. The proof follows in the same faith as Proposition 3.3. We define the operator

$$\Gamma_\varepsilon[\ell] := \Gamma[(\kappa^\varepsilon * \ell)], \quad \text{where} \quad (\kappa^\varepsilon * \ell) := \int_0^\cdot \kappa^\varepsilon(\cdot - s) \ell_s \, ds.$$

Therefore, solutions to (3.4) are equivalent to finding almost sure fixed points of Γ_ε . A simple consequence of Proposition 3.1, is that Γ_ε is also continuous.

Corollary 3.4 (Continuity of Γ_ε). *Let ℓ^n and ℓ be a sequence of adapted W^0 -measurable processes that take values in M such that $\ell^n \rightarrow \ell$ almost surely in M . Then $\Gamma_\varepsilon[\ell^n] \rightarrow \Gamma_\varepsilon[\ell]$ almost surely in M .*

Proof. By Proposition 3.1, it is sufficient to show that the map $\tilde{\ell} \mapsto \kappa^\varepsilon * \tilde{\ell}$ is continuous on M . It is clear that if we implicitly define the value of $\kappa^\varepsilon * \tilde{\ell}$ to be 1 at ∞ , then it is an element of M . Let $\tilde{\ell}^n$ and $\tilde{\ell}$ be deterministic functions in M such that $\tilde{\ell}^n \rightarrow \tilde{\ell}$ in M . That is, we have pointwise convergence on the continuity points of $\tilde{\ell}$. As $\kappa \in \mathcal{W}^{1,1}(\mathbb{R}_+)$, it has a continuous representative. So without loss of generality, we take κ to be this representative. Hence κ is bounded on compacts, so an easy application of the Dominated Convergence Theorem gives

$$\lim_{n \rightarrow \infty} (\kappa^\varepsilon * \tilde{\ell}^n)_t = \lim_{n \rightarrow \infty} \int_0^t \kappa^\varepsilon(t-s) \tilde{\ell}_s^n \, ds = \int_0^t \kappa^\varepsilon(t-s) \tilde{\ell}_s \, ds = (\kappa^\varepsilon * \tilde{\ell})_t \quad \square$$

As convolution with non-negative functions preserves monotonicity, we further deduce that Γ_ε is also monotonic by Lemma 3.2.

Corollary 3.5. *Let ℓ^1 and ℓ^2 be W^0 -measurable processes with paths in M such that $\ell^1 \leq \ell^2$ almost surely, then $\Gamma_\varepsilon[\ell^1] \leq \Gamma_\varepsilon[\ell^2]$ almost surely.*

With monotonicity and continuity of the operator Γ_ε in hand, we have all the necessary results to deduce the existence of solutions to (3.4).

Proposition 3.6. *There exists a càdlàg W^0 -measurable process $\underline{L}^\varepsilon$ which solves (3.4) and for any other càdlàg W^0 -measurable process L^ε which satisfies (3.4), we have $\underline{L}^\varepsilon \leq L^\varepsilon$ almost surely.*

Proof. By employing Corollary 3.4 and Corollary 3.5, this proof is verbatim to that of Proposition 3.3. \square

The purpose of κ^ε in (3.4) is two-fold. Firstly, it smoothens the effect of the feedback component on the system, hence preventing the system from jumping and making it continuous. Secondly, it delays the effect of L_t^ε of the system. Intuitively, one would expect that the system with instantaneous feedback, i.e. (3.1), will be dominated by that with delayed feedback. Furthermore, intuitively as we decrease ε , then the system with the smaller value of ε should be dominated by one with a larger value. This is because as ε decreases, the rate at which the feedback is felt by the system increases.

Lemma 3.7. *For any $\varepsilon, \tilde{\varepsilon} > 0$ such that $\tilde{\varepsilon} < \varepsilon$, it holds that*

$$\underline{L}^\varepsilon \leq \underline{L}^{\tilde{\varepsilon}} \quad \text{and} \quad \underline{L}^{\tilde{\varepsilon}} \geq \underline{L}^\varepsilon$$

almost surely.

Proof. For any deterministic functions $\tilde{\ell}^1, \tilde{\ell}^2 \in M$ such that $\tilde{\ell}^1 \leq \tilde{\ell}^2$, a straightforward computation shows that $(\kappa^\varepsilon * \tilde{\ell}^1) \leq \tilde{\ell}^2$. Furthermore, for any $t > 0$, we have:

$$\begin{aligned} \int_0^t \kappa^\varepsilon(s) \tilde{\ell}_{t-s}^1 ds &= \int_0^t \varepsilon^{-1} \kappa(s\varepsilon^{-1}) \tilde{\ell}_{t-s}^1 ds \\ &= \int_0^{\frac{t}{\varepsilon}} \tilde{\varepsilon} \kappa(\tilde{s}\tilde{\varepsilon}^{-1}) \tilde{\ell}_{t-\tilde{s}}^1 d\tilde{s} \\ &\leq \int_0^t \kappa^\varepsilon(\tilde{s}) \tilde{\ell}_{t-\tilde{s}}^1 d\tilde{s} \\ &\leq \int_0^t \kappa^\varepsilon(\tilde{s}) \tilde{\ell}_{t-\tilde{s}}^2 d\tilde{s}. \end{aligned}$$

The second equality follows from employing the substitution $\tilde{s}\tilde{\varepsilon}^{-1} = s\varepsilon^{-1}$, and the third inequality follows from the fact that $\tilde{\varepsilon}\varepsilon^{-1} < 1$, hence $\frac{t}{\varepsilon} < t$ and $\tilde{\ell}_{t-\tilde{s}}^1 \leq \tilde{\ell}_{t-s}^1$. The claim now follows from the monotonicity from Proposition 3.1 and Lemma 3.2. \square

3.2 Convergence of minimal solutions

From now on, we will fix a sequence of positive real numbers $(\varepsilon_n)_{n \geq 1}$ that converge to zero. As we have established that $\underline{L}^\varepsilon$ is a decreasing process in ε by Lemma 3.7, we shall exploit this structure to construct a solution to (3.1). This will be a W^0 -measurable solution that will be dominated by every other W^0 -measurable solution. Therefore, we may conclude that this solution must coincide with \underline{L} on a set of full measure.

Theorem 3.8 (Almost sure convergence). *Let $(\varepsilon_n)_{n \geq 1}$ be a sequence of positive real numbers that converges to zero. Let $(\underline{X}^\varepsilon, \underline{L}^\varepsilon)$ denote the W^0 -measurable solution to (3.4) constructed in Proposition 3.6, and $(\underline{X}, \underline{L})$ denote the W^0 -measurable solution to (3.1) constructed in Proposition 3.3. Then by considering the extended system*

$$\tilde{X}^{\varepsilon_n} := \begin{cases} X_{0-} & t \in [-1, 0), \\ \underline{X}_t^{\varepsilon_n} & t \geq 0, \end{cases} \quad \tilde{X} := \begin{cases} X_{0-} & t \in [-1, 0), \\ \underline{X}_t & t \geq 0, \end{cases}$$

we have $\text{Law}(\tilde{X}^{\varepsilon_n} \mid \mathcal{F}^{W^0}) \rightarrow \text{Law}(\tilde{X} \mid \mathcal{F}^{W^0})$ almost surely in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$. Furthermore, $\underline{L}^{\varepsilon_n}$ converges to \underline{L} almost surely in M and \underline{L} satisfies the physical jump condition.

Proof. As $(\varepsilon_n)_{n \geq 1}$ is a bounded sequence of reals converging to zero, we may find a decreasing subsequence $(\varepsilon_{n_j})_{j \geq 1}$ which converges to zero. We fix a $\mathbb{D} \subset \mathbb{R}_+$ that is countable and dense in \mathbb{R}_+ and by Lemma 3.7 we may find a $\Omega_0 \in \mathcal{F}^{W^0}$ such that $L^{\varepsilon_n \vee \varepsilon_m} \leq L^{\varepsilon_n \wedge \varepsilon_m}$ for any $n, m \in \mathbb{N}$. By the boundedness of L^ε and Lemma 3.7,

$$\ell_t := \lim_{j \rightarrow \infty} L_t^{\varepsilon_{n_j}} \mathbb{1}_{\Omega_0}$$

is well defined for any $t \in \mathbb{D}$. Furthermore by Lemma 3.7, we may deduce that

$$\ell_t := \lim_{n \rightarrow \infty} L_t^{\varepsilon_n} \mathbb{1}_{\Omega_0}$$

for any $t \in \mathbb{D}$. It is clear by construction that ℓ_t is $\mathcal{F}_t^{W^0}$ -measurable. Lastly, we define

$$L_t = \lim_{s \downarrow t, s \in \mathbb{D}} \ell_s.$$

It is immediate that L is a càdlàg W^0 -measurable process. Following the similar procedure as at the end of Proposition 3.1 with the obvious changes, we obtain that $L^{\varepsilon_n} \rightarrow L$ almost surely in M . For simplicity, we will denote X^L by simply X and let

$$\tilde{X} := \begin{cases} X_{0-} & t \in [-1, 0), \\ X_t & t \geq 0. \end{cases}$$

Then $\tilde{X}^{\varepsilon_n} \rightarrow \tilde{X}$ almost surely in $D_{\mathbb{R}}$. As $\Delta \tilde{X}_t \leq 0$ for every t almost surely and

$$\mathbb{P} \left[\inf_{s \in (\tau_0(\tilde{X}), \tau_0(\tilde{X})+h)} \left\{ \tilde{X}_s - \tilde{X}_{\tau_0(\tilde{X})} \right\} \geq 0 \right] = 0$$

for any $h \geq 0$, we have that τ_0 is M_1 -continuous at almost every path of \tilde{X} . Therefore we deduce (X, L) is a W^0 -measurable solution to (3.1). By Lemma 3.7, we have that $L \leq \underline{L}$ almost surely. By Proposition 3.3, we must have $L = \underline{L}$ almost surely and hence $L^{\varepsilon_n} \rightarrow \underline{L}$ almost surely in M . As $\tilde{X}^{\varepsilon_n}$ converges to \tilde{X} almost surely in $D_{\mathbb{R}}$, then by the Conditional Dominated Convergence Theorem $\text{Law}(\tilde{X}^{\varepsilon_n} \mid \mathcal{F}^{W^0}) \rightarrow \text{Law}(\tilde{X} \mid \mathcal{F}^{W^0})$ in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$. By Lemma A.9 and [25, Proposition 3.5], we have

$$\Delta \underline{L}_t = \inf \{x \geq 0 : \nu_{t-}[0, \alpha x] < x\} \quad \text{a.s.}$$

for all $t \geq 0$. □

Remark 3.9 (Propagation of minimality). This result is parallel to Theorem 6.6 in [9], which states that minimal solutions to the finite particle system approximation will converge in probability to the limiting equation provided a unique physical solution exists. The above shows that the W^0 -measurable minimal solutions to the smoothed system will converge to the W^0 -measurable minimal solution of the limiting system *without* needing to assume the existence of a unique physical solution.

All of the results in this section only required non-negativity of the initial condition. Moreover, we only established the existence of solutions to (3.1) and (3.4) but made no comments and have no results regarding the number of solutions in such a general setting. However, if we assume that the initial condition satisfies Assumption 2.2 (vi), then there is a unique solution to (3.4). In other words, the $\underline{L}^{\varepsilon}$ we constructed is the only solution. Furthermore, if we further assume that the initial condition satisfies

$$\inf \{x > 0 : \nu_{t-}[0, \alpha x] < x\} = 0,$$

then 0 is an almost sure continuity point of \underline{X} . Therefore these observations along with Theorem 3.8 allow us to deduce the following result.

Corollary 3.10. *Let $(\varepsilon_n)_{n \geq 1}$ be a sequence of positive real numbers that converges to zero. We suppose that the initial condition, X_{0-} , satisfies Assumption 2.2 (vi) and that $\inf \{x > 0 : \nu_{t-}[0, \alpha x] < x\} = 0$. Then $\text{Law}(X^{\varepsilon_n} \mid \mathcal{F}^{W^0}) \rightarrow \text{Law}(\underline{X} \mid \mathcal{F}^{W^0})$ almost surely in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$. Furthermore, L^{ε_n} converges to \underline{L} almost surely in M and \underline{L} satisfies the physical jump condition.*

4 Rates of convergence

One of the limitations of the previous arguments is that they fail to yield a rate at which the convergence will occur. For simple systems, that is in the case of no drift, no common noise and a volatility parameter set to 1, we employ a coupling argument to derive the speed of convergence, which depends on the regularity of L .

The regularity of the loss process, L , has been established in the literature, [11, 17], for a suitable class of initial conditions. In this setting, we not only have almost-sure convergence of the stochastic process along a subsequence, but we will have uniform convergence on any time domain before the time that the regularity of L decays. These results are in some sense parallel to those presented in [21], with the crucial difference that we are looking at the rate of convergence of systems with smoothed loss to the limiting system, as opposed to the convergence of timestepping schemes that approximate the limiting system. Note also that [22, 8] provide convergence orders

for timestepping schemes in the case of Hölder continuous L , but convergence without order in the case jump case.

To be precise, we will be considering the following system of equations

$$\begin{cases} X_t^\varepsilon = X_{0-} + W_t - \alpha \mathfrak{L}_t^\varepsilon, \\ \tau^\varepsilon = \inf \{t \geq 0 : X_t^\varepsilon \leq 0\}, \\ L_t^\varepsilon = \mathbb{P}(\tau^\varepsilon \leq t), \\ \mathfrak{L}_t^\varepsilon = \int_0^t \kappa^\varepsilon(t-s) L_s^\varepsilon ds, \end{cases} \quad \begin{cases} X_t = X_{0-} + W_t - \alpha L_t, \\ \tau = \inf \{t \geq 0 : X_t \leq 0\}, \\ L_t = \mathbb{P}(\tau \leq t), \end{cases} \quad (4.1)$$

where $t \geq 0$, W is a standard Brownian motion, κ is a function from \mathbb{R} to \mathbb{R} satisfying Assumption 2.2 and $\text{supp}(\kappa) \subset [0, 1]$.

4.1 Theoretical estimates on rates of convergence

The main result of this section is the following:

Proposition 4.1. *Let $(X, L)_{t \geq 0}$ be a physical solution to (4.1) with initial condition X_{0-} . Suppose further that X_{0-} admits a bounded initial density V_{0-} s.t.*

$$V_{0-}(x) \leq Cx^\beta \mathbb{1}_{\{x \leq x_*\}} + D \mathbb{1}_{\{x > x_*\}} \quad \forall x > 0,$$

where $C, D, x_*, \beta > 0$ are constants with $\beta < 1$. Then, for any $t_0 \in (0, t_{\text{explode}})$ there exists a constant $\tilde{K} = \tilde{K}(t_0)$ s.t.

$$\sup_{s \in [0, t_0]} |L_s - L_s^\varepsilon| \leq \tilde{K} \varepsilon^{\beta/2},$$

where

$$t_{\text{explode}} := \sup\{t > 0 : \|L\|_{H^1(0, t)} < +\infty\} \in (0, +\infty]. \quad (4.2)$$

Proof. By assumption, we are in the setting of [17, Theorem 1.8]. Hence, we have a unique solution, L , to (4.1) up to the time t_{explode} defined as in (4.2). Also, for all $t_0 \in (0, t_{\text{explode}})$ there exists $K = K(t_0)$ s.t. $L \in \mathcal{S}(\frac{1-\beta}{2}, K, t_0)$ where

$$\mathcal{S}\left(\frac{1-\beta}{2}, K, t_0\right) := \{l \in H^1(0, t_0) : l'_t \leq Kt^{-\frac{1-\beta}{2}} \text{ for almost all } t \in [0, t_0]\}$$

Step 1: Regularity of L . Choose $t_0 \in (0, t_{\text{explode}})$. As $L \in H^1(0, t_0)$, for Lebesgue a.e. $t, s \in (0, t_0)$ we may write

$$L_t - L_s = \int_s^t L'_s ds \leq K(1-\gamma)^{-1}(t^{1-\gamma} - s^{1-\gamma}),$$

where the last inequality is from $L \in \mathcal{S}(\gamma, K, t_0)$ with $\gamma = (1-\beta)/2$. This implies

$$\frac{|L_t - L_s|}{|t-s|^{1-\gamma}} \leq \frac{K(t^{1-\gamma} - s^{1-\gamma})}{(1-\gamma)|t-s|^{1-\gamma}} \leq \frac{K}{1-\gamma}.$$

The last inequality is due to the subadditivity of concave functions. Therefore, L_t is almost everywhere $\frac{\beta+1}{2}$ -Hölder continuous.

