Chapter 1

Smooth manifolds

As mentioned in the preface, spacetime is, on the set-theoretic level, a collection of points, called events. We want to do calculus on spacetime, because firstly, the field equation determining the geometry of spacetime is a partial differential equation on spacetime, and secondly, the geodesic equation describing motion of matter in spacetime is an ordinary differential equation on spacetime. It was discovered in the early 20^{th} century that the appropriate structure to be used to describe spacetime is that of a Lorentzian smooth manifold. In this chapter, we will first learn about the notion of a smooth manifold.

Roughly speaking, an m-dimensional smooth manifold is a set M which can be covered by patches such that in each patch one can introduce coordinates and use them to do calculus. As coordinates are ad-hoc, we need to make sure that only certain 'admissible' coordinates are allowed, so that the definition of smoothness is independent of the choice of coordinates. It turns out that this also endows the manifold with a topology, so that smooth manifolds are special types of topological spaces.

1.1 Charts and atlases

We have prior experience with using coordinates, for example, for the surface of a sphere, one could use polar and azimuthal angles. But then we realise that in order to obtain an injective mapping from the sphere to the set of parameter ranges of the coordinates, we must work with patches on the sphere, instead of the whole sphere in one go. So we anticipate that also in the case of a manifold, it will need to be covered by patches, and coordinates need to be set up in each patch.

Before we introduce the notion of a manifold, we explain what we mean by putting coordinates in a patch. The aim of this section is to introduce the notions of a 'chart' and an 'atlas'. Roughly speaking, a chart describes the local coordinates set up in a patch of the manifold, and an atlas is a collection of such charts so that the union of the patches of the charts covers the whole manifold. In order that there is no conflict later on when defining smooth objects on the manifold, we will demand that the charts that make up an atlas are 'compatible'.

First, let us recall the notion of a topological space.

Definition 1.1. (Topological space.)
A *topological space* $(M, \mathcal{O})$ is a set M together with a collection $\mathcal{O}$ of subsets of M, such that the following hold:

(T1) $\varnothing, M \in \mathcal{O}$.

(T2) Whenever U_i, $i \in I$, belong to $\mathcal{O}$, also $\bigcup_{i \in I} U_i \in \mathcal{O}$.

(T3) Whenever $U, V \in \mathcal{O}$, also $U \cap V \in \mathcal{O}$.

The elements of $\mathcal{O}$ are called *open sets*, and $\mathcal{O}$ itself is referred to as a *topology on* M.

Recall that $\mathbb{R}^m$ can be equipped with its usual Euclidean topology.

Example 1.1. (Euclidean space $\mathbb{R}^m$.)
Let $\mathbb{R}^m = \{\mathbf{x} = (x^1, \cdots, x^m) : x^\ell \in \mathbb{R},\ 1 \leqslant \ell \leqslant m\}$ be the real vector space with componentwise operations of vector addition and multiplication by real scalars. For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, their Euclidean inner product is given by $\langle \mathbf{x}, \mathbf{y} \rangle = x^1 y^1 + \cdots + x^m y^m$. For $\mathbf{x} \in \mathbb{R}^m$, we define the ball $B(\mathbf{x}, r)$ with center $\mathbf{x}$ and radius $r > 0$ by $B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^m : \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle < r^2\}$. We define $\mathcal{O}$ to be the collection of subsets U of $\mathbb{R}^m$ with the property that whenever $\mathbf{x} \in U$, there exists an $r > 0$ such that $B(\mathbf{x}, r) \subset U$. Then $\mathcal{O}$ is a topology on $\mathbb{R}^m$. $\diamond$

Definition 1.2. (Chart.)
Let M be a set. An *m-chart* on M is a pair (U, φ), where $U \subset M$, the map $\varphi : U \to \mathbb{R}^m$ is injective, and $\varphi(U)$ is an open subset of $\mathbb{R}^m$.

Henceforth, we will often drop the specification 'm' in 'm-chart', and simply refer to 'charts' for M, with the understanding that for a given M, the m is fixed. A chart allows us to talk about the *coordinates* of a point $p \in U \subset M$, with respect to the chart (U, φ), as the m-tuple of numbers $\varphi(p) \in \mathbb{R}^m$.

Example 1.2. (A chart for $\mathbb{R}^m$.) Let $U = \mathbb{R}^m$, and $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be the identity map id. Then $(\mathbb{R}^m, \mathrm{id})$ is a chart on $\mathbb{R}^m$. $\diamond$

We will see later on that locally a manifold looks like $\mathbb{R}^m$ (its tangent space).

In Example 1.1, there is a distinguished point, namely the origin, but as nature does not provide natural coordinate systems, we also introduce $\mathbb{R}^m$ which has forgotten its origin, namely an affine space.

Example 1.3. (Affine space.)
An *affine space of dimension* m consists of

- a set M (of 'points'),
- an m-dimensional vector space V (whose vectors 'translate' points of M),
- a map $M \times V \ni (p, \mathbf{v}) \mapsto p + \mathbf{v} \in M$,

such that the following hold:

(A1) for all $p \in M$, $\mathbf{u}, \mathbf{v} \in V$, $p + (\mathbf{u} + \mathbf{v}) = (p + \mathbf{u}) + \mathbf{v}$,
(A2) for all $p \in M$, $p + \mathbf{0} = p$,
(A3) for all $p, q \in M$, there exists a unique $\mathbf{v}_{pq} \in V$ such that $q = p + \mathbf{v}_{pq}$.

