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RESEARCH ARTICLE

Density of monochromatic infinite subgraphs II

Jan Corsten[1], Louis DeBiasio[2] and Paul McKenney[3]

[1]Department of Mathematics, London School of Economics and Political Science (LSE), London, United Kingdom; E-mail: jan.corsten92@gmail.com.
[2]Department of Mathematics, Miami University, Oxford, OH, United States; E-mail: debiasld@miamioh.edu (corresponding author).
[3]Department of Mathematics, Miami University, Oxford, OH, United States; E-mail: pmckenney@gmail.com.

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Abstract
In 1967, Gerencsér and Gyárfás [16] proved a result which is considered the starting point of graph-Ramsey theory: In every 2-coloring of K_n, there is a monochromatic path on $\lceil (2n+1)/3 \rceil$ vertices, and this is best possible. There have since been hundreds of papers on graph-Ramsey theory with some of the most important results being motivated by a series of conjectures of Burr and Erdős [2, 3] regarding the Ramsey numbers of trees (settled in [31]), graphs with bounded maximum degree (settled in [5]), and graphs with bounded degeneracy (settled in [23]). In 1993, Erdős and Galvin [13] began the investigation of a countably infinite analogue of the Gerencsér and Gyárfás result: What is the largest d such that in every 2-coloring of $K_{\mathbb{N}}$ there is a monochromatic infinite path with upper density at least d? Erdős and Galvin showed that $2/3 \leq d \leq 8/9$, and after a series of recent improvements, this problem was finally solved in [7] where it was shown that $d = (12+\sqrt{8})/17$.

This paper begins a systematic study of quantitative countably infinite graph-Ramsey theory, focusing on infinite analogues of the Burr-Erdős conjectures. We obtain some results which are analogous to what is known in finite case, and other (unexpected) results which have no analogue in the finite case.

Contents

1. Introduction

It was proven by Ramsey [28] that for every graph G and every positive integer r, there exists a positive integer N such that in every r-coloring[1] of K_N, there is a monochromatic copy of G. The smallest possible choice for N is called the *r-color Ramsey number* and is denoted by $R_r(G)$. Determining Ramsey numbers of different (families of) graphs is one of the central topics in combinatorics. In this paper, we are interested in similar problems for countably infinite graphs. (We will not consider uncountably infinite graphs and thus always mean 'countably infinite' when we write 'infinite' from now on.)

Let $K_{\mathbb{N}}$ be the graph on vertex set $\mathbb{N}$ with edge set edge set $\binom{\mathbb{N}}{2}$ (typically $\mathbb{N}$ denotes the set of positive integers, and we typically begin counting at 1; however, there are certain situations where it is convenient to let $\mathbb{N}$ denote the non-negative integers or to start counting at 0, but this distinction will never have an impact on the results). Ramsey [28] also proved that in every r-coloring of $K_{\mathbb{N}}$, there is a monochromatic copy of $K_{\mathbb{N}}$. Thus, in order to make the problem quantitative, we will thus consider the density of the monochromatic graphs we are looking for. The *upper density* of a graph G with $V(G) \subseteq \mathbb{N}$ is defined as

$$\overline{\mathrm{d}}(G) = \limsup_{t\to\infty} \frac{|V(G) \cap \{0, 1, 2, \ldots, t\}|}{t}.$$

The *lower density*, denoted $\underline{\mathrm{d}}(G)$, is defined analogously in terms of the lim inf, and we speak of the *density* whenever lower and upper densities coincide.

Erdős and Galvin [13] described a 2-coloring of $K_{\mathbb{N}}$ in which every graph having finitely many isolated vertices and bounded maximum degree has lower density 0; thus, we typically restrict our attention to upper densities. However, this does raise the question of whether there is any graph G (with finitely many isolated vertices) having the property that in every 2-coloring of $K_{\mathbb{N}}$, there is a monochromatic copy of G with positive lower density. We will return to this question later and prove that, surprisingly, such graphs exist in a strong sense.

Given a graph G and an r-coloring of ϕ of $K_{\mathbb{N}}$, the *Ramsey upper density of G with respect to φ*, denoted $\overline{\mathrm{Rd}}_{\varphi}(G)$, is the supremum of $\overline{\mathrm{d}}(G)$ over all monochromatic copies of G in the coloring φ of $K_{\mathbb{N}}$.

[1]Throughout the paper, an r-coloring of a graph K will always mean an r-coloring of the edges of K

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The r-color Ramsey upper density of G, denoted $\overline{\mathrm{Rd}}_r(G)$, is the infimum of $\overline{\mathrm{Rd}}_\varphi(G)$ over all r-colorings φ of $K_\mathbb{N}$. If $r = 2$, we drop the subscript.

Possibly the first such (implicitly) quantitative result is due to Rado [27] who proved that every r-coloring of $K_\mathbb{N}$ contains r vertex-disjoint monochromatic infinite paths which together cover all of $\mathbb{N}$. In particular, one of the paths must have upper density at least $1/r$ and hence $\overline{\mathrm{Rd}}_r(P_\infty) \geq 1/r$, where P_∞ is the (one-way) infinite path. For two colors, this was improved by Erdős and Galvin [13] who proved that $2/3 \leq \overline{\mathrm{Rd}}(P_\infty) \leq 8/9$. More recently, DeBiasio and McKenney [10] improved the lower bound to $3/4$ and conjectured the correct value to be $8/9$. Progress towards this conjecture was made by Lo, Sanhueza-Matamala and Wang [25], who raised the lower bound to $(9 + \sqrt{17})/16 \approx 0.82019$. Corsten, DeBiasio, Lamaison and Lang [7] finally proved that $\overline{\mathrm{Rd}}(P_\infty) = (12 + \sqrt{8})/17 \approx 0.87226$, thereby settling the problem for two colors. In this paper, we initiate a systematic study of Ramsey densities for other infinite graphs. An independent systematic study was undertaken by Lamaison [21], who fortunately focused on a different aspect of the general problem (locally-finite graphs), and thus, the two papers have very little overlap.

1.1. Graphs with positive Ramsey upper density

The problem of estimating the Ramsey numbers of sparse finite graphs has received a lot of attention. The problem was motivated by a series of conjectures proposed by Burr and Erdős [2, 3], starting with graphs of bounded maximum degree.

Conjecture 1.1 (Burr–Erdős [2]). *For all* $\Delta \in \mathbb{N}$*, there exists some* $c = c(\Delta) > 0$ *such that every* 2*-colored* K_n *contains a monochromatic copy of every graph* G *with at most* cn *vertices and* $\Delta(G) \leq \Delta$.

Theorem 1.1 was solved by Chvatál, Rödl, Szemerédi, Trotter [5] in an early application of the regularity lemma. Since then, there has been many improvements to the constant $c(\Delta)$ (see [6] for a more detailed history). Allen, Brightwell and Skokan [1] proved that this constant can be significantly improved to $c = 1/(2\chi(G) + 4) \geq 1/(2\Delta + 6)$ for graphs of small bandwith (see [1] for the precise statement of their result), where $\chi(G)$ denotes the chromatic number of G.

Our first theorem proves an analogue of this for infinite graphs. It turns out that much weaker conditions on the degrees suffice. Given $k \geq 2$, we say that a graph G is one-way k-locally finite if there exists a partition of $V(G)$ into k independent sets $V_1, \ldots, V_k$ with $|V_1| \geq \ldots \geq |V_k|$ such that for all $1 \leq i < j \leq k$ and all $v \in V_j$, $d(v, V_i) < \infty$. Note that every vertex in V_k has finite degree, but it is possible for any vertex in $V_1 \cup \cdots \cup V_{k-1}$ to have infinite degree. A good example of a one-way 2-locally finite graph exhibiting this property is the *infinite bipartite half graph*, which is the graph on $\mathbb{N} = A \cup B$, where A is the set of positive odd integers and B is the set of positive even integers and uv is an edge if and only if $u < v$ and u is odd and v is even. Further note that one-way k-locally finite graphs have chromatic number at most k and, if G is locally finite (that is every vertex has finite degree) with $\chi(G) < \infty$, then G is one-way $\chi(G)$-locally finite.

Theorem 1.2. *Let* $k, r \in \mathbb{N}$ *and let* G *be an infinite, one-way* k*-locally finite graph.*

(i) *If* $k = 2$*, then* $\overline{\mathrm{Rd}}_r(G) \geq 1/r$.
(ii) *If* $k \geq 2$*, then* $\overline{\mathrm{Rd}}(G) \geq \frac{1}{2(k-1)}$.
(iii) *If* $r, k \geq 3$*, then* $\overline{\mathrm{Rd}}_r(G) \geq \frac{1}{\sum_{i=0}^{(k-2)r+1}(r-1)^i} \geq \frac{1}{r^{(k-2)r+1}}$.

Since graphs with $\Delta(G) = \Delta < \infty$ have $\chi(G) \leq \Delta + 1$, we get that $\overline{\mathrm{Rd}}(G) \geq \frac{1}{2\Delta}$ and $\overline{\mathrm{Rd}}_r(G) \geq 1/r^{\Delta r}$ for every $r \geq 3$ (which answers a question from [10]). However, we are able to prove a slightly stronger result for 2 colors.

Corollary 1.3. *If* G *is an infinite graph with* $\Delta(G) = \Delta < \infty$*, then* $\overline{\mathrm{Rd}}(G) \geq \frac{1}{2(\Delta-1)}$.

A graph G is *d-degenerate* if there is an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that for all $i \geq 1$, $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq d$. The *degeneracy* of G, denoted $\mathrm{degen}(G)$, is the smallest non-negative

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integer d such that G is d-degenerate; if no such integer exists, say $\text{degen}(G) = \infty$. Note that if G is d-degenerate, then $\chi(G) \le d + 1 \le \Delta(G) + 1$. Also note that a graph can have finite degeneracy, but infinite maximum degree.

Conjecture 1.4 (Burr–Erdős [2]). *For all $d \in \mathbb{N}$, there exists some $c = c(d) > 0$ such that every 2-colored K_n contains a copy of every d-degenerate graph on at most cn vertices.*

Theorem 1.4 was recently confirmed by Lee [23]. It would be very interesting to prove an analogue of this for infinite graphs.

Problem 1.5. For all $d \in \mathbb{N}$, does there exist some $c = c(d) > 0$ such that $\overline{\text{Rd}}(G) \ge c$ for every infinite graph G with degeneracy at most d? A weaker version of this question is for all infinite graphs G with finite degeneracy, does there exist some $c = c(G) > 0$ such that $\overline{\text{Rd}}(G) \ge c$?

As we will discuss in the next section, we obtain a positive answer to a weaker version of this question.

1.2. Ramsey-dense graphs

We say that an infinite graph G is *r-Ramsey-dense* if in every r-coloring of $K_{\mathbb{N}}$, there is a monochromatic copy of G with positive upper density. If $r = 2$, we drop the prefix and just say G is *Ramsey-dense*. Note that if G is Ramsey-dense, this does not necessarily imply that $\overline{\text{Rd}}(G) > 0$ as there are infinitely many colorings, so the infimum of the upper densities over all colorings can be 0. Indeed, we shall see below that the so-called Rado graph $\mathcal{R}$ is an example of an infinite graph which is Ramsey-dense yet $\overline{\text{Rd}}(\mathcal{R}) = 0$. However, every infinite graph G with $\overline{\text{Rd}}(G) > 0$ is Ramsey-dense.

Ramsey-dense graphs are another natural analogue of graphs with linear Ramsey number. We will describe a simple property guaranteeing that a graph is Ramsey-dense and then show that every Ramsey-dense graph is not far from having this property.

A set $X \subseteq V(G)$ is called *dominating* if every vertex $v \in V(G) \setminus X$ has a neighbor in X. We call a set $X \subseteq V(G)$ *ruling* if X is finite and all but finitely many vertices $v \in V(G) \setminus X$ have a neighbor in X. We say that an infinite graph G is t-ruled if there are at most t disjoint minimal ruling sets. The *ruling number* of a graph G, denoted by $\text{rul}(G)$, is the smallest $t \in \mathbb{N}$ such that G is t-ruled; if no such t exists, we say G is infinitely ruled, or $\text{rul}(G) = \infty$. Equivalently, $\text{rul}(G)$ is the matching number of the hypergraph whose edges are all minimal ruling sets. Note that a graph G is 0-ruled if and only if there is no finite dominating set and finitely-ruled (i.e., t-ruled for some $t \in \mathbb{N}$) if and only if there is a finite set $S \subseteq V(G)$ such that $G[S^c]$ has no finite dominating sets.

Theorem 1.6. *If G is an infinite graph with $\text{rul}(G) < \infty$, then G is r-Ramsey-dense for all $r \in \mathbb{N}$.*

This has a few interesting corollaries. Since locally finite graphs have ruling number 0, we immediately get the following.

Corollary 1.7. *If G is a locally finite, infinite graph, then G is r-Ramsey-dense for all $r \in \mathbb{N}$.*

The *Rado graph* is the graph $\mathcal{R}$ with vertex-set $\mathbb{N}$ defined by placing an edge between $m < n$ if and only if the mth digit in the binary expansion of n is 1. The Rado graph has many interesting properties; for example, it is isomorphic to the infinite random graph (that is, the graph on $\mathbb{N}$ in which every edge is present independently with probability $1/2$) with probability 1. It is easy to verify that the Rado graph does not have any finite dominating sets and hence $\text{rul}(\mathcal{R}) = 0$.

Corollary 1.8. *The Rado graph $\mathcal{R}$ is r-Ramsey-dense for all $r \in \mathbb{N}$.*

However, we will show that $\overline{\text{Rd}}(\mathcal{R}) = 0$ (see Theorem 2.5). Another corollary asserts that graphs with bounded degeneracy are Ramsey-dense.

Corollary 1.9. *If G is an infinite graph with bounded degeneracy, then G is r-Ramsey-dense for all $r \in \mathbb{N}$.*

By Theorem 1.6, it suffices to show that every d-degenerate infinite graph G is d-ruled.

Fact 1.10. Let $d \in \mathbb{N}$. If G is d-degenerate, then $\mathrm{rul}(G) \leq d$.

Proof. Suppose for contradiction, there is a d-degenerate infinite graph G with $\mathrm{rul}(G) > d$ for some $d \in \mathbb{N}$. Let $S_1, \ldots, S_{d+1}$ be disjoint minimal ruling sets and let $S_0 \subseteq V(G) \setminus (S_1 \cup \ldots \cup S_{d+1})$ be the set of vertices which do not have a neighbor in some S_i. Note that $S := S_0 \cup S_1 \cup \ldots \cup S_{d+1}$ is finite. Therefore, there is a vertex $u \in \mathbb{N} \setminus S$ which comes after all vertices in S in a d-degenerate ordering of $V(G)$ and hence $\deg(u, S) \leq d$. However, by construction, u has a neighbor in each of $S_1, \ldots, S_{d+1}$, a contradiction. □

Problem 1.11. Is there a Ramsey-dense graph G with $\mathrm{rul}(G) = \infty$?

If the answer is no, then together with Theorem 1.6, this would give a complete characterization of Ramsey-dense graphs. We will give a partial answer to the question by showing that if $\mathrm{rul}(G) = \infty$ and additionally the sizes of the minimal ruling sets do not grow too fast, then G is not Ramsey-dense (see Theorem 2.15).

1.3. Trees

Another famous conjecture of Burr and Erdős [3] concerns the Ramsey number of trees. A graph is *acyclic* if it contains no finite cycles, a *forest* is an acyclic graph, and a *tree* is a connected acyclic graph.

Conjecture 1.12 (Burr–Erdős [3]). *Let $n \in \mathbb{N}$ and let T be a tree on at most $\frac{n}{2} + 1$ vertices. Every 2-colored K_n contains a monochromatic copy of T.*

Theorem 1.12 was solved for large n by Zhao [31]. The following result provides an analogue of this in infinite graphs and can be seen to be best possible. Note that Theorem 1.2 already implies that $\overline{\mathrm{Rd}}(T) \geq 1/2$ for every infinite locally finite forest T.

Theorem 1.13. $\overline{\mathrm{Rd}}(T) \geq 1/2$ *for every infinite forest T.*

We further show that $\overline{\mathrm{Rd}}(T_\infty) = 1/2$, where T_∞ is the infinite tree in which every vertex has infinite degree and there are also infinite locally finite trees T with $\overline{\mathrm{Rd}}(T) = 1/2$ (see Theorem 2.2).

Erdős, Faudree, Rousseau and Schelp [14] showed that if T is a tree on more than $\lceil 3n/4 \rceil$ vertices, then there exists a 2-coloring of K_n which contains no monochromatic copy of T. Furthermore, they showed that this bound can be acheived by certain trees such as the tree obtained by joining the center of $K_{1,n/4}$ with a path on $n/2 - 1$ vertices (see also [30]). In other words, $3/4$ is the largest proportion of vertices that a single connected graph can cover in an arbitrary 2-coloring of K_n. We now consider an analogous question for infinite graphs.

Say that a graph G is *Ramsey-cofinite* if in every 2-coloring of $K_\mathbb{N}$ there exists a monochromatic copy of G such that $V(G)$ is cofinite. It is clear that any graph G with infinitely many isolated vertices is Ramsey-cofinite. Say that a graph G is *Ramsey-lower-dense* if in every 2-coloring of $K_\mathbb{N}$ there is a monochromatic copy of G with positive lower density. As mentioned earlier, Erdős and Galvin proved that for any graph G with finitely many isolated vertices and bounded maximum degree, then G is not Ramsey-lower-dense, and thus, G is not Ramsey-cofinite.

Surprisingly, we show that there exist connected graphs which are Ramsey-cofinite. In fact, we are able to completely characterize all acyclic graphs which are Ramsey-cofinite. Say that a graph G is *weakly expanding* if for all $k \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ such that for all independent sets A in G with $|A| \geq \ell$, we have $|N(A)| > k$. Say that a graph G is *strongly contracting* if there exists $k \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}$, there exists an independent set A in G with $|A| \geq \ell$ such that $|N(A)| \leq k$. Note that every infinite graph is either strongly contracting or weakly expanding. Finally, let $\mathcal{T}^*$ be the family of forests T having one vertex t of infinite degree, every other vertex has degree at most d for some $d \in \mathbb{N}$, t is adjacent to infinitely many leaves and infinitely many non-leaves, and cofinitely many vertices of T have distance at most 2 to t (in particular, if T is not connected, then T has one infinite component and finitely many finite components).

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Theorem 1.14. *Let T be a forest.*

(i) *If T is strongly contracting, has no finite dominating set, and $T \notin \mathcal{T}^*$, then T is Ramsey-cofinite.*
(ii) *If T is weakly expanding, has a finite dominating set, or $T \in \mathcal{T}^*$, then T is not Ramsey-lower-dense (and thus, T is not Ramsey-cofinite).*

To get a better sense of what Theorem 1.14 says in terms of trees, say that a graph G has *unbounded leaf degree* if for every $\ell \in \mathbb{N}$, there exists $v \in V(G)$ such that v is adjacent to at least ℓ leaves; otherwise, say that G has *bounded leaf degree*. A tree is strongly contracting if and only if it has unbounded leaf degree, and a tree is weakly expanding if and only if it has bounded leaf degree.

In light of Theorem 1.14, it would be natural to ask if there is any connected graph T such that there is a spanning monochromatic copy of T in every 2-coloring of $K_{\mathbb{N}}$; however, this is not possible. Clearly, if T is an infinite star, it does not have this property, so suppose T is not an infinite star and 2-color the edges of $K_{\mathbb{N}}$ by fixing a vertex v, coloring all edges incident with v red, and coloring all other edges blue. Every monochromatic copy of T must be blue and therefore not be spanning.