Step 2: Decomposition of L into an integral form. We may write L as

$$L_t = \int_0^t \kappa^\varepsilon(t-s) L_s ds + \left[1 - \int_0^t \kappa^\varepsilon(t-s) ds\right] L_t + \int_0^t \kappa^\varepsilon(t-s)(L_t - L_s) ds.$$

Observe

$$\left[1 - \int_0^t \kappa^\varepsilon(t-s) \, ds\right] L_t \leq \frac{2K\varepsilon^{1-\gamma}}{1-\gamma} \quad \text{and} \quad \int_0^t \kappa^\varepsilon(t-s)(L_t - L_s) \, ds \leq \frac{K\varepsilon^{1-\gamma}}{1-\gamma}.$$

Therefore

$$L_t = \int_0^t \kappa^\varepsilon(t-s) L_s \, ds + \Psi^\varepsilon(t) \quad \text{where } |\Psi^\varepsilon(t)| \leq \frac{3K\varepsilon^{1-\gamma}}{1-\gamma} \, \forall t \in [0, t_0]. \quad (4.3)$$

Step 3: Comparison between the delayed loss and instantaneous loss. By Lemma 3.7, we have that $L \geq L^\varepsilon$, in the same spirit as [17, Proposition 3.1],

$$0 \leq L_t - L_t^\varepsilon \leq c_1 \int_0^t \frac{L_u - L_u^\varepsilon}{\sqrt{t-u}} L'_u \, du \leq c_1 \int_0^t \int_0^u \kappa^\varepsilon(u-s) \frac{L_s - L_s^\varepsilon}{\sqrt{t-u}} L'_u \, ds \, du + c_1 \int_0^t \frac{\Psi^\varepsilon(s) L'_s}{\sqrt{t-s}} \, ds,$$

where $c_1 = \alpha\sqrt{2/\pi}$ and the second inequality follows by (4.3). As $L \in \mathcal{S}(\gamma, K, t_0)$,

$$0 \leq |L_t - L_t^\varepsilon| \leq Kc_1 \int_0^t \int_0^u \frac{\kappa^\varepsilon(u-s) |L_s - L_s^\varepsilon|}{u^\gamma \sqrt{t-u}} \, ds \, du + Kc_1 \int_0^t \frac{|\Psi^\varepsilon(s)|}{s^\gamma \sqrt{t-s}} \, ds.$$

By (4.3), we may find a constant $C_{K,t_0,\alpha}$ such that the second term above is bounded by $C_{K,t_0,\alpha}\varepsilon^{1-\gamma}$. Therefore,

$$0 \leq |L_t - L_t^\varepsilon| \leq Kc_1 \int_0^t |L_s - L_s^\varepsilon| \rho^\varepsilon(t, s) \, ds + C_{K,t_0,\alpha}\varepsilon^{1-\gamma}, \quad (4.4)$$

where

$$\rho^\varepsilon(t, s) = \int_s^t \frac{\kappa^\varepsilon(u-s)}{u^\gamma \sqrt{t-u}} \, du.$$

Step 4: Bounds on $\rho^\varepsilon(t, s)$

As ρ^ε depends on t and s , we may not immediately apply Grönwall's lemma or any of its generalisations. To this end, we first construct upper bounds to relax the dependence of ρ^ε on t and s via the function κ . In the case when $0 \leq t-s \leq \varepsilon$

$$\begin{aligned} \rho^\varepsilon(t, s) &= \int_s^t \frac{\kappa^\varepsilon(u-s)}{u^\gamma \sqrt{t-u}} \, du = \int_0^{\frac{t-s}{\varepsilon}} \frac{\kappa(\tilde{u})}{(\varepsilon\tilde{u}+s)^\gamma \sqrt{t-s-\varepsilon\tilde{u}}} \, d\tilde{u} \leq \frac{\|\kappa\|_{L^\infty}}{s^\gamma \varepsilon^{1/2}} \int_0^{\frac{t-s}{\varepsilon}} \frac{d\tilde{u}}{\sqrt{\frac{t-s}{\varepsilon} - \tilde{u}}} \\ &= \frac{2\|\kappa\|_{L^\infty} (t-s)^{1/2}}{s^\gamma \varepsilon} \leq \frac{2\|\kappa\|_{L^\infty}}{s^\gamma (t-s)^{1/2}}, \end{aligned}$$

where we used the substitution $\tilde{u} = (u-s)\varepsilon^{-1}$. In the case when $t-s > \varepsilon$, as the support of κ^ε is in $[0, \varepsilon]$

$$\begin{aligned} \rho^\varepsilon(t, s) &= \int_s^t \frac{\kappa^\varepsilon(u-s)}{u^\gamma \sqrt{t-u}} \, du = \int_s^{s+\varepsilon} \frac{\kappa^\varepsilon(u-s)}{u^\gamma \sqrt{t-u}} \, du \\ &\leq \frac{\|\kappa\|_{L^\infty}}{s^\gamma \varepsilon} \int_s^{s+\varepsilon} \frac{du}{\sqrt{t-u}} = \frac{2\|\kappa\|_{L^\infty}}{s^\gamma} \left[\frac{(t-s)^{1/2} - (t-s-\varepsilon)^{1/2}}{\varepsilon} \right]. \end{aligned}$$

Step 5: Grönwall type argument

Now that we have sufficiently decoupled κ from ρ^ε , we may put (4.4) into a form where we may apply a generalised Grönwall Lemma. By step 4 case 1 and (4.4), for $t \leq \varepsilon$,

$$|L_t - L_t^\varepsilon| \leq Kc_1 \int_0^t 2\|\kappa\|_{L^\infty} s^{-\gamma} (t-s)^{-1/2} |L_s - L_s^\varepsilon| \rho^\varepsilon(t, s) \, ds + C_{K,t_0,\alpha}\varepsilon^{1-\gamma}.$$

By the second case of step 4 and (4.4), we have for $t > \varepsilon$

$$\begin{aligned} |L_t - L_t^\varepsilon| &\leq Kc_1 \int_0^{t-\varepsilon} |L_s - L_s^\varepsilon| \rho^\varepsilon(t, s) \, ds + Kc_1 \int_{t-\varepsilon}^t |L_s - L_s^\varepsilon| \rho^\varepsilon(t, s) \, ds + C_{K, t_0, \alpha} \varepsilon^{1-\gamma} \\ &\leq 2Kc_1 \|\kappa\|_{L^\infty} \int_0^{t-\varepsilon} \left[\frac{(t-s)^{1/2} - (t-s-\varepsilon)^{1/2}}{\varepsilon} \right] s^{-\gamma} |L_s - L_s^\varepsilon| \, ds \\ &\quad + 2Kc_1 \|\kappa\|_{L^\infty} \int_{t-\varepsilon}^t (t-s)^{-1/2} s^{-\gamma} |L_s - L_s^\varepsilon| \, ds + C_{K, t_0, \alpha} \varepsilon^{1-\gamma} \\ &\leq 2Kc_1 \|\kappa\|_{L^\infty} \sum_{j \geq 2} \tilde{C}_j \varepsilon^{j-1} \int_0^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} s^{-\gamma} |L_s - L_s^\varepsilon| \, ds \\ &\quad + 2Kc_1 \|\kappa\|_{L^\infty} \int_0^t (t-s)^{-1/2} s^{-\gamma} |L_s - L_s^\varepsilon| \, ds + C_{K, t_0, \alpha} \varepsilon^{1-\gamma}, \end{aligned}$$

where the last line follows from applying Taylor's Theorem and the Monotone Convergence Theorem. We note $\tilde{C}_j := (2j-2)!/[j!(j-1)!2^{2j-1}]$ is summable. Now we turn our attention onto the expression in the penultimate line. In the case when $\varepsilon < t \leq 2\varepsilon$,

$$\tilde{C}_j \varepsilon^{j-1} \int_0^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} s^{-\gamma} \, ds \leq \tilde{C}_j \varepsilon^{j-1} \varepsilon^{\frac{-2j+1}{2}} \int_0^{t-\varepsilon} s^{-\gamma} \, ds \leq \tilde{C}_j \varepsilon^{\frac{-1}{2}} \int_0^\varepsilon s^{-\gamma} \, ds \leq \frac{\tilde{C}_j \varepsilon^{\frac{1}{2}-\gamma}}{1-\gamma}$$

where the first inequality follows from the fact that $(-2j+1)/2 < 0$ as $j \geq 2$ and $t-s \in [\varepsilon, t]$ for $s \in [0, t-\varepsilon]$. In the case when $t > 2\varepsilon$, we observe that

$$\tilde{C}_j \varepsilon^{j-1} \int_0^\varepsilon (t-s)^{\frac{-2j+1}{2}} s^{-\gamma} \, ds \leq \tilde{C}_j \varepsilon^{j-1} \varepsilon^{\frac{-2j+1}{2}} \int_0^\varepsilon s^{-\gamma} \, ds \leq \frac{\tilde{C}_j \varepsilon^{\frac{1}{2}-\gamma}}{1-\gamma},$$

and

$$\begin{aligned} \tilde{C}_j \varepsilon^{j-1} \int_\varepsilon^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} s^{-\gamma} \, ds &\leq \tilde{C}_j \varepsilon^{j-1} \varepsilon^{-\gamma} \int_\varepsilon^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} \, ds \\ &= \tilde{C}_j \varepsilon^{j-1-\gamma} \frac{2}{2j-3} (t-s)^{\frac{-2j+3}{2}} \Big|_{s=\varepsilon}^{t-\varepsilon} \\ &\leq \frac{2\tilde{C}_j \varepsilon^{j-1-\gamma} \varepsilon^{\frac{-2j+3}{2}}}{2j-3} = \frac{2\tilde{C}_j \varepsilon^{1/2-\gamma}}{2j-3} \leq \frac{2\tilde{C}_j \varepsilon^{1/2-\gamma}}{1-\gamma}. \end{aligned}$$

Therefore, we have shown that

$$\tilde{C}_j \varepsilon^{j-1} \int_0^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} s^{-\gamma} \, ds \leq \frac{3\tilde{C}_j \varepsilon^{1/2-\gamma}}{1-\gamma}$$

for all $t > \varepsilon$. As L and L^ε are bounded by 1, we have independent of t being greater or less than ε ,

$$\begin{aligned} |L_t - L_t^\varepsilon| &\leq 2Kc_1 \|\kappa\|_{L^\infty} \int_0^t (t-s)^{-1/2} s^{-\gamma} |L_s - L_s^\varepsilon| \, ds + \frac{12Kc_1 \|\kappa\|_{L^\infty} \varepsilon^{1/2-\gamma} \sum_{j \geq 2} \tilde{C}_j}{1-\gamma} + C_{K, t_0, \alpha} \varepsilon^{1-\gamma} \\ &= 2Kc_1 \|\kappa\|_{L^\infty} \int_0^t (t-s)^{-1/2} s^{-\gamma} |L_s - L_s^\varepsilon| \, ds + C_{K, t_0, \alpha, \gamma} \varepsilon^{1/2-\gamma}, \end{aligned}$$

for any $t \in [0, t_0]$. Lastly, by Proposition A.10, using $\tilde{\beta} = 1/2$ and $\tilde{\alpha} = 1-\gamma$, then $\tilde{\alpha} + \tilde{\beta} - 1 > 0$ as $\gamma < 1/2$ and

$$\begin{aligned} |L_t - L_t^\varepsilon| &\leq C_{K, t_0, \alpha, \gamma} \varepsilon^{1/2-\gamma} \sum_{n \geq 0} (2Kc_1 \|\kappa\|_{L^\infty})^n C_n t_0^{n(1/2-\gamma)} \\ &= C_{K, t_0, \alpha, \gamma} \varepsilon^{\beta/2} \sum_{n \geq 0} (2Kc_1 t_0^{\beta/2} \|\kappa\|_{L^\infty})^n C_n, \end{aligned}$$

where in the last equality we used $\gamma = (1-\beta)/2$. This completes the proof. \square

The works of Fasano et al. [13, 14], Di Benedetto et al. [12], and Chayes et al. [6, 5] extensively investigate the supercooled Stefan problem, focusing on the existence of a unique solution without blow-ups for all time or until the entire liquid freezes. Recently, Delarue et al. [11] established global uniqueness for the system described in (4.1), under the condition that the density of the initial condition X_{0-} undergoes only a finite number of changes in monotonicity. Under this assumption, the loss is continuously differentiable on $(0, t_{\text{explode}})$ for t_{explode} defined in (4.2). Moreover, if the initial density has sufficient regularity, the loss will be continuously differentiable down to $t = 0$. Motivated by these results, we next investigate the rate of convergence when the loss function is differentiable.

Proposition 4.2. *Suppose we have a unique physical solution (X, L) to (4.1) such that $L \in C^1([0, t_{\text{explode}}))$ for $t_{\text{explode}} \in (0, \infty]$. Then for any $t_0 \in (0, t_{\text{explode}})$, there exists a constant $\tilde{K} = \tilde{K}(t_0)$ such that*

$$\sup_{s \in [0, t_0]} |L_s - L_s^\varepsilon| \leq \tilde{K} \varepsilon^{1/2}.$$

Proof. See appendix. □

4.2 Numerical simulations

Lastly, we investigate the convergence rate of the smoothed loss function towards the singular loss function through numerical simulations. The estimates from the previous section, for the case without common noise, give upper bounds on the rate at which the smoothed system will approach the singular system, prior to the decline in regularity of the singular loss function. The proofs employed in the analysis utilised relatively crude upper bounds, prompting the question of whether the obtained rates are optimal.

To the best of our knowledge, there is no existing literature on the regularity of the loss process in the presence of common noise. Consequently, the theoretical methods employed earlier may not be applicable in this scenario. Nevertheless, we can still explore the convergence rate numerically in this context as well. We consider the simplest setting with common noise,

$$\begin{cases} X_t^\varepsilon = X_{0-} + (1 - \rho^2)^{1/2} W_t + \rho W_t^0 - \alpha \mathcal{L}_t^\varepsilon, \\ \tau^\varepsilon = \inf \{t \geq 0 : X_t^\varepsilon \leq 0\}, \\ L_t^\varepsilon = \mathbb{P} \left[\tau^\varepsilon \leq t \mid \mathcal{F}_t^{W^0} \right], \\ \mathcal{L}_t^\varepsilon = \int_0^t \kappa^\varepsilon(t-s) L_s^\varepsilon ds, \end{cases} \quad \begin{cases} X_t = X_{0-} + (1 - \rho^2)^{1/2} W_t + \rho W_t^0 - \alpha L_t, \\ \tau = \inf \{t \geq 0 : X_t \leq 0\}, \\ L_t = \mathbb{P} \left[\tau \leq t \mid \mathcal{F}_t^{W^0} \right], \end{cases} \quad (4.5)$$

where $\rho \in [0, 1)$ is a fixed constant. We propose a numerical scheme that employs a particle system approximation to compute both the limiting and smoothed loss functions. Instead of employing numerical integration to compute the mollified loss of X^ε , the system will feel the impulse from a particle hitting the boundary at a random time in the future sampled from a random variable whose probability density function is the mollification kernel. The scheme is given in Algorithm 1.