Let $p \in M$ be fixed and let $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$ be a basis for V. Given any $q \in M$, there exists a unique vector $\mathbf{v}_{pq} \in V$ such that $q = p + \mathbf{v}_{pq}$. This vector $\mathbf{v}_{pq}$ can be expressed in terms of the basis vectors, giving unique coordinates $\varphi(q) := (x^1, \cdots, x^m) \in \mathbb{R}^m$. Thus, $q = p + \mathbf{v}_{pq} = p + x^i \mathbf{e}_i$. Clearly, φ is injective, and with $U := M$, $\varphi(U) = \mathbb{R}^m$. So (M, φ) is a chart on M. $\diamond$

Exercise 1.1. Show that if $p, q, r \in M$, then $\mathbf{v}_{pr} = \mathbf{v}_{pq} + \mathbf{v}_{qr}$.

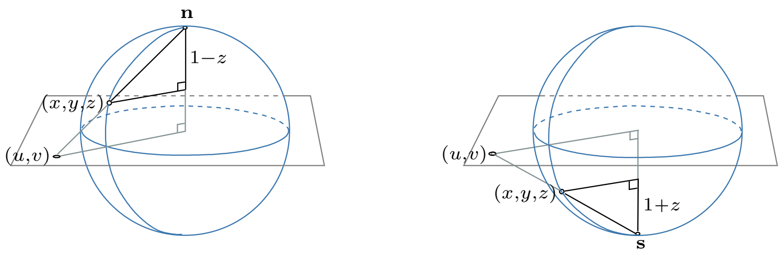
Example 1.4. (Charts on the sphere S^2.) Let

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

and $\mathbf{n} = (0, 0, 1)$, $\mathbf{s} = (0, 0, -1)$ denote the north and south poles in S^2. Set $U_\mathbf{n} = S^2 \backslash \{\mathbf{n}\}$, $U_\mathbf{s} = S^2 \backslash \{\mathbf{s}\}$, and define the 'stereographic' projections

$$S^2 \backslash \{\mathbf{n}\} = U_\mathbf{n} \ni (x, y, z) \longmapsto \varphi_\mathbf{n}(x, y, z) = \frac{1}{1 - z}(x, y) \in \mathbb{R}^2,$$

$$S^2 \backslash \{\mathbf{s}\} = U_\mathbf{s} \ni (x, y, z) \longmapsto \varphi_\mathbf{s}(x, y, z) = \frac{1}{1 + z}(x, y) \in \mathbb{R}^2.$$



Charts $(U_\mathbf{n}, \varphi_\mathbf{n})$ and $(U_\mathbf{s}, \varphi_\mathbf{s})$ on S^2. Looking at the two similar triangles in the left picture, we have $(1 - z) : 1 = x : u$ and $(1 - z) : 1 = y : v$. Analogously, from the two similar triangles in the right picture, $(1 + z) : 1 = x : u$ and $(1 + z) : 1 = y : v$. This holds irrespective of the sign of z.

Then $(U_\mathbf{n}, \varphi_\mathbf{n})$ and $(U_\mathbf{s}, \varphi_\mathbf{s})$ are charts on S^2. $\diamond$

Exercise 1.2. Show that the inverse $\varphi_{\mathbf{n}}^{-1} : \mathbb{R}^2 \to S^2 \backslash \{\mathbf{n}\}$ of the map $\varphi_{\mathbf{n}}$ is given by $(u,v) \stackrel{\varphi_{\mathbf{n}}^{-1}}{\longmapsto} (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1})$. *Hint:* $u^2+v^2 = \frac{x^2+y^2}{(1-z)^2} = \frac{1-z^2}{(1-z)^2}$.

Exercise 1.3. Let $H^2 = \{(x,y,t) \in \mathbb{R}^3 : x^2+y^2-t^2 = -1, t>0\}$. The line joining the south pole $\mathbf{s} = (0,0,-1)$ to a point $\mathbf{p} \in H^2$ meets the xy-plane at $\mathbf{x}(\mathbf{p}) \in \mathbb{R}^2$. Show that $\varphi_{\mathbf{s}}(\mathbf{p}) = \frac{1}{1+t}(x,y)$, $\mathbf{p} = (x,y,t) \in H^2$, and $(H^2, \varphi_{\mathbf{s}})$ is a chart on H^2.

Exercise 1.4. Show that $(\mathbb{R}, x \mapsto x^3)$ is a chart on $\mathbb{R}$.

Exercise 1.5. (Cylinder.)
Consider the cylinder C in $\mathbb{R}^3$ given by $C = S^1 \times \mathbb{R} = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$. Let $U_{x+} := \{(x,y,z) \in C : x > 0\}$, $\varphi_{x+}(x,y,z) = (y,z)$. Show that (U_{x+}, φ_{x+}) is a chart on C. Similar charts (U_{x-}, φ_{x-}), (U_{y+}, φ_{y+}) and (U_{y-}, φ_{y-}) can be defined analogously, so that the union of $U_{x+}, U_{x-}, U_{y+}, U_{y-}$ contains C.

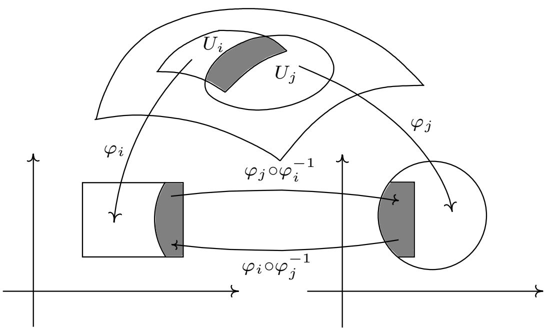
A collection of charts for M (with the same m) will form an atlas provided they cover the set M and satisfy a compatibility condition.