Completely characterizing all graphs which are Ramsey-cofinite is still an open question and is discussed in Section 8.4.

1.4. Bipartite Ramsey densities

Gyárfás and Lehel [17] and independently Faudree and Schelp [15] proved that every 2-colored $K_{n,n}$ contains a monochromatic path with at least $2\lceil n/2\rceil$ vertices (that is, roughly half the vertices of the graph). They further proved that this is best possible. We will prove an analogue of this for infinite graphs. Here, $K_{\mathbb{N},\mathbb{N}}$ is the infinite complete bipartite graph with one part being all even positive integers and the other part being all odd positive integers.

Theorem 1.15. *Every 2-colored $K_{\mathbb{N},\mathbb{N}}$ contains a monochromatic path of upper density at least $1/2$.*

Pokrovskiy [26] proved that the vertices of every 2-colored complete bipartite graph $K_{n,n}$ can be partitioned into three monochromatic paths. Soukup [29] proved an analogue of this for infinite graphs which holds for multiple colors: The vertices of every r-colored $K_{\mathbb{N},\mathbb{N}}$ can be partitioned into $2r-1$ monochromatic paths. He also presents an example where this is best possible. However, in his example, all but finitely many vertices can be covered by r monochromatic paths. Our next result shows that this is always possible for two colors.

Theorem 1.16. *The vertices of every 2-colored $K_{\mathbb{N},\mathbb{N}}$ can be partitioned into a finite set and at most two monochromatic paths.*

Theorem 1.15 is an immediate consequence of Theorem 1.16. We will provide an example which demonstrates that Theorems 1.15 and 1.16 are best possible (see Theorem 2.6). We believe that a similar statement is true for multiple colors.

Conjecture 1.17. *Let $r \in \mathbb{N}$ with $r \geq 2$. The vertices of every $(r-1)$-colored $K_{\mathbb{N},\mathbb{N}}$ can be partitioned into a finite set and at most $r-1$ monochromatic paths.*

Theorem 2.6 also shows that Theorem 1.17 is best possible, if true.

The main motivation for the above question had to do with a potential relationship to the problem of determining the value of $\overline{\mathrm{Rd}}_r(P_\infty)$ for $r \geq 3$. Very recently, Day and Lo [8] proved a result which implies that if the above conjecture is true (in fact, if a weaker conjecture is true), then for all $r \geq 3$, $\overline{\mathrm{Rd}}_r(P_\infty) \geq \frac{1}{r-1}$. In particular, Theorem 1.15 combined with their result implies that $\overline{\mathrm{Rd}}_3(P_\infty) = \frac{1}{2}$. They also showed that their weaker conjecture is true for $r = 3$ and $r = 4$, which additionally gives $\overline{\mathrm{Rd}}_4(P_\infty) = \frac{1}{3}$.

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1.5. Overview

We begin by summarizing our main results, and then describe where in the paper these results may be found.

(i) Let G be a countably infinite, (one-way) locally finite graph with chromatic number $\chi < \infty$ (in particular, the infinite bipartite half-graph has this property). Every 2-coloring of $K_{\mathbb{N}}$ contains a monochromatic copy of G with upper density at least $\frac{1}{2(\chi-1)}$.

(ii) Let G be a countably infinite graph having the property that there exists a finite set $X \subseteq V(G)$ such that $G - X$ has no finite dominating set (in particular, graphs with bounded degeneracy have this property, as does the infinite random graph). Every finite coloring of $K_{\mathbb{N}}$ contains a monochromatic copy of G with positive upper density.

(iii) For every countably infinite tree T, every 2-coloring of $K_{\mathbb{N}}$ contains a monochromatic copy of T of upper density at least $1/2$, and this is best possible. This is a perfect analogue of the corresponding result in the finite case which says that every 2-colored K_n contains a monochromatic copy of every tree on at most $\frac{n}{2} + 1$ vertices.

(iv) There exists connected graphs G such that every 2-coloring of $K_{\mathbb{N}}$ contains a monochromatic copy of G which covers all but finitely many vertices of $\mathbb{N}$. In fact, we classify all trees with this property. This result is particularly surprising in part because it has no analogue in the finite case (since for every connected graph G on more than $\lceil \frac{3n}{4} \rceil$ vertices, there is a 2-coloring of K_n with no monochromatic copy of G). In the process, we prove two results which may have independent interest: we give a characterization of graphs which are a spanning subgraph of every infinitely connected graph, and a characterization of graphs which can be cofinitely embedded into every graph with infinitely many vertices of cofinite degree.

In Section 2, we collect a variety of examples which are used to for instance obtain upper bounds on the upper Ramsey density of certain graphs. In Section 3, we discuss ultrafilters and a general embedding strategy that we will use to prove our results about one-way locally finite graphs in Section 4 and graphs of bounded ruling number in Section 5. In Section 6, we prove some additional results about graphs with bounded degeneracy. In Section 7, we prove Theorem 1.16. In Section 8, we prove Theorems 1.13 and 1.14 together with a variety of supporting results which may be of independent interest. In Section 9, we discuss a more general extension of the notion of a graph being Ramsey-dense. Finally, we end with some open problems in Section 10.

1.6. Notation

For a positive integer n, we let $[n] = \{1, 2, \ldots, n\}$.

A subset X of an infinite set Y is called cofinite in Y if $Y \setminus X$ is finite. If Y is clear from context, we will call X cofinite and write $X^c = Y \setminus X$. We write $A \subseteq^* B$ to mean that $A \setminus B$ is finite.

Given an edge-colored graph G and a color c, we write G_c for the spanning subgraph of G with all edges of color c. Given a vertex $v \in V(G)$, we define $N(v)$ to be the set of neighbors of v, and given a color c, we define $N_c(v) \subseteq N(v)$ to be the set of vertices which are adjacent to v via an edge of color c. Given $S \subseteq V(G)$, we write $N(S) = \bigcup_{v \in S} N(v)$ and $N^\cap(S) = \bigcap_{v \in S} N(v)$, and given a color c, we define $N_c(S) = \bigcup_{v \in S} N_c(v)$ and $N_c^\cap(S) = \bigcap_{v \in S} N_c(v)$.

If f is a function, we write dom f and ran f for the domain and range of f, respectively. (This notation is useful because we are often constructing an embedding of a graph G into a graph H, and at each step, we have a function from some subset of $V(G)$ to some subset of $V(H)$.)

The following well-known fact follows from the definition of upper- and lower-density. For disjoint sets $A, B \subseteq \mathbb{N}$, we have

$$\underline{d}(A) + \underline{d}(B) \leq \underline{d}(A \cup B) \leq \underline{d}(A) + \overline{d}(B) \leq \overline{d}(A \cup B) \leq \overline{d}(A) + \overline{d}(B). \tag{1.1}$$

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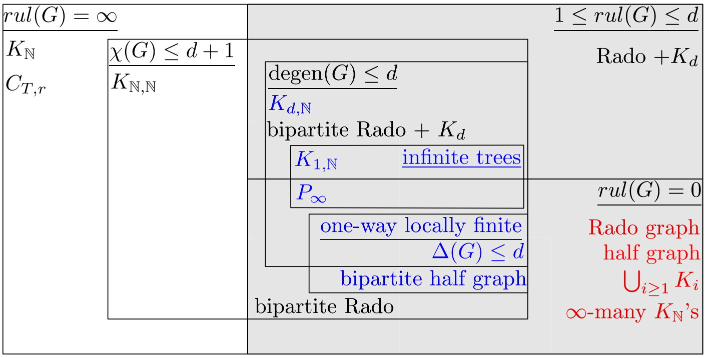

Figure 1. *The lightly shaded area represents graphs which are Ramsey-dense. The blue text represents graphs G for which $\overline{\text{Rd}}(G) > 0$. The red text represents graphs G which are Ramsey-dense, but $\overline{\text{Rd}}(G) = 0$.*

2. Examples

2.1. Basics

First, we present some examples to get a better understanding how the different parameters discussed in this paper are related.

The *infinite half graph* is the graph on $\mathbb{N}$ such that uv is an edge if and only if $u < v$ and v is even. Given a complete bipartite graph G between two disjoint infinite sets A and B, the *half graph coloring* of G is obtained by taking a bijection f from A to the odd integers and a bijection g from B to the even integers and coloring an edge uv with $u \in A$ and $v \in B$ red if $g(u) < f(v)$ and blue otherwise. Note that in this coloring, both the red and the blue graph are isomorphic to the infinite bipartite half graph.

The *bipartite Rado graph* is the graph $\mathcal{R}_2$ with vertex-set $\mathbb{N} \setminus \{1\}$ defined by placing an edge between $m < n$ if and only if the mth digit in the binary expansion of n is 1 and m and n differ in the first bit (i.e., m and n have different parity).

Example 2.1.

(i) There is a graph G with $\text{rul}(G) = 0$, but $\chi(G) = \infty$, and thus, $\text{degen}(G) = \infty$ (half graph, Rado graph, infinitely many disjoint $K_{\mathbb{N}}$'s).
(ii) There is a graph G with $\chi(G) = 2$, but $\text{rul}(G) = \infty$, and thus, $\text{degen}(G) = \infty$ ($K_{\mathbb{N},\mathbb{N}}$).
(iii) There is a graph G with $\text{rul}(G) = 0$ and $\chi(G) = 2$, but $\text{degen}(G) = \infty$ (bipartite Rado graph).
(iv) There is a one-way 2-locally finite graph G (with $\text{rul}(G) = 0$ and $\chi(G) = 2$), but $\text{degen}(G) = \infty$ (bipartite half graph).
(v) There is a locally finite graph G with $\text{rul}(G) = 0$ but $\chi(G) = \infty$, and thus, $\text{degen}(G) = \infty$ (infinite collection of disjoint finite cliques of increasing size).
(vi) There is a graph which is d-degenerate (and d-ruled) but not one-way k-locally-finite for any k ($K_{d,\mathbb{N}}$, T_∞).

2.2. Upper bounds on upper densities

Example 2.2. Let $r \in \mathbb{N}$.

(i) Let $D \ge 2$. If T is an infinite D-ary tree, then $\overline{\text{Rd}}_r(T) \le \frac{1}{r}(1 + \frac{1}{D})$.
(ii) There exists a locally finite, infinite tree T such that $\overline{\text{Rd}}_r(T) \le 1/r$.

Proof. Partition $\mathbb{N}$ by residues mod r – that is, $\mathbb{N} = A_0 \cup \ldots \cup A_{r-1}$, where $A_i = \{n \in \mathbb{N} : n \equiv i \pmod r\}$. We define an r-coloring as follows: if $m \in A_i$ and $n > m$, color the edge mn with color i. Note that if $n \not\equiv i \pmod r$, then n has at most $\lceil (n-1)/r \rceil$ neighbors of color i.

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(i) Let T be an infinite D-ary tree and suppose we have a copy of T of color i. For all $n \in \mathbb{N}$, let V'_n be the set of vertices in $V(T) \cap [n]$ which are not congruent to $i (\mathrm{mod}\ r)$. Since every vertex $m \in V'_n$ can only have successors (of color i) in $A_i \cap [n-1]$, we must have $D \cdot |V'_n| \leq \lceil (n-1)/r \rceil$. So we have

$$\frac{|V(T) \cap [n]|}{n} \leq \frac{\lceil \frac{n}{r} \rceil + |V'_n|}{n} \leq \frac{\lceil \frac{n}{r} \rceil + \frac{1}{D}\lceil \frac{n-1}{r} \rceil}{n} \xrightarrow{n \to \infty} \frac{1}{r}(1 + \frac{1}{D}),$$

and thus, $\overline{\mathrm{Rd}}_r(T) \leq \frac{1}{r}(1 + \frac{1}{D})$.

(ii) Let $0 < d_1 < d_2 < \ldots$ be an increasing sequence of integers. Let T be a tree in which every vertex on level i has degree d_i. We can repeat the argument from case (i), except now we have $|V'_n|/n \to 0$ as $n \to \infty$ and thus $\frac{|V(T) \cap [n]|}{n} \leq \frac{\lceil \frac{n}{r} \rceil + |V'_n|}{n} \xrightarrow{n \to \infty} \frac{1}{r}$. □

Note that when $r = 2$, there are connected graphs G in which every vertex has infinite degree but $\overline{\mathrm{Rd}}_2(G) = \overline{\mathrm{Rd}}(G) \geq 1/2$ (Theorem 1.13, for instance). The following example shows that there is an unexpected change in behavior as we go from 2 colors to 3 colors.

Example 2.3. Let $r \in \mathbb{N}$ with $r \geq 3$. If G is a graph with finitely many components and finitely many vertices of finite degree, then $\overline{\mathrm{Rd}}_r(G) = 0$.

Proof. Let $\epsilon > 0$ be given, let c be the number of components of G, and let k be an integer with $k > c/\epsilon$. Partition $\mathbb{N}$ by residues mod k – that is, $\mathbb{N} = A_0 \cup \ldots \cup A_{k-1}$, where $A_i = \{n \in \mathbb{N} : n \equiv i \pmod{k}\}$. For all $0 \leq i \leq k-1$, color all edges inside A_i with green, and for all $0 \leq i < j \leq k-1$, color the edges between A_i and A_j with the half graph coloring where the vertices in A_i have cofinite red degree to A_j and the vertices in A_j have cofinite blue degree to A_i. Note that we have only used three colors, but this can be considered as an r-coloring for all $r \geq 3$.

Note that for all $0 \leq i \leq k-1$, every vertex in A_i has finite red degree to $A_0 \cup \cdots \cup A_i$. If there is a red copy of G, then let $0 \leq i \leq k-1$ be maximum such that $V(G) \cap A_i$ is infinite. But this is a contradiction because every vertex in $V(G) \cap A_i$ has finite red degree. Similarly, for all $0 \leq i \leq k-1$, every vertex in A_i has finite blue degree to $A_i \cup \cdots \cup A_{k-1}$. If there is a blue copy of G, then let $0 \leq i \leq k-1$ be minimum such that $V(G) \cap A_i$ is infinite. But this is a contradiction because every vertex in $V(G) \cap A_i$ has finite blue degree. Therefore, every monochromatic copy of G is green and thus has upper density at most $c/k < \epsilon$. □

Example 2.4. For every nontrivial connected graph G, $\overline{\mathrm{Rd}}(G) \leq \frac{1}{\chi(G)-1}$. In particular, if $\chi(G) = \infty$, then $\overline{\mathrm{Rd}}(G) = 0$.

Proof. Assume first that $\chi(G) < \infty$ and partition $\mathbb{N}$ by residues mod $\chi(G) - 1$. Color all edges inside the sets red and all edges between the sets blue. There is no blue copy of G, so every copy of G lies entirely inside one of the sets, all of which have density $\frac{1}{\chi(G)-1}$.

If $\chi(G) = \infty$, this construction shows that $\overline{\mathrm{Rd}}(G) \leq 1/(k-1)$ for every $k \geq 2$, and therefore, $\overline{\mathrm{Rd}}(G) = 0$. □

Corollary 2.5.

(i) $\overline{\mathrm{Rd}}(\mathcal{R}) = 0$ *(where $\mathcal{R}$ is the Rado graph).*
(ii) *There exists a locally finite graph G such that* $\overline{\mathrm{Rd}}(G) = 0$.

Proof. (i) Since $\mathcal{R}$ contains an infinite clique, we have $\chi(\mathcal{R}) = \infty$, and thus, the result follows from Theorem 2.4.

(ii) Let G be a graph on vertex set $\mathbb{N}$ where $[\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}]$ induces a clique for all $n \in \mathbb{N}$. G is locally finite, connected, and contains a clique of order n for all $n \in \mathbb{N}$. So $\chi(G) = \infty$, and thus, the result follows from Theorem 2.4. □

Example 2.6. For all $r \in \mathbb{N}$, there is an r-coloring of $K_{\mathbb{N},\mathbb{N}}$ in which every monochromatic path has upper density at most $1/r$. In particular, it is not possible to cover all but finitely many vertices with less than r monochromatic paths.

Proof. Let A and B be the parts of $K_{\mathbb{N},\mathbb{N}}$ and partition both of them into r parts $A_1, \ldots, A_r$ and $B_1, \ldots, B_r$, each of density $1/(2r)$. For all $i, j \in [r]$, color every edge between A_i and B_j by $(i - j)$ mod r. It is easy to see that every part is incident to exactly one other part of each color, and therefore, every monochromatic path can cover at most two parts, finishing the proof. □

2.3. Lower density

As mentioned in the introduction, Erdős and Galvin proved that for all positive integers Δ, there exists a 2-coloring of $K_{\mathbb{N}}$ such that if G is a graph with maximum degree at most Δ and finitely many isolated vertices, then every monochromatic copy of G has lower density 0. We now show that a broader class of graphs has this property.

Recall that a graph G is weakly expanding if for all $k \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ such that for all independent sets A in G with $|A| \geq \ell$, we have $|N(A)| > k$. Note that if G is weakly expanding, then there is an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, if A is an independent set in G with $|A| \geq f(k)$, then $|N(A)| > k$. Also note that if G is weakly expanding, then G has finitely many isolated vertices. To better understand this definition, we collect some useful properties which imply that that a graph is weakly expanding.

Fact 2.7. G is weakly expanding if

(i) G has finite independence number, or
(ii) G has bounded maximum degree and finitely many isolated vertices, or
(iii) G is a tree with bounded leaf degree, or
(iv) for all $n \in \mathbb{N}$, G has finitely many vertices of degree n.

The following is a modification of the example used by Erdős and Galvin [13] to prove the result mentioned about about graphs with bounded maximum degree and finitely many isolated vertices.

Example 2.8 (Forward interval coloring). If G is a graph which is weakly expanding, then G is not Ramsey-lower-dense.

We note that the forthcoming Theorem 8.7 shows that if G is strongly contracting, then there is a confinite monochromatic copy of G in every forward interval coloring.

Proof. Suppose G is weakly expanding and let f be the function guaranteed by the definition.

Let a_n be an increasing sequence of natural numbers such that $a_0 = 1$ and for all $k \geq 1$,

$$a_k > k(a_{k-1} + f(a_{k-1})). \tag{2.1}$$

For all $u, v \in \mathbb{N}$ with $u < v$, color the edge uv red if $u \in [a_{2n-1}, a_{2n})$ and blue if $u \in [a_{2n}, a_{2n+1})$ for some $n \in \mathbb{N}$.

Suppose there is a, say, blue copy of G in this 2-coloring with vertex set U. We must have that $A_n := U \cap [a_{2n-1}, a_{2n})$ induces an independent set for all $n \in \mathbb{N}$, and because of the coloring, we have $N_B(A_n) \subseteq [0, a_{2n-1})$, and thus, $|N_B(A_n)| \leq a_{2n-1}$. Thus, by the definition of weakly expanding, we have $|A_n| < f(a_{2n-1})$. We conclude that for all $n \in \mathbb{N}$,

$$|U \cap [0, a_{2n})| = |U \cap [0, a_{2n-1})| + |A_n| < a_{2n-1} + f(a_{2n-1}) \overset{2.1}{<} \frac{1}{2n} a_{2n},$$

and thus, $\underline{d}(U) = 0$. □

We conclude with two more examples.

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Example 2.9 (Backward interval coloring). If G is a graph with a finite dominating set (i.e., $\mathrm{rul}(G) > 0$), then G is not Ramsey-lower-dense.

We note that the forthcoming Theorem 3.6 shows that if G has no finite dominating set, then there is a spanning monochromatic copy of G in every backwards interval coloring.