By setting ρ to zero, the algorithm approximates the loss in the setting without common noise. To compute the limiting loss function we set ς to zero. In the case when $\rho = 0$ and $\varsigma = 0$, we recover the numerical scheme proposed in [21, 22]. In the numerical experiments below, we used $10^{6.5}$ particles and a uniform time mesh of width $\Delta_t := \min_i \{\varepsilon_i\} / 10$, with $\{\varepsilon_i\}_i$ the set of delay values, so that $t_i = i \times \Delta_t$ in Algorithm 1.

Overall, given sufficient regularity of the loss function, a rate of convergence close to 1 is observed. In the other cases studied with Hölder initial data, with the possibility of there being a jump after the test interval, and with common noise, the rate of

Algorithm 1: Discrete time Monte Carlo scheme for simulation of the smoothed loss process with common noise

Require: N – number of interacting particles
Require: n – number of time steps: $0 < t_1 < t_2 < \dots < t_n$
Require: ε – the strength of the delay

- 1 Draw one sample of W^0
- 2 Draw N samples of X_{0-} , W , and ς (r.v. with distribution $\kappa^\varepsilon(t) dt$)
- 3 **for** $i = 1 : n$ **do**
- 4 $\hat{L}_{t_i}^\varepsilon = \frac{1}{N} \sum_{m=1}^N \mathbb{1}_{(-\infty, 0]}(\min_{t_j < t_i} \{\hat{X}_{t_j}^{(m)}\})$
- 5 **for** $k = 1 : N$ **do**
- 6 Update
 $\hat{X}_{t_i}^{(k)} = X_{0-}^{(k)} + (1 - \rho^2)^{1/2} W_{t_i}^{(k)} + \rho W_{t_i}^0 - \frac{\alpha}{N} \sum_{m=1}^N \mathbb{1}_{(-\infty, 0]}(\min_{t_j < t_i - \varsigma^{(m)}} \{\hat{X}_{t_j}^{(m)}\})$
- 7 **end**
- 8 **end**

convergence appears to be between $1/2$ and 1 . See Appendix B for further analysis regarding the rate of convergence and how Δ_t affects the estimated rate.

4.3 Initial density vanishing at zero and no discontinuity or common noise

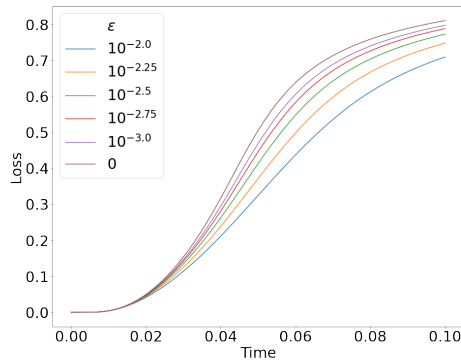
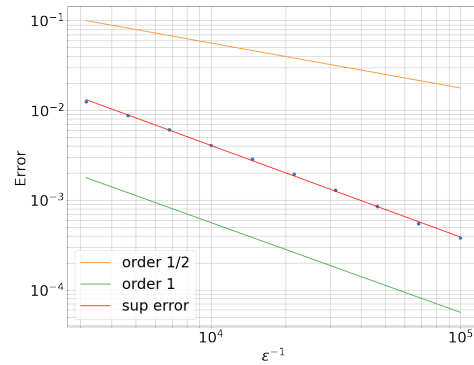
Two different initial conditions were examined in our experimental analysis, and no discontinuity was observed in either case. In the first simulation, we set X_{0-} to follow a uniform distribution on $[0.25, 0.35]$, with $\alpha = 0.5$. In the second scenario, X_{0-} was generated from a gamma distribution with parameters $(2.1, \frac{1}{2})$, with $\alpha = 1.3$. Interestingly, the data from Fig. 1 indicate a convergence rate of 1 in both cases. This exceeds the theoretically guaranteed convergence rate of $1/2$.

4.4 Setting with discontinuity and without common noise

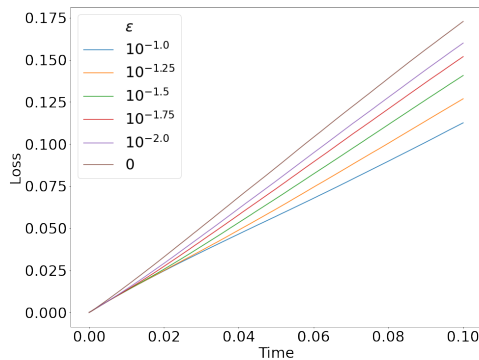
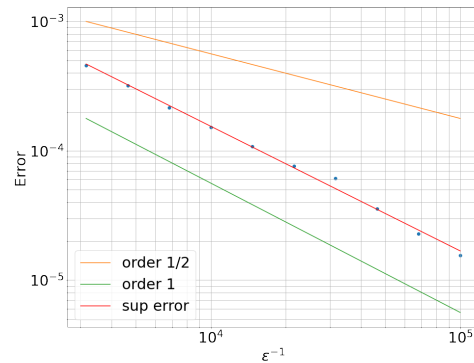
To simulate a setting where we would see a systemic event, we changed the parameters of the Gamma distribution such that most of the mass will be near the boundary and made α sufficiently large. In Fig. 2, we conducted simulations using two different initial conditions. In the first case, we set X_{0-} to follow a Gamma distribution with parameters $(1.2, 0.5)$ and set $\alpha = 0.9$. In the second case, X_{0-} was generated from a Gamma distribution with parameters $(1.4, 0.5)$, and $\alpha = 2$. Within this particular setup, we observe a convergence rate between $1/2$ and 1 prior to the first jump, while the theory only gives a rate below 0.5 . The rate does not display a clear dependence on the characteristics of the density of X_{0-} near the boundary, in contrast to the theoretical estimates; see Table 3 for more details on the data.

4.5 Simulations with common noise

In both experiments in this subsection, we assigned X_{0-} a uniform distribution over the interval $[0.25, 0.35]$, the same as in subsection 4.3, and set $\alpha = 0.5$ and $\rho = 0.5$, with different common noise paths for each experiment. In the first simulation, the common noise path increases to 1 over the time domain. This led to the loss process becoming rougher than the loss in the previous setting. In the second simulation, the common noise path decreases to -1 . This induces a systemic event due to the rapid loss of mass. Despite the differences between the scenarios, we observed a similar rate of convergence between $1/2$ and 1 as illustrated in Fig. 3.


 (a) $X_{0-} \sim^d \text{Uniform}[0.25, 0.35]$, $\alpha = 0.5$.


(b) Rate of Convergence.


 (c) $X_{0-} \sim^d \Gamma(2.1, \frac{1}{2})$, $\alpha = 1.3$.


(d) Rate of Convergence.

Figure 1: Initial density vanishing at zero with no discontinuity or common noise.

A Technical lemmas

Lemma A.1. Suppose that $\tilde{P}^{\varepsilon_n} = \text{Law}(\tilde{X}^{\varepsilon_n}, \tilde{Y}^{\varepsilon_n})$ converges weakly in $\mathcal{P}(D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}})$ to \tilde{P}^* , where $\tilde{X}^{\varepsilon_n}$ and $\tilde{Y}^{\varepsilon_n}$ are the extensions of X^{ε_n} and Y^{ε_n} , respectively. Let X^* and Y^* be the canonical processes on $D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$ such that for $(\eta, \omega) \in D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}}$, $X^*(\eta, \omega) = \eta$ and $Y^*(\eta, \omega) = \omega$. Then, under \tilde{P}^* , Y^* is a martingale with respect to the filtration generated by (X^*, Y^*) with quadratic variation

$$\langle Y^* \rangle_t = \begin{cases} 0 & t \in [-1, 0), \\ \int_0^t \sigma(s, X_s^*)^2 ds & t \in [0, T], \\ \int_0^T \sigma(s, X_s^*)^2 ds + (t - T) & t \in (T, \bar{T}]. \end{cases}$$

Proof. Set \tilde{P}^* to be the limit point of $(\tilde{P}^{\varepsilon_n})_{n \geq 0}$ and

$$\mathbb{T}^{\tilde{P}^*} := \left\{ t \in [-1, \bar{T}] : \tilde{P}^*(\eta_t = \eta_{t-}) = 1 \right\}.$$

Contagious McKean–Vlasov problems with common noise

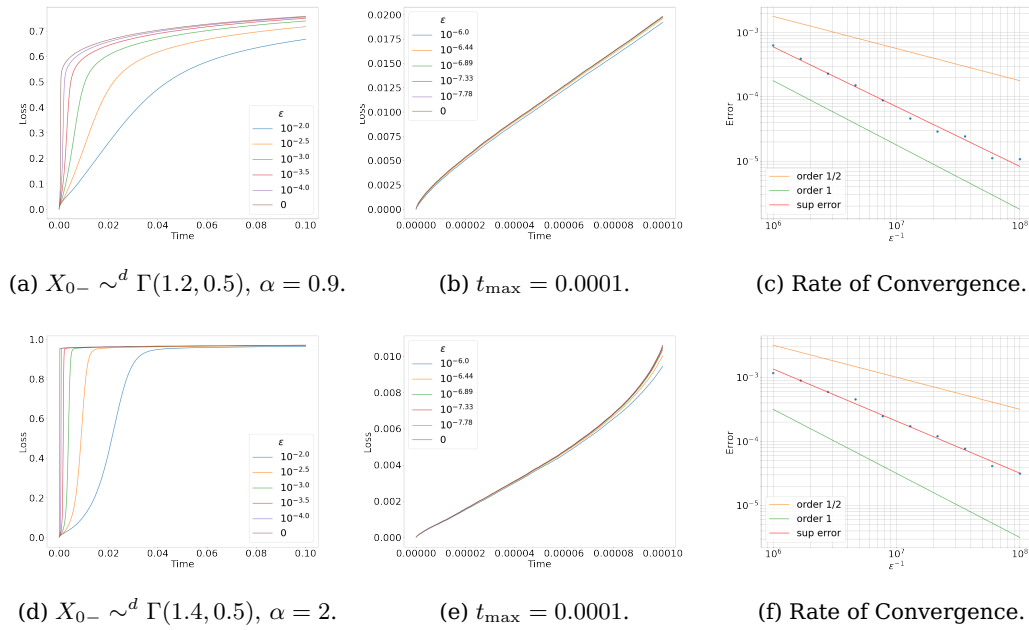


Figure 2: Initial density vanishing at zero with discontinuity and no common noise.

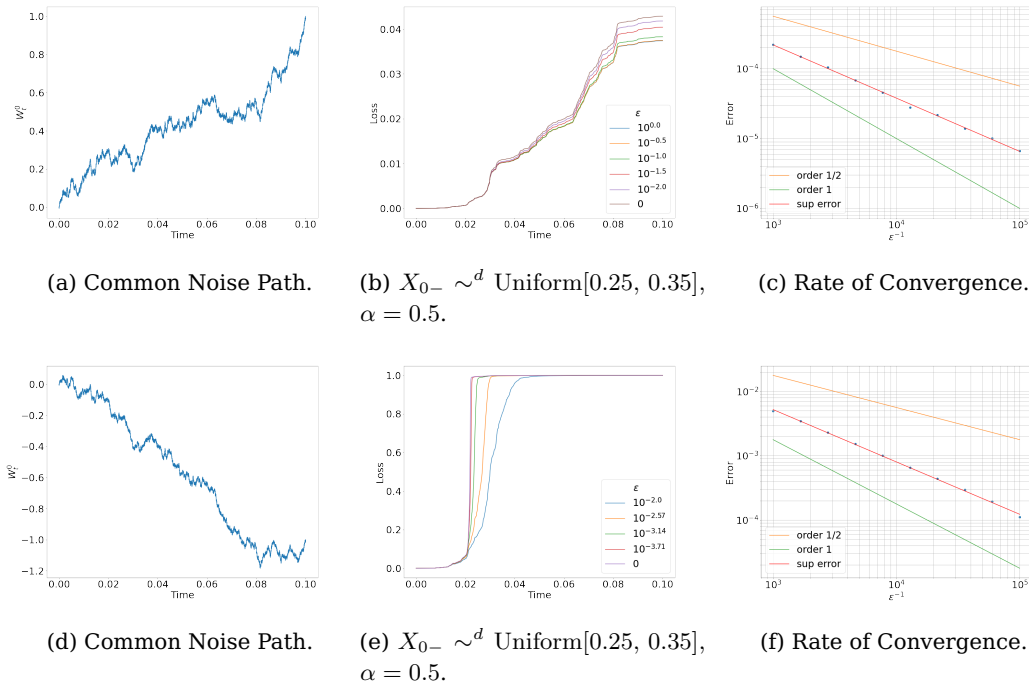


Figure 3: Initial density vanishing at zero with common noise.

Now for any $s_0, t_0 \in \mathbb{T}^{\tilde{P}^*}$ with $s_0 < t_0$ and $\{s_i\}_{i=1}^k \subset [-1, s_0] \cap \mathbb{T}^{\tilde{P}^*}$, we define the function

$$F : D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}} \rightarrow \mathbb{R}, \quad (\eta, \omega) \mapsto (\omega_{t_0} - \omega_{s_0}) \prod_{i=1}^k f_i(\eta_{s_i}, \omega_{s_i}),$$

for arbitrary $f_i \in \mathcal{C}_b(D_{\mathbb{R}} \times \mathcal{C}_{\mathbb{R}})$. To show that Y^* is a martingale, it is sufficient to show that $\mathbb{E}^{\tilde{P}^*}[F(X^*, Y^*)] = 0$.

As $\tilde{P}^{\varepsilon_n} \Rightarrow \tilde{P}^*$, then by Skorokhod's Representation Theorem, see [2, Theorem 7.6], there exist $((x^n, y^n))_{n \geq 1}$ and (x, y) defined on the same background space such that (x^n, y^n) converges to (x, y) almost surely in $(D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_{\infty})$ with $\text{Law}(x^n, y^n) = \tilde{P}^{\varepsilon_n}$ and $\text{Law}(x, y) = \tilde{P}^*$. Now for any $p > 1$,

$$\mathbb{E}[|F(x^n, y^n)|^p] \leq C \mathbb{E} \left[\sup_{s \leq T} |\tilde{\mathcal{Y}}_s^{\varepsilon_n}|^p \right] \leq C,$$

where C is a constant that changes from line to line and depends only on p, σ, T and the f_i 's but is independent of ε . Therefore, $F(x^n, y^n)$ is L^p -bounded uniformly in n . For $t \in \{t_0, s_0, s_1, \dots, s_k\}$, t is an almost sure continuity point of x . Therefore, by the properties of M_1 -convergence, (x_t^n, y_t^n) converges to (x_t, y_t) almost surely. Hence, we have almost sure convergence of $F(x^n, y^n)$ to $F(x, y)$. Vitali's Convergence Theorem states that almost sure convergence and uniform integrability imply convergence of means, hence

$$\mathbb{E}^{\tilde{P}^*}[F(X^*, Y^*)] = \mathbb{E}[F(x, y)] = \lim_{n \rightarrow \infty} \mathbb{E}[F(x^n, y^n)] = \lim_{n \rightarrow \infty} \mathbb{E}[F(\tilde{X}^{\varepsilon_n}, \tilde{\mathcal{Y}}^{\varepsilon_n})] = 0,$$

where the last equality follows from the fact that $\mathbb{E}[F(\tilde{X}^{\varepsilon_n}, \tilde{\mathcal{Y}}^{\varepsilon_n})] = 0$ for all n as $\tilde{\mathcal{Y}}^{\varepsilon_n}$ is a martingale. Therefore, by a monotone class theorem argument, Y^* is a continuous local martingale.