Definition 1.3. (Atlas.)
Let M be a set. A collection of m-charts $\{(U_i, \varphi_i) : i \in I\}$ on M is called an *m-atlas* if it has the following properties:

(A1) $\bigcup_{i \in I} U_i = M$.

(A2) For all $i,j \in I$, $\varphi_i(U_i \cap U_j)$ is open in $\mathbb{R}^m$.

(A3) For all $i,j \in I$, $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is C^∞.
(The maps $\varphi_j \circ \varphi_i^{-1}$ are called *chart transition maps.*)



Compatible charts (U_i, φ_i) and (U_j, φ_j).

Just as with charts, we will often drop the m, and speak simply of an atlas, instead of an m-atlas. Note that $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a bijection. Recall that a function $f : U \to \mathbb{R}^n$ is C^∞ on an open subset U of $\mathbb{R}^m$ if the components f^i of f, $1 \leqslant i \leqslant n$, have at each point of U, all partial derivatives of all orders with respect to the variables x^j, $1 \leqslant j \leqslant m$, which are also continuous on U.

The single charts in Examples 1.2, 1.3 and Exercise 1.4 are all atlases in a trivial manner.

Example 1.5. (S^2 revisited.)
The charts $(U_\mathbf{n}, \varphi_\mathbf{n})$ and $(U_\mathbf{s}, \varphi_\mathbf{s})$ from Example 1.4 form an atlas for S^2. Firstly, $U_\mathbf{n} \cup U_\mathbf{s} = S^2$. Secondly, $\varphi_\mathbf{n}(U_\mathbf{n} \cap U_\mathbf{s}) = \mathbb{R}^2 \backslash \{(0,0)\} = \varphi_\mathbf{s}(U_\mathbf{n} \cap U_\mathbf{s})$. Finally, $\varphi_\mathbf{s} \circ \varphi_\mathbf{n}^{-1}$, $\varphi_\mathbf{n} \circ \varphi_\mathbf{s}^{-1} : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2 \backslash \{(0,0)\}$, the chart transition maps, are both given by $(u,v) \mapsto \frac{(u,v)}{u^2+v^2}$, which is C^∞. ◇

Exercise 1.6. We revisit Exercise 1.3. Defining the projection map $p : H^2 \to \mathbb{R}^2$ by $p(x,y,t) = (x,y)$ for all $(x,y,t) \in H^2$, it is easy to see that p is injective and $p(H^2) = \mathbb{R}^2$. Thus (H^2, p) is a chart for H^2. Show that the charts $(H^2, \varphi_\mathbf{s})$ and (H^2, p), form an atlas for H^2.

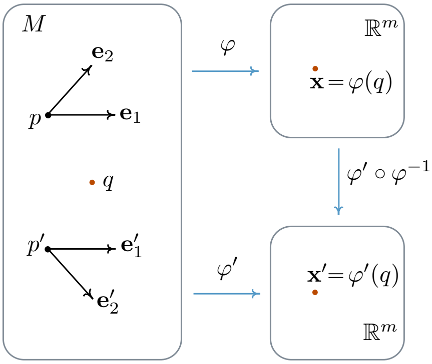
Exercise 1.7. Show that the four charts in Exercise 1.5 form an atlas for the cylinder C in $\mathbb{R}^3$.

Example 1.6. (A non-atlas.) With $m = 1$ in Example 1.2, we get the chart $(\mathbb{R}, \varphi_1)$, with the chart map $\varphi_1(x) := x$ for $x \in \mathbb{R}$. On the other hand, in Exercise 1.4, we had seen that $(\mathbb{R}, \varphi_2 := (x \mapsto x^3))$ is yet another chart on $\mathbb{R}$. While they individually form atlases $\mathcal{A}_1 := \{(\mathbb{R}, \varphi_1)\}$ and $\mathcal{A}_2 := \{(\mathbb{R}, \varphi_2)\}$, their union $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2 = \{(\mathbb{R}, \varphi_1), (\mathbb{R}, \varphi_2)\}$ does not form an atlas. Indeed, not all chart transition maps are smooth. Although $\varphi_2 \circ \varphi_1^{-1} = (x \mapsto x^3)$ is smooth on $\mathbb{R}$, $\varphi_1 \circ \varphi_2^{-1} = (x \mapsto x^{1/3})$ is not C^∞ everywhere on $\mathbb{R}$ since it is not differentiable at $x = 0$. ◇

The previous example motivates the following definition.

Definition 1.4. (Compatible atlases.)
Let M be a set. Two m-atlases $\mathcal{A}_1, \mathcal{A}_2$ are *compatible* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is also an m-atlas on M.

Example 1.7. (Affine space revisited.) It is clear from the chart map definition given in Example 1.3, that different choices of points p, and of bases $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$, will lead to different coordinates. Consider points $p, p' \in M$, and bases $B = \{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$, $B' = \{\mathbf{e}'_1, \cdots, \mathbf{e}'_m\}$ for V, giving the chart maps φ, φ'. Are the atlases $\{(M, \varphi)\}$ and $\{(M, \varphi')\}$ compatible? To investigate this, we compute the chart transition map $\varphi' \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^m$.



Change of coordinates.