Proof. Let a_n be an increasing sequence of natural numbers and let $A_i = [a_i, a_{i+1})$ for all $i \in \mathbb{N}$. For all $u \in A_i$ and $v \in A_j$ with $u < v$, color the edge uv red if j is odd and blue if j is even. Let A^0 be the union of all even indexed intervals and let A^1 be the union of all odd indexed intervals. We note that every vertex in A^0 has finite blue degree to A^1 and every vertex in A^1 has finite red degree to A^0.

Let D be a finite dominating set in G and suppose there is a monochromatic, say, blue copy of G with vertex set V. Since D is finite, there exists an index t such that $D \subseteq A_1 \cup A_2 \cup \cdots \cup A_t$. Now for all i such that $2i+1 > t$, there are no blue edges from A_{2i+1} to D contradicting the fact that D is a dominating set. So G has finite intersection with say A^1, and thus, if a_n is increasing fast enough, G has lower density 0. □

Example 2.10. Let G be a connected graph. If $\chi(G) \geq 3$, or G is bipartite with one part finite, then G is not Ramsey-lower-dense.

Proof. Let $\{X, Y\}$ be a partition of $\mathbb{N}$ into two sets of lower density 0 (for instance, as we did in Example 2.8 and Example 2.9). Color all edges inside X or inside Y with blue, and color all edges between X and Y red. Note that since G is connected, any blue copy of G is completely contained in X or Y and thus has lower density 0.

If $\chi(G) \geq 3$, then there is no red copy of G, and we are done. If G is bipartite and one of the parts is finite, then G intersects either X or Y in only finitely many vertices, and thus, any red copy of G will have lower density 0. □

2.4. The Rado graph, 0-ruled and 0-coruled graphs

If G and H are two graphs, then we write $G \preceq H$ if G is isomorphic to a spanning subgraph of H. Clearly, $\preceq$ is reflexive and transitive.

We say that an infinite graph G has the *extension property* if for every pair of disjoint finite sets $F, F' \subseteq V(G)$, there is a vertex $v \in V(G) \setminus (F \cup F')$ such that v is adjacent to every $w \in F$ and not adjacent to any $w' \in F'$. The following well-known theorem (see [4]) shows why this property is useful.

Theorem 2.11. *Any two infinite graphs satisfying the extension property are isomorphic.*

Furthermore, it is not hard to see that the Rado graph $\mathcal{R}$ and (with probability 1) the infinite random graph (every edge is present independently with probability 1/2) both satisfy the extension property. Hence, with probability 1, the infinite random graph is isomorphic to the Rado graph.

Observe that G is 0-ruled if and only if G satisfies the 'non-adjacency' half of the extension property above (i.e., if for every finite $F' \subseteq V(G)$ there is a vertex $v \in V(G) \setminus F'$ such that v is not adjacent to any $w' \in F'$). We will call G *0-coruled* if G satisfies only the 'adjacency' half of extension property (i.e., for every finite $F \subseteq V(G)$, there is a $v \in V(G) \setminus F$ such that v is adjacent to every $w \in F$). Using this, it is easy (and very similar to the proof of Theorem 2.11) to prove the following proposition.

Proposition 2.12. *G is 0-ruled if and only if $G \preceq \mathcal{R}$ and G is 0-coruled if and only if $\mathcal{R} \preceq G$.*

Note that for finite graphs, $G \preceq H$ and $H \preceq G$ imply $G \cong H$, and thus, $\preceq$ is a partial order (on isomorphism classes of graphs), but this is not the case for infinite graphs. A simple example is letting G be an infinite clique together with infinitely many disjoint copies of some finite graph F and letting H be two disjoint infinite cliques together with infinitely many disjoint copies of some finite graph F). Another example comes from the fact that the infinite half graph is both 0-ruled and 0-coruled, but is not isomorphic to $\mathcal{R}$. We ask the following question out of curiosity.[2]

[2]We asked this question on MathOverflow and received evidence [24] suggesting that there may not be a simple answer.

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Problem 2.13. Under what conditions on G and H does $G \preceq H$ and $H \preceq G$ imply that $G \cong H$?

The *Rado coloring* of $E(K_{\mathbb{N}})$ is the 2-coloring ρ defined by setting $\rho(\{s,t\})$ to be the sth bit in the binary expansion of t for all $s, t \in \mathbb{N}$ with $s < t$. For instance, $\rho(\{2,14\}) = 1$ since the 2nd bit (reading right to left) in the binary expansion of 14 is 1, and $\rho(\{5,14\}) = 0$ since the 5th bit (reading right to left, and appending extra 0's to the left as necessary) in the binary expansion of 14 is 0. Also note that the Rado coloring can be described by coloring all of the edges of the Rado graph with color 1 and coloring all of the edges in the complement of the Rado graph with color 0.

The key property of the Rado coloring is that for any $F \subseteq \mathbb{N}$ and $i \in \{0,1\}$, we have

$$d\left(\bigcap_{v \in F} N_i(v)\right) = 2^{-|F|}. \tag{2.2}$$

We first use the Rado coloring to make the following observation about complete multipartite graphs.

Proposition 2.14. *Let K be an infinite complete multipartite graph and let $n \in \mathbb{N}$.*

(i) *If K has at least two infinite parts, or infinitely many vertices in finite parts, then K is not Ramsey-dense.*
(ii) *If K has exactly one infinite part and exactly n vertices in finite parts, then*

$$\frac{1}{2^{2n-1}} \leq \overline{\mathrm{Rd}}(K) \leq \frac{1}{2^n}.$$

Proof. Take the Rado coloring of $K_{\mathbb{N}}$.

If K has at least two infinite parts, or infinitely many vertices in finite parts, then K contains a spanning copy of $K_{\mathbb{N},\mathbb{N}}$; let (A,B) be such a spanning copy of $K_{\mathbb{N},\mathbb{N}}$. Let $a_1, a_2, \ldots$ be the elements of A. Then B is contained in the neighborhood of $a_1, \ldots, a_n$, and hence has density $\leq 2^{-n}$, for each n, by (2.2). Hence, B must have density 0. The same goes for A.

Now suppose K has exactly one infinite part and exactly n vertices in finite parts. Then by (2.2), we have $\overline{\mathrm{Rd}}(K) \leq \frac{1}{2^n}$ since the infinite part is the intersection of the neighborhoods of the n vertices in finite parts.

To see $\overline{\mathrm{Rd}}(K) \geq \frac{1}{2^{2n-1}}$, we are given an arbitrary 2-coloring of $K_{\mathbb{N}}$, and we choose an arbitrary vertex v. Either $\overline{d}(N_R(v)) \geq 1/2$ or $\overline{d}(N_B(v)) \geq 1/2$, and we choose the color with largest upper density. We repeat this process inside the chosen neighborhood for $2n-1$ steps, at which point we have n vertices whose common neighborhood in color, say, red has upper density at least $\frac{1}{2^{2n-1}}$ and the infinite part of K can be embedded in such a way that it spans this set. □

The following result suggests that the question of whether G is Ramsey-dense or not may depend on the rate of growth of the ruling sets in G.

Theorem 2.15. *Let G be an infinite graph. If G has pairwise-disjoint ruling sets F_n ($n \in \mathbb{N}$) satisfying $|F_n| \leq \log_2(n)$ for all sufficiently large n, then G is not Ramsey-dense.*

Proof. Consider the Rado coloring of $K_{\mathbb{N}}$. Suppose now that V is the vertex set of a monochromatic copy of G, say with color i. Then for each N, we have

$$V \subseteq^* \bigcap_{n=1}^{N} \bigcup_{v \in F_n} N_i(v).$$

Note that

$$d\left(\bigcap_{n=1}^{N} \bigcup_{v \in F_n} N_i(v)\right) = \prod_{n=1}^{N}(1 - 2^{-|F_n|}).$$

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Hence,

$$\overline{\mathrm{d}}(V) \leq \prod_{n=1}^{\infty}(1-2^{-|F_n|}).$$

It is well known that an infinite product $\prod_{n=1}^{\infty} \alpha_n$, with $\alpha_n \in (0,1)$, converges to 0 if and only if

$$\sum_{n=1}^{\infty} \log(\alpha_n) = -\infty.$$

In our case, we have $|F_n| \leq \log_2(n)$ for all sufficiently large n, so

$$\log(1-2^{-|F_n|}) \leq \log\left(1-\frac{1}{n}\right) \leq -1/n.$$

By the limit comparison test and the divergence of the harmonic series, we have $d(V) = 0$. □

3. Ultrafilters and embedding

The concept of ultrafilters will play an important role in this paper.

Definition 3.1. Given a set X, a set system $\mathscr{U} \subseteq 2^X$ is called an *ultrafilter* if

(i) $X \in \mathscr{U}$ and $\emptyset \notin \mathscr{U}$,
(ii) If $A \in \mathscr{U}$ and $A \subseteq B \subseteq X$, then $B \in \mathscr{U}$,
(iii) If $A, B \in \mathscr{U}$, then $A \cap B \in \mathscr{U}$ and
(iv) For all $A \subseteq X$, either $A \in \mathscr{U}$ or $X \setminus A \in \mathscr{U}$, or
(iv)′ $\mathscr{U}$ is maximal among all families satisfying (i) - (iii).

A family satisfying (i)–(iii) is called a *filter*. Conditions (iv) and $(iv)'$ are equivalent for filters (see [18, Chapter 11, Lemma 2.3]), and we will make use whichever is more convenient for the current application. Let us list some additional properties of ultrafilters.

Proposition 3.2. *If $\mathscr{U}$ is an ultrafilter on X, we have*

(i) *If $A_1, \ldots, A_n \in \mathscr{U}$, then $A_1 \cap \ldots \cap A_n \in \mathscr{U}$.*
(ii) *If $A_1, \ldots, A_n$ are pairwise disjoint and $A_1 \cup \ldots \cup A_n \in \mathscr{U}$, then there is exactly one $i \in [n]$ with $A_i \in \mathscr{U}$.*

Informally, we think of sets $A \in \mathscr{U}$ as 'large' sets. A common example of an ultrafilter is the so-called trivial ultrafilter $\mathscr{U}_x := \{A \subseteq X : x \in A\}$ for $x \in X$. It is not hard to see that an ultrafilter is trivial if and only if it contains a finite set.

We say that an ultrafilter $\mathscr{U}$ on $\mathbb{N}$ is *positive* if every set $A \in \mathscr{U}$ has positive upper density in $\mathbb{N}$. Positive ultrafilters play a crucial role in the proof of Theorem 1.6.

Proposition 3.3. *If $X \subseteq \mathbb{N}$ is infinite, then there exists a nontrivial ultrafilter $\mathscr{U}$ on X. There exists a positive ultrafilter $\mathscr{U}$ on $\mathbb{N}$.*

Proof. To prove the first part of the theorem, apply Zorn's lemma to

$$\{\mathcal{F} \subseteq 2^{\mathbb{N}} : \mathcal{F} \text{ contains all cofinite sets and satisfies (i) - (iii) in Theorem 3.1}\}$$

to get a maximal such family $\mathscr{U}$, which must be an ultrafilter. Finally, if A is finite, $\mathscr{U}$ contains the cofinite set A^c, and hence, $A \notin \mathscr{U}$.

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To prove the second part, apply Zorn's lemma to

$$\{\mathcal{F} \subseteq 2^{\mathbb{N}} : \mathcal{F} \text{ contains all sets of lower density 1 and satisfies (i) - (iii) in Theorem 3.1}\}$$

to get a maximal such family $\mathcal{U}$, which must be an ultrafilter. Furthermore, if $A \subseteq \mathbb{N}$ has upper density 0, then $\mathbb{N} \setminus A$ has lower density 1 (see (1.1)) and consequently $A \notin \mathcal{U}$. □

Definition 3.4 (Vertex-coloring induced by $\mathcal{U}$). Let $r \geq 2$ be an integer and suppose the edges of an infinite graph G are colored with r colors. Let $\mathcal{U}$ be a nontrivial ultrafilter on $V(G)$. Define a coloring $c_{\mathcal{U}} : V(G) \to [r]$ where $c_{\mathcal{U}}(v) = i$ if and only if $N_i(v) \in \mathcal{U}$. Since $V(G) \setminus \{v\} \in \mathcal{U}$ for all $v \in V(G)$, it follows from Theorem 3.2 (ii) that $c_{\mathcal{U}}$ is well defined. We call $c_{\mathcal{U}}$ the vertex-coloring induced by $\mathcal{U}$.

The following two propositions allow us to use ultrafilters to embed the desired subgraphs in the proof of Theorem 1.2 and Theorem 1.6.

Proposition 3.5. *Let $k \geq 2$ be an integer, let G be a one-way k-locally finite graph and let H be a graph such that $\{U_1, \ldots, U_k\}$ is a partition of $V(H)$ with $|U_1| = \cdots = |U_k| = \infty$ and for all $i \in [k]$ and any finite subset $W \subseteq U_1 \cup \cdots \cup U_{i-1}$, the set of common neighbors of W in U_i is infinite. Then, there is an embedding f of G into H such that $U_1 \subseteq \operatorname{ran} f$.*

Given a k-partite graph G with parts $V_1, \ldots, V_k$ and a set $S \subseteq V(G)$, the *left neighborhood cascade* of S is the tuple $(S_1, \ldots, S_k)$, where $S_k = S \cap V_k$, and for all $1 \leq i \leq k-1$, $S_i = (S \cup \bigcup_{j=i+1}^{k} N(S_j)) \cap V_i$.

Proof. Let $V_1 \cup V_2 \cup \cdots \cup V_k$ be a partition of $V(G)$ into independent sets which witness the fact that G is one-way k-locally-finite (in particular, V_1 is infinite). We will construct an embedding f iteratively in finite pieces. Initially, f is the empty embedding. Then, for each $n \in \mathbb{N}$, we will proceed as follows: let

$$S_n = \{\min(V_i \setminus \operatorname{dom} f) : i \in [k] \text{ with } V_i \setminus \operatorname{dom} f \neq \emptyset\}.$$

That is, S_n contains the smallest not yet embedded vertex of each V_i which is not completely embedded yet. Let $(T_{1,n}, \ldots, T_{k,n})$ be left neighborhood cascade of S_n in G. We will now extend f to cover $\bigcup_{i\in[k]} T_{i,n}$. Observe that $T_{i,n}$ is disjoint from $\operatorname{dom} f$ for all $i \in [k]$ since we embedded the whole left neighborhood cascade in every previous step. Since V_1 is infinite, $T_{1,n}$ is nonempty. Let $T'_{1,n} \subseteq U_1 \setminus \operatorname{ran} f$ be the set of $|T_{1,n}|$ smallest vertices in $U_1 \setminus \operatorname{ran} f$ and extend f by embedding $T_{1,n}$ into $T'_{1,n}$ arbitrarily. By assumption, $T'_{1,n}$ has infinitely many common neighbors in U_2. Since $\operatorname{ran} f$ is finite, we can select a set $T'_{2,n} \subseteq (U_2 \cap N^{\cap}(T'_{1,n}) \setminus \operatorname{ran} f$ of size $|T_{2,n}|$. Extend f by embedding $T_{2,n}$ into $T'_{2,n}$ arbitrarily. Similarly, we can extend f by embedding $T_{i,n}$ into appropriate sets $T'_{i,n}$ for all $i = 3, \ldots, k$.

Since we maintain a partial embedding of G into H throughout the process and every vertex of G will eventually be embedded (by choice of S_n which contains the smallest not yet embedded vertex of $V(G)$), the resulting function f defines an embedding of G into H. Since we cover the smallest not-yet covered vertex of U_1 in each step, we further have $U_1 \subseteq \operatorname{ran} f$. □

Proposition 3.6. *Let H be a graph having the property that for every finite set of vertices $W \subseteq V(H)$, the set of common neighbors of W is infinite. If G is an infinite 0-ruled graph, then there is a surjective embedding of G into H.*

Proof. Let $v_1, v_2, \ldots$ be an enumeration of $V(G)$ and let $u_1, u_2, \ldots$ be an enumeration of $V(H)$. Let $f(v_1) = u_1$. Now suppose $\operatorname{dom} f = \{v_1, \ldots, v_n\}$ for some $n \in \mathbb{N}$. Let u_{i_n} be the vertex of smallest index in $V(H) \setminus \operatorname{ran} f$. Since G is 0-ruled, there exists a vertex v_p with $p > n$ such that v_p has no neighbors in $\{v_1, \ldots, v_n\}$. We set $f(v_p) = u_{i_n}$, and if $p > n+1$, we do the following for all $n+1 \leq i \leq p-1$: since $\{f(v_1), \ldots, f(v_{i-1}), f(v_p)\}$ has infinitely many common neighbors, we may choose a vertex $u \in V(H) \setminus \operatorname{ran} f$ which is adjacent to every vertex in $\{f(v_1), \ldots, f(v_{i-1}), f(v_p)\}$ and set $f(v_i) = u$. Continuing in this way, we obtain an embedding of G into H. Since on each step, the vertex of smallest index in $V(H) \setminus \operatorname{ran} f$ becomes part of the range of f, the embedding is surjective. □

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4. Graphs of bounded chromatic number

In this section, we will prove Theorem 1.2. First, note that if G is one-way k-locally-finite, then G is 0-ruled.

Proof of Theorem 1.2. (i) We are given an infinite one-way 2-locally-finite graph G and an r-coloring of the edges of $K_{\mathbb{N}}$. Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$. Let $c_{\mathcal{U}}$ be the vertex-coloring induced by $\mathcal{U}$, and for all $i \in [r]$, let A_i be the set of vertices receiving color i. We may suppose without loss of generality that $\overline{\mathrm{d}}(A_1) \geq 1/r$ (see (1.1)). If $A_1 \in \mathcal{U}$, then the set of common neighbors of S in A_1 of color 1 is infinite for every finite set $S \subseteq A_1$. Thus, we can apply Proposition 3.6 to embed G in color 1 in such a way that A_1 is covered. If $A_1 \notin \mathcal{U}$, then every finite set $S \subseteq A_1$ has infinitely many common neighbors of color 1 in A_1^c. Hence, by applying Proposition 3.5, we can find a monochromatic copy of G in color 1 such that A_1 is covered. Either way, we have a monochromatic copy of G of upper density at least $\overline{\mathrm{d}}(A_1) \geq 1/r$.

(ii) We are given an infinite one-way k-locally-finite graph G and an 2-coloring of the edges of $K_{\mathbb{N}}$. Let $\mathcal{U}_1$ be a nontrivial ultrafilter on $\mathbb{N}$. Let $c_{\mathcal{U}_1}$ be the vertex-coloring induced by $\mathcal{U}_1$, and for all $i \in [2]$, let $A_{1,i}$ be the set of vertices receiving color i. Choose $i_1 \in [2]$ so that $A_{1,i_1} \in \mathcal{U}_1$ and let $i_1' = 3 - i_1$. If $A_{1,i_1'}$ is finite, then stop; otherwise, let $\mathcal{U}_2$ be a nontrivial ultrafilter on $W_2 = A_{1,i_1'}$ and let $c_{\mathcal{U}_2}$ be the vertex-coloring of W_2 induced by $\mathcal{U}_2$. For all $i \in [2]$, let $A_{2,i}$ be the set of vertices receiving color i. Choose i_2 so that $A_{2,i_2} \in \mathcal{U}_2$ and let $i_2' = 3 - i_2$. Let $W_3 := A_{2,i_2'}$ and continue in this manner until the point at which (a) $A_{t,i_t'}$ is finite for some t, or (b) there exists t and $j \in [2]$ such that there exists a set $J \subseteq [t]$ where $|J| = k - 1$ and $A_{j,i_j} \in \mathcal{U}_j$ for all $j \in J$. Note that by pigeonhole, we must have $t \leq 2k - 3$ in either case.