Recall $x^n \rightarrow x$ almost surely in $(D_{\mathbb{R}}, M_1)$, hence we have pointwise convergence at the continuity points of x , see [33, Theorem 12.5.1]. As σ is in $\mathcal{C}^{1,2}$ by Assumption 2.2, there exists a set of full probability such that $\sigma(s, x_s^n) \rightarrow \sigma(s, x_s)$ for a set of s 's that have full Lebesgue measure in $[0, T]$. Furthermore, as σ is bounded, by the Bounded Convergence Theorem

$$\int_0^t \sigma(s, x_s^n)^2 ds \rightarrow \int_0^t \sigma(s, x_s)^2 ds \quad (\text{A.1})$$

almost surely for any $t \in [0, T]$. Set

$$\langle Y \rangle_t = \begin{cases} 0 & t \in [-1, 0), \\ \int_0^t \sigma(s, X_s)^2 ds & t \in [0, T], \\ \int_0^T \sigma(s, X_s)^2 ds + (t - T) & t \in (T, \bar{T}], \end{cases}$$

where Y is any one of Y^*, y^n or $\tilde{\mathcal{Y}}^{\varepsilon_n}$ and X is the respective X^*, x^n or $\tilde{X}^{\varepsilon_n}$. Employing (A.1),

$$F(x^n, (y^n)^2 - \langle y^n \rangle) \rightarrow F(x, y^2 - \langle y \rangle) \quad \text{almost surely.}$$

Also, by the above and the boundedness of σ by Assumption 2.2, $F(x^n, (y^n)^2 - \langle y^n \rangle)$ is L^p -bounded uniformly in n . Hence, by Vitali's Convergence Theorem,

$$\begin{aligned} \mathbb{E}^{\tilde{P}^*}[F(X^*, (Y^*)^2 - \langle Y^* \rangle)] &= \mathbb{E}[F(x, y^2 - \langle y \rangle)] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[F(x^n, (y^n)^2 - \langle y^n \rangle)] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[F(\tilde{X}^{\varepsilon_n}, (\tilde{\mathcal{Y}}^{\varepsilon_n})^2 - \langle \tilde{\mathcal{Y}}^{\varepsilon_n} \rangle)] = 0, \end{aligned}$$

where the last equality follows from the fact that $(\tilde{\mathcal{Y}}^{\varepsilon_n})^2 - \langle \tilde{\mathcal{Y}}^{\varepsilon_n} \rangle$ is a true martingale due to the boundedness of σ . This completes the proof. \square

Lemma A.2. Consider the process $Z_t = M_t + tX$ for $t \in [-1, \bar{T}]$, where M_t is a continuous local martingale with $c_M(t-s) \leq \langle M \rangle_t - \langle M \rangle_s \leq C_M(t-s)$ for any $0 \leq s < t$ almost surely, and X is a non-negative random variable such that $\mathbb{E}[X] < \infty$. Then, for any stopping time τ where $\tau \geq 0$ almost surely, then

$$\mathbb{P} \left[\inf_{s \in (\tau, (\tau+h) \wedge \bar{T})} \{Z_s - Z_\tau\} \geq 0, \tau < \bar{T} \right] = 0,$$

for any $h > 0$.

Proof. In the case when M is simply a Brownian motion, the result readily follows from the Strong Markov Property and the standard properties of Brownian motion. As M is a continuous local martingale, we may view it as a (random) time-changed Brownian motion. We exploit this fact to show the claim. To begin, fix a $\Delta \in (0, h)$, $\lambda > 0$ and set $\bar{\tau} := \tau \wedge (\bar{T} - \Delta)$. Then, conditioning on the event $E := \{\tau \leq \bar{T} - \Delta\} \cap \{X \leq \lambda\}$ and its complement, we have

$$\begin{aligned} \mathbb{P} \left[\inf_{s \in (\tau, (\tau+h) \wedge \bar{T})} \{Z_s - Z_\tau\} > -\Delta, \tau < \bar{T} \right] &\leq \mathbb{P} \left[\inf_{s \in (\bar{\tau}, \bar{\tau}+\Delta)} \{M_s - M_{\bar{\tau}} + (s - \bar{\tau})\lambda\} > -\Delta \right] \\ &\quad + \mathbb{P} \left[E^c, \tau < \bar{T} \right]. \end{aligned} \tag{A.2}$$

Focussing on the first term, we observe

$$\mathbb{P} \left[\inf_{s \in (\bar{\tau}, \bar{\tau}+\delta)} \{M_s - M_{\bar{\tau}} + (s - \bar{\tau})\lambda\} > -\Delta \right] \leq \mathbb{P} \left[\inf_{s \in (\bar{\tau}, \bar{\tau}+\delta)} \{M_s - M_{\bar{\tau}}\} > -\Delta(1 + \lambda) \right]$$

By the Dubins-Schwarz Theorem, M is a time-changed Brownian motion. Therefore there exists a Brownian Motion B such that

$$\mathbb{P} \left[\inf_{s \in (\bar{\tau}, \bar{\tau}+\delta)} \{M_s - M_{\bar{\tau}}\} > -\Delta(1 + \lambda) \right] = \mathbb{P} \left[\inf_{s \in (\bar{\tau}, \bar{\tau}+\delta)} \{B_{\langle M \rangle_s - \langle M \rangle_{\bar{\tau}}} \} > -\Delta(1 + \lambda) \right]$$

Now as $\bar{\tau} = \tau \wedge (\bar{T} - \Delta) > 0$ almost surely, $\langle M \rangle_s - \langle M \rangle_{\bar{\tau}} \geq c_M(s - \bar{\tau})$ for any $s > \bar{\tau}$ almost surely. So

$$\mathbb{P} \left[\inf_{s \in (\bar{\tau}, \bar{\tau}+\delta)} \{B_{\langle M \rangle_s - \langle M \rangle_{\bar{\tau}}} \} > -\Delta(1 + \lambda) \right] \leq \mathbb{P} \left[\inf_{s \in (0, \Delta)} \{B_{c_M s}\} > -\Delta(1 + \lambda) \right]$$

By the reflection principle of Brownian motion, we have

$$\begin{aligned} \mathbb{P} \left[\inf_{s \leq \Delta} B_{c_M s} \leq -\Delta(1 + \lambda) \right] &= 2\mathbb{P} [B_{\gamma c_M} \leq -\Delta(1 + \lambda)] \\ &= 2(2\pi)^{-1/2} \int_{-\infty}^{-\Delta^{1/2} c_M^{-1/2} (1+\lambda)} e^{-\frac{y^2}{2}} dy \geq 1 - 2\Delta^{1/2} (1 + \lambda) (2\pi c_M)^{-1/2}. \end{aligned}$$

In conclusion, we have shown

$$\begin{aligned} \mathbb{P} \left[\inf_{s \in (\bar{\tau}, \bar{\tau}+\Delta)} \{M_s - M_{\bar{\tau}} + (s - \bar{\tau})\lambda\} > -\Delta \right] &\leq \mathbb{P} \left[\inf_{s \in (0, \Delta)} \{B_{c_M s}\} > -\Delta(1 + \lambda) \right] \\ &\leq 2\Delta^{1/2} (1 + \lambda) (2\pi c_M)^{-1/2} \end{aligned}$$

Setting $\lambda = \Delta^{-1/4}$, then by continuity of measure and the above, the expression in (A.2) converges to 0 as we send Δ to zero. This completes the proof. \square

Lemma A.3 (Convergence of Stopping Times). *Consider a sequence of functions $(z^n)_{n \geq 1}$ in $D_{\mathbb{R}}$ converging towards some $z \in D_{\mathbb{R}}$ with respect to the M_1 -topology. We assume that z has the following crossing property:*

$$\forall h > 0 \quad \tau_0(z) < \bar{T} \quad \implies \quad \inf_{s \in (\tau_0(z), (\tau_0(z)+h) \wedge \bar{T})} \{z_s - z_{\tau_0(z)}\} < 0, \quad (\text{A.3})$$

where τ_0 is defined as in (2.14) and $\Delta z_t \leq 0$ for all $t \in [-1, \bar{T}]$. Then we have

$$\lim_{n \rightarrow \infty} \tau_0(z^n) = \tau_0(z).$$

Proof. The proof is composed of two steps. We shall show that $\limsup_{n \rightarrow \infty} \tau_0(z^n) \leq \tau_0(z) \leq \liminf_{n \rightarrow \infty} \tau_0(z^n)$. Hence, we will have equality and the claim follows.

Step 1: $\limsup_{n \rightarrow \infty} \tau_0(z^n) \leq \tau_0(z)$

We define the set of continuity points of z to be $\mathbb{T}^z := \{t \in [-1, \bar{T}] : z_t = z_{t-}\}$. We remark that \mathbb{T}^z is co-countable by [2, Section 13]. As $\tau_0(z) < \bar{T}$, by (A.3) for any fixed $m \in \mathbb{N}$ there exists a $t \in (\tau_0(z), (\tau_0(z) + 1/m) \wedge \bar{T}) \cap \mathbb{T}^z$ such that $z_t < 0$. Now, as t is a continuity point of z , by [33, Theorem 12.5.1], we have that $z_t^n \rightarrow z_t$ in \mathbb{R} as $n \rightarrow \infty$. Therefore, for large n , $z_t^n < 0$ hence

$$\limsup_{n \rightarrow \infty} \tau_0(z^n) \leq t \leq \tau_0(z) + \frac{1}{m}.$$

As $m \in \mathbb{N}$ was arbitrary, the claim follows.

Step 2: $\liminf_{n \rightarrow \infty} \tau_0(z^n) \geq \tau_0(z)$

As $z^n \rightarrow z$ in the M_1 -topology, we may find a sequence of parametric representations $((u^n, r^n))_{n \geq 1}$ of $(z^n)_{n \geq 1}$ which converges uniformly to a parametric representation (u, r) of z , see [33, Theorem 12.5.1]. Therefore, we may find an $s^n \in [0, 1]$ such that $(u_{s^n}^n, r_{s^n}^n) = (z_{\tau_0(z^n)}^n, \tau_0(z^n))$. By Step 1, since $\tau_0(z) < \bar{T}$, we have

$$\liminf_{n \rightarrow \infty} \tau_0(z^n) \leq \limsup_{n \rightarrow \infty} \tau_0(z^n) \leq \tau_0(z) < \bar{T}.$$

Therefore, by the finiteness of $\liminf_{n \rightarrow \infty} \tau_0(z^n)$ and compactness of $[0, 1]$, we may find a subsequence n_k such that $\tau_0(z^{n_k}) \rightarrow \liminf_{n \rightarrow \infty} \tau_0(z^n)$ and $s_{n_k} \rightarrow s$ for some $s \in [0, 1]$. By the uniform convergence of the parametric representations

$$\begin{aligned} \liminf_{k \rightarrow \infty} z_{\tau_0(z^{n_k})}^{n_k} &= \liminf_{k \rightarrow \infty} u_{s^{n_k}}^{n_k} = u_s, \\ \liminf_{k \rightarrow \infty} \tau_0(z^{n_k}) &= \liminf_{k \rightarrow \infty} r_{s^{n_k}}^{n_k} = r_s. \end{aligned}$$

As $r_s = \liminf_{n \rightarrow \infty} \tau_0(z^n)$, we may find $\gamma \in [0, 1]$ such that $u_s = \gamma z(\liminf_{n \rightarrow \infty} \tau_0(z^n)) + (1 - \gamma) z(\liminf_{n \rightarrow \infty} \tau_0(z^n))$. We also note $u_s \leq 0$ as $\liminf_{k \rightarrow \infty} z_{\tau_0(z^{n_k})}^{n_k} \leq 0$. Lastly, as $\Delta z_t \leq 0$ for all t , we have $z(\liminf_{n \rightarrow \infty} \tau_0(z^n)) \leq 0$. Therefore, $\tau_0(z) \leq \liminf_{n \rightarrow \infty} \tau_0(z^n)$. This completes the proof. \square

Lemma A.4 (Functional Continuity II). *Let $\mu \in \mathcal{P}(D_{\mathbb{R}})$ be any measure such that*

$$\mu \left(\inf_{s \in (\tau_0(\eta), (\tau_0(\eta)+h) \wedge \bar{T})} \{\eta_s - \eta_{\tau_0(\eta)}\} \geq 0, \tau_0(\eta) < \bar{T} \right) = 0, \quad (\text{A.4})$$

for any $h > 0$. Then, for any sequence of measures $(\mu^n)_{n \geq 1}$ such that $\mu^n \implies \mu$ in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$, we have

$$\nu_t^{\mu^n} := \mu^n(\eta_t \in \cdot, \tau_0(\eta) > t) \implies \nu_t^\mu := \mu(\eta_t \in \cdot, \tau_0(\eta) > t),$$

in $\mathbf{M}_{\leq 1}(\mathbb{R})$, the space of sub-probability measures on \mathbb{R} endowed with the topology of weak convergence, for and $t \geq 0$ such that $\mu(\eta_t = \eta_{t-}) = 1$ and $\mu(\tau_0(\eta) = t) = 0$.

Proof. The proof is an application of the Continuous Mapping Theorem, [2, Theorem 2.7]. We only need to construct μ -almost sure continuous maps.