Given $\mathbf{x} = (x^1, \cdots, x^m) \in \mathbb{R}^m$, $q := \varphi^{-1}(\mathbf{x}) \in M$, and we wish to find the coordinates $\mathbf{x}'$ of this q using the point p' and the basis B'. So we need to write $q = p' + \mathbf{v}_{p'q}$, and find $\mathbf{x}'$ by expanding $\mathbf{v}_{p'q}$ using the basis B'. We have $\mathbf{v}_{p'q} = \mathbf{v}_{p'p} + \mathbf{v}_{pq}$, and as $q = \varphi^{-1}(\mathbf{x})$, also $\mathbf{v}_{pq} = x^i \mathbf{e}_i$. Introduce the change of basis matrix $A = [A_i^j] \in \mathrm{GL}_m(\mathbb{R})$, where A_i^j denotes the entry in the j^{th} row and i^{th} column of A, defined by $\mathbf{e}_i = A_i^j \mathbf{e}'_j$. Also, let $\mathbf{b} = (b^1, \cdots, b^m) \in \mathbb{R}^m$ be defined by $\mathbf{v}_{p'p} = b^j \mathbf{e}'_j$. Then

$$\mathbf{v}_{p'q} = \mathbf{v}_{p'p} + \mathbf{v}_{pq} = b^j \mathbf{e}'_j + x^i \mathbf{e}_i = b^j \mathbf{e}'_j + x^i (A_i^j \mathbf{e}'_j) = (b^j + A_i^j x^i)\, \mathbf{e}'_j,$$

and so the chart transition map $\varphi' \circ \varphi^{-1}$ is the affine linear map given by $\varphi' \circ \varphi^{-1}(\mathbf{x}) = \mathbf{b} + A\mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^m$, which is C^∞. Its inverse $\varphi \circ (\varphi')^{-1}$ is given by $\varphi \circ (\varphi')^{-1}(\mathbf{x}) = -A^{-1}\mathbf{b} + A^{-1}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^m$. So the atlases $\{(M, \varphi)\}$ and $\{(M, \varphi')\}$ are compatible. ◇

Exercise 1.8. Show that the set $\mathbb{R}^m \times \mathrm{GL}_m(\mathbb{R})$ is a group with the composition $(\mathbf{b}_2, A_2) \cdot (\mathbf{b}_1, A_1) = (\mathbf{b}_2 + A_2\mathbf{b}_1, A_2A_1)$ for $(\mathbf{b}_2, A_2), (\mathbf{b}_1, A_1)$ in $\mathbb{R}^m \times \mathrm{GL}_m(\mathbb{R})$.

Exercise 1.9. Prove that compatibility is an equivalence relation on the collection of all atlases on a set M.

Thus to specify a manifold[1], we should work with compatible atlases (so that in hindsight, there will be no conflict when we define smooth objects on the manifold), and given that compatibility is an equivalence relation, we just need to commit to one particular atlas for the set M at hand. We make this a definition.

[1] The name ‘manifold’ comes from the German word ‘mannigfaltigkeit’ used by Riemann in his doctoral thesis, which contained, among other things, a discussion of multi-valued complex functions and their (now called) Riemann-surfaces.

Definition 1.5. (Smooth manifold, dimension of a manifold.)
A *smooth manifold* $(M, [\mathcal{A}])$ is a set M together with an equivalence class $[\mathcal{A}]$ of compatible m-atlases on M. We call m the *dimension* of the smooth manifold M. A chart from an atlas in $[\mathcal{A}]$ is said to be *admissible* for the smooth manifold. We refer to $[\mathcal{A}]$ as a *smooth structure on* M.

Thus the atlases given in Examples 1.2, 1.3, 1.4 and Exercise 1.5 can be used to make the respective set M into a smooth manifold. As we will use it frequently, we will call $\mathbb{R}^m$ with the atlas comprising the single chart $(\mathbb{R}^m, \text{id})$ as the *smooth manifold* $\mathbb{R}^m$ *with the standard smooth structure*.

In Example 1.6, the two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are not compatible. Hence, $(\mathbb{R}, [\mathcal{A}_1])$ and $(\mathbb{R}, [\mathcal{A}_2])$ are two different smooth manifolds.

Example 1.8. (Sphere.) Consider the sphere as a smooth manifold with the smooth structure given by the atlas in Example 1.5. In this example, we give a different compatible atlas, using one of the charts as the familiar one with spherical polar coordinates. It can be shown that the map

$$(0, \pi) \times (0, 2\pi) \ni (\theta, \phi) \mapsto \big((\sin\theta)\cos\phi, (\sin\theta)\sin\phi, \cos\theta\big) \in S^2 \subset \mathbb{R}^3$$

is injective, and hence a bijection onto its image

$$U = S^2 \backslash \{(x, y, z) \in \mathbb{R}^3 : y = 0 \text{ and } x \geqslant 0\}.$$

For a point $p \in U$, the angle $\theta(p)$ is called the *polar angle of* p, and the angle $\phi(p)$ is called the *azimuthal angle of* p. For a point $p \in U$, we define the map φ on U by $\varphi(p) := (\theta(p), \phi(p)) \in (0, \pi) \times (0, 2\pi)$, where if $p = (x, y, z)$, then $\theta(p) = \cos^{-1} z$ and

$$\phi(p) := \measuredangle(x, y) := \begin{cases} \cos^{-1} \frac{x}{\sqrt{x^2+y^2}} & \text{if } y > 0, \\ \pi - \sin^{-1} \frac{y}{\sqrt{x^2+y^2}} & \text{if } x < 0, \\ 2\pi - \cos^{-1} \frac{x}{\sqrt{x^2+y^2}} & \text{if } y < 0. \end{cases}$$