If we are in case (a), then by (1.1), one of the sets $A_{1,i_1}, A_{2,i_2}, \ldots, A_{t,i_t}$ has upper density at least $\frac{1}{t}$, say A_{j,i_j}. Now by applying Theorem 3.6 with color i_j, we get the desired monochromatic copy of G covering A_{j,i_j} with upper density at least $\frac{1}{t} \geq \frac{1}{2k-3} > \frac{1}{2(k-1)}$. If we are in case (b), suppose without loss of generality that $j = 1$. Set $W_{t+1} := W_t \setminus A_{t,1}$. Since $\{A_{1,1}, A_{2,1}, \ldots, A_{t,1}, W_{t+1}\}$ is a partition of $\mathbb{N}$, we have by (1.1) that one of the sets $A_{1,1}, A_{2,1}, \ldots, A_{t,1}, W_{t+1}$ has upper density at least $\frac{1}{2(k-1)}$. If, say, $\overline{\mathrm{d}}(A_{\ell,i_\ell}) \geq \frac{1}{2(k-1)}$ for some $\ell \in [t]$, then applying Proposition 3.6 with color i_ℓ gives the desired monochromatic copy of G covering A_ℓ; otherwise, $\overline{\mathrm{d}}(W_{t+1}) \geq \frac{1}{2(k-1)}$, and applying Proposition 3.5 with color 2 gives the desired monochromatic copy of G covering W_{t+1}.

(iii) The process is very similar to (ii), in that we repeatedly choose ultrafilters until the leftover vertices are finite, or we are guaranteed that some color appears $k - 1$ times from every set at the end of the process (see Figure 3). However, the formal proof is a bit more technical.

We will use the following notation. Given $i_1, i_2 \in \mathbb{N}$, and $L_1 \in \mathbb{N}^{i_1}$ and $L_2 \in \mathbb{N}^{i_2}$, we write $L_1 \prec L_2$ if L_1 is an initial segment of L_2. Furthermore, given $L = (j_1, \ldots, j_i) \in \mathbb{N}^i$ for some $i \in \mathbb{N}$, we define $L^- := (j_1, \ldots, j_{i-1})$.

Suppose the edges of $K_{\mathbb{N}}$ are colored with r colors and let $q = (k-2)r + 1$. We will define sets A_L for $L \in \bigcup_{i=0}^{q} [r-1]^i$ and colorings $\chi_1 : \{A_L : L \in \bigcup_{i=0}^{q-1} [r-1]^i\} \to [r]$ and $\chi_2 : \bigcup_{i=1}^{q} [r-1]^i \to [r]$ with the following properties.

(a) The sets A_L, $L \in \bigcup_{i=0}^{q} [r-1]^i$, are pairwise disjoint, and their union is cofinite.
(b) For every $L \in \bigcup_{i=1}^{q} [r-1]^i$, A_L is empty or every finite set $S \subseteq A_L$ has infinitely many common neighbors of color $\chi_1(A_L)$ in A_L.
(c) For every $L \in \bigcup_{i=1}^{q} [r-1]^i$, A_L is empty or every finite set $S \subseteq \bigcup_{L \prec L'} A_{L'}$ has infinitely many common neighbors of color $\chi_2(L)$ in A_{L^-}.

We will construct these sets and colorings recursively. In the process, we will also construct sets B_L and ultrafilters $\mathcal{U}_L$ on B_L for every $L \in \bigcup_{i=0}^{q} [r-1]^i$.

Let $B_{()} = \mathbb{N}$ and let $\mathcal{U}_{()}$ be a nontrivial ultrafilter on $B_{()}$, where $()$ denotes the empty sequence. Let $c_{\mathcal{U}_{()}}$ be the vertex-coloring induced by $\mathcal{U}_{()}$. Let c be the color so that $A_{()}$, the set of vertices of color c, is in $\mathcal{U}_{()}$ and let $\chi_1(A_{()}) = c$. Let $[r] \setminus \{c\} = \{j_1, \ldots, j_{r-1}\}$ and, for $i \in [r-1]$, let $B_{(i)}$ be the set of vertices receiving color j_i and let let $\chi_2((i)) = j_i$.

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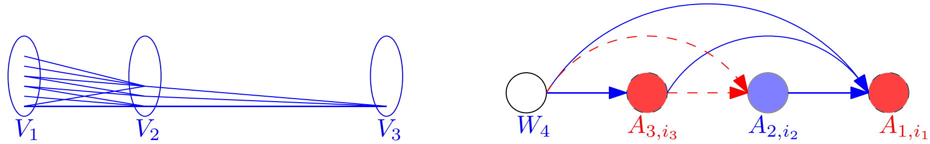

(a) A one-way 3-locally-finite graph G

(b) An arrow from X to Y of color i indicates that any finite set of vertices in X has infinitely many common neighbors in Y of color i. If X is filled with color i, then any finite set of vertices in X has infinitely many common neighbors in X of color i.

Figure 2. *An example of the proof of Theorem 1.2.(ii). In this example, G will be embedded in blue into* $W_4 \cup A_{3,i_3} \cup A_{1,i_1}$ *such that* $W_4 \subseteq V(G)$.

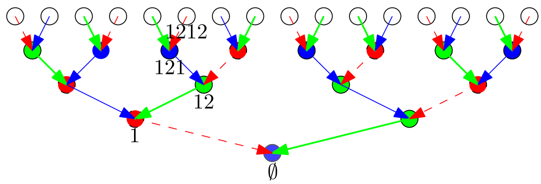

Figure 3. *An example of the proof of Theorem 1.2.(iii) with* $r = 3$ *and* $k = 3$. *Here, we have highlighted the sequence* $A_{(1,2,1,2)}, A_{(1,2,1)}, A_{(1,2)}, A_{(1)}, A_\emptyset$ *and note that some color, in this case red, must appear at least twice, which means we can embed G into* $A_{(1,2,1,2)} \cup A_{(1,2,1)} \cup A_\emptyset$ *in such a way that* $A_{(1,2,1,2)}$ *is covered.*

In the next step, we proceed as follows for every $i_0 \in [r-1]$. If $B_{(i_0)}$ is finite, let $A_{(i_0)} = B_{(i_0,i)} = \emptyset$ for every $i \in [r-1]$. Otherwise, let $\mathscr{U}_{(i_0)}$ be a nontrivial ultrafilter on $B_{(i_0)}$ and let $c_{\mathscr{U}_{(i_0)}}$ be the vertex-coloring induced by $\mathscr{U}_{(i_0)}$. Let c be the color so that $A_{(i_0)}$, the set of vertices of color c, is in $\mathscr{U}_{(i_0)}$ and let $\chi_1(A_{(i_0)}) = c$. Let $[r] \setminus \{c\} = \{j_1, \dots, j_{r-1}\}$ and, for $i \in [r-1]$, let $B_{(i_0,i)}$ be the set of vertices receiving color j_i and let $\chi_2((i_0, i)) = j_i$.

We proceed like this until we define the sets B_L for every $L \in [r-1]^q$ and let $A_L := B_L$ for all $L \in [r-1]^q$. It is easy to see from the ultrafilter properties that the above properties hold.

Therefore, for every $L \in \bigcup_{i=0}^{q-1} [r-1]^i$, A_L is empty or can be covered by a monochromatic copy of G by Theorem 3.6. Furthermore, for every $L \in [r-1]^q$ for which A_L is nonempty, we find $k-1$ sets $L_1 \prec \dots \prec L_{k-1} \prec L$ of the same color w.r.t. χ_2 by the pigeonhole principle. Therefore, applying Theorem 3.5 to $U_k := A_{L_1^-}, \dots, U_2 := A_{L_{k-1}^-}, U_1 := A_L$, we find a monochromatic copy of G covering A_L. Since, there are $C := \sum_{i=0}^{q} (r-1)^i$ sets A_L, one of them has upper density at least $1/C$. □

Let G be a graph with $\Delta := \Delta(G) < \infty$. Since $\chi(G) \le \Delta(G) + 1$, we immediately obtain as a corollary that $\overline{\mathrm{Rd}}(G) \ge \frac{1}{2\Delta}$. However, with a bit more work, we obtain the following corollary.

Corollary 4.1. *Let G be an infinite graph. If* $2 \le \Delta := \Delta(G) < \infty$, *then* $\overline{\mathrm{Rd}}(G) \ge \frac{1}{2(\Delta-1)}$.

First, we note the following fact (this result also appears in [23, Theorem 1(i)]).

Proposition 4.2. *Let* $r \in \mathbb{N}$. *If G is a graph with infinitely many components, then* $\overline{\mathrm{Rd}}_r(G) \ge 1/r$.

Proof. Let G be a graph with infinitely many components and note that by merging components if necessary, we may assume that G has infinitely many components, each of which has infinitely many vertices.

By Ramsey's theorem, it is possible to partition any r-colored $K_\mathbb{N}$ into monochromatic infinite cliques and a finite set. Indeed, greedily take disjoint monochromatic copies of $K_\mathbb{N}$ in which the smallest vertex is minimal. Either the process ends with a finite set of uncovered vertices, or the process continues for infinitely many steps and the union misses infinitely many vertices. However, now there

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is a monochromatic copy of $K_{\mathbb{N}}$ whose minimal vertex must be smaller than one of the monochromatic cliques in our collection, a contradiction.

Without loss of generality, suppose the cliques of color 1 have upper density at least $1/r$. Since G has infinitely many components, G can be surjectively embedded into the cliques of color 1. □

Proof of Theorem 4.1. First, note that if G has infinitely many components, then we are done by Theorem 4.2. If $\chi(G) \leq \Delta$, then we are done by Theorem 1.2; so suppose that G has finitely many components and $\chi(G) = \Delta + 1$. Now by Brooks theorem, either $\Delta = 2$ and G contains finitely many components which are odd cycles, or $\Delta \geq 3$ and G contains finitely many components which are cliques on $\Delta + 1$ vertices. Note that in either case, every infinite component of G (of which there is at least one) has chromatic number at most Δ. Let $V_2 \subseteq V(G)$ be the vertex-set of the finitely many components which are odd cycles or cliques of size $\Delta + 1$, and let $V_1 = V(G) \setminus V_2$.

We are given a 2-coloring of $K_{\mathbb{N}}$. If there is a red clique R and a blue clique B each of size $|V_2|$, we can apply Theorem 1.2 to $G[V_1]$ (which is one-way Δ-locally finite) and $K_{\mathbb{N}}[(R \cup B)^c]$ to get a monochromatic copy of $G[V_1]$ of upper density at least $\frac{1}{2(\Delta-1)}$. Together with either R or B, this gives the desired copy of G.

So suppose that there is no, say, red clique of order $|V_2|$. If $\Delta \geq 3$, we repeat the proof of Theorem 1.2(ii); however, in each iteration, $i_j = 1$ (here, blue is 1 and red is 2); otherwise, there would be an infinite red clique. Thus, we can stop when $t = \chi(G) - 1 \leq \Delta$ and get a monochromatic copy of G of upper density at least $\frac{1}{\Delta+1} \geq \frac{1}{2(\Delta-1)}$. Finally, if $\Delta = 2$, we repeat the proof of Theorem 1.2(ii), but after the first step, we have $A_{1,1} \in \mathscr{U}_1$ and $W_2 = A_{1,2}$. If $\overline{\mathrm{d}}(A_{1,1}) \geq 1/2$, then we are done as usual. So suppose $\overline{\mathrm{d}}(A_{1,2}) \geq 1/2$. If there is an infinite red matching in $A_{1,2}$, then these edges can be used to make the odd cycles comprising V_2 and then V_1 can be embedded as usual. Otherwise, $A_{1,2}$ does not contain an infinite red matching, and thus, there is a cofinite subset of $A_{1,2}$ which induces a blue clique into which we can embed G. □

Finally, we note the following strengthening of Theorem 1.2 which generalizes a result of Elekes, D. Soukup, L. Soukup and Szentmiklóssy [12] who proved a similar statement for powers of cycles.

Theorem 4.3. *Let $k, r \in \mathbb{N}$ and let G be a one-way k-locally finite graph. In every r-coloring of the edges of $K_{\mathbb{N}}$, there exists a collection of*

$$f(r,k) = \begin{cases} r & \text{if } k = 2 \\ \sum_{i=0}^{(k-2)r+1} (r-1)^i & \text{if } k \geq 3 \end{cases}$$

vertex-disjoint, monochromatic copies of G whose union covers all but finitely many vertices.

Remark 4.4. The proof of Theorem 1.2 immediately shows that for every one-way k-locally finite graph G and every r-colored $K_{\mathbb{N}}$, there is a collection of at most $f(r,k)$ monochromatic copies of G *covering* a cofinite subset of $\mathbb{N}$, where $f(r,k)$ is as in the statement of Theorem 4.3. In order to obtain a *partition* as required by Theorem 4.3, we need to guarantee that these copies can be chosen to be disjoint. To do so, instead of applying Theorems 3.5 and 3.6, we will embed the graphs simultaneously doing one step of the embedding algorithms of Theorems 3.5 and 3.6 at a time always making sure not to repeat vertices (which is possible since we have infinitely many choices in every step but only finitely many embedded vertices). Otherwise, the proof is exactly the same and therefore we will omit it.

5. Graphs of bounded ruling number

In this section, we will prove Theorem 1.6.

Proof of Theorem 1.6. Let G be a finitely ruled graph and suppose $K_{\mathbb{N}}$ is colored with r-colors for some $r \in \mathbb{N}$. Let $\mathscr{U}$ be a positive ultrafilter on $\mathbb{N}$ and denote by V_i the set of vertices of color i in the vertex-coloring induced by $\mathscr{U}$. Suppose without loss of generality that $V_1 \in \mathscr{U}$. Since G is finitely ruled, there is a finite set $S \subseteq V(G)$ such that $G[S^c]$ does not have any finite dominating set, and in particular, $G[S^c]$ is 0-ruled.

We will now construct the embedding $f : V(G) \to \mathbb{N}$. First, embed S into an arbitrary clique of color 1 in V_1 of size $|S|$ (such a clique can be found be iteratively applying the ultrafilter property). Let $V_1^0 = N_1^{\cap}(f(S)) \cap V_1$ and note that $V_1^0 \in \mathscr{U}$ and hence satisfies the assumptions of Theorem 3.6. Therefore, $G[S^c]$ can be surjectively embedded into V_1^0, and we can extend f to an embedding of G. Since $V_1^0 \subseteq f(V(G))$ has positive upper density, we are done. □

6. Graphs of bounded degeneracy

Given $k \in \mathbb{N}$ and a graph G, we say that $X \subseteq V(G)$ is *k-wise intersecting* if for all $S \subseteq X$ with $|S| \leq k$, $N^{\cap}(S)$ is infinite. We say that $X \subseteq V(G)$ is *k-wise self-intersecting* if for all $S \subseteq X$ with $|S| \leq k$, $X \cap N^{\cap}(S)$ is infinite. We say that a graph G is *k-wise intersecting* if $V(G)$ is k-wise intersecting (and consequently k-wise self-intersecting). Finally, if G is an r-colored graph for some $r \in \mathbb{N}$, we say that $X \subseteq V(G)$ is *k-wise (self-)intersecting in color i* if X is k-wise (self-)intersecting in G_i.

The following is related to Proposition 3.6.

Proposition 6.1. *Let $d \in \mathbb{N}$ and let G be an infinite, 0-ruled, d-degenerate graph. If H is a $(d+1)$-wise intersecting graph, then we can surjectively embed G into H.*

Proof. Do the same as in the proof of Theorem 3.6, but since G is d-degenerate, when we get to the second phase of the embedding step, where we embed all vertices from $\{v_{n+1}, \ldots, v_{p-1}\}$ into H one at a time, we note that each vertex v_i is adjacent to at most $d+1$ vertices in $\{v_1, \ldots, v_{i-1}\} \cup \{v_p\}$, so it is possible to choose an image for v_i in H. □

In the proofs of Theorem 1.2 and Theorem 1.6, we implicitly proved the following. However, for completeness, we will give a short proof.

Proposition 6.2. *Let $r \in \mathbb{N}$. For every r-coloring of $K_{\mathbb{N}}$, there is a set X with upper density at least $1/r$ and a color $i \in [r]$ such that for every $k \in \mathbb{N}$, X is k-wise intersecting in color i. Moreover, there is a set Y with positive upper density and a color $i \in [r]$ such that for every $k \in \mathbb{N}$, Y is k-wise self-intersecting in color i.*

Proof. Let $\mathcal{U}$ be a positive ultrafilter on $\mathbb{N}$ (that is, $\mathcal{U}$ is an ultrafilter such that every set in $\mathcal{U}$ has positive upper density). For all $i \in [r]$, let V_i be the set of vertices $v \in \mathbb{N}$ such that $N_i(v) \in \mathcal{U}$. Let X be the set in $\{V_1, \ldots, V_r\}$ with the largest upper density and let Y be the set in $\{V_1, \ldots, V_r\}$ which is in $\mathcal{U}$. □

Note that by Proposition 6.1, for the purposes of embedding 0-ruled, d-degenerate graphs, we do not need the set Y described above to be k-wise self-intersecting for all $k \in \mathbb{N}$. Thus, we can ask if it is possible to find a set Y which is $(d+1)$-wise self-intersecting and has upper density bounded below by some function of d. While we have not been able to address this question, we now give an example which provides an upper bound on the upper density of such a set. This example is due to Chris Lambie-Hanson [22].

Proposition 6.3. *For all $k \in \mathbb{N}$, there exists a 2-coloring of $K_{\mathbb{N}}$ such that every monochromatic k-wise self-intersecting set has upper density at most $1/2k$.*

Proof. Let $k \in \mathbb{N}$ and partition $\mathbb{N}$ into sets $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ of equal asymptotic density $1/2k$. Let $A = A_1 \cup \cdots \cup A_k$ and $B = B_1 \cup \cdots \cup B_k$. The coloring is as follows. Given $a \in A$ and $b \in B$, we color $\{a, b\}$ red if $a < b$ and blue otherwise. Given $a, a' \in A$, we color $\{a, a'\}$ red if a and a' are in the same set A_i, and blue otherwise. Given $b, b' \in B$, we color $\{b, b'\}$ blue if b, b' are in the same set B_i, and red otherwise.

The colors are clearly symmetric, so it suffices to consider a red k-wise self-intersecting set X. We claim that X is contained in a single A_i.

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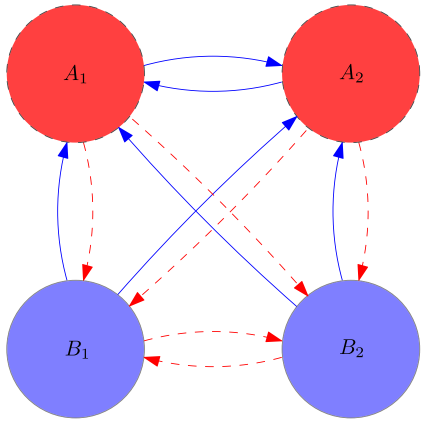

Figure 4. *An example of the coloring from Theorem 6.3 in the case when $k = 2$. The shaded areas denote cliques of the respective colors, and a blue/solid (red/dashed) arrow from one part to another indicates that vertices in the first part have cofinitely many blue (red) neighbors in the second part.*

Note that for all $b \in B_i$, $N_R(b) \cap A$ is finite and $N_R(b) \cap B_i = \emptyset$. Thus, if $X \cap B_i \neq \emptyset$, $X \cap B_j \neq \emptyset$ for some $j \neq i$. Applying the same argument with elements of B_i and B_j, we see that $X \cap B_h \neq \emptyset$ for some $h \neq i, j$, and continuing, we get $X \cap B_\ell \neq \emptyset$ for all $\ell = 1, \ldots, k$. But then taking F to be a subset of X consisting of one vertex from each B_ℓ, we see that $N_R^\cap(F)$ is finite, a contradiction.