Step 1: Projection of measures from $D_{\mathbb{R}}$ to $\mathbb{R} \times \{0, 1\}$

Consider the map

$$(X_t, \mathbb{1}_{\{\tau_0(\cdot)\}}) : D_{\mathbb{R}} \rightarrow \mathbb{R} \times \{0, 1\}, \quad \eta \mapsto (\eta_t, \mathbb{1}_{\{\tau_0(\eta) > t\}}) \quad (\text{A.5})$$

(A.5) is a μ -almost sure continuous map. Choose a $\eta \in D_{\mathbb{R}}$ such that $\eta_t = \eta_{t-}$, $\tau_0(\eta) \neq t$ and is in the complement of the event in (A.4). Such η 's have full measure under μ . M_1 -convergence implies pointwise convergence at continuity points, [33, Theorem 12.5.1], therefore X_t is M_1 -continuous for every such η . Also by Lemma 2.13, since (A.4) holds, τ_0 is an M_1 -continuous map at η . As $\tau_0(\eta) \neq t$, $\mathbb{1}_{\{\tau_0(\cdot)\}}$ is M_1 -continuous at η . Hence, $(X_t, \mathbb{1}_{\{\tau_0(\cdot)\}})$ is a μ -almost sure continuous map. By the Continuous Mapping Theorem,

$$(X_t, \mathbb{1}_{\{\tau_0(\cdot)\}})^{\#} \mu^n \implies (X_t, \mathbb{1}_{\{\tau_0(\cdot)\}})^{\#} \mu \quad \text{in } \mathcal{P}(D_{\mathbb{R}} \times \{0, 1\}).$$

Step 2: Weak convergence of sub-probability measures

For any $f \in \mathcal{C}_b(\mathbb{R})$, define the map

$$\hat{f} : \mathbb{R} \times \{0, 1\} \rightarrow \mathbb{R}, \quad (x, y) \mapsto f(x) \mathbb{1}_{\{1\}}(y).$$

It is clear $\hat{f} \in \mathcal{C}_b(\mathbb{R} \times \{0, 1\})$. By step 1,

$$\langle (X_t, \mathbb{1}_{\{\tau_0(\cdot)\}})^{\#} \mu^n, \hat{f} \rangle \rightarrow \langle (X_t, \mathbb{1}_{\{\tau_0(\cdot)\}})^{\#} \mu, \hat{f} \rangle.$$

But by definition, $\langle (X_t, \mathbb{1}_{\{\tau_0(\cdot)\}})^{\#} \mu^n, \hat{f} \rangle = \nu_t^{\mu^n}(f)$ and $\langle (X_t, \mathbb{1}_{\{\tau_0(\cdot)\}})^{\#} \mu, \hat{f} \rangle = \nu_t^{\mu}(f)$. So $\nu_t^{\mu^n}(f) \rightarrow \nu_t^{\mu}(f)$. The conclusion now follows by Portmanteau's Theorem. \square

Lemma A.5 (Weak convergence of sub-probability measures). *Suppose that $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \tilde{\mathbf{P}}^*$ on $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$ for a positive sequence $(\varepsilon_n)_{n \geq 1}$ which converges to zero. Set*

$$\mathbb{T} := \left\{ t \in [-1, \bar{T}] : \mathbb{E} \left[\tilde{\mathbf{P}}^*(\eta_t = \eta_{t-}) \right] = 1, \mathbb{E} \left[\tilde{\mathbf{P}}^*(\tau_0(\eta) = t) \right] = 0 \right\}.$$

Then for any $t \in \mathbb{T}$,

$$\nu_t^{\varepsilon_n} \implies \nu_t^* := \tilde{\mathbf{P}}^*(\eta_t \in \cdot, \tau_0(\eta) > t) \quad \text{in } \mathbf{M}_{\leq 1}(\mathbb{R}).$$

Proof. By definition of \mathbb{T} and Lemma 2.11. for any $t \in \mathbb{T}$ there is a set of μ 's of full $\text{Law}(\mathbf{P}^*)$ -measure such that

$$\mu(\eta_t \neq \eta_{t-}) = \mu(\tau_0(\eta) = t) = 0, \quad \mu \left(\inf_{s \in (\tau_0(\eta), (\tau_0(\eta) + h) \wedge \bar{T})} \{\eta_s - \eta_{\tau_0(\eta)}\} \geq 0, \tau_0(\eta) < \bar{T} \right) = 0. \quad (\text{A.6})$$

As $\tilde{\mathbf{P}}^{\varepsilon_n} \implies \tilde{\mathbf{P}}^*$, by Skorokhod's Representation Theorem there exists a $(\mathbf{Q}^n)_{n \geq 1}$, \mathbf{Q}^* such that $\mathbf{Q}^n \rightarrow \mathbf{Q}^*$ almost surely, $\text{Law}(\mathbf{Q}^n) = \text{Law}(\tilde{\mathbf{P}}^{\varepsilon_n})$, $\text{Law}(\mathbf{Q}^*) = \text{Law}(\tilde{\mathbf{P}}^*)$ and \mathbf{Q}^* satisfies (A.6) almost surely. Set

$$\nu_t^{\mathbf{Q}^n} = \mathbf{Q}^n(\eta_t \in \cdot, \tau_0(\eta) > t), \quad \text{and} \quad \nu_t^{\mathbf{Q}^*} = \mathbf{Q}^*(\eta_t \in \cdot, \tau_0(\eta) > t)$$

By Lemma A.4, $\nu_t^{\mathbf{Q}^n} \rightarrow \nu_t^{\mathbf{Q}^*}$ almost surely in $\mathbf{M}_{\leq 1}(\mathbb{R})$. Now, for any $F \in \mathcal{C}_b(\mathbf{M}_{\leq 1}(\mathbb{R}))$, by the Dominated Convergence Theorem

$$\lim_{n \rightarrow \infty} \mathbb{E} [F(\nu_t^{\varepsilon_n})] = \lim_{n \rightarrow \infty} \mathbb{E} [F(\nu_t^{\mathbf{Q}^n})] = \mathbb{E} [F(\nu_t^{\mathbf{Q}^*})] = \mathbb{E} [F(\nu_t^*)].$$

The result now follows by Portmanteau's Theorem. \square

Lemma A.6 (Functional Continuity III). *Let $\mu \in \mathcal{P}(D_{\mathbb{R}})$ be any measure such that*

$$\mu \left(\inf_{s \in (\tau_0(\eta), (\tau_0(\eta)+h) \wedge \bar{T})} \{\eta_s - \eta_{\tau_0(\eta)}\} \geq 0, \tau_0(\eta) < \bar{T} \right) = 0, \quad (\text{A.7})$$

for any $h > 0$ and let $g(t, x, \nu)$ be any function satisfying Assumption 2.2 i. Then $\int_0^t g(s, \eta_s^n, \nu_s^{\mu^n}) ds$ converges to $\int_0^t g(s, \eta_s, \nu_s^\mu) ds$ for any $t \geq 0$ whenever $(\eta^n, \mu^n) \rightarrow (\eta, \mu)$ in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk}) \times (D_{\mathbb{R}}, M_1)$ along a sequence for which $\sup_{n \geq 1} \langle \mu^n, \sup_{s \leq \bar{T}} |\tilde{\eta}_s|^p \rangle < \infty$ for some $p > 1$ and any $t \geq 0$. For any measure $m \in \mathcal{P}(D_{\mathbb{R}})$, $\nu_s^m := m(\tilde{\eta}_s \in \cdot, \tau_0(\tilde{\eta}) > s)$.

Proof. By Assumption 2.2,

$$\left| g(s, \eta_s^n, \nu_s^{\mu^n}) \right| \leq C(1 + \sup_{m \geq 1} \sup_{u \leq \bar{T}} |\eta_u^m| + \sup_{m \geq 1} \langle \mu^m, \sup_{u \leq \bar{T}} |\tilde{\eta}_u| \rangle) \quad (\text{A.8})$$

The right-hand side of (A.8) is finite because $\eta^n \rightarrow \eta$ in $(D_{\mathbb{R}}, M_1)$ and by our assumption. Thus, it is sufficient to show that $g(s, \eta_s^n, \nu_s^{\mu^n})$ converges to $g(s, \eta_s, \nu_s^\mu)$ on a set of full Lebesgue measure. The conclusion then follows from the Dominated Convergence Theorem.

By Assumption 2.2,

$$\left| g(s, \eta_s^n, \nu_s^{\mu^n}) - g(s, \eta_s, \nu_s^\mu) \right| \leq C |\eta_s^n - \eta_s| + C(1 + |\eta_s| + \langle \nu_s^{\mu^n}, |\cdot| \rangle) d_0(\nu_s^{\mu^n}, \nu_s^\mu). \quad (\text{A.9})$$

For any $s \in \mathbb{T}^\mu := \{t \in [-1, \bar{T}] : \mu(\eta_t = \eta_{t-}) = 1, \mu(\tau_0 = t) = 0\}$, the first term converges to zero as $\eta^n \rightarrow \eta$ in $(D_{\mathbb{R}}, M_1)$. For the second term in (A.9), it suffices to show $d_0(\nu_s^{\mu^n}, \nu_s^\mu) \rightarrow 0$, since $(1 + |\eta_s| + \langle \nu_s^{\mu^n}, |\cdot| \rangle)$ is bounded uniformly in n by our assumption.

Recall that

$$d_0(\nu_s^{\mu^n}, \nu_s^\mu) = \sup \left\{ \left| \langle \nu_s^{\mu^n} - \nu_s^\mu, \psi \rangle \right| : \psi \in \mathcal{C}_{d_0} \right\}, \quad (\text{A.10})$$

where $\mathcal{C}_{d_0} := \{\psi \in \mathcal{C}(\mathbb{R}) : \|\psi\|_{\text{Lip}} \leq 1, |\psi(0)| \leq 1\}$. Fix a $\delta > 0$, then for any $\lambda > 1$, by the Arzèrlà-Ascoli Theorem, there exists a finite family of functions $\psi_1, \dots, \psi_m \in \mathcal{C}_{d_0}$ supported on $[-\lambda - 1, \lambda + 1]$, for $m = m(\lambda) \in \mathbb{N}$ such that for any $\psi \in \mathcal{C}_{d_0}$,

$$\sup_{x \in [-\lambda, \lambda]} |\psi(x) - \psi_i(x)| < \delta/2 \quad (\text{A.11})$$

for some $i \in \{1, \dots, m\}$. Fixing any $\psi \in \mathcal{C}_{d_0}$ and choosing a ψ_i such that (A.11) holds, we have

$$\begin{aligned} \left| \langle \nu_s^{\mu^n} - \nu_s^\mu, \psi \rangle \right| &\leq \left| \int_0^\lambda (\psi - \psi_i) d(\nu_s^{\mu^n} - \nu_s^\mu) \right| + \left| \int_\lambda^\infty (\psi - \psi_i) d(\nu_s^{\mu^n} - \nu_s^\mu) \right| \\ &\quad + \left| \langle \nu_s^{\mu^n} - \nu_s^\mu, \psi_i \rangle \right|. \end{aligned}$$

By our choice of ψ_i , the first term is bounded by δ uniformly in n . By the linear growth condition for functions in \mathcal{C}_{d_0} ,

$$\left| \int_\lambda^\infty (\psi - \psi_i) d(\nu_s^{\mu^n} - \nu_s^\mu) \right| \leq C \langle \nu_s^{\mu^n} + \nu_s^\mu, |\cdot| \mathbb{1}_{[\lambda, \infty)} \rangle$$

for a constant C independent of n . Consequently, by the definition of $\nu_s^{\mu^n}$,

$$\begin{aligned} \langle \nu_s^{\mu^n}, |\cdot| \mathbb{1}_{[\lambda, \infty)} \rangle &\leq \int_{D_{\mathbb{R}}} |\tilde{\eta}_s| \mathbb{1}_{\{\tilde{\eta}_s > \lambda\}} d\mu^n(\tilde{\eta}) \leq \langle \mu^n, \sup_{u \leq \bar{T}} |\tilde{\eta}_u|^p \rangle^{\frac{1}{p}} \mu^n \left(\sup_{u \leq \bar{T}} |\tilde{\eta}_u| > \lambda \right)^{\frac{p-1}{p}} \\ &\leq \frac{1}{\lambda^{p-1}} \langle \mu^n, \sup_{u \leq \bar{T}} |\tilde{\eta}_u|^p \rangle. \end{aligned}$$

The second inequality follows from Hölder's inequality, while the last inequality follows from Markov's inequality. We may also employ the same argument to upper bound the $\langle \nu_s^\mu, |\cdot| \mathbb{1}_{[\lambda, \infty)} \rangle$ term in the above. Hence, by these upper bounds and as $\mu^n \Rightarrow \mu$, we deduce that

$$\left| \int_{\lambda}^{\infty} (\psi - \psi_i) d(\nu_s^{\mu^n} - \nu_s^{\mu}) \right| \leq \frac{C}{\lambda^{p-1}} \sup_{m \geq 1} \langle \mu^m, \sup_{u \leq T} |\tilde{\eta}_u|^p \rangle,$$

where C is independent of n . Returning to (A.10), we have shown that

$$d_0(\nu_s^{\mu^n}, \nu_s^{\mu}) \leq \delta + \frac{C}{\lambda^{p-1}} \sup_{m \geq 1} \langle \mu^m, \sup_{u \leq T} |\tilde{\eta}_u|^p \rangle + \sum_{i=1}^m |\langle \nu_s^{\mu^n} - \nu_s^{\mu}, \psi_i \rangle|.$$

The middle term will vanish uniformly in n as $\lambda \rightarrow \infty$. Thus, we fix a λ sufficiently large such that the middle term is bounded by δ uniformly in n . As λ is now fixed, so is m . By Lemma A.4, $\nu_s^{\mu^n} \Rightarrow \nu_s^{\mu}$ in $\mathbf{M}_{\leq 1}(\mathbb{R})$; hence, as the ψ_i 's have compact support, they are continuous and bounded. Therefore, the final term will be smaller than δ for all n sufficiently large. Consequently, for all n sufficiently large, $d_0(\nu_s^{\mu^n}, \nu_s^{\mu}) \leq C\delta$ for some constant C independent of n . Hence, $d_0(\nu_s^{\mu^n}, \nu_s^{\mu}) \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof. \square

Lemma A.7. Fix any $t < T$. There is a constant $C > 0$ independent of ε and t such that for any $\gamma < 1 \wedge (T - t)$ we have

$$\mathbb{P} \left[\nu_t^\varepsilon [0, \alpha_t z + C\gamma^{1/3} + \alpha_t(L_t^\varepsilon - \mathfrak{L}_t^\varepsilon) + (\alpha_{t+\gamma} - \alpha_t)] \geq z, \forall z \leq L_{t+\gamma}^\varepsilon - L_t^\varepsilon - C\gamma^{1/3} \right] \geq 1 - C\gamma^{1/3}$$

Proof. To begin, fix a $\gamma > 0$ such that $\gamma < 1 \wedge (T - t)$ and fix a $z \in \mathbb{R}$. Then we denote by E_1^z the event

$$\left\{ X_t^\varepsilon - \gamma C(1 + \sup_{u \leq t+\gamma} |X_u^\varepsilon| + \mathbb{E}[\sup_{u \leq t+\gamma} |X_u^\varepsilon| W^0]) - \sup_{u \leq \gamma} |\mathcal{Y}_{t+u}^\varepsilon - \mathcal{Y}_t^\varepsilon| - \alpha_t z - \alpha_t(L_t^\varepsilon - \mathfrak{L}_t^\varepsilon) - (\alpha_{t+\gamma} - \alpha_t) \leq 0, \tau^\varepsilon > t \right\}$$

where C is the constant from the linear growth condition on b . Now fix $x \leq L_{t+\gamma}^\varepsilon - L_t^\varepsilon$. By the continuity of the loss process, [18, Theorem 2.4], there exists a $s \leq \gamma$ such that $x = L_{t+s}^\varepsilon - L_t^\varepsilon$. Employing the integration by parts formula, we observe for any $u \in [t, t+s]$

$$\begin{aligned} \int_t^u \alpha_v d\mathfrak{L}_v^\varepsilon &= \alpha_u \mathfrak{L}_u^\varepsilon - \alpha_t \mathfrak{L}_t^\varepsilon - \int_t^u \alpha'(v) \mathfrak{L}_v^\varepsilon dv \leq \alpha_u \mathfrak{L}_u^\varepsilon - \alpha_t \mathfrak{L}_t^\varepsilon \\ &\leq \alpha_{t+s} L_{t+s}^\varepsilon - \alpha_t \mathfrak{L}_t^\varepsilon \leq \alpha_t L_{t+s}^\varepsilon + (\alpha_{t+s} - \alpha_t) - \alpha_t \mathfrak{L}_t^\varepsilon, \end{aligned}$$

where to establish upper bounds we use the fact that α is non-negative and non-decreasing, and $\mathfrak{L}_v^\varepsilon \leq L_v^\varepsilon \leq 1$ for any $v \geq 0$. Therefore for any $u \in [t, t+s]$

$$\begin{aligned} X_u^\varepsilon &= X_t^\varepsilon + (X_u^\varepsilon - X_t^\varepsilon) = X_t^\varepsilon + \int_t^u b(v, X_v^\varepsilon, \nu_v^\varepsilon) dv + \mathcal{Y}_u - \mathcal{Y}_t - \int_t^u \alpha_v d\mathfrak{L}_v^\varepsilon \\ &\geq X_t^\varepsilon - \gamma C(1 + \sup_{u \leq t+\gamma} |X_u^\varepsilon| + \mathbb{E}[\sup_{u \leq t+\gamma} |X_u^\varepsilon| W^0]) - \sup_{u \leq \gamma} |\mathcal{Y}_{t+u}^\varepsilon - \mathcal{Y}_t^\varepsilon| \\ &\quad - \{\alpha_t L_{t+s}^\varepsilon + (\alpha_{t+s} - \alpha_t) - \alpha_t \mathfrak{L}_t^\varepsilon \pm \alpha_t L_t^\varepsilon\} \\ &\geq X_t^\varepsilon - \gamma C(1 + \sup_{u \leq t+\gamma} |X_u^\varepsilon| + \mathbb{E}[\sup_{u \leq t+\gamma} |X_u^\varepsilon| W^0]) - \sup_{u \leq \gamma} |\mathcal{Y}_{t+u}^\varepsilon - \mathcal{Y}_t^\varepsilon| \\ &\quad - \alpha_t x - (\alpha_{t+s} - \alpha_t) - \alpha_t(L_t^\varepsilon - \mathfrak{L}_t^\varepsilon) \end{aligned}$$