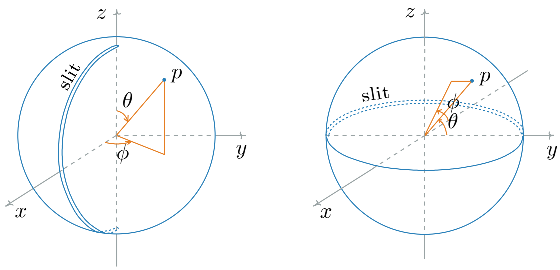
Here $\cos^{-1} : (-1, 1) \to (0, \pi)$ and $\sin^{-1} : (-1, 1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ are the inverse trigonometric functions. It can be checked that ϕ is well-defined and that the map $\mathbb{R}^2 \backslash \{(x, y) \in \mathbb{R}^2 : x \geqslant 0\} \ni (x, y) \mapsto \measuredangle(x, y)$ is C^∞. Using this, it can be checked that (U, φ) is an admissible chart: e.g., if $v > 0$, then

$$\varphi_{\mathbf{n}}(U_{\mathbf{n}} \cap U) \ni (u, v) \overset{\varphi \circ \varphi_{\mathbf{n}}^{-1}}{\longmapsto} \Big(\cos^{-1} \frac{u^2+v^2-1}{u^2+v^2+1}, \cos^{-1} \frac{u}{\sqrt{u^2+v^2}}\Big)$$

is C^∞, and for all $(\theta, \phi) \in (0, \pi) \times (0, 2\pi)$, we have

$$\varphi(U \cap U_{\mathbf{n}}) \ni (\theta, \phi) \overset{\varphi_{\mathbf{n}} \circ \varphi^{-1}}{\longmapsto} \Big(\frac{(\sin\theta)\cos\phi}{1 - \cos\theta}, \frac{(\sin\theta)\sin\phi}{1 - \cos\theta}\Big)$$

is C^∞.



The chart U covers S^2 except for a 'slit', namely the intersection of S^2 with the half plane $\{(x, y, z) : y = 0,\ x \geqslant 0\}$. In order to cover S^2, we can take another chart (V, ψ), defined in a similar manner, by taking a differently placed slit, in a plane perpendicular to the one containing the original slit. Then V together with U, covers S^2. More explicitly, V covers S^2 except for the intersection of S^2 with the half plane $\{(x, y, z) : z = 0,\ x \leqslant 0\}$. V is the image of the map

$$(0, \pi) \times (0, 2\pi) \ni (\theta, \phi) \mapsto \big(-(\sin\theta)\cos\phi, \cos\theta, (\sin\theta)\sin\phi\big) \in S^2 \subset \mathbb{R}^3,$$

and this map is obtained by taking the polar angle with the positive y-axis, and the azimuthal angle with the negative x-axis counterclockwise in the $y = 0$ plane; see the picture above. ◇

Exercise 1.10. Consider the square $S := \{(x, y) \in \mathbb{R}^2 : |x| + |y| = 1\}$. Show that $\mathcal{A} := \{(U_+, \varphi_+), (U_-, \varphi_-), (V_+, \psi_+), (V_-, \psi_-)\}$ is an atlas for S, where

$$\begin{aligned} U_+ &:= \{(x, y) \in S : x > 0\} & \varphi_+(x, y) &= y, \\ U_- &:= \{(x, y) \in S : x < 0\} & \varphi_-(x, y) &= y, \\ V_+ &:= \{(x, y) \in S : y > 0\} & \psi_+(x, y) &= x, \\ V_- &:= \{(x, y) \in S : y < 0\} & \psi_-(x, y) &= x. \end{aligned}$$

Thus $(S, [\mathcal{A}])$ is a smooth manifold. So a smooth manifold may not necessarily 'appear' smooth.

We now discuss four 'spacetimes' by just looking at the underlying smooth structure. Later on, our examples below will be made 'Lorentzian' manifolds, that is, smooth manifolds with some added structure. While we are not yet ready to specify the added structure, we nevertheless introduce these by only describing their smooth structures. Thus these are 'pre-' spacetimes for now, ripe for becoming legitimate spacetimes later.

Example 1.9. (Minkowski spacetime.) Let V be a 4-dimensional real vector space. Suppose that M is an affine space over V. Take any $p \in M$, and a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ for V, and let $\varphi : M \to \mathbb{R}^4$ be the corresponding chart as in Example 1.3. Let $\mathcal{A} := \{(M, \varphi)\}$. Then $(M, [\mathcal{A}])$ is a smooth manifold, referred to as the *Minkowski spacetime*. ◇

Example 1.10. (Cylindrical spacetime.) Let $M = \mathbb{R} \times S^1$ with the atlas $\mathcal{A}$ comprising the charts defined in Exercise 1.5. Then $(M, [\mathcal{A}])$ is a smooth manifold, referred to as the *cylindrical spacetime.* ◇

Example 1.11. (FLRW spacetime.) Let $I := (0, \infty)$. Then $M := I \times \mathbb{R}^3$, with the atlas $\{(I \times \mathbb{R}^3, \mathrm{id}_{I\times\mathbb{R}^3})\}$ is a 4-dimensional smooth manifold, called the FLRW *spacetime* (after Friedman, Lemaitre, Robertson, Walker). ◇

Exercise 1.11. (Product of smooth manifolds.) Let M be an m-dimensional smooth manifold, with an atlas $\mathcal{A}_M = \{(U_i, \varphi_i), i \in I\}$. Let N be an n-dimensional smooth manifold, with an atlas $\mathcal{A}_N = \{(V_j, \psi_j), j \in J\}$. Define for $i \in I$, $j \in J$, the maps $\varphi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^{m+n}$ by $(\varphi_i \times \psi_j)(p, q) = (\varphi_i(p), \psi_j(q))$ for all $p \in U_i$, $q \in V_j$. Show that $\{(U_i \times V_j, \varphi_i \times \psi_j), i \in I, j \in J\}$ is an atlas for $M \times N$, making it an $(m + n)$-dimensional smooth manifold.