So we must have $X \subseteq A$. But note that $N_R(a) \cap A \subseteq A_i$ for all $a \in A_i$. Hence, X must be contained in A_i for some i. □

It is not immediately clear that there exists a d-wise intersecting graph with bounded degeneracy. So we now give a construction of a family of d-wise intersecting graphs which are d-degenerate (and 0-ruled).

Proposition 6.4. *For every $d \in \mathbb{N}$, there is an infinite graph H_d which is d-wise intersecting, d-degenerate and 0-ruled.*

Proof. Let $n_0 = d$. For all $i \geq 0$, let $S_1, S_2, \ldots, S_{\binom{n_i}{d}}$ be an enumeration of all the d-element subsets of $[n_i]$, let $n_{i+1} = n_i + \binom{n_i}{d}$, and let

$$E_{i+1} = \bigcup_{1 \leq j \leq \binom{n_i}{d}} \{\{n_i + j, v\} : v \in S_j\}.$$

Let H_d be the graph on vertex set $\mathbb{N}$ with edge set $\bigcup_{j \in \mathbb{N}} E_j$.

By the construction, it is clear that H_d is d-wise intersecting and d-degenerate. To see that G is 0-ruled, note that for any finite set $X \subseteq \mathbb{N}$ and any d-element set $Y \subseteq \mathbb{N} \setminus X$, there are infinitely many vertices which are adjacent to every vertex in Y and none of the vertices in X. Thus, G cannot have a finite dominating set. □

Note that, in particular, H_d contains a spanning copy of every $(d-1)$-degenerate 0-ruled graph (by Proposition 6.1). Denote by $\rho(d)$ the smallest Ramsey upper density of a d-degenerate infinite graph and by $\tau(d)$ the largest $\tau \geq 0$ such that every 2-colored complete graph contains a monochromatic d-wise self-intersecting subgraph of density at least d. The above propositions imply

$$\tau(d-1) \geq \rho(d-1) \geq \tau(d) \geq \rho(d)$$

for every $d \geq 2$. In particular, we have $\tau(d) > 0$ for every $d \in \mathbb{N}$ if and only if $\rho(d) > 0$ for every $d \in \mathbb{N}$. So in order to answer Theorem 1.5 positively for 0-ruled graphs, it would suffice to answer Theorem 1.5 positively for H_d for all d. Note that $H_1 = T_\infty$ and thus Theorem 1.13 gives a positive answer for the case $d = 1$.

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We conclude this section with a few comments about Theorem 1.5.

In light of Theorem 1.2, if there was a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $d \in \mathbb{N}$, every d-degenerate graph is one-way $f(d)$-locally finite, then we would have a positive answer to Theorem 1.5; however, this is not the case, as there are d-degenerate graphs which are not one-way k-locally-finite for any k. For instance, the graph H_d constructed above is d-degenerate but, since every vertex has infinite degree, is not one-way k-locally-finite for any k. Also $K_{d,\mathbb{N}}$ is d-degenerate but not one-way k-locally-finite for any k (although in this case, we know $\overline{\mathrm{Rd}}(K_{d,\mathbb{N}}) \geq \frac{1}{2^{2d-1}}$).

Theorem 1.5 is about all d-degenerate graphs. However, the discussion in this section is about 0-ruled, d-degenerate graphs. It seems possible that answering Theorem 1.5 positively for 0-ruled, d-degenerate graphs could imply a positive answer for all d-degenerate graphs (cf. the proof of Theorem 1.6).

Problem 6.5. If Theorem 1.5 were true for all 0-ruled d-degenerate graphs (in particular, H_d), would this imply that Theorem 1.5 was true for all d-degenerate graphs?

Let $d \geq 2$ and say that a digraph D is *d-directed* if every d-set in $V(D)$ has a common out-neighbor; that is, for all $S \subseteq V(D)$ with $|S| = d$, there exists $w \in V(D)$ (where it is possible for $w \in S$) such that for all $v \in S$, $(v, w) \in E(D)$. For example, the digraph $D = (\{a, b, c\}, \{(a, b), (b, c), (c, a), (a, a), (b, b), (c, c)\})$ is 2-directed, but not 3-directed.

In order to get a monochromatic d-wise self-intersecting set with upper density at least some fixed amount in an arbitrary 2-coloring of $K_{\mathbb{N}}$, we likely have to solve the following problem.[3]

Problem 6.6. Given a 2-coloring of the edges of a complete (finite) digraph K (including loops), is it possible to cover $V(K)$ with at most $2d$ monochromatic d-directed graphs? (If not $2d$, some other bound depending only on d?)

The reason is that given any 2-coloring of a complete digraph K (plus loops), we can create a corresponding 2-coloring of $K_{\mathbb{N}}$ as follows. Split $\mathbb{N}$ into infinite sets A_i, one for each vertex i of K. Color the edges inside A_i according to the color of the loop on i. Now if both directed edges (i, j) and (j, i) are the same color, give all edges between A_i and A_j that color; if not, then color the bipartite graph between A_i and A_j with the bipartite half graph coloring (where (i, j) being red means that the vertices in A_i have cofinite red degree to A_j). Then any d-wise self-intersecting set B must be the union of some collection of A_i's whose corresponding vertices i make up a monochromatic $2d$-directed set in K.

7. Bipartite Ramsey densities

In this section, we prove Theorem 1.16. An infinite graph G is said to be *infinitely connected* if G remains connected after removing any finite set of vertices. Note that every vertex of an infinitely connected graph has infinite degree. Given some set of vertices $S \subseteq V(G)$, we say that S is *infinitely connected* if $G[S]$ is infinitely connected. Similarly, we call a set $S \subseteq V(G)$ *infinitely linked* if for all distinct $u, v \in S$, there are infinitely many internally vertex-disjoint paths in G from u to v (note that the internal vertices of these paths need not be contained in the set S). Note that every infinitely connected set is also infinitely linked, but the converse is not true (for example, both parts of $K_{\mathbb{N},\mathbb{N}}$ are infinitely linked but not connected). Further note that if $S_1, \dots, S_k$ are sets, each of which is infinitely linked, then there are disjoint paths $P_1, \dots, P_k$ such that $P_1 \cup \cdots \cup P_k$ covers $S_1 \cup \cdots \cup S_k$.

If G is a colored graph and c is a color, we say that G is *infinitely connected in c* if G_c (the spanning subgraph of G with all edges of color c) is infinitely connected. A set $S \subseteq V(G)$ is *infinitely connected in color c* (*infinitely linked in color c*) if S is infinitely connected (infinitely linked) when restricted to G_c. S is called monochromatic infinitely connected (infinitely linked) if it is infinitely connected in some color c.

The following proposition directly implies Theorem 1.16 which implies Theorem 1.15.

[3]Since we first posted this paper, this problem has essentially been resolved [9] (although, determining the minimum number of d-directed graphs needed to cover $V(K)$ is still open).

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Proposition 7.1. *Every 2-colored $K_{\mathbb{N},\mathbb{N}}$ can be partitioned into a finite set and two monochromatic infinitely linked sets X and Y.*

Proof. Let V_1, V_2 be the parts of the bipartite graph and let $\mathcal{U}_1, \mathcal{U}_2$ be nontrivial ultrafilters on V_1 and V_2. For $i = 1, 2$, let $B_i \subseteq V_i$ be the blue vertices in the induced vertex-coloring and let $R_i = V_i \setminus B_i$ be the red vertices.

Case 1 ($|R_1| = |R_2| = |B_1| = |B_2| = \infty$). If there are infinitely many disjoint red paths between R_1 and R_2, then $X := R_1 \cup R_2$ is infinitely linked in red. Indeed, if $v_1, v_2 \in R_1$ or $v_1, v_2 \in R_2$, then they have infinitely many common red neighbors (by the properties of the ultrafilter). If $v_1 \in R_1$ and $v_2 \in R_2$, we will construct infinitely many internally disjoint paths between $x_0 := v_1$ and $x_5 := v_2$ as follows: let $P = x_2 \dots x_3$ be a red path so that $x_2 \in R_1$ and $x_3 \in R_2$, and let x_1 be a common red neighbor of x_0 and x_2 (of which we have infinitely many as above) and x_4 be a common neighbor of x_3 and x_5. It is clear that $x_0 x_1 x_2 \dots x_3 x_4 x_5$ defines a red path and that we can construct infinitely many internally disjoint paths like this. If there are only finitely many disjoint red paths between R_1 and R_2, then there is a finite set S so that, in particular, $X := (R_1 \cup R_2) \setminus S$ induces a complete blue bipartite graph with parts of infinite size and hence is infinitely linked in blue. Similarly, there is a set $Y \subseteq B_1 \cup B_2$ which is cofinite in $B_1 \cup B_2$ and infinitely linked in red or infinitely linked in blue.

Case 2. Suppose without loss of generality that R_1 is finite. It is easy to verify that $X = B_1 \cup B_2$ is infinitely linked in blue and $Y := R_2$ is infinitely linked in red. □

As mentioned in the introduction, the above result combined with a recent result of Day and Lo [8] implies that $\overline{\mathrm{Rd}}_3(P_\infty) = \frac{1}{2}$.

8. Trees

8.1. General embedding results

Given $k \in \mathbb{N}$, we say that a connected graph T has *radius* at most k if there exists $u \in V(T)$ such that for all $v \in V(T)$, there is a path of length at most k from u to v; if no such k exists, we say that T has *unbounded radius*.

Lemma 8.1. *Let T be a graph. A spanning copy of T can be found in every infinitely connected graph H if and only if T is a forest and (i) T has a component of unbounded radius or (ii) T has infinitely many components.*

In order to simplify the proof of Lemma 8.1, we first prove the following structural result about trees with unbounded radius. An *increasing star* is a tree obtained by taking an infinite collection of disjoint finite paths of unbounded length and joining one endpoint of each of the paths to a new vertex v. Note that an increasing star has unbounded radius, no infinite path, and exactly one vertex of infinite degree (which is called the *center*). Also note that an increasing star has distinct vertices $v_0, v_1, v_2, \dots$ and internally disjoint paths $P_1, P_2, \dots$ such that for all $i \geq 1$, P_i is a path from v_0 to v_i and the length of P_{i+1} is greater than the length of P_i.

Fact 8.2. Let T be a tree of unbounded radius. Either for all $v \in V(T)$, there is an infinite path in T starting with v or there exists $v_0 \in V(T)$ such that T contains an increasing star having v_0 as the center.

Proof. Let T be a tree, let $v \in V(T)$, and suppose there is no infinite path in T starting with v (since T is connected, this implies that there is no infinite path in T at all). Since T has unbounded radius, we can do the following: let Q_1 be a path from v to a leaf u_1, which has some length k_1. Now there must exist a path Q_2 of length $k_2 > k_1$ from v to a leaf u_2, and so on. This process gives an infinite set of leaves U and an increasing sequence $k_1, k_2, \dots$ such that there is a path from v to u_i of length k_i. Now we apply the Star-Comb lemma [11, Lemma 8.2] to the set U. Since T has no infinite path, there must exist a subdivision of an infinite star with center v_0 such that all the leaves, call them U', are in U. We claim that for all k, there exists a path from v_0 to U' which has length greater than k, which would prove the lemma. If not, then there exists k such that every path from v_0 to U' has length at most k. However,

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since there is a path from v to v_0, this would imply that there exists a k' such that every path from v to U' has length at most k'. But this contradicts the fact that the lengths of the paths from v to U' form an increasing sequence. □

Proof of Theorem 8.1. First, suppose that a spanning copy of T can be found in every infinitely connected graph H. It is known that there exist infinitely connected graphs with arbitrarily high girth (see [11, Chapter 8, Exercise 7]); for instance, let H_0 be a cycle of length k, and for all $i \geq 1$, let H_i be the graph obtained by adding a vertex x_i and internally disjoint paths of length $\lceil k/2 \rceil$ from x_i to every vertex in H_{i-1}. Then let $H = \cup_{i \geq 0} H_i$. So H is infinitely connected and has girth k. This proves that T is acyclic (i.e., T is a forest) because otherwise there would be an infinitely connected graph in which every cycle is longer than the shortest cycle in T.

Let H be the infinite blow-up of a one-way infinite path (i.e., replace each vertex with an infinite independent set and each edge with a complete bipartite graph). Clearly, H is infinitely connected. If T is a spanning subgraph of H, then T has a component of unbounded radius or T has infinitely many components.

Thus, T must be a forest with a component of unbounded radius or infinitely many components.

Next suppose T is a forest with a component of unbounded radius or infinitely many components. If T has infinitely many components $T_1, T_2, \ldots$, we may select for all $i \geq 1$, $t_i \in V(T_i)$ and add the edge $t_i t_{i+1}$ for all $i \geq 1$ to get a tree with unbounded radius which contains T as a spanning subgraph. If T has finitely many components $T_1, \ldots, T_k$, at least one of which has unbounded radius, we may for all $i \in [k-1]$ add an edge from $t_i \in V(T_i)$ to $t_{i+1} \in V(T_{i+1})$ to get a tree with unbounded radius which contains T as a spanning subgraph. Thus, it suffices to prove the result when T is a tree with unbounded radius.

By Theorem 8.2, there exists a vertex t_0 such that either there is an infinite path starting with t_0 (in which case we say T is of *Type 1*), or an increasing star having t_0 as the center (in which case we say T is of *Type 2*). Now starting with t_0, fix an enumeration of $V(T) = \{t_0, t_1, t_2 \ldots\}$ such that for all $i \geq 1$, $T[\{t_0, \ldots, t_i\}]$ is connected (in fact, for all $i \geq 1$, t_i has exactly one neighbor in $\{t_0, \ldots, t_{i-1}\}$). Also fix an enumeration of $V(H) = \{v_0, v_1, v_2, \ldots\}$. We will build an embedding f of T into H recursively, in finite pieces, at each stage ensuring that we add the first vertices of $V(T) \setminus \mathrm{dom}\, f$ and $V(H) \setminus \mathrm{ran}\, f$ into the domain and range of f respectively.

Initially, let $f(t_0) = v_0$ (we think of t_0 as being the root of the tree and v_0 as the embedding of the root in H) and let $t_{last} := t_0$ and $v_{last} := v_0$. We now show that Algorithm 1 gives the desired embedding.

Algorithm 1

while True **do**
  if $V(H) \setminus \mathrm{ran}\, f \neq \emptyset$ **then**
    Let $next$ be the smallest index such that $v_{next} \in V(H) \setminus \mathrm{ran}\, f$.
    Let $P_{next} \subseteq H$ be a finite path from v_{next} to v_{last} which is internally disjoint from $\mathrm{ran}\, f$.
    Let V_{next} be a set of $|V(P_{next})| - 1$ vertices in $V(T) \setminus \mathrm{dom}\, f$ such that $\{t_{last}\} \cup V_{next}$ induces a path in T.
    Extend f by embedding V_{next} into $V(P_{next}) \setminus \{v_{last}\}$.
    if T is of Type 1 **then**
      Set $t_{last} := f^{-1}(v_{next})$ and $v_{last} := v_{next}$.
  if $V(T) \setminus \mathrm{dom}\, f \neq \emptyset$ **then**
    Let $next$ be the smallest index such that $t_{next} \in V(T) \setminus \mathrm{dom}\, f$.
    Let $back < next$ be the unique index such that t_{back} is adjacent to t_{next}.
    Embed t_{next} into an arbitrary vertex in $N_H(f(t_{back})) \setminus \mathrm{ran}\, f$.
    if T is of Type 1 and $t_{back} = t_{last}$ **then**
      Set $t_{last} := t_{next}$

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Note that if T is of Type 2, then $t_{last} = t_0$ and $v_{last} = v_0$ throughout the process.

To see that f is a well-defined surjective embedding of T into H, first note that we can always follow lines 4 and 12 of Algorithm 1 since H is infinitely connected and in particular every vertex has infinite degree. Line 5 is always possible since there is either an infinite path starting at v_0 or an increasing star having v_0 as the center. Line 11 is always possible by the enumeration of $V(T)$. So f is well defined.

We alternate between embedding the vertex t of smallest index from T which has not yet been embedded into an available vertex from H in such that way that the parent t' of t has already been embedded and $f(t)$ is adjacent to $f(t')$, and embedding a path $t_0t_1 \dots t_\ell$ to a vertex such that $f(t_0)f(t_1)\dots f(t_\ell)$ is a path in H and $f(t_\ell)$ is the vertex of smallest index from $V(H)$ which has yet to be mapped to. So f will be a surjective embedding of T. □

Now we prove another useful lemma.

Lemma 8.3. *Let T be a tree with at least one vertex of infinite degree. If H is a graph in which every vertex has infinite degree, then for all $v \in V(H)$, H contains a copy of T covering $N_H(v)$.*

Proof. Let $t_1 \in V(T)$ be a vertex of infinite degree and let $v_1 = v$ from the statement of the theorem (again we think of t_1 as being the root of the tree and v_1 as the embedding of the root in H). We will build an embedding f of T into H recursively, in finite pieces, at each stage adding one more child of every previously embedded $t \in T$ (unless all children have been embedded already). The embedding strategy is very similar to that in the proof of Theorem 8.1. Initially, let $f(t_1) = v_1$. We will use the following Algorithm 2.

Algorithm 2

1: **while** True **do**
2:     **for** $t \in \operatorname{dom} f$ **do**
3:         **if** $S := N_T(t) \setminus \operatorname{dom} f$ is non-empty **then**
4:             Embed $\min(S)$ into $\min(N_H(f(t)) \setminus \operatorname{ran} f)$.

First, note that we can always follow line 4 of Algorithm 1 since every vertex in H has infinite degree. Let $f : V(T) \to V(H)$ be the function produced by Algorithm 2. We need to prove that f is well defined, an embedding of T and that $N_H(v) \subseteq \operatorname{dom} f$.

Since we always embed the smallest not yet embedded neighbor of every previously embedded $t \in V(T)$ in line 4, every other vertex will be embedded eventually as well. Therefore, f is well defined. Furthermore, by construction of f, it defines a proper embedding (whenever a new vertex $t \in T$ is embedded, its parent t' is already embedded, and we make sure that $f(t)$ is adjacent to $f(t')$). Finally, note that we are infinitely often in line 4 when $t = t_1$ since $N_T(t_1)$ is infinite. Since we always choose the smallest available vertex in $N_H(v) \setminus \operatorname{ran} f$, it follows that $N_H(v) \subseteq \operatorname{ran} f$. □

8.2. Upper density of monochromatic trees

In this section, we will deduce Theorem 1.13 from Lemma 8.1, Lemma 8.3, and the following two lemmas.

Lemma 8.4. *For any 2-coloring of $K_{\mathbb{N}}$, there are sets R and S such that*

(i) *$R \cup S$ is cofinite,*
(ii) *if R is infinite, then it is infinitely connected in red, and*
(iii) *if S is infinite, then it is infinitely connected in one of the colors.*

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Lemma 8.5. *Let H be a 2-colored $K_{\mathbb{N}}$. There exists a set $A \subseteq \mathbb{N}$, a vertex $v \in A$, and a color c such that every vertex in $F := H_c[A]$ has infinite degree and $\overline{d}(N_F(v)) \geq 1/2$.*

It is now easy to prove Theorem 1.13.