Therefore as L^ε is W^0 measurable and conditioning on W^0 fixes L^ε , we have

$$\mathbb{P} [E_1^x | W^0] \geq \mathbb{P} \left[\inf_{t \leq u \leq t+s} X_u^\varepsilon \leq 0, \tau^\varepsilon > t \mid W^0 \right] = L_{t+s}^\varepsilon - L_t^\varepsilon = x.$$

Now, for any fixed $z \leq L_{t+\gamma}^\varepsilon - L_t^\varepsilon - 2\gamma^{1/3}$, set $z_0 = z + 2\gamma^{1/3}$. We define the event

$$E_2 := \left\{ \gamma C \left(1 + \sup_{u \leq t+\gamma} |X_u^\varepsilon| + \mathbb{E} \left[\sup_{u \leq t+\gamma} |X_u^\varepsilon| | W^0 \right] \right) + \sup_{u \leq \gamma} |\mathcal{Y}_{t+u}^\varepsilon - \mathcal{Y}_t^\varepsilon| \geq \gamma^{1/3} \right\}$$

Then on the event $E_1^{z_0} \cap E_2^c$

$$\begin{aligned} X_t^\varepsilon - \alpha_t z_0 &\leq \gamma C \left(1 + \sup_{u \leq t+\gamma} |X_u^\varepsilon| + \mathbb{E} \left[\sup_{u \leq t+\gamma} |X_u^\varepsilon| | W^0 \right] \right) + \sup_{u \leq \gamma} |\mathcal{Y}_{t+u}^\varepsilon - \mathcal{Y}_t^\varepsilon| \\ &\quad + \alpha_t (L_t^\varepsilon - \mathfrak{L}_t^\varepsilon) + (\alpha_{t+\gamma} - \alpha_t) \\ &\leq \gamma^{1/3} + \alpha_t (L_t^\varepsilon - \mathfrak{L}_t^\varepsilon) + (\alpha_{t+\gamma} - \alpha_t). \end{aligned}$$

Therefore on the same event

$$X_t^\varepsilon - \alpha_t z = X_t^\varepsilon - \alpha_t z_0 + 2\alpha_t \gamma^{1/3} \leq (1 + 2\alpha_t) \gamma^{1/3} + \alpha_t (L_t^\varepsilon - \mathfrak{L}_t^\varepsilon) + (\alpha_{t+\gamma} - \alpha_t)$$

Consequently, we deduce

$$\begin{aligned} \nu_t^\varepsilon[0, \alpha_t z + (1 + 2\alpha_t) \gamma^{1/3} + \alpha_t (L_t^\varepsilon - \mathfrak{L}_t^\varepsilon) + (\alpha_{t+\gamma} - \alpha_t)] &\geq \mathbb{P} \left[E_1^{z_0} \cap E_2^c | W^0 \right] \\ &\geq \mathbb{P} \left[E_1^{z_0} | W^0 \right] - \mathbb{P} \left[E_2 | W^0 \right] \\ &\geq z_0 - \mathbb{P} \left[E_2 | W^0 \right]. \end{aligned}$$

Therefore if we have control over the mass $\mathbb{P} \left[E_2 | W^0 \right]$, we may estimate the mass with respect to ν_t that is near the boundary. Therefore, defining the event $E_3 := \{ \mathbb{P} \left[E_2 | W^0 \right] \leq \gamma^{1/3} \}$ we deduce on E_3

$$\nu_t^\varepsilon[0, \alpha_t z + (1 + 2\alpha_t) \gamma^{1/3} + \alpha_t (L_t^\varepsilon - \mathfrak{L}_t^\varepsilon) + (\alpha_{t+\gamma} - \alpha_t)] \geq z_0 - \gamma^{1/3} \geq z.$$

The last inequality follows from the fact that $z_0 = z + 2\gamma^{1/3}$. Now we only need to find a C independent of ε , γ and t such that $\mathbb{P}[E_3^c] \leq C\gamma^{1/3}$. By application of Markov's inequality twice

$$\begin{aligned} \mathbb{P} \left[E_3^c \right] &\leq \gamma^{-1/3} \mathbb{P} \left[E_2 \right] \\ &\leq \gamma^{-1/3} \mathbb{P} \left[\gamma C \left(1 + \sup_{u \leq t+\gamma} |X_u^\varepsilon| + \mathbb{E} \left[\sup_{u \leq t+\gamma} |X_u^\varepsilon| | W^0 \right] \right) \geq 2^{-1} \gamma^{1/3} / 2 \right] \\ &\quad + \gamma^{-1/3} \mathbb{P} \left[\sup_{u \leq \gamma} |\mathcal{Y}_{t+u}^\varepsilon - \mathcal{Y}_t^\varepsilon| \geq 2^{-1} \gamma^{1/3} / 2 \right] \\ &\leq 2C\gamma^{1/3} \mathbb{E} \left[1 + \sup_{u \leq t+\gamma} |X_u^\varepsilon| + \mathbb{E} \left[\sup_{u \leq t+\gamma} |X_u^\varepsilon| | W^0 \right] \right] + 2^8 \gamma^{-3} \mathbb{E} \left[\sup_{u \leq \gamma} |\mathcal{Y}_{t+u}^\varepsilon - \mathcal{Y}_t^\varepsilon|^8 \right] \\ &\leq c_1 (\gamma^{1/3} + \gamma) \leq 2c_1 \gamma^{1/3}, \end{aligned}$$

where c_1 depends on the constant from Proposition 2.8, the constant from Burkholder–Davis–Gundy to bound the second term and the uniform bounds on σ , but is notably independent of ε . Therefore, setting $C = \max\{1 + 2\alpha(T), c_1\}$ completes the proof. \square

Lemma A.8. Suppose that $\tilde{\mathbf{P}}^{\varepsilon_n} \Rightarrow \tilde{\mathbf{P}}^*$ on $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{wk})$ for a positive sequence $(\varepsilon_n)_{n \geq 1}$ which converges to zero. Set

$$\mathbb{T} := \left\{ t \in [-1, \bar{T}] : \mathbb{E} \left[\tilde{\mathbf{P}}^*(\eta_t = \eta_{t-}) \right] = 1, \mathbb{E} \left[\tilde{\mathbf{P}}^*(\tau_0(\eta) = t) \right] = 0 \right\}.$$

Then for any $t \in \mathbb{T} \cap [0, T)$ and $\gamma > 0$ such that $t + \gamma \in \mathbb{T} \cap [0, T)$ we have

$$\mathbb{P} \left[\nu_t[0, \alpha(t)z + C\gamma^{1/3} + \alpha(t + \gamma) - \alpha(t)] \geq z, \forall z \leq L_{t+\gamma} - L_t - C\gamma^{1/3} \right] \geq 1 - C\gamma^{1/3}$$

Proof. As $\tilde{\mathbf{P}}^{\varepsilon_n} \Rightarrow \tilde{\mathbf{P}}^*$, by employing Skorokhod's Representation Theorem, there exists a $(\mu^n)_{n \geq 1}$ and μ such that $\text{Law}(\mu^n) = \text{Law}(\tilde{\mathbf{P}}^{\varepsilon_n})$, $\text{Law}(\mu) = \text{Law}(\tilde{\mathbf{P}}^*)$ and $\mu^n \rightarrow \mu$ almost surely in $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$. As $\text{Law}(\mathbf{P}^*)$ -almost every measure μ satisfy (A.4) by Lemma 2.11, then by Lemma A.4 $\nu_t^{\mu^n} \rightarrow \nu_t^{\mu}$ almost surely for any $t \in \mathbb{T}$. Furthermore for any $t \in \mathbb{T} \cap [0, T)$, by Lemma 2.13 and Corollary 2.14, we have

$$\mu^n(\tau_0(\eta) \leq t) \rightarrow \mu(\tau_0(\eta) \leq t) \quad \text{and} \quad \int_0^t \kappa^{\varepsilon_n}(t-s) \mu^n(\tau_0(\eta) \leq s) \, ds \rightarrow \int_0^t \kappa^{\varepsilon_n}(t-s) \mu(\tau_0(\eta) \leq s) \, ds$$

almost surely. Therefore for simplicity and notational convenience for the remainder of this proof, we may suppose

$$\begin{aligned} \nu_t^{\varepsilon_n} &\rightarrow \nu_t^* && \text{a.s. in } \mathbf{M}_{\leq 1}(\mathbb{R}), \\ L_t^{\varepsilon_n} &\rightarrow L_t^* && \text{a.s. in } \mathbb{R}, \\ \mathfrak{L}_t^{\varepsilon_n} &\rightarrow L_t^* && \text{a.s. in } \mathbb{R}. \end{aligned}$$

Recall by Lemma A.7,

$$\mathbb{P} \left[\nu_t^{\varepsilon} [0, \alpha_t z + C\gamma^{1/3} + \alpha_t(L_t^{\varepsilon} - \mathfrak{L}_t^{\varepsilon}) + (\alpha_{t+\gamma} - \alpha_t)] \geq z, \forall z \leq L_{t+\gamma}^{\varepsilon} - L_t^{\varepsilon} - C\gamma^{1/3} \right] \geq 1 - C\gamma^{1/3},$$

for any $\varepsilon > 0$. It is well known that the Levy-Prokhorov metric, d_L , metricizes weak convergence, [30, Theorem 1.11]. Fixing $\delta_1, \delta_2, \delta_3, \delta_4 > 0$, we define the event

$$A_n := \{ |L_t^{\varepsilon_n} - L_t^*| < \delta_1, |L_{t+\gamma}^{\varepsilon_n} - L_{t+\gamma}^*| < \delta_2, |\mathfrak{L}_t^{\varepsilon_n} - L_t^{\varepsilon_n}| < \delta_3, d_L(\nu_t^{\varepsilon_n}, \nu_t^*) < \delta_4 \}$$

Therefore

$$\begin{aligned} 1 - C\gamma^{1/3} &\leq \mathbb{P} \left[\nu_t^{\varepsilon_n} [0, \alpha_t z + C\gamma^{1/3} + \alpha_t(L_t^{\varepsilon_n} - \mathfrak{L}_t^{\varepsilon_n}) + (\alpha_{t+\gamma} - \alpha_t)] \geq z, \forall z \leq L_{t+\gamma}^{\varepsilon_n} - L_t^{\varepsilon_n} - C\gamma^{1/3} \right] \\ &\quad + \mathbb{P}[A_n^c] \\ &\leq \mathbb{P} \left[\delta_4 + \nu_t^* (-\delta_4, \alpha_t z + C\gamma^{1/3} + \alpha_t \gamma_3 + (\alpha_{t+\gamma} - \alpha_t) + \delta_4) \geq z, \forall z \leq L_{t+\gamma}^* - L_t^* - C\gamma^{1/3} - \delta_1 - \delta_2 \right] \\ &\quad + \mathbb{P}[A_n^c] \end{aligned}$$

Sending $\varepsilon_n \rightarrow 0$, then $\mathbb{P}[A_n^c] \rightarrow 0$ by the Dominated Convergence Theorem as we have almost sure convergence. Lasts by sending $\delta_1 \rightarrow 0, \delta_2 \rightarrow 0, \delta_3 \rightarrow 0, \delta_4 \rightarrow 0$ one at a time and in order, then by employing continuity of measure we may conclude

$$\mathbb{P} \left[\nu_t^* [0, \alpha_t z + C\gamma^{1/3} + (\alpha_{t+\gamma} - \alpha_t)] \geq z, \forall z \leq L_{t+\gamma}^* - L_t^* - C\gamma^{1/3} \right] \geq 1 - C\gamma^{1/3} \quad \square$$

Lemma A.9. Suppose that $\tilde{\mathbf{P}}^{\varepsilon_n} \Rightarrow \tilde{\mathbf{P}}^*$ on $(\mathcal{P}(D_{\mathbb{R}}), \mathfrak{T}_{M_1}^{\text{wk}})$ for a positive sequence $(\varepsilon_n)_{n \geq 1}$ which converges to zero. Then we have

$$L_t^* \leq \inf \{ x \geq 0; \nu_{t-}^*([0, \alpha(t)x]) < x \} \quad (\text{A.12})$$

almost surely for any $t \in [0, T)$.

Proof. It is clear that Lemma A.9 holds for any $t \in \mathbb{T}$. Hence we must only show the upper bound for $t \notin \mathbb{T}$. We first consider the case when $t \in (0, T) \cap \mathbb{T}^c$. The case when $t = 0$ will be treated separately. Now as \mathbb{T} is dense in $[0, T]$, we may find a $(t_n)_{n \geq 1}, (t_n + \gamma_n)_{n \geq 1} \subset \mathbb{T}$ such that $t_n \uparrow t, t_n + \gamma_n \downarrow t$ and $\gamma_n < 2^{-3n}$. Now by the Borel-Cantelli Lemma, we have a set of full measure such that

$$\nu_{t_n}^* [0, \alpha_{t_n} z + C\gamma_n^{1/3} + (\alpha_{t_n + \gamma_n} - \alpha_{t_n})] \geq z, \forall z \leq L_{t_n + \gamma_n} - L_{t_n} - C\gamma_n^{1/3}, \quad (\text{A.13})$$

for all (possibly stochastic) n large. Furthermore, by the dominated convergence theorem, we have

$$\nu_{t_n}^* \rightarrow \nu_{t-}^*, \quad (\text{A.14})$$

Law($\tilde{\mathbf{P}}^*$)-almost surely in $\mathbf{M}_{\leq 1}(\mathbb{R})$ as for any $\phi \in \mathcal{C}_b(\mathbb{R})$

$$\lim_{n \rightarrow \infty} \nu_{t_n}^*(\phi) = \lim_{n \rightarrow \infty} \int_{D_{\mathbb{R}}} \phi(\eta_{t_n}) \mathbb{1}_{\{\tau_0(\eta) > t_n\}} d\tilde{\mathbf{P}}^*(\eta) = \int_{D_{\mathbb{R}}} \phi(\eta_{t-}) \mathbb{1}_{\{\tau_0(\eta) \geq t\}} d\tilde{\mathbf{P}}^*(\eta).$$

So on an event of full Law(\mathbf{P}^*) measure where (A.13) and (A.14) holds, by Portmanteau Theorem for any $\gamma > 0$, $z < \Delta L_t$

$$\nu_{t-}^*[0, \alpha(t)z + \gamma] \geq \limsup_{n \rightarrow \infty} \nu_{t_n}^*[0, \alpha(t)z + \gamma] \geq \limsup_{n \rightarrow \infty} \nu_{t_n}^*[0, \alpha_{t_n}z + C\gamma_n^{1/3} + (\alpha_{t_n} + \gamma_n - \alpha_{t_n})] \geq z.$$

This holds as $z < \Delta L_t^*$ and $L_{t_n + \gamma_n}^* - L_{t_n}^* - C\gamma_n^{1/3} \rightarrow \Delta L_t$. Sending γ to zero shows the claim for every $t > 0$.