Example 1.12. (Schwarzschild[2] spacetime.) Let $m > 0$, and $I := (2m, \infty)$. Let $M = \mathbb{R} \times I \times S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$. Taking the atlases $\{(\mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$, $\{(I, \mathrm{id}_I)\}$, and $\{(U, \varphi), (V, \psi)\}$ (Example 1.8), for $\mathbb{R}, I, S^2$, respectively, we see that M is a smooth manifold using the construction based on Exercise 1.11. We call this 4-dimensional smooth manifold the *Schwarzschild spacetime.* ◇

1.2 Topology on a smooth manifold

We will want to talk about continuous maps between smooth manifolds, for example a 'worldline' in a spacetime (Definition 1.2). The way we equip a smooth manifold with a topology is by insisting that the chart maps are homeomorphisms (Theorem 1.2). This is the motivation for the following definition.

Definition 1.6. (Open set in a smooth manifold.)
Let $(M,[\mathcal{A}])$ be an m-dimensional smooth manifold and $\{(U_i, \varphi_i), i \in I\} \in [\mathcal{A}]$. A set $U \subset M$ is *open* if for all $i \in I$, $\varphi_i(U \cap U_i)$ is open in $\mathbb{R}^m$, where $\mathbb{R}^m$ is given its standard Euclidean topology, described by the Euclidean metric

$$d(\mathbf{x}, \mathbf{y}) := \sqrt{\sum_{i=1}^{m} (x^i - y^i)^2} \quad \mathbf{x} = (x^1, \cdots, x^m),\ \mathbf{y} = (y^1, \cdots, y^m) \in \mathbb{R}^m.$$

Proposition 1.1. *Definition 1.6 of an open set is well-defined, that is, it does not depend on the choice of the atlas in* $[\mathcal{A}]$*.*

Proof. Let $\mathcal{A}_1 = \{(U_i, \varphi_i), i \in I\}$ and $\mathcal{A}_2 = \{(V_j, \psi_j), j \in J\}$ be atlases in $[\mathcal{A}]$. Let $U \subset M$, and suppose for each $i \in I$, $A_i := \varphi_i(U \cap U_i)$ is open in

[2] After Karl Schwarzschild (1873–1916), a German physicist and astronomer.

$\mathbb{R}^m$. Let $j \in J$. We must show that $\psi_j(U \cap V_j)$ is open in $\mathbb{R}^m$. We have

$$\psi_j(U \cap V_j) = \psi_j\big((U \cap M) \cap V_j\big) = \psi_j\big(U \cap \big(\bigcup_i U_i\big) \cap V_j\big) = \psi_j\big(\bigcup_i (U \cap U_i \cap V_j)\big)$$
$$= \bigcup_i \psi_j(U \cap U_i \cap V_j). \qquad (\star)$$

Set $B_i := \varphi_i(U_i \cap V_j)$. Then B_i is open, since the charts (U_i, φ_i) and (V_j, ψ_j) belong to the atlas $\mathcal{A}_1 \cup \mathcal{A}_2$. The intersection of this open B_i with the open set $A_i = \varphi_i(U \cap U_i)$, is open. Now $A_i \cap B_i = \varphi_i(U \cap U_i \cap V_j)$. (Indeed, $\supset$ is trivially true, and $\subset$ follows from the injectivity of φ_i on U_i.) Consider the C^∞ (and in particular, continuous) map $\varphi_i \circ \psi_j^{-1} : \psi_j(U_i \cap V_j) \to \varphi_i(U_i \cap V_j)$. As the open set $A_i \cap B_i = \varphi_i(U \cap U_i \cap V_j)$ is contained in the open set $\varphi_i(U_i \cap V_j) \subset \mathbb{R}^m$, it follows that $(\varphi_i \circ \psi_j^{-1})^{-1}(A_i \cap B_i)$ is an open subset of the open set $\psi_j(U_i \cap V_j) \subset \mathbb{R}^m$, that is,

$$(\varphi_i \circ \psi_j^{-1})^{-1}(A_i \cap B_i) = \psi_j(\varphi_i^{-1}(\varphi_i(U \cap U_i \cap V_j))) = \psi_j(U \cap U_i \cap V_j)$$

is open in $\mathbb{R}^m$. So $\psi_j(U \cap V_j) \overset{(\star)}{=} \bigcup_i \psi_j(U \cap U_i \cap V_j)$ is open in $\mathbb{R}^m$. □

We show that calling such sets ‘open’ is justified, as they form a topology on the manifold.

Theorem 1.1. *Let $(M, [\mathcal{A}])$ be an m-dimensional smooth manifold. Then the collection $\mathcal{O} := \{U \subset M : U$ is open in $M\}$ is a topology on M.*

Proof. Let $\{(U_i, \varphi_i), i \in I\} \in [\mathcal{A}]$. Then $\varnothing = \varphi_i(\varnothing \cap U_i)$ is open in $\mathbb{R}^m$ for all $i \in I$, and so $\varnothing \in \mathcal{O}$. Also, for all $i \in I$, $\varphi_i(U_i) = \varphi_i(M \cap U_i)$ is open in $\mathbb{R}^m$ since (U_i, φ_i) is a chart, and so $M \in \mathcal{O}$.

Let $U, V \in \mathcal{O}$. Then for all $i \in I$, $\varphi_i((U \cap V) \cap U_i) = \varphi_i(U \cap U_i) \cap \varphi_i(V \cap U_i)$ ($\subset$ is always true for any map, and $\supset$ holds by the injectivity of φ_i). Being the intersection of open sets, $\varphi_i((U \cap V) \cap U_i)$ is open in $\mathbb{R}^m$ for all $i \in I$, and consequently, $U \cap V \in \mathcal{O}$.