Proof of Theorem 1.13. It clearly suffices to prove the result for trees, so let T be an infinite tree and suppose the edges of $K_{\mathbb{N}}$ are colored with two colors. If T does not have an infinite path, it must have at least one vertex of infinite degree (by König's infinity lemma [20]), and therefore, the theorem follows immediately from Theorems 8.5 and 8.3. So suppose T has an infinite path. By Theorem 8.4, there is an infinite set A with $\overline{d}(A) \geq 1/2$ and a color c, so that the induced subgraph on A is infinitely connected in c. By Theorem 8.1, there is a monochromatic copy of T spanning A, and we are done. □

It remains to prove the two lemmas.

Proof of Theorem 8.4. Fix a 2-coloring of $K_{\mathbb{N}}$. We define a sequence of sets R_α, S_α, for all ordinals α, as follows. Let $S_0 = \mathbb{N}$. For each α, let R_α be the set of vertices in S_α whose blue neighborhood has finite intersection with S_α, and let $S_{\alpha+1} = S_\alpha \setminus R_\alpha$. If λ is a limit ordinal, then we let S_λ be the intersection of the sets S_α, for $\alpha < \lambda$.

Note that the sets R_α are pairwise disjoint, and hence, there is some countable ordinal γ such that $R_\alpha = \emptyset$ for all $\alpha \geq \gamma$. Let γ^* be the minimal ordinal such that R_{γ^*} is finite; it follows then that $R_\beta = \emptyset$ for all $\beta > \gamma^*$. Set

$$R = \bigcup \{R_\alpha \mid \alpha < \gamma^*\}.$$

(Note that γ^* may be 0, in which case $R = \emptyset$.)

Suppose that R is infinite. Then $\gamma^* > 0$ and R_α is infinite for all $\alpha < \gamma^*$. Let $u, v \in R$ with $u \in R_\alpha$ and $v \in R_\beta$ for some $\alpha \leq \beta < \gamma^*$. It follows that the red neighborhoods of both u and v are cofinite in R_β. Since R_β is infinite, this implies that there is a red path of length 2 connecting u and v, even after removing a finite set of vertices. Hence, R is infinitely connected in red.

Set $S = S_{\gamma^*+1}$. Then $R \cup S = \mathbb{N} \setminus R_{\gamma^*}$, so $R \cup S$ is cofinite. Moreover, since $R_{\gamma^*+1} = \emptyset$, it follows that for every $v \in S$, the blue neighborhood of v has infinite intersection with S. Now suppose that S is not infinitely connected in blue. Then there is a finite set $F \subseteq S$ and a partition $S \setminus F = X \cup Y$ such that X and Y are both nonempty, and every edge between X and Y is red. Note that X and Y must both be infinite, since if $x_0 \in X$ and $y_0 \in Y$, then $X \cup F$ and $Y \cup F$ must contain the blue neighborhoods of x_0 and y_0 (both of which are infinite), respectively. But then the red graph restricted to $X \cup Y = S \setminus F$ is infinitely connected. □

Proof of Theorem 8.5. Fix a 2-coloring of $K_{\mathbb{N}}$. Similarly as in the proof of Theorem 8.4, we will construct sets $R_\alpha, B_\alpha, S_\alpha$ for all ordinals α. Let $S_0 = \mathbb{N}$. For each α, let R_α be the set of vertices in S_α whose blue neighborhood has finite intersection with S_α, let B_α be the set of vertices in S_α whose red neighborhood has finite intersection with S_α, and let $S_{\alpha+1} = S_\alpha \setminus (R_\alpha \cup B_\alpha)$. If λ is a limit ordinal, then we let S_λ be the intersection of the sets S_α, for $\alpha < \lambda$. As in the proof of Theorem 8.4, the following properties hold.

(i) There is a unique ordinal γ^* such that $R_\alpha \cup B_\alpha$ is infinite for all $\alpha < \gamma^*$, finite for $\alpha = \gamma^*$ and empty for all $\alpha > \gamma^*$. We denote $R = \bigcup_{\alpha<\gamma^*} R_\alpha$ and $B = \bigcup_{\alpha<\gamma^*} B_\alpha$.
(ii) $S_\alpha = S_{\alpha'}$ for all ordinals $\alpha, \alpha' > \gamma^*$. We denote $S = S_{\gamma^*+1}$.
(iii) R, B, S are pairwise disjoint, and $R \cup B \cup S$ is cofinite.
(iv) If $v \in R_\gamma$ for some ordinal γ, then v has finitely many blue neighbors in $S \cup \bigcup_{\alpha\geq\gamma} R_\alpha$.
(v) If $v \in B_\gamma$ for some ordinal γ, then v has finitely many red neighbors in $S \cup \bigcup_{\alpha\geq\gamma} B_\alpha$.
(vi) Every $v \in S$ has infinitely many neighbors of both colors in S.

If $R \cup B$ is empty, then let $A = S$ and choose an arbitrary vertex $v \in S$. Since A is cofinite in $\mathbb{N}$, either the blue or the red neighborhood of v in A has upper density at least $1/2$. Since every vertex in A has infinite degree in both colors, we are done.

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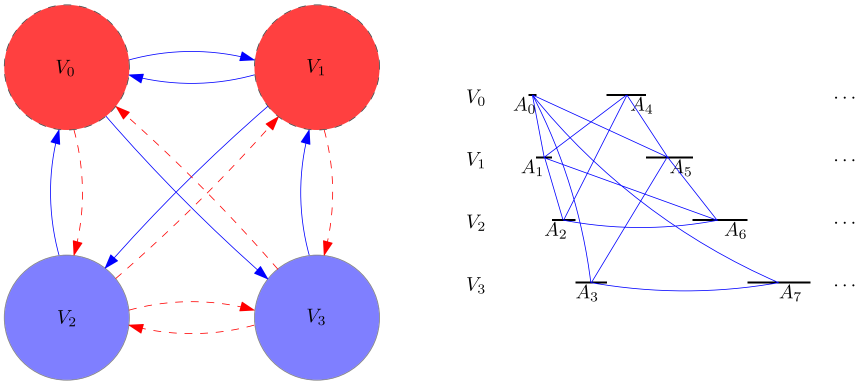

Figure 5. *The shaded areas denote cliques of the respective colors and a blue/solid (red/dashed) arrow from V_i to V_j indicates that vertices in V_i have cofinitely many blue (red) neighbors in $V)j$. On the right, we have an example of the relevant edges in the case where we are embedding a blue copy of $T \in \mathcal{T}^*$ with root t in V_0.*

If $R \cup B$ is nonempty, it must be infinite (by the way R and B are defined). Since $R \cup B \cup S = (R \cup S) \cup (B \cup S)$ is cofinite, we may assume without loss of generality that R is nonempty and $\underline{\mathrm{d}}(R \cup S) \geq 1/2$. Let $A = R \cup S$ and let $v \in R_0$ be arbitrary (if R is nonempty, then R_0 must be infinite). Clearly, every vertex in A has infinite red degree in A, and since v has only finitely many blue neighbors in A, we are done. □

8.3. Ramsey-cofinite forests

In this section, we will prove Theorem 1.14.

We already know from Theorems 2.9 and 2.8 that if T is weakly expanding or has a finite dominating set, then T is not Ramsey-lower-dense (and thus T is not Ramsey-confinite). So all that remains to prove Theorem 1.14(ii) is to show that every forest in $\mathcal{T}^*$ is not Ramsey-lower-dense. Recall that $\mathcal{T}^*$ is the family of forests T having one vertex t of infinite degree, every other vertex has degree at most d for some $d \in \mathbb{N}$, t is adjacent to infinitely many leaves and infinitely many non-leaves, and cofinitely many vertices of T have distance at most 2 to t (in particular, if T is not connected, then T has one infinite component and finitely many finite components).

Proposition 8.6. *If $T \in \mathcal{T}^*$, then T is not Ramsey-lower-dense.*

Proof. Let $T \in \mathcal{T}^*$, let t be the vertex of infinite degree in T, and let d be the maximum degree of $T - \{t\}$ as guaranteed by the definition.

We begin by partitioning $\mathbb{N}$ into intervals $A_0, A_1, A_2, \ldots$ as follows: Let $a_0 = 1$, and for all $i \geq 1$, let $a_i = i \cdot d \cdot a_{i-1}$. Then for all $i \geq 0$, let $A_i = [a_i, a_{i+1})$. Now for all $r \in \{0, 1, 2, 3\}$, let $V_r = A_r \cup A_{4+r} \cup A_{8+r} \cup \ldots$.

Now color all edges which are inside V_0 or V_1 red, and all edges inside V_2 or V_3 blue. Color all edges between V_0 and V_1 blue and all edges between V_2 and V_3 red. Finally, for all $i \in \{0, 1\}$, $j \in \{2, 3\}$, we color the complete bipartite graphs $K(V_i, V_j)$ according to Figure 5 as follows: Suppose first that there is a red arrow from V_i to V_j (and thus a blue arrow from V_j to V_i). Let $A_s \subseteq V_i$ and $A_t \subseteq V_j$. If $s < t$, then color all edges between A_s and A_t red. If $t < s$, then color all edges between A_t and A_s blue. If there is a blue arrow from V_i to V_j (and thus a red arrow from V_j to V_i), we do the opposite.

Assume there is a monochromatic copy T' of T, let f be the corresponding embedding, and let $V_i' = f(T) \cap V_i$ for all $i \in \{0, 1, 2, 3\}$. By symmetry, we may assume that t is embedded in V_0; and let $v = f(t)$.

Suppose first that T is embedded in the blue subgraph. Since v has finitely many blue neighbors in V_2 and cofinitely many vertices have distance at most 2 to v, all but finitely many vertices of V_2' are neighbors of vertices in V_1' in $f(T)$. So for all sufficiently large q (i.e., large enough so that $N_{T'}(v) \cap V_2 \subseteq A_2 \cup A_6 \cup \cdots \cup A_{4q-2}$), we have

$$|V_2' \cap A_{4q+2}| \leq d|V_1' \cap (A_1 \cup A_5 \cup \cdots \cup A_{4q+1})| \leq d \cdot a_{4q+1} \leq \frac{a_{4q+2}}{4q+2},$$

and thus,

$$\begin{aligned}\underline{d}(\operatorname{ran} f) &\leq \lim_{q\to\infty} |\operatorname{ran} f \cap (A_1 \cup \ldots \cup A_{4q+2})|/(a_1 + \ldots + a_{4q+2}) \\ &\leq \lim_{q\to\infty} (a_1 + \ldots + a_{4q+1} + \tfrac{a_{4q+2}}{4q+2})/(a_1 + \ldots + a_{4q+2}) = 0.\end{aligned}$$

Suppose next that T is embedded in the red subgraph. Since v has no red neighbors in V_1, all but finitely many vertices of V_1' are neighbors of vertices in V_2' in $f(T)$. So for all $q \geq 1$, we have

$$|V_1' \cap A_{4q+1}| \leq d|V_2' \cap (A_2 \cup A_6 \cup \cdots \cup A_{4q-2})| \leq d \cdot a_{4q-2} \leq \frac{a_{4q+1}}{4q+1},$$

and thus, $\underline{d}(f(T)) = 0$. Since f was an arbitrary embedding, this shows that T is not Ramsey lower dense. □

We now turn to part (i) of Theorem 1.14; that is, if T is a forest which is strongly contracting, has no finite dominating set, and $T \notin \mathcal{T}^*$, then T is Ramsey-cofinite. We begin with a lemma which allows us to embed forests which are strongly contracting into graphs with infinitely many vertices of cofinite degree. Here, it is not important that we are embedding forests, and we will state and prove the lemma more generally. Recall that a graph F is strongly contracting if there exists $k \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}$, there exists an independent set A in F with $|A| \geq \ell$ such that $|N(A)| \leq k$ (and in particular, a forest is strongly contracting if and only if it has finitely many components and unbounded leaf degree).

Lemma 8.7. *Let F be a graph. A cofinite copy of F can be found in every graph H having infinitely many vertices of cofinite degree if and only if F is strongly contracting.*

In order to simplify the proof of the lemma, we first prove a structural result regarding strongly contracting graphs.

Proposition 8.8. *If F is strongly contracting, then there exists non-negative integers $\ell \leq k$, an infinite independent set U, a family of disjoint sets $\{U_i \subseteq U : i \in \mathbb{N}\}$ with $|U_i| \geq i$ for all $i \in \mathbb{N}$, an ℓ-set V_0, and a family of disjoint $(k-\ell)$-sets $\{V_i \subseteq V(F) \setminus (U \cup V_0) : i \in \mathbb{N}\}$ such that for all $i \in \mathbb{N}$, $F[U_i, V_0 \cup V_i]$ is a complete bipartite graph (note that the family $\{V_0 \cup V_i : i \in \mathbb{N}\}$ is a k-uniform sunflower with a core of order ℓ).*

Proof. Since F is strongly contracting, there exists a non-negative integer k', and without loss of generality, there exists disjoint independent sets $U_1', U_2', \ldots$ such that for all $i \in \mathbb{N}$, $|U_i'| \geq i2^{k'}$ and $|N(U_i')| \leq k'$. Now by pigeonhole, there exists a subset $U_i \subseteq U_i'$ with $|U_i| \geq i$ and a nonempty subset $V_i \subseteq N(U_i)$ such that $F[U_i, V_i]$ is a complete bipartite graph. Indeed, since $|N(U_i')| \leq k$, there are at most $2^{k'}$ different possible neighborhoods for vertices in U_i'. By pigeonhole, there are at least i vertices in U_i' with the same neighborhood. Now by pigeonhole again, there exists an integer $0 \leq k \leq k'$ and an infinite subsequence $\{i_j\}_{j\in\mathbb{N}}$ such that $|U_{i_j}| \geq i_j$ and $|V_{i_j}| = k$ and $F[U_{i_j}, V_{i_j}]$ is a complete bipartite graph. Finally, by the infinite version of the sunflower lemma (see [19, Page 107, Problem 1][4]), there exists a subsequence $\{j_h\}_{h\in\mathbb{N}}$ such that the family $\{V_{i_{j_h}} : h \in \mathbb{N}\}$ is a sunflower. □

[4] The proof is short. If $k = 1$, then it is clear, so let $k \geq 2$ and suppose it is true for $(k-1)$-uniform hypergraphs. If there is an infinite matching, we are done; so suppose not. Since there is no infinite matching, there is a finite vertex cover and thus a vertex v of infinite degree. Now apply induction to the $(k-1)$-uniform link graph of v.

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Proof of Theorem 8.7. Note that in Example 2.8, both the red graph and the blue graph have the property that there are infinitely many vertices of cofinite degree. So if a cofinite copy of F can be found in every graph H having infinitely many vertices of cofinite degree, then F is not weakly expanding (i.e., F is strongly contracting).

Now suppose F is strongly contracting. By Theorem 8.8, there exists non-negative integers k and ℓ, an infinite independent set U, a family of disjoint sets $\{U_i \subseteq U : i \in \mathbb{N}\}$ with $|U_i| \geq i$ for all $i \in \mathbb{N}$, an ℓ-set V_0, and a disjoint family of k-sets $\{V_i \subseteq V(F) \setminus (U \cup V_0) : i \in \mathbb{N}\}$ such that for all $i \in \mathbb{N}$, $F[U_i, V_0 \cup V_i]$ is a complete bipartite graph.

Since there are infinitely many vertices V' in H with cofinite degree, we may choose an infinite clique $K \subseteq H$ such that $V(K) \subseteq V'$. If $V(H) \setminus V(K)$ is finite. Then we are done (since it is clear that we can surjectively embed F into the clique K), so suppose not. Let $X \subseteq V(K)$ such that X and $V(K) \setminus X$ are both infinite. Let $y_1, y_2, \dots$ be an enumeration of $Y := \mathbb{N} \setminus V(K)$. For all $i \in \mathbb{N}$, let $X_i = \{x \in X : \forall j \geq i, \{x, y_j\} \in E(H)\}$. Note that if there exists $i \in \mathbb{N}$ such that X_i is infinite, then we have a copy of $K_{\mathbb{N},\mathbb{N}}$ which covers all but finitely many vertices in Y, and from here, we can easily get the desired embedding. So suppose that X_i is finite for all $i \in \mathbb{N}$. For all $x \in X$, there exists $i \in \mathbb{N}$ such that $x \in X_i$, so let $\phi(x) = \min\{i \in \mathbb{N} : x \in X_i\}$. Let $x_1, x_2, \dots$ be an enumeration of X such that for all $i \leq j$, $\phi(x_i) \leq \phi(x_j)$. Let $X'_0 = \{x_1, \dots, x_\ell\}$ and for all $i \geq 1$, let $X'_i = X'_0 \cup \{x_{\ell+(i-1)k+1}, \dots, x_{\ell+ik}\}$ and let $\phi(i) = \phi(x_{\ell+ik})$. Finally, for all $i \geq 1$, let $Y_i = \{y_{\phi(i)}, \dots, y_{\phi(i+1)-1}\}$ (note that $Y_i = \emptyset$ if and only if $\phi(i) = \phi(i+1)$). Let $I = \{i \geq 1 : Y_i \neq \emptyset\}$ and note that since every X_i is finite, we have that I is infinite. Also note that for all $i \in I$, $[X'_i, Y_i]$ is a complete bipartite graph.

From the properties of F mentioned above, there is an injection $f : I \to \mathbb{N}$ such that $\mathbb{N} \setminus f(I)$ is infinite and for all $i \in I$, $|U_{f(i)}| \geq |Y_i|$, and thus, there exists $U'_{f(i)} \subseteq U_{f(i)}$ such that $|U'_{f(i)}| = |Y_i|$. Now for all $i \in I$, we embed $U'_{f(i)}$ to Y_i and embed $V_{f(i)}$ to X'_i. Since $\mathbb{N} \setminus f(I)$ is infinite, there are infinitely many remaining vertices in F, and these can be embedded to $K - X$ (since K is a clique and U is an independent set). □

Note that Theorem 8.7 allows us to focus on colorings in which all but finitely many vertices have infinite degree in both colors. Given such a coloring, we will need to separate into a few cases depending on the structure of T.

Proposition 8.9. *Let T be an infinite tree with unbounded leaf degree, no finite dominating set, and $T \notin \mathcal{T}^*$. Then either T has unbounded radius, or there exists a vertex $t \in V(T)$ such that t is adjacent to infinitely many non-leaves and*

(i) *there are infinitely many paths of length 3 starting at t which are pairwise vertex-disjoint apart from t, or*
(ii) *there is a vertex in $T - \{t\}$ of infinite degree, or*
(iii) *the neighbors of t have unbounded degrees.*

Proof. Suppose that T has unbounded leaf degree, no finite dominating set, bounded radius, and $T \notin \mathcal{T}^*$. We first show that there is a vertex $t \in V(T)$ which is adjacent to infinitely many non-leaves. Let $s_0 \in V(T)$ and think of it as the root. If s_0 is adjacent to infinitely many non-leaves, we are done; so assume it is adjacent to finitely many non-leaves S_1. Since T has no finite dominating set, $V(T) \setminus N(s_0)$ is infinite. In particular, some $s_1 \in S_1$ has an infinite subtree. If s_1 is adjacent to infinitely many non-leaves, we are done; so assume it is adjacent to finitely many non-leaves S_2 other than s_1. Since there is no finite dominating set, some $s_2 \in S_2$ has an infinite subtree. We keep iterating until we find a vertex s_i adjacent to infinitely many non-leaves. Since T has bounded radius, this process must finish eventually.