In the case when $t = 0$, we have by Lemma A.7

$$\mathbb{P} \left[\nu_{0-}^\varepsilon[0, \alpha_0 z + C\gamma^{1/3} + (\alpha_\gamma - \alpha_0)] \geq z, \forall z \leq L_\gamma^\varepsilon - C\gamma^{1/3} \right] \geq 1 - C\gamma^{1/3}.$$

As $\nu_{0-}^\varepsilon = \nu_{0-}$, where ν_{0-} is a deterministic measure, and ν_{0-}^* is almost surely distributed as ν_{0-} almost surely, then we have by Lemma A.8

$$\mathbb{P} \left[\nu_{0-}^*[0, \alpha(0)z + C\gamma^{1/3} + \alpha(\gamma) - \alpha(0)] \geq z, \forall z \leq L_\gamma^* - C\gamma^{1/3} \right] \geq 1 - C\gamma^{1/3}.$$

for $\gamma \in \mathbb{T}$. Now the rest of the proof follows similar arguments as above by choosing $\gamma_n \in \mathbb{T}$ such that $\gamma_n \downarrow 0$. \square

Proposition A.10 (Grönwall Type Inequality I). *Suppose $a, \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}_+$ such that $a \geq 0$, $0 < \tilde{\beta} < 1$, $\tilde{\alpha} > 0$. Suppose $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T$, $g(t) \leq M$ (constant), and suppose $u(t)$ is nonnegative and bounded on $0 \leq t < T$ with*

$$u(t) \leq a + g(t) \int_0^t (t-s)^{\tilde{\beta}-1} s^{\tilde{\alpha}-1} u(s) ds$$

on this interval. Then

$$u(t) \leq a \left[1 + \sum_{n \geq 1} g_t^n C_n t^{n(\tilde{\alpha} + \tilde{\beta} - 1)} \right], \quad 0 \leq t < T,$$

where $C_0 := 1$, $C_{n+1} := B((n+1)\tilde{\alpha} + n\tilde{\beta} - n, \tilde{\beta}) C_n$, with $B(\tilde{\alpha}, \tilde{\beta}) := \int_0^1 (1-\tilde{s})^{\tilde{\beta}-1} \tilde{s}^{\tilde{\alpha}-1} d\tilde{s}$.

Proof. Let $B\phi_t = g_t \int_0^t (t-s)^{\tilde{\beta}-1} s^{\tilde{\alpha}-1} \phi_s ds$, $t \geq 0$, for localising integrable functions ϕ . Then $u_t \leq a(t) + Bu_t$ implies

$$u_t \leq \sum_{k=0}^{n-1} B^k a + B^n u_t.$$

Let us prove that

$$B^n(1)_t \leq_n t^{n(\tilde{\alpha} + \tilde{\beta} - 1)} g_t^n \tag{A.15}$$

and $B^n u_t \rightarrow 0$ as $n \rightarrow +\infty$ for each t in $0 \leq t < T$.

Step 1: $B^n(1)_t \leq C_n t^{n(\tilde{\alpha} + \tilde{\beta} - 1)} g_t^n$. For $n = 1$,

$$\begin{aligned} B(1)_t &= g_t \int_0^t (t-s)^{\tilde{\beta}-1} s^{\tilde{\alpha}-1} ds, \quad \text{set } \tilde{s} = s/t \\ &= g_t \int_0^1 t^{\tilde{\beta}-1} (1-\tilde{s})^{\tilde{\beta}-1} t^{\tilde{\alpha}-1} \tilde{s}^{\tilde{\alpha}-1} t d\tilde{s} \\ &= g_t t^{\tilde{\alpha} + \tilde{\beta} - 1} B(\tilde{\alpha}, \tilde{\beta}). \end{aligned}$$

Now, suppose the claim is true for $n = k$, then for $n = k + 1$

$$\begin{aligned} B^{k+1}(1)_t &= g_t \int_0^t (t-s)^{\tilde{\beta}-1} s^{\tilde{\alpha}-1} B^k(1)_s \, ds, \\ &\leq g_t \int_0^t (t-s)^{\tilde{\beta}-1} s^{\tilde{\alpha}-1} s^{k(\tilde{\alpha}+\tilde{\beta}-1)} g_s^k C_k \, ds, \quad \text{by above} \\ &\leq C_k g_t^{k+1} \int_0^t (t-s)^{\tilde{\beta}-1} s^{\tilde{\alpha}-1} s^{k(\tilde{\alpha}+\tilde{\beta}-1)} \, ds, \quad \text{set } \tilde{s} = s/t \\ &= C_k g_t^{k+1} \int_0^1 t^{\tilde{\beta}-1} (1-\tilde{s})^{\tilde{\beta}-1} t^{\tilde{\alpha}-1} \tilde{s}^{\tilde{\alpha}-1} t^{k(\tilde{\alpha}+\tilde{\beta}-1)} \tilde{s}^{k(\tilde{\alpha}+\tilde{\beta}-1)} t \, d\tilde{s}, \\ &= C_{k+1} g_t^{k+1} t^{(k+1)(\tilde{\alpha}+\tilde{\beta}-1)}. \end{aligned}$$

Hence the claim is true by the principle of induction.

Step 2: We observe that B is monotone, that is if $\phi_1 \leq \phi_2 \forall t \in [0, T]$, then by the nonnegativity of g we have $B(\phi_1) \leq B(\phi_2)$. Also by the linearity of integration, we see also that B is a linear operator. Therefore,

$$B(u)_t = g(t) \int_0^t (t-s)^{\tilde{\beta}-1} s^{\tilde{\alpha}-1} u(s) \, ds \leq \|u\|_{L^\infty} g(t) \int_0^t (t-s)^{\tilde{\beta}-1} s^{\tilde{\alpha}-1} \, ds = \|u\|_{L^\infty} B(1)_t$$

Therefore, by linearity, monotonicity and step 1

$$B^n(u)_t \leq \|u\|_{L^\infty} B^n(1)_t \leq \|u\|_{L^\infty} C_n g_t^n t^{n(\tilde{\alpha}+\tilde{\beta}-1)}$$

Step 3: Summability of C_n . By Gautschi's inequality, [16], we have that for all $x > 0$ and $s \in (0, 1)$

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}$$

Therefore,

$$\begin{aligned} \frac{C_{n+1}}{C_n} &= B\left((n+1)\tilde{\alpha} + n\tilde{\beta} - n, \tilde{\beta}\right), \\ &= \frac{\Gamma\left((n+1)\tilde{\alpha} + n\tilde{\beta} - n\right) \Gamma(\tilde{\beta})}{\Gamma\left((n+1)\tilde{\alpha} + (n+1)\tilde{\beta} - n\right)}, \\ &= \Gamma(\tilde{\beta}) \left[\frac{\Gamma\left((n+1)\tilde{\alpha} + (n+1)\tilde{\beta} - (n+1) + 1 - \tilde{\beta}\right)}{\Gamma\left((n+1)\tilde{\alpha} + (n+1)\tilde{\beta} - (n+1) + 1\right)} \right], \\ &\leq \Gamma(\tilde{\beta})(n+1)^{-\tilde{\beta}}(\tilde{\alpha} + \tilde{\beta} - 1)^{-\tilde{\beta}} \quad \text{by Gautschi's Inequality.} \end{aligned}$$

Hence $C_{n+1}/C_n \rightarrow 0$ and $n \rightarrow +\infty$. Hence by the ratio test, we have that C_n is summable.

Step 4: Summability of $B^n(u)_t$. By step 3, we have that

$$\frac{C_{n+1} \|u\|_{L^\infty} t^{(n+1)(\tilde{\alpha}+\tilde{\beta}-1)} g_t^{n+1}}{C_n \|u\|_{L^\infty} t^{n(\tilde{\alpha}+\tilde{\beta}-1)} g_t^n} = \|u\|_{L^\infty} t^{\tilde{\alpha}+\tilde{\beta}-1} g_t \frac{C_{n+1}}{C_n} \xrightarrow{n \rightarrow +\infty} 0$$

Therefore by the ratio test then the comparison test, we have that $B^n(u)_t$ is summable. Hence $B^n(u)_t \rightarrow 0$ as $n \rightarrow +\infty$.

Step 5: As $u_t \leq a + B(u)_t$, then it is clear by the Principle of Induction, by using the

monotonicity and linearity of B , we have $u_t \leq \sum_{j=1}^{N-1} aB^j(1)_t + B^N(u)_t$. Hence taking limiting as $N \rightarrow \infty$, by step 1 and step 4, we conclude $u_t \leq \sum_{j \geq 0} aC_j t^{j(\bar{\alpha} + \bar{\beta} - 1)} g_t^j$. The proof is now complete. \square

Proof of Proposition 4.2. This proof is analogous to that of Proposition 4.1. Most of the details have been skipped for brevity. Choose $t_0 \in (0, t_{\text{explode}})$.

Step 1: Regularity of L and decomposition into integral form. As $L \in \mathcal{C}^1([0, t_{\text{explode}}))$, by the Fundamental Theorem of Calculus we have $L_t - L_s \leq K(t-s)$ for any $t, s \in [0, t_{\text{explode}})$ with $t > s$ and $K = \sup_{u \leq t_0} |L'_u|$. Now we may write L as

$$L_t = \int_0^t \kappa^\varepsilon(t-s) L_s \, ds + \left[1 - \int_0^t \kappa^\varepsilon(t-s) \, ds \right] L_t + \int_0^t \kappa^\varepsilon(t-s) (L_t - L_s) \, ds.$$

Observe

$$\left[1 - \int_0^t \kappa^\varepsilon(t-s) \, ds \right] L_t \leq 2K\varepsilon \quad \text{and} \quad \int_0^t \kappa^\varepsilon(t-s) (L_t - L_s) \, ds \leq K\varepsilon.$$

Therefore

$$L_t = \int_0^t \kappa^\varepsilon(t-s) L_s \, ds + \Psi^\varepsilon(t) \quad \text{where } |\Psi^\varepsilon(t)| \leq 3K\varepsilon \quad \forall t \in [0, t_0].$$

Step 2: Comparison between the delayed loss and the instantaneous loss. As in Proposition 4.1, we have

$$0 \leq |L_t - L_t^\varepsilon| \leq Kc_1 \int_0^t \int_0^u \frac{\kappa^\varepsilon(u-s) |L_s - L_s^\varepsilon|}{\sqrt{t-u}} \, ds \, du + Kc_1 \int_0^t \frac{|\Psi^\varepsilon(s)|}{\sqrt{t-s}} \, ds.$$

Note as $|\Psi^\varepsilon(t)| \leq 3K\varepsilon$ for all $t \in [0, t_0]$, we see that the second term above is bounded above by $C_{K,t_0}\varepsilon$. Therefore,

$$\begin{aligned} 0 \leq |L_t - L_t^\varepsilon| &\leq Kc_1 \int_0^t \int_s^t \frac{\kappa^\varepsilon(u-s) |L_s - L_s^\varepsilon|}{\sqrt{t-u}} \, du \, ds + C_{K,t_0}\varepsilon \\ &= Kc_1 \int_0^t |L_s - L_s^\varepsilon| \rho^\varepsilon(t, s) \, ds + C_{K,t_0}\varepsilon, \end{aligned} \tag{A.16}$$

where

$$\rho^\varepsilon(t, s) = \int_s^t \frac{\kappa^\varepsilon(u-s)}{\sqrt{t-u}} \, du.$$

Step 3: Bounds on $\rho^\varepsilon(t, s)$. As in Proposition 4.1, the presence of κ in ρ^ε makes the function too general to do any analysis, hence we shall construct polynomial bounds on ρ^ε . Then we can apply generalised versions of Grönwall's Lemma. Recall $t \geq s$,

Case 1: $t - s \leq \varepsilon$

$$\begin{aligned} \rho^\varepsilon(t, s) &= \int_s^t \frac{\kappa^\varepsilon(u-s)}{\sqrt{t-u}} \, du \quad \text{let } \tilde{u} = \frac{u-s}{\varepsilon} \\ &= \int_0^{\frac{t-s}{\varepsilon}} \frac{\kappa(\tilde{u})}{\sqrt{t-s-\varepsilon\tilde{u}}} \, d\tilde{u} \leq \frac{\|\kappa\|_{L^\infty}}{\varepsilon^{1/2}} \int_0^{\frac{t-s}{\varepsilon}} \frac{d\tilde{u}}{\sqrt{\frac{t-s}{\varepsilon} - \tilde{u}}} \\ &= \frac{2\|\kappa\|_{L^\infty} (t-s)^{1/2}}{\varepsilon} \leq \frac{2\|\kappa\|_{L^\infty}}{(t-s)^{1/2}} \end{aligned}$$

Case 2: $t - s > \varepsilon$

As the support of κ^ε is in $[0, \varepsilon]$

$$\begin{aligned}\rho^\varepsilon(t, s) &= \int_s^t \frac{\kappa^\varepsilon(u-s)}{\sqrt{t-u}} du = \int_s^{s+\varepsilon} \frac{\kappa^\varepsilon(u-s)}{\sqrt{t-u}} du \\ &\leq \frac{\|\kappa\|_{L^\infty}}{\varepsilon} \int_s^{s+\varepsilon} \frac{du}{\sqrt{t-u}} = 2\|\kappa\|_{L^\infty} \left[\frac{(t-s)^{1/2} - (t-s-\varepsilon)^{1/2}}{\varepsilon} \right].\end{aligned}$$

Step 4: Grönwall type argument. Now that we have sufficiently simplified ρ^ε , we may put (A.16) into a form where we may apply Grönwall's inequality. By step 4 case 1 and (A.16), we have for $t \leq \varepsilon$

$$|L_t - L_t^\varepsilon| \leq Kc_1 \int_0^t 2\|\kappa\|_{L^\infty} (t-s)^{-1/2} |L_s - L_s^\varepsilon| \rho^\varepsilon(t, s) ds + C_{K, t_0} \varepsilon.$$

By step 4 case 2 and (A.16), we have for $t > \varepsilon$

$$\begin{aligned}|L_t - L_t^\varepsilon| &\leq Kc_1 \int_0^{t-\varepsilon} |L_s - L_s^\varepsilon| \rho^\varepsilon(t, s) ds + Kc_1 \int_{t-\varepsilon}^t |L_s - L_s^\varepsilon| \rho^\varepsilon(t, s) ds + C_{K, t_0} \varepsilon \\ &\leq 2Kc_1 \|\kappa\|_{L^\infty} \int_0^{t-\varepsilon} \left[\frac{(t-s)^{1/2} - (t-s-\varepsilon)^{1/2}}{\varepsilon} \right] |L_s - L_s^\varepsilon| ds \\ &\quad + 2Kc_1 \|\kappa\|_{L^\infty} \int_{t-\varepsilon}^t (t-s)^{-1/2} |L_s - L_s^\varepsilon| ds + C_{K, t_0} \varepsilon \\ &\leq 2Kc_1 \|\kappa\|_{L^\infty} \sum_{j \geq 2} \tilde{C}_j \varepsilon^{j-1} \int_0^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} |L_s - L_s^\varepsilon| ds \\ &\quad + 2Kc_1 \|\kappa\|_{L^\infty} \int_0^t (t-s)^{-1/2} |L_s - L_s^\varepsilon| ds + C_{K, t_0} \varepsilon,\end{aligned}$$

where the second term in the last line captures the higher order terms from Taylor's Theorem. The Monotone Convergence Theorem allows us to swap integrals and sums. We now consider

$$\tilde{C}_j \varepsilon^{j-1} \int_0^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} ds.$$

We shall proceed in two cases.