Let $V_j \in \mathcal{O}$ for all $j \in J$. Then we have that for all $i \in I$,

$$\varphi_i\big(\big(\bigcup_j V_j\big) \cap U_i\big) = \bigcup_j \varphi_i(V_j \cap U_i),$$

is open in $\mathbb{R}^m$, as it is the union of open sets $\varphi_i(V_j \cap U_i)$ in $\mathbb{R}^m$. Hence, $\bigcup_j V_j \in \mathcal{O}$. □

Definition 1.7. (Topology induced by a smooth structure.)
Let $(M, [\mathcal{A}])$ be an m-dimensional smooth manifold. Then the collection $\mathcal{O} := \{U \subset M : U$ is open (Definition 1.6) in $M\}$ is called the *topology induced on M by the smooth structure* $[\mathcal{A}]$.

Remark 1.1. Often in the literature, a smooth manifold is defined by first introducing the concept of a 'topological manifold', where one starts with a topological space which can be covered by charts which are homeomorphisms to open subsets of $\mathbb{R}^m$. We have not adopted this route, since such an approach forces one to begin with a topology. But we now reconcile our definition with this prevalent one in the following result. $\circledast$

Theorem 1.2.
Let $(M, [\mathcal{A}])$ be an m-dimensional smooth manifold, and let

$$\mathcal{O} := \{U \subset M : U \text{ is open in } M\}$$

be the topology induced on M by the smooth structure $[\mathcal{A}]$. Suppose that $\{(U_i, \varphi_i), i \in I\} \in [\mathcal{A}]$. Then for each $i \in I$, $\varphi_i : U_i \to \varphi_i(U_i)$ is a homeomorphism.

Proof. Let $i \in I$. As (U_i, φ_i) is a chart, we know that $\varphi_i(U_i)$ is open in $\mathbb{R}^m$ and that $\varphi_i : U_i \to \varphi_i(U_i)$ is a bijection. We only need to show the continuity of φ_i and φ_i^{-1}. Let $V \subset \varphi_i(U_i)$ be open. Then $\varphi_i^{-1}V \subset U_i$. We must show that this is an open set in M. For any $j \in I$, we have $\varphi_j((\varphi_i^{-1}V) \cap U_j) = (\varphi_j \circ \varphi_i^{-1})(V \cap \varphi_i(U_i \cap U_j))$. As V and $\varphi_i(U_i \cap U_j)$ are open in $\mathbb{R}^m$, so is their intersection. Thus $(\varphi_j \circ \varphi_i^{-1})(V \cap \varphi_i(U_i \cap U_j))$, being the inverse image under the (C^∞ and hence) continuous map $(\varphi_j \circ \varphi_i^{-1})^{-1}$ of the open set $V \cap \varphi_i(U_i \cap U_j)$ ($\subset \varphi_i(U_i \cap U_j)$), is open. Hence $\varphi_j((\varphi_i^{-1}V) \cap U_j)$ is open for all $j \in I$, that is, $\varphi_i^{-1}V$ is open in M. So $\varphi_i : U_i \to \varphi_i(U_i)$ is continuous.

Let $U \subset U_i$ be open. We want to show that $\varphi_i(U) = (\varphi_i^{-1})^{-1}U$ is open in $\mathbb{R}^m$ (and hence open in $\varphi_i(U_i)$). The fact that U is open means in particular that $\varphi_i(U \cap U_i)$ is open in $\mathbb{R}^m$. But $\varphi_i(U \cap U_i) = \varphi_i(U)$, since $U \subset U_i$. Thus the inverse map $\varphi_i^{-1} : \varphi_i(U_i) \to U_i$ is also continuous. $\square$

Exercise 1.12. Let $\mathbb{R}^m$ be equipped with the standard smooth structure. Show that the topology induced by this smooth structure coincides with the standard Euclidean topology.

Exercise 1.13. Consider the double cone $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\} \subset \mathbb{R}^3$. Show that C cannot carry a smooth structure $[\mathcal{A}]$ making it a 2-dimensional smooth manifold such that the topology induced by $[\mathcal{A}]$ on C coincides with the subspace topology on C (as a subset of $\mathbb{R}^3$ with its standard Euclidean topology). If we delete the point $\mathbf{0} = (0, 0, 0)$ from C, i.e., we consider $C_* := C\backslash\{\mathbf{0}\}$, then we do get a smooth manifold, for example by taking an atlas comprising two charts, namely $(\{(x, y, z) \in C_* : z > 0\}, \pi)$ and $(\{(x, y, z) \in C_* : z < 0\}, \pi)$, where the chart map π in each case is just the restriction to these chart domains of the projection map onto the xy-plane: $\mathbb{R}^2 \ni (x, y, z) \mapsto (x, y) \in \mathbb{R}^2$.

Remark 1.2. (Hausdorff and second countable assumptions on $\mathcal{O}$.)
In order to do analysis, it is desirable to have two additional properties enjoyed by the topology $\mathcal{O}$:

(H) A topology $\mathcal{O}$ on a set M is *Hausdorff* if for every $p, q \in M$, there exist $U, V \in \mathcal{O}$ such that $p \in U$, $q \in V$, and $U \cap V = \varnothing$. Thus distinct points possess disjoint neighbourhoods, a type of 'separation axiom'. Such a property is quite basic, since otherwise limits of sequences are not guaranteed to be unique.

(S) A *basis for* $\mathcal{O}$ is a collection $\mathcal{B} = \{B_i : i \in I\}$ of open sets such that every open set in $\mathcal{O}$ is a union of elements from $\mathcal{B}$. A topology $\mathcal{O}$ on a set M is *second countable* if there exists a countable basis for $\mathcal{O}$. When wanting to do 'integration' on manifolds, this property will be needed in order to construct a so-called 'partition of unity', which will essentially mean that we can use m-charts to set up Riemann integrals of functions defined on the manifold, and patch these contributions to obtain an integral of the function defined on the whole manifold.