Let $t \in V(T)$ which is adjacent to infinitely many non-leaves and let S be the non-leaves adjacent to t. Assume that there are only finitely many paths of length 3 starting at t which are pairwise vertex-disjoint apart from t and that there is no $t' \in V(T) \setminus \{t\}$ of infinite degree (otherwise we are done). Let $S' \subseteq S$ be those vertices whose only children are leaves and let $S'' = S \setminus S'$. By the assumption, we have that S'' is finite. We claim that for each $i \in \mathbb{N}$, there is a vertex $t' \in S'$ of degree at least i. Assume for contradiction this is not the case and let $d := \max_{s \in S'} \deg(s)$. Note that since T has bounded radius and

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no vertex of infinite degree other than t, there are only finitely many vertices which are successors of vertices in S''. Thus, since T has unbounded leaf degree, t must be adjacent to infinitely many leaves. Therefore, $T \in \mathcal{T}^*$, a contradiction. □

The main difficulty in the proof of Theorem 1.14 will be dealing with the trees of bounded radius described in Theorem 8.9. The following lemma deals with that case.

Lemma 8.10. *Let H be a 2-colored $K_{\mathbb{N}}$ in which every vertex has infinite degree in both colors. Let T be a tree of bounded radius containing a vertex t such that t is adjacent to infinitely many non-leaves and*

(i) *there are infinitely many paths of length 3 starting at t which are pairwise vertex-disjoint apart from t, or*
(ii) *there is a vertex in $T - \{t\}$ of infinite degree, or*
(iii) *the neighbors of t have unbounded degrees.*

Then there is a cofinite monochromatic copy of T in H.

Assuming Theorem 8.10 (the proof of which we delay for the moment), we now prove Theorem 1.14.

Proof of Theorem 1.14. Part (ii) follows from Theorems 2.9, 2.8 and 8.6.

So let T be a forest which is strongly contracting, has no finite dominating set, and $T \notin \mathcal{T}^*$.

Let H be a 2-colored $K_{\mathbb{N}}$, and assume the colors are red and blue. If there are infinitely many vertices of cofinite red degree or infinitely many vertices of cofinite blue degree, then we are done by Lemma 8.7; so (by removing finitely many vertices) we may assume every vertex in H has infinite red degree and infinite blue degree.

First, suppose that T has a component of unbounded radius or infinitely many components. If the blue subgraph H_B is infinitely connected, we can find a monochromatic spanning copy of T in H_B by Lemma 8.1. If H_B is not infinitely connected, there exists a finite set X such that $H_B - X$ is not connected. So there exists a partition $\{Y, Z\}$ of $\mathbb{N} - X$ such that all edges between Y and Z are red. Note that for all $v \in Y$, $N_B(v) \subseteq Y \cup X$ and for all $v \in Z$, $N_B(v) \subseteq Z \cup X$. Since all vertices have infinite blue degree and X is finite, this implies that both Y and Z are infinite. So $Y \cup Z$ is cofinite and $H_R[Y, Z]$ induces a red copy of $K_{\mathbb{N},\mathbb{N}}$. Since T has no finite dominating set, both parts of its bipartition are infinite, and we can surjectively embed T into $H_R[Y, Z]$.

Finally, suppose that T has finitely many components $T_1, \ldots, T_k$ all of which have bounded radius. For all $i \in [k]$, let $t_i \in V(T_i)$, and for all $2 \le i \le k$, add the edge $t_1 t_k$ to get a tree T' which has bounded radius, is strongly contracting, has no finite dominating set, $T' \notin \mathcal{T}^*$, and T' contains T as a spanning subgraph. Therefore, by Theorem 8.9, T' satisfies the hypotheses of Theorem 8.10, and thus, we can find a monochromatic copy of T' (and consequently T) which spans cofinitely many vertices of H. □

It remains to prove Theorem 8.10.

Proof of Theorem 8.10. Let T and t be as in the statement and let H be a 2-colored $K_{\mathbb{N}}$ which is as in the statement. Note that $\deg(t) = \infty$, and there are infinitely many paths of length 2 starting at t which are vertex disjoint apart from t ($\star$). Let $v \in \mathbb{N}$ be an arbitrary vertex. Let B be the set of blue neighbors of v and let R be the set of red neighbors of v. Furthermore, let B' be the set of vertices in B with finitely many red neighbors in R and let R' be the set of vertices in R with finitely many blue neighbors in B. Let $R'' = R \setminus R'$ and $B'' = B \setminus B'$.

Case 1 (B' or R' is finite.)

Suppose without loss of generality that R' is finite. We will find an embedding f of T into the blue subgraph H_B covering the cofinite set $\{v\} \cup B \cup R''$. We build f iteratively in finite pieces maintaining a partial embedding of T whose domain is connected. Note that by keeping dom f connected, we ensure that every not yet embedded vertex is adjacent to at most one vertex in dom f. Initially, we set $f(t) = v$. Then we repeatedly follow the following two steps.

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Step 1. Let $s \in V(T) \setminus \operatorname{dom} f$ be the smallest not yet embedded vertex. Since every vertex has infinite blue degree, it is easy to extend f so that $\operatorname{dom} f$ remains connected and $s \in \operatorname{dom} f$ (by adding a path to s).

Step 2. Let $u \in (B \cup R'') \setminus \operatorname{ran} f$ be the smallest not yet covered vertex. If $u \in B$, we can simply choose a not-yet embedded neighbor of t and embed u into it (since t has infinite degree). If $u \in R''$, it has infinitely many blue neighbors in B (by definition of R'), and therefore, there is a blue path vxu of length 2 for some $x \in B \setminus \operatorname{ran} f$. By ($\star$) we can extend f to cover x and u so that $\operatorname{dom} f$ remains connected.

Routinely, this defines an embedding of T into H_B covering $B \cup R'$.

Case 2 (B' and R' are infinite.)

We further split into subcases depending on the structure of T.

Case 2.1 (T has infinitely many paths of length 3 starting at t which are pairwise vertex-disjoint apart from t): We will find an embedding f of T into the blue subgraph H_B covering $\mathbb{N}$. We build f iteratively in finite pieces maintaining a partial embedding of T whose domain is connected. Initially, we set $f(t) = v$. Then we repeat the following two steps.

Step 1. We extend f to include the smallest not yet embedded vertex as above.

Step 2. Let $u \in \mathbb{N} \setminus \operatorname{ran} f$ be the smallest not yet covered vertex. If $u \in B \cup R''$, we proceed as above; so suppose $u \in R'$. Note that u (as every other vertex) has infinitely many blue neighbors, and by definition of R', only finitely many of those can lie outside R. Furthermore, every vertex $u' \in B'$ has only finitely many red neighbors in R, and thus, u and u' have infinitely many common blue neighbors in R. It follows that there is a blue path P of length 3 from v to u so that $(P \setminus \{v\}) \cap \operatorname{ran} f = \emptyset$. By the case assumption, we can extend f to cover this path.

Routinely, the resulting function f is an embedding of T into $\mathbb{N}$. Observe that the only difference to the previous case is how we deal with vertices in R'. This will be similar in the following cases and we therefore skip some details.

Case 2.2 (T has a vertex $t' \in V(T) \setminus \{t\}$ of infinite degree): Given some integer $d \geq 1$, we say that a path P is d-good in blue (red) if it has length d, starts at v and ends at some $v' \in \mathbb{N} \setminus \{v\}$, is monochromatic in blue (red) and v' has only finitely many red (blue) neighbors in R' (B'). We will first show that (i) for every positive integer $d \neq 2$, there is a red d-good path and a blue d-good path, and (ii) there is a red 2-good path or a blue 2-good path.

If $d = 1$, any vertex in B' forms a blue d-good path with v. Since any two vertices in B' have infinitely many common blue neighbors in R, we can extend this path to a 3-good path. We can proceed like this to get a d-good path in blue path for any odd d. If $d \geq 4$ is even, we start by building a blue path of length 2 to some $u \in R'$ and take some $v' \in B'$ not yet in the path. Since u has infinitely many blue neighbors in R and every vertex in B' has only finitely many red neighbors in R, u and v' have a common blue neighbor not yet in the path, giving us a d-good path in blue. We can extend this path now as before to any even length $d \geq 4$. We can proceed similarly for red paths. Finally suppose $d = 2$. If there are $u_1 \in R$ and $u_2 \in R'$ such that u_1u_2 is red, then vu_1u_2 is 2-good in red, and we are done. Otherwise, every $u \in R$ has only blue neighbors in R'. We can thus form a blue path from v to R of length 2, which is 2-good in blue.

Let now d be the distance from t to t' in T and let P be a d-good path (say in blue). Embed t into v and the unique path from t to t' into P (and call this partial embedding f). Let $v' = f(t')$ and remove the finite set of vertices in R' which is not in the blue neighborhood of v'. We then extend f in finite pieces exactly as in the previous case apart from when $u \in R'$, where we simply embed an available neighbor of t' into u.

Case 2.3 (for all $i \in \mathbb{N}$, there is a vertex $t' \in N_T(t)$ of degree at least i): We may assume we are not in Case 2.2 and thus there is an infinite set $S \subseteq N_T(t)$ of vertices with distinct degrees. Furthermore, we may assume we are not in Case 2.1, and thus, cofinitely many vertices of T have distance at most 2 to t. Let T_0 be the finite subtree rooted at t which consists of all paths from t to leaves of distance at least 3.

Let $u_1, u_2, \ldots$ be an enumeration of $N_T(t)$. Let $y_1, y_2, \ldots$ be an enumeration of R. Let $B_1 \subseteq B'$ such that B_1 is infinite and $B' \setminus B_1$ is infinite, then set $B_2 = B \setminus B_1$.

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For all $x \in B_1$ there exists $\phi(x) \in \mathbb{N}$ such that $y_{\phi(x)-1} \notin N_B(x)$ and $y_j \in N_B(x)$ for all $j \geq \phi(x)$ (if $Y \subseteq N_B(x)$, then $\phi(x) = 1$). For all $i \geq 1$, let $X^i = \{x \in B_1 : \phi(x) = i\}$. If $|X^i| = \infty$ for some $i \in \mathbb{N}$, then reset $B_1 := X^i$ and $B_2 := B \setminus B_1$, enumerate B_1 as $x_1, x_2, \ldots$. Otherwise, X^i is finite for all i and thus there is a natural enumeration $x_1, x_2, \ldots$ of B_1 such that $\phi(x_i) \leq \phi(x_j)$ whenever $i \leq j$.

Initially, we set $f(t) = v$, and we embed T_0 in H_B using the fact that every vertex in H has infinite blue degree. Now every vertex in $V(T) \setminus \mathrm{dom}\, f$ has distance at most 2 to r. Now we repeat the following two steps.

Step 1. Let x_i and x_j (with $i < j$) be the two smallest vertices in $B_1 \setminus \mathrm{ran}\, f$ and let $u_{n_i} \in N_T(t) \setminus \mathrm{dom}\, f$ such that u_{n_i} has at least $\phi(x_j) - \phi(x_i)$ children in T (which is possible by the case). Set $f(u_{n_i}) = x_i$ and embed the (finitely many) vertices in $N_T(u_{n_i}) \setminus \{t\}$ to the smallest vertices in $\{y_{\phi(x_i)}, y_{\phi(x_i)+1}, \ldots\} \setminus \mathrm{ran}\, f$.

Step 2. Injectively embed all vertices from $\{u_1, u_2, \ldots, u_{n_i+1}\} \setminus \mathrm{dom}\, f$ (which is nonempty since $u_{n_i+1} \notin \mathrm{dom}\, f$) to the smallest vertices in $B_2 \setminus \mathrm{ran}\, f$. Now, using the fact that every vertex in H has infinite blue degree, iteratively embed the children of each vertex $u_{n_\ell} \in \{u_{n_{i-1}+1}, \ldots, u_{n_i-1}\}$ anywhere in $N_B(f(u_{n_\ell})) \setminus (\{x_j\} \cup \mathrm{ran}\, f)$. Now move to Step 1 (and notice that x_j will become the smallest vertex in $B_1 \setminus \mathrm{ran}\, f$).

The resulting function f is an embedding of T into H covering a cofinite set. □

8.4. General graphs

In the previous section, we completely characterize forests which are Ramsey-cofinite. We know that if a graph F is Ramsey-cofinite, then F is bipartite, strongly contracting, and has no finite dominating set (by Theorems 2.8 to 2.10). However, from the proof in the previous section, we know that if G is bipartite, strongly contracting, and has no finite dominating set and we are given a 2-coloring of $K_{\mathbb{N}}$ such that one of the colors is not infinitely connected, then there is a cofinite monochromatic copy of G. So this raises the question of completely characterizing all graphs which are Ramsey-cofinite. However, given the information from the previous sections, we can narrow this down to a much more specific question.

Problem 8.11. Characterize the graphs G which are bipartite, strongly contracting, and have no finite dominating set such that there exists a cofinite monochromatic copy in every 2-coloring of $K_{\mathbb{N}}$ in which both colors are infinitely connected.

The following is an easy to state sufficient condition (we are aware of a more general sufficient condition which contains Theorem 1.14(i), but as we don't believe the more general condition is necessary, we go for simplicity instead).

Theorem 8.12. *If G is bipartite, strongly contracting, and has arbitrarily long paths whose internal vertices have degree 2, then G is Ramsey-cofinite.*

This follows because if we are given a 2-coloring of $K_{\mathbb{N}}$ in which both colors are infinitely connected, then at least one of those colors contains an infinite clique. So we can use the following lemma.

Lemma 8.13. *Let F be a connected graph. A spanning copy of F can be found in every infinitely connected graph H with an infinite clique if F has arbitrarily long paths whose internal vertices have degree 2.*

Proof. Assume that F has arbitrarily long induced paths and that H is infinitely connected with an infinite clique $K \subseteq H$. We will construct an embedding f of F into H iteratively in finite pieces. For each $i \in \mathbb{N}$, we will do the following two steps: First, let $t \in V(T) \setminus \mathrm{dom}\, f$ be the smallest not-yet embedded vertex and embed it into an arbitrary vertex $u \in V(K) \setminus \mathrm{ran}\, f$. Second, let $v \in V(H) \setminus \mathrm{ran}\, f$ be the smallest not-yet covered vertex and let P be a finite path in H which starts and ends in K, contains v and avoids $\mathrm{ran}\, f$ (such a path exist since H is infinitely connected). Let P' be an induced path in T of the same length as P which avoids $\mathrm{dom}\, f$ (such a path exists since T has arbitrary long induced paths). Extend f by embedding P' into P. Note that all neighbors of the internal vertices of P' will be embedded, and the endpoints of P' are in K. Therefore, we maintain a partial embedding throughout the process. Since we eventually embed every $t \in V(T)$, the resulting function f is an embedding of T into H. Since we eventually cover every $v \in V(H)$, this embedding is surjective. □

9. Ramseyness of coideals

9.1. Ideals and coideals

An *ideal* on a set X is a collection $\mathcal{I}$ of subsets of X such that (1) for any $B \in \mathcal{I}$ and $A \subseteq B$, we have $A \in \mathcal{I}$, and (2) for any $A, B \in \mathcal{I}$, we have $A \cup B \in \mathcal{I}$. We call an ideal $\mathcal{I}$ on X *proper* if $X \notin \mathcal{I}$. If $\mathcal{I}$ is an ideal, then we write $\mathcal{I}^+$ for its complement $\mathcal{P}(X) \setminus \mathcal{I}$, and we call $\mathcal{I}^+$ a *coideal*.

In this section, we will primarily be concerned with ideals on countable sets, and in particular ideals on $\mathbb{N}$. Some commonly used examples of ideals on $\mathbb{N}$ are

(i) $\mathrm{fin} = \{A \subseteq \mathbb{N} \mid |A| < \infty\}$,
(ii) $\mathcal{Z}_0 = \{A \subseteq \mathbb{N} \mid d(A) = 0\}$,
(iii) $\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} \mid \sum_{n \in A} 1/n < \infty\}$.

In general, we view an ideal $\mathcal{I}$ on X as a way of measuring which subsets of X are ‘small’. In this light, we only consider an ideal $\mathcal{I}$ to be nontrivial if $\mathcal{I}$ is proper and contains the finite subsets of X, since at the very least, the finite subsets of X should be ‘small’, and X itself should not be ‘small’.

Let G be a graph and $\mathcal{I}^+$ be a coideal on $\mathbb{N}$. We say that G is $\mathcal{I}^+$*-Ramsey* if, for every finite coloring of $K_{\mathbb{N}}$, there is a monochromatic copy of G whose vertex set is in $\mathcal{I}^+$.

The first thing to note is that we may reexpress one of the central notions of this paper using the above terminology; namely, a graph G is Ramsey-dense if and only if G is $\mathcal{Z}_0^+$-Ramsey. For another example, Ramsey’s theorem says that $K_{\mathbb{N}}$ is fin^+-Ramsey. In the remainder of this section, we investigate the relationship between coideals $\mathcal{I}^+$ and graphs G such that G is $\mathcal{I}^+$-Ramsey. We hope that the results to follow will help the reader to better understand some of the characteristics of Ramsey-dense graphs, while simultaneously establishing a more general setting for the kind of questions we are interested in, where different notions of ‘small’ other than ‘asymptotic density zero’ are considered. Before continuing, we note three easy observations.

Fact 9.1. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on $\mathbb{N}$ and suppose $\mathcal{I} \subseteq \mathcal{J}$. For any graph G, if G is $\mathcal{J}^+$-Ramsey, then G is $\mathcal{I}^+$-Ramsey.

Fact 9.2. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and let $A \in \mathcal{I}^+$. For any partition $\{A_1, \ldots, A_k\}$ of A, there exists $i \in [k]$ such that $A_i \in \mathcal{I}^+$.

Fact 9.3. For every nontrivial ideal $\mathcal{I}$, there is an ultrafilter $\mathcal{U} \subseteq \mathcal{I}^+$.

Proof. Let $\mathcal{F} = \{I^c : I \in \mathcal{I}\}$ and observe that $\mathcal{F}$ satisfies $(i) - (iii)$ in Theorem 3.1 (such a family is called a *filter*). Hence, we can apply Zorn’s lemma as in Theorem 3.3 to get an ultrafilter $\mathcal{U}$ containing $\mathcal{F}$. Then, for every $I \in \mathcal{I}$, we have $I^c \in \mathcal{F} \subseteq \mathcal{U}$, and thus, $I \notin \mathcal{U}$; hence, $\mathcal{U} \subseteq \mathcal{I}^+$. □

9.2. Finitely-ruled graphs and the ideal nwd

Recall that one of the motivating problems of this paper is to characterize the Ramsey-dense graphs – or in other words, the graphs G such that G is $\mathcal{Z}_0^+$-Ramsey. In Theorem 9.4, we provide a characterization of those graphs G for which G is $\mathcal{I}^+$-Ramsey for *every* nontrivial ideal $\mathcal{I}$ on $\mathbb{N}$. Interestingly, this characterization reduces to one particular ideal, and one particular 2-coloring, both of which we will describe now.