Case 1: $\varepsilon < t \leq 2\varepsilon$

$$\tilde{C}_j \varepsilon^{j-1} \int_0^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} ds \leq \tilde{C}_j \varepsilon^{j-1} \varepsilon^{\frac{-2j+1}{2}} \int_0^{t-\varepsilon} ds = \tilde{C}_j \varepsilon^{\frac{1}{2}}$$

where the first inequality follows from the fact that $(-2j+1)/2 < 0$ as $j \geq 2$ and $t-s \in [\varepsilon, t]$ for $s \in [0, t-\varepsilon]$.

Case 2: $t > 2\varepsilon$

We observe that

$$\tilde{C}_j \varepsilon^{j-1} \int_0^\varepsilon (t-s)^{\frac{-2j+1}{2}} ds \leq \tilde{C}_j \varepsilon^{j-1} \varepsilon^{\frac{-2j+1}{2}} \int_0^\varepsilon ds = \tilde{C}_j \varepsilon^{\frac{1}{2}}$$

and

$$\begin{aligned}\tilde{C}_j \varepsilon^{j-1} \int_{\varepsilon}^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} ds &= \tilde{C}_j \varepsilon^{j-1} \frac{2}{2j-3} (t-s)^{\frac{-2j+3}{2}} \Big|_{s=\varepsilon}^{t-\varepsilon} \\ &= \frac{2\tilde{C}_j \varepsilon^{j-1}}{2j-3} \left[\varepsilon^{\frac{-2j+3}{2}} - (t-\varepsilon)^{\frac{-2j+3}{2}} \right] \\ &\leq \frac{2\tilde{C}_j \varepsilon^{j-1} \varepsilon^{\frac{-2j+3}{2}}}{2j-3} \\ &= \frac{2\tilde{C}_j \varepsilon^{1/2}}{2j-3} \leq 2\tilde{C}_j \varepsilon^{1/2},\end{aligned}$$

where the last inequality follows from the fact that $j \geq 2$. Therefore, we have

$$\begin{aligned}\tilde{C}_j \varepsilon^{j-1} \int_0^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} ds &= \tilde{C}_j \varepsilon^{j-1} \int_0^{\varepsilon} (t-s)^{\frac{-2j+1}{2}} ds + \tilde{C}_j \varepsilon^{j-1} \int_{\varepsilon}^{t-\varepsilon} (t-s)^{\frac{-2j+1}{2}} ds \\ &\leq 3\tilde{C}_j \varepsilon^{1/2},\end{aligned}$$

for all $t > \varepsilon$. As L and L^ε are bounded by 1, so independent of t being greater or less than ε we have

$$\begin{aligned}|L_t - L_t^\varepsilon| &\leq 2Kc_1 \|\kappa\|_{L^\infty} \int_0^t (t-s)^{-1/2} |L_s - L_s^\varepsilon| ds + 12Kc_1 \|\kappa\|_{L^\infty} \varepsilon^{1/2} \sum_{j \geq 2} \tilde{C}_j + C_{K,t_0} \varepsilon \\ &= 2Kc_1 \|\kappa\|_{L^\infty} \int_0^t (t-s)^{-1/2} s^{-\gamma} |L_s - L_s^\varepsilon| ds + C_{K,t_0} \varepsilon^{1/2},\end{aligned}$$

for any $t \in [0, t_0]$. Lastly by Proposition A.10 using $\tilde{\beta} = 1/2$ and $\tilde{\alpha} = 1$, then $\tilde{\alpha} + \tilde{\beta} - 1 > 0$ as $\gamma < 1/2$ and

$$|L_t - L_t^\varepsilon| = C_{K,t_0} \varepsilon^{1/2} \sum_{n \geq 0} (2Kc_1 t_0^{1/2} \|\kappa\|_{L^\infty})^n C_n.$$

This completes the proof. \square

B Further numerical analysis

In Section 4.2, we considered six examples to compare the theoretical rate of convergence with that obtained numerically. The parameters used for each simulation are given in Table 1.

Table 1: Parameters of numerical simulations in Section 4.2.

Simulation	CC1 ¹	CC2 ²	DC1 ³	DC2 ⁴	CNC1 ⁵	CNC2 ⁶
Initial Condition	Unif[0.25, 0.35]	$\Gamma(2.1, 0.5)$	$\Gamma(1.2, 0.5)$	$\Gamma(1.4, 0.5)$	Unif[0.25, 0.35]	Unif[0.25, 0.35]
α	0.5	1.3	0.9	2	0.5	0.5
Δ_t	10^{-6}	10^{-6}	10^{-9}	10^{-9}	10^{-6}	10^{-6}
t_{\max}	0.1	0.1	10^{-4}	10^{-4}	0.1	2×10^{-2}

With the chosen parameters, we generated the convergence graphs in Fig. 1, Fig. 2 and Fig. 3 from $\text{Error}(\varepsilon_n)$, where $\varepsilon_n := \varepsilon \times \Delta^n$, with ε and Δ as positive constants, and

¹Continuous case 1

²Continuous case 2

³Discontinuous case 1

⁴Discontinuous case 2

⁵Common noise case 1: with increasing path

⁶Common noise case 2: with decreasing path

Error is the corresponding difference between the smoothed and limiting loss functions. Assuming a power law relationship between the error and the parameter ε ,

$$\text{Error}(\varepsilon) \approx A\varepsilon^\beta,$$

where A and β are constants, we performed a linear regression on $\text{Log Error}(\varepsilon)$ versus $\text{Log } \varepsilon$, which determined the line of best fit shown in the plots. The slope, shown in Table 2, represents our best estimate for the rate of convergence for each setting.

Table 2: Gradient of the regression line.

Setting	CC1	CC2	DC1	DC2	CNC1	CNC2
Rate	1.0202	0.9635	0.9295	0.8126	0.7621	0.8144

To assess if the estimated slope corresponds to an asymptotic value, we also conducted an alternative analysis of the rate of convergence. By computing the ratio between two consecutive errors, we observe

$$\frac{\text{Error}(\varepsilon_{n+1})}{\text{Error}(\varepsilon_n)} \approx \frac{A\Delta^{n\beta+\beta}\varepsilon_n^\beta}{A\Delta^{n\beta}\varepsilon_n^\beta} \approx \Delta^\beta,$$

and taking logarithms with base Δ , we may deduce

$$\text{Log}_\Delta \left(\frac{\text{Error}(\varepsilon_{n+1})}{\text{Error}(\varepsilon_n)} \right) \approx \beta.$$

By using the relationship that $\text{Log}_a b = \text{Log}_c b / \text{Log}_c a$ for any $a, b, c > 0$, we obtain approximate expressions for β as follows:

$$\begin{aligned} \beta_n &:= \text{Log}_\Delta \left(\frac{\text{Error}(\varepsilon_{n+1})}{\text{Error}(\varepsilon_n)} \right) \\ &= \frac{\text{Log}(\text{Error}(\varepsilon_{n+1})) - \text{Log}(\text{Error}(\varepsilon_n))}{\text{Log}(\Delta)} = \frac{\text{Log}(\text{Error}(\varepsilon_{n+1})) - \text{Log}(\text{Error}(\varepsilon_n))}{\text{Log}(\varepsilon_{n+1}) - \text{Log}(\varepsilon_n)}. \end{aligned} \quad (\text{B.1})$$

where Log represents the logarithm with respect to any base. From Table 3, it is evident that in the cases of CC1 and CC2, the rate of convergence approaches 1 asymptotically. However, for all other scenarios, there appears to be no distinct pattern or clear convergence of the gradients, which generally lie between 1/2 and 1.

Table 3: Gradient between adjacent points in the Log-Log plots in Fig. 1, Fig. 2 and Fig. 3.

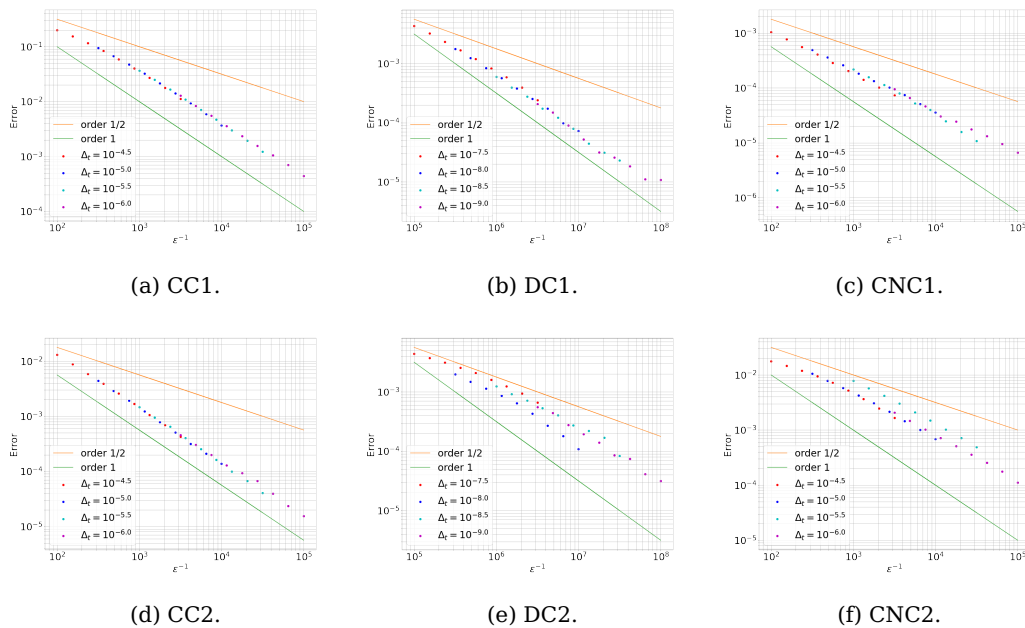
Simulation	ε_n								
	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
CC1	0.9395	0.9596	1.0348	0.9428	0.9649	1.0884	1.0954	1.1309	0.9805
CC2	0.9315	1.0231	0.9101	0.8885	0.9012	0.5896	1.3951	1.1744	1.0025
DC1	0.9195	1.0594	0.8032	1.0674	1.2587	0.9238	0.3265	1.5408	0.0566
DC2	0.5304	0.7907	0.5235	1.1918	0.7092	0.6909	0.8841	1.2225	0.5127
CNC1	0.7646	0.7054	0.8223	0.8060	0.9489	0.4754	0.8792	0.6219	0.8243
CNC2	0.7258	0.7915	0.8005	0.8219	0.8305	0.7787	0.7749	0.8241	1.0809

Furthermore, we investigated the sensitivity of the convergence rate analysis to the choice of Δ_t . It is clear that for meaningful approximations to the smoothed system it is needed that Δ_t is sufficiently small compared to ε , which necessitates extremely small time steps and makes the simulation of the particle systems computationally costly.

To assess whether Δ_t is sufficiently small, we generated in Fig. 4 and Table 4 rate of convergence plots with different values of Δ_t . For each Δ_t , we selected several values of ε that were uniformly spaced (after taking logarithms) within the interval $[\Delta_t \times 10^{-2.5}, \Delta_t \times 10^{-1}]$. The findings indicate that the estimated rate of convergence remains consistent with respect to variations in Δ_t .

Table 4: Gradient of the line of best fit in Fig. 4 (if plotted).

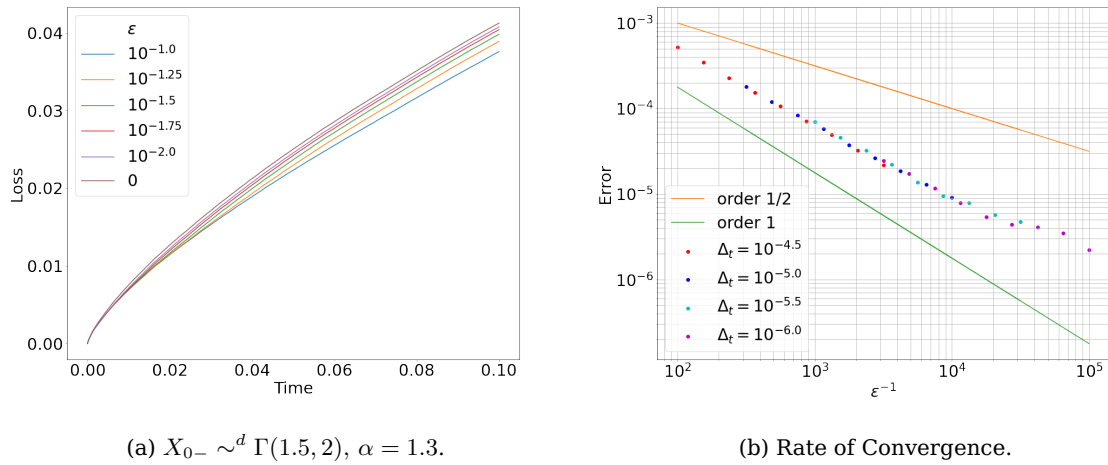
Simulation	Δ_t							
	$10^{-4.5}$	10^{-5}	$10^{-5.5}$	10^{-6}	$10^{-7.5}$	10^{-8}	$10^{-8.5}$	10^{-9}
CC1	0.836	0.944	0.987	0.967	—	—	—	—
CC2	0.988	1.014	1.039	0.970	—	—	—	—
DC1	—	—	—	—	0.821	0.940	0.968	0.909
DC2	—	—	—	—	0.539	0.829	0.728	0.850
CNC1	0.775	0.750	0.875	0.751	—	—	—	—
CNC2	0.685	0.793	0.806	0.830	—	—	—	—

Figure 4: Sensitivity of rate of convergence with respect to changes in Δ_t .

Finally, in Fig. 5 and Table 5, we investigated a scenario where the initial condition is Hölder continuous near the boundary without any observed jump discontinuity in the simulations, specifically, $X_{0-} \sim^d \Gamma(1.5, 2)$ with $\alpha = 1.3$. By [11, Theorem 1.1], the limiting loss function is 1/2-Hölder continuous at 0. The rate of convergence appears to be between 1/2 and 1 in this setting.

Table 5: Gradient of the line of best fit in Fig. 5.

Δ_t	$10^{-4.5}$	10^{-5}	$10^{-5.5}$	10^{-6}
Gradient	0.913	0.863	0.798	0.66


 Figure 5: Sensitivity of the rate of convergence with respect to changes in Δ_t .

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Acknowledgments. AP thanks Philipp Jettkant for discussions on this material.

Data availability statement

No data were created or analysed in this work. Data sharing is not applicable to this article.

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