Unfortunately, for a smooth manifold, neither of these properties are guaranteed to hold for the topology $\mathcal{O}$ from Theorem 1.1. So, in order to proceed without pitfalls, we will make a standing assumption that whenever we talk of a smooth manifold in this book, we will mean in addition that the associated topology $\mathcal{O}$ is Hausdorff and second countable. The standard topology of the Euclidean space $\mathbb{R}^m$ generated by the 2-norm $\|\cdot\|$ satisfies the second countability assumption since the open balls with centers all of whose components are rational numbers, and whose radius is also a rational number, form a countable basis. Now, if the manifold can be covered by an atlas in the smooth structure containing countably many charts, then it follows that (since the chart maps are homeomorphisms) the images of members of the countable basis for $\mathbb{R}^m$ under the inverse of the chart maps will form a countable basis for the topology of the manifold. All the examples of smooth manifolds considered in this book will be of this type. $\circledast$

Exercise 1.14. Let U be an open subset of a smooth manifold M given by an atlas $\mathcal{A}$. Let $\mathcal{A}_U := \{(U \cap V, \psi|_{U\cap V}) : (V, \psi) \in \mathcal{A}\}$. Show that $\mathcal{A}_U$ is an atlas for U. Prove that if (W, σ) is admissible for M, then $(U \cap W, \sigma|_{U\cap W})$ is admissible for $(U, [\mathcal{A}_U])$. U is then said to be given the *smooth structure induced by* $(M, [\mathcal{A}])$. In particular, if (U, φ) is an admissible chart for M, then $[\mathcal{A}_U] = [\{(U, \varphi)\}]$.

As a spacetime M is the collection of all events, the life of a particle can be modelled by a curve in M by stringing together all the events encountered by the particle in its lifetime. Let $I \subset \mathbb{R}$ be an interval and M be a smooth manifold. A continuous map $\gamma : I \to M$ is called a *curve* or a *worldline*.

1.3 Smooth maps

The point of the definition of a smooth manifold is to enable the consideration of smooth objects on it, for example, a real-valued smooth function (think of temperature), a 'vector field', etc., in an unambiguous way.

Definition 1.8. (Smooth map.)
Let M, N be smooth manifolds, with dimensions m, n, respectively. A map $f : M \to N$ is said to be *smooth* if for all $p \in M$,

- there exists an admissible chart (U, φ) for M such that $p \in U$,
- there exists an admissible chart (V, ψ) for N such that $f(U) \subset V$ (in particular $f(p) \in V$),
- $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n$ is C^∞ on $\varphi(U) \subset \mathbb{R}^m$.

If M is a smooth manifold, and $\mathbb{R}$ has the standard smooth structure, then we use the notation $C^\infty(M)$ to denote the set of all smooth maps $f : M \to \mathbb{R}$.

For a smooth manifold M, the identity map $\mathrm{id}_M : M \to M$ is smooth.

Example 1.13. (Chart maps are smooth.)
Let (U, φ) be a chart from an atlas defining the smooth manifold M. We now consider U itself to be a smooth manifold, described by the trivial atlas $\{(U, \varphi)\}$. Then $\varphi(U) \subset \mathbb{R}^m$ is an open subset of $\mathbb{R}^m$. We consider $\varphi(U)$ as a smooth manifold described by the atlas comprising the single chart $(\varphi(U), \mathrm{id}_{\varphi(U)})$. We claim that the chart map $\varphi : U \to \varphi(U)$ is smooth. For each $p \in U$, we take the admissible chart (U, φ) for U containing p, and the admissible chart $(V := \varphi(U), \mathrm{id}_{\varphi(U)})$ for the smooth manifold $\varphi(U)$. Then $\varphi(U) = V$. Moreover, $\mathrm{id}_{\varphi(U)} \circ \varphi \circ \varphi^{-1} = \mathrm{id}_{\varphi(U)} : \varphi(U) \to \varphi(U) \subset \mathbb{R}^m$, which is clearly C^∞. As $p \in U$ was arbitrary, $\varphi : U \to \varphi(U)$ is smooth. $\diamond$

Exercise 1.15. Let M, N be smooth manifolds and $f : M \to N$ be a smooth map. Show that f is continuous.

Exercise 1.16. Let M_1, M_2, M_3 be smooth manifolds, and let $f_{12} : M_1 \to M_2$, $f_{23} : M_2 \to M_3$ be smooth maps. Prove that $f_{23} \circ f_{12} : M_1 \to M_3$ is smooth.

Exercise 1.17. Let M, N be smooth manifolds, and $M \times N$ be the smooth manifold described in Exercise 1.11. Let the projection map $\pi_M : M \times N \to M$ be given by $M \times N \ni (p, q) \mapsto p \in M$. Given a $q \in N$, let the injection map $i_q : M \to M \times N$ be given by $M \ni p \mapsto (p, q) \in M \times N$.

- Show that π_M is smooth. (Similarly, $M \times N \ni (p, q) \mapsto q \in N$ is smooth.)
- Show that i_q is smooth. (Also, for $p \in M$, $N \ni q \mapsto (p, q) \in M \times N$ is smooth.)

In particular, Exercise 1.17 has the following consequences. Firstly, given any $g \in C^\infty(M)$, the map $M \times N \ni (p,q) \mapsto g(p) \in \mathbb{R}$, is an element of $C^\infty(M \times N)$, as it is the composition of the smooth