Note that every positive integer n has a binary expansion in which the leftmost digit is a 1, the *truncated binary expansion* of n is what remains after removing the leftmost digit from the binary expansion (for instance, the truncated binary expansion of 19 is 0011). Given $s, t \in \mathbb{N}$, we say that t

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extends s, if $s \leq t$ and the truncated binary expansion of t contains the truncated binary expansion of s as its initial segment (for instance, 19 extends 7 since 0011 contains 11 as its initial segment, reading from right to left). Given $s \in \mathbb{N}$, we write $\langle s \rangle$ for the set of $t \in \mathbb{N}$ which extend s (for instance, $\langle 1 \rangle = \mathbb{N}$, $\langle 2 \rangle$ is the positive even integers, and $\langle 3 \rangle$ is the positive odd integers). The ideal nwd consists of all sets $A \subseteq \mathbb{N}$ such that for every $s \in \mathbb{N}$, there exists an extension t of s such that $A \cap \langle t \rangle = \emptyset$.[5] It is straightforward to check that nwd is a nontrivial ideal (that is, a proper ideal containing all of the finite subsets of $\mathbb{N}$).

Recall that the Rado coloring was defined in Section 2.4.

Theorem 9.4. *The following are equivalent for any countably infinite graph G.*

(i) *For every nontrivial ideal $\mathcal{I}$ on $\mathbb{N}$, G is $\mathcal{I}^+$-Ramsey.*
(ii) *In the Rado coloring of $K_{\mathbb{N}}$, there is a monochromatic copy of G such that $V(G) \in \mathrm{nwd}^+$.*
(iii) *G is finitely-ruled.*

Proof of Theorem 9.4. (i $\implies$ ii) nwd is a nontrivial ideal on $\mathbb{N}$, so in every 2-coloring of the edges of $K_{\mathbb{N}}$ (and in particular, the Rado coloring), there is a monochromatic copy of G with $V(G) \in \mathrm{nwd}^+$.

(ii $\implies$ iii) Suppose G is infinitely-ruled and there exists a monochromatic copy of G in the Rado coloring of $K_{\mathbb{N}}$ with color $i \in \{0, 1\}$. We show that $V(G) \in \mathrm{nwd}$.

Let F_n ($n \in \mathbb{N}$) be pairwise-disjoint, finite ruling sets in G, and fix $s \in \mathbb{N}$. Then there is some n such that for all $t \in F_n$, $t > s$. Let $u \in \mathbb{N}$ with $u > \max\{t \mid t \in F_n\}$ such that u extends s and the tth bit of u is $1 - i$ for all $t \in F_n$. This means that no vertex in $\langle u \rangle$ is adjacent to any vertex in F_n in color i. Since F_n is a ruling set, this implies that $V(G) \cap \langle u \rangle$ is finite. Since $V(G) \cap \langle u \rangle$ is finite, there exists $u' > u$ such that u' extends u and $V(G) \cap \langle u' \rangle = \emptyset$, and thus, $V(G) \in \mathrm{nwd}$.

(iii $\implies$ i) Let F be a finite subset of $V(G)$ for which $G \setminus F$ is 0-ruled, and let $n = |F|$. Fix a proper ideal $\mathcal{I}$ on X containing the finite subsets of X, and let $\mathcal{U}$ be an ultrafilter contained in $\mathcal{I}^+$ (which exists by Theorem 9.3). Now we proceed exactly as in the proof of Theorem 1.6. □

We see that Theorem 1.6 is a special case of Theorem 9.4 since, in particular, Theorem 9.4 shows that every finitely-ruled graph G is $\mathcal{Z}_0^+$-Ramsey. Problem 1.11 asks whether the converse is true; that is, if G is $\mathcal{Z}_0^+$-Ramsey, is G finitely-ruled? Theorem 9.4 might be viewed as evidence towards this conclusion, since it shows that this is true at least for the coideal nwd^+ in place of $\mathcal{Z}_0^+$. However, we might view Theorem 9.4 as evidence in the opposite direction, since one would expect there to be some infinite graph G which distinguishes the coideals nwd^+ and $\mathcal{Z}_0^+$ as nwd and $\mathcal{Z}_0$ have very different properties as ideals. Of course, this is all just speculation.

9.3. Infinitely ruled graphs and relative density zero ideals

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. The ideal $\mathcal{Z}_f$ is defined to be the set of all $A \subseteq \mathbb{N}$ such that $|A \cap \{1, \ldots, n\}|/f(n) \to 0$ as $n \to \infty$. The ideal $\mathcal{Z}_0$ is one example, where we take f to be the identity function. The reader may check that $\mathcal{Z}_f \subseteq \mathcal{Z}_g$ whenever $f \leq g$, though of course the converse does not hold.

In Theorem 2.15, we showed that if G is infinitely ruled and the ruling sets grow slowly enough, then G is not $\mathcal{Z}_0^+$-Ramsey (i.e., G is not Ramsey dense). In this section, we will give an example of a family $\mathcal{G}$ of infinitely ruled graphs (where the size of the ruling sets may go to infinity at any prescribed rate, no matter how slowly) such that for all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n)/n \to 0$ and for all $G \in \mathcal{G}$, G is $\mathcal{Z}_f^+$-Ramsey.

Let T be a tree with a fixed root r. Given vertices $u, v \in V(T)$, we say that v is an *extension* of u if u lies on the unique path from r to v. We say that two vertices in T are *compatible* if one is an extension

[5]The notation nwd stands for 'nowhere dense', and typically the ideal nwd is studied on the set $\mathbb{Q}$; however, for consistency with the rest of the paper, we state all of the results in terms of the set $\mathbb{N}$. That being said, it is possible to show that the set $\mathbb{N}$, when given the topology generated by the sets $\langle s \rangle$, is homeomorphic to the space $\mathbb{Q} \cap [0, 1)$, and under this homeomorphism, the sets in nwd correspond to those subsets of $\mathbb{Q} \cap [0, 1)$ which are not dense in any subinterval of $[0, 1)$.

of the other. The *compatibility graph of* (T, r), $C_{T,r}$, is the graph with vertex set $V(T)$ and edges $\{u, v\}$ for all compatible vertices u and v. (This is sometimes called the *downward closure* of (T, r).)

An *antichain* in T is a set of pairwise-incompatible vertices, and we call an antichain A *maximal* if there is no antichain B such that A is a proper subset of B. Note that every finite maximal antichain in T is a ruling set in $C_{T,r}$; in particular, if T is locally finite, then the sets $R_n = \{v \in V(T) \mid d_T(v, r) = n\}$ form finite ruling sets in $C_{T,r}$.

We say that T is *perfect* if every vertex has at least two incompatible extensions. If T is locally finite and perfect, then $|R_n| \to \infty$ as $n \to \infty$, but the growth of this sequence may be arbitrarily slow.

Theorem 9.5. *Let $f : \mathbb{N} \to \mathbb{N}$ be any function satisfying $f(n)/n \to 0$, and let T be any locally finite, perfect tree with fixed root r. Then $C_{T,r}$ is $\mathcal{Z}_f^+$-Ramsey.*

Theorem 9.5 is immediately implied by the following two results.

Proposition 9.6. *Suppose $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(n)/n \to 0$. Then for any finite coloring of $K_{\mathbb{N}}$, there is a monochromatic complete multipartite subgraph, with finite parts, whose vertex set is in $\mathcal{Z}_f^+$.*

Proposition 9.7. *Let T be a perfect tree with a fixed root r, and let M be a complete, infinite multipartite graph, with finite parts. Then there is a copy of $C_{T,r}$ in M which spans all but finitely many of the parts of M.*

Proof of Theorem 9.6. Consider the ideal $\mathcal{I}_f$ consisting of all sets $A \subseteq \mathbb{N}$ satisfying

$$\sup_n |A \cap \{1, \dots, n\}|/f(n) < \infty$$

(Note that the assumption on f implies that $\mathcal{I}_f$ is proper.) Clearly, $\mathcal{Z}_f \subseteq \mathcal{I}_f$. We moreover note that if $A_1, A_2, \dots$ is a sequence from $\mathcal{I}_f^+$ satisfying $A_1 \supseteq A_2 \supseteq \cdots$, then there is a set $A \in \mathcal{I}_f^+$ satisfying $|A \setminus A_n| < \infty$ for all n. (For instance, A may be constructed by letting $A = \bigcup_n A_n \cap [k_n, k_{n+1}]$, where k_n is chosen recursively to satisfy $|A_n \cap [k_n, k_{n+1}]|/f(k_{n+1}) \geq n$.)

Now fix an r-coloring χ of $K_{\mathbb{N}}$. We will recursively construct sets $A_1, A_2, \dots \in \mathcal{I}_f^+$, and an r-coloring ρ of $\mathbb{N}$, such that $A_1 \supseteq A_2 \supseteq \cdots$ and for all n and $m \in A_n$, $\chi(\{n, m\}) = \rho(n)$. This construction goes as follows. First, we set $A_1 = \mathbb{N}$. Now, given A_n, note that the sets $A_n \cap N_i(n)$, for $i = 1, \dots, r$, partition $A_n \setminus \{n\}$, and hence at least one must be in $\mathcal{I}_f^+$ (by Theorem 9.2). We choose one to be A_{n+1} and define $\rho(n)$ to be the associated color i.

Now by the above-mentioned property of $\mathcal{I}_f^+$, we may find a single set $A' \in \mathcal{I}_f^+$ such that $|A' \setminus A_n| < \infty$ for all n. Now let $A = A' \cap \rho^{-1}(i)$ for some $i \in [r]$ such that $A \in \mathcal{I}_f^+$. Hence, for all $n \in A$, there are only finitely-many $m \in A$ for which $\chi(\{n, m\}) \neq i$. In other words, the graph $K_i[A]$ consisting of edges of color i induced on vertex set A has the property that every vertex has cofinite degree. This allows us to construct, recursively, a sequence $b_0 < b_1 < \cdots$ such that for all $n, m \in A$ with $n \leq b_k$ and $m \geq b_{k+1}$, $\chi(\{n, m\}) = i$. Let

$$A_0^* = A \cap \bigcup_{k=0}^{\infty} [b_{2k}, b_{2k+1})$$

and $A_1^* = A \setminus A_0$. Then each of A_0^* and A_1^* is the vertex set of a monochromatic multipartite graph with finite parts, and moreover, at least one is in $\mathcal{I}_f^+$, and hence, $\mathcal{Z}_f^+$ (by Fact 9.1, since $\mathcal{Z}_f \subseteq \mathcal{I}_f$). □

Proof of Theorem 9.7. Let $r_1 := r$ and let $P = r_1 r_2 \dots r_k$ be the shortest path from r_1 to a vertex r_k such that r_k has at least two successors (if r_1 itself has two successors, then $r_k = r_1$ and $P = r_1$ is just a trivial path). Note that every vertex r_i on the path P is a maximal antichain (and thus a ruling set in $C_{T,r}$). Also, every vertex $v \in V(C_{T,r}) \setminus V(P)$ is part of a maximal infinite independent set which we denote $I(v)$.

Let $K := K_{1,\dots,1,\mathbb{N},\mathbb{N},\dots}$ be the complete multipartite graph with k parts of order 1 and infinitely many infinite parts. Clearly, K can be embedded into M in such that way that K spans all but finitely many parts of M. We will show that $C_{T,r}$ can be surjectively embedded into K which will then complete the proof.

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First, we embed the path $P = r_1 \dots r_k$ into the parts of order 1. Let $v_1, v_2, \dots$ be an enumeration of the remaining vertices of K. Note that this ordering induces an ordering $V^1, V^2, \dots$ of the infinite parts themselves (meaning that if $v_n \in V^j$, then $\{v_1, v_2, \dots, v_n\} \subseteq \bigcup_{i=1}^{j} V_i$) and an ordering v_1^i, v_2^i of each V^i. Finally, let $u_1, u_2, \dots$ be an enumeration of $V(T) \setminus V(P)$ such that $T[\{r_1, \dots, r_k, u_1, \dots, u_i\}]$ is connected for all $i \geq 1$.

Initially, we set $f(u_1) = v_1$ (where $v_1 \in V^1$) and then we repeat the following two steps.

Step 1. Let $v_j^i \in V^i$ be the smallest vertex in $V(K) \setminus \operatorname{ran} f$. If $j = 1$ (i.e., $V^i \cap \operatorname{ran} f = \emptyset$, move to Step 2. Otherwise, let $u \in \operatorname{dom} f$ such that $f(u) \in V^i$. Now let $u' \in I(u) \setminus \operatorname{dom} f$ and set $f(u') = v_j^i$).

Step 2. Let m be the largest index such that $u_{m-1} \in \operatorname{dom} f$ and let $U' = \{u_1', u_2', \dots, u_\ell'\} := \{u_1, \dots, u_m\} \setminus \operatorname{dom} f$. Let n be the largest index such that $V^n \cap \operatorname{ran} f \neq \emptyset$. Now for all $i \in [\ell]$, set $f(u_i') = v_1^{n+i}$ (where v_1^{n+i} is the first vertex in V^{n+i}). Note that for all $i \geq 1$, v_1^{n+i} is adjacent to every vertex in $\operatorname{ran} f$.

At the end of each instance of Step 1 and Step 2, we have covered the first available vertex in $V(K) \setminus \operatorname{ran} f$, and we have embedded an entire interval $\{u_1, u_2, \dots, u_m\}$ in the ordering of $V(T)$, including the first available vertex in $V(T) \setminus \operatorname{dom} f$. Thus, we have defined a surjective embedding of $C_{T,r}$ into K, which completes the proof. □

In the proof of Theorem 9.6, we have a graph with vertex set $A \in \mathcal{I}_f^+$ in which every vertex has cofinite degree. We use this to show that A can be partitioned into two infinite complete multipartite graphs with all parts finite, one of which, call it M, must have vertex set in $\mathcal{I}_f^+$. We then show that if T is perfect tree with fixed root r, then $C_{T,r}$ can be embedded into M. This raises the following two questions.

Problem 9.8.

(i) Characterize all graphs which can be cofinitely embedded into every graph in which every vertex has cofinite degree.
(ii) Characterize all graphs which can be cofinitely embedded into every infinite complete multipartite graph with finite parts.

10. Conclusion and open problems

10.1. Graphs of bounded chromatic number/maximum degree/degeneracy

Let G be a graph with $\Delta := \Delta(G) \geq 2$. We know that $2 \leq \chi(G) \leq \Delta(G) + 1$, and we proved that $\frac{1}{2(\Delta-1)} \leq \overline{\mathrm{Rd}}(G) \leq \frac{1}{\chi(G)-1}$. It would be interesting to know whether these bounds can be improved in general. From Theorem 2.2.(ii), we know that the bound in Theorem 1.2.(i) cannot be improved without further restrictions.

Problem 10.1. If possible, improve the bounds in Theorem 1.2.(ii),(iii).

We make the following conjecture which would imply Theorem 1.5 in the case where G has no finite dominating set (see Theorem 6.5 and Theorem 6.6).

Conjecture 10.2. *Let $d \in \mathbb{N}$ and let H_d be the graph defined in Theorem 6.4. There exists $c = c(d) > 0$ such that $\overline{\mathrm{Rd}}(H_d) \geq c$. More weakly, $\overline{\mathrm{Rd}}(H_d) > 0$.*

We know that Theorem 10.2 is true when $d = 1$ because $H_1 = T_\infty$ and $\overline{\mathrm{Rd}}(T_\infty) = 1/2$. Even solving the conjecture when $d = 2$ would be a big step forward.

Note that for all $r \geq 3$, Theorem 2.3 shows that $\overline{\mathrm{Rd}}_r(H_d) = 0$ for all $d \in \mathbb{N}$. This stands in contrast to the finite case where Lee's proof [23] shows that Theorem 1.4 holds for any number of colors. However, if G is d-degenerate and locally finite, then Theorem 1.2.(iii) implies that $\overline{\mathrm{Rd}}_r(G) \geq 1/r^{dr}$. So for more than 2 colors, the only interesting case is d-degenerate graphs with infinitely many vertices of finite degree and some vertices of infinite degree.

10.2. Ramsey-dense graphs and graphs with positive upper Ramsey density

We know that every 0-ruled graph is Ramsey-dense, and we know that there exist 0-ruled graphs G with $\overline{\mathrm{Rd}}(G) = 0$, but all such graphs G that we know of have $\chi(G) = \infty$ (see Theorem 2.5). So we ask the following question.

Problem 10.3. Does there exist a graph G which is 0-ruled and $\chi(G) < \infty$, but $\overline{\mathrm{Rd}}(G) = 0$? For instance, is $\overline{\mathrm{Rd}}(\mathcal{R}_2) = 0$? (where $\mathcal{R}_2$ is the bipartite Rado graph)

Note that for $r \geq 3$, Theorem 2.3 provides examples of 0-ruled graphs G with $\chi(G) < \infty$, but $\overline{\mathrm{Rd}}_r(G) = 0$.

In Theorem 1.11, we ask if there are Ramsey-dense graphs with $\mathrm{rul}(G) = \infty$. A more expansive series of questions is the following.

Problem 10.4.

(i) Characterize all Ramsey-dense graphs.
(ii) Characterize graphs G with $\overline{\mathrm{Rd}}(G) > 0$.
(iii) Characterize graphs G which are Ramsey-dense, but $\overline{\mathrm{Rd}}(G) = 0$.

In Theorem 9.5, we proved that every graph in a certain class of graphs with infinite ruling number is $\mathcal{Z}_f^+$-Ramsey. So we raise the following problem. Also, see Theorem 9.8.

Problem 10.5. Characterize all graphs G having the property that for all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n)/n \to 0$, G is $\mathcal{Z}_f^+$-Ramsey.

10.3. Ramsey-lower-dense and Ramsey-cofinite graphs

One of the main results in the paper is a characterization of all Ramsey-cofinite forests. The most interesting open problem here is the following (cf. Theorem 8.11).

Problem 10.6. Characterize all Ramsey-cofinite graphs.

In Theorem 1.14, we proved that every forest is either Ramsey-cofinite or is not Ramsey-lower-dense. We suspect that this is true of every graph (in [10, Problem 8.10] we asked the weaker question of whether $\overline{\mathrm{Rd}}(G) < 1$ implies that G has Ramsey-lower-density 0).

Conjecture 10.7. *For every graph G, if G is not Ramsey-cofinite, then G is not Ramsey-lower-dense.*

10.4. Ramsey-upper-density of trees

We showed that $\overline{\mathrm{Rd}}(T) \geq 1/2$ for every infinite tree T, and we showed that this result is tight for some trees such as T_∞. Lamaison [21] obtained sharp results on Ramsey upper densities of locally finite trees.

It would be interesting to extend some of these results to more colors. By Theorem 2.3, we know that for all $r \geq 3$, if T is a tree with finitely many vertices of finite degree, then $\overline{\mathrm{Rd}}_r(T) = 0$. However, we know from Theorem 1.2, that if T is an infinite, locally finite tree, then $\overline{\mathrm{Rd}}_r(T) \geq 1/r$ (which is best possible for some trees); or more generally, if T is an infinite, one-way k-locally finite tree for some $k \geq 2$, then $\overline{\mathrm{Rd}}_r(T) \geq 1/r^{(k-2)r+1}$. So for more than 2 colors, one should focus on trees with infinitely many vertices of finite degree which are not one-way k-locally finite for any $k \geq 2$.

10.5. Bipartite graphs

We conjectured (Theorem 1.17) that the vertices of every r-colored $K_{\mathbb{N},\mathbb{N}}$ can be partitioned into a finite set and at most r monochromatic paths. We proved this for $r = 2$ and mentioned how a result of Day and Lo [8] implies that if Theorem 1.17 is true, then for all $r \geq 3$, $\overline{\mathrm{Rd}}_r(P_\infty) \geq \frac{1}{r-1}$ (which is now an open problem for $r \geq 5$).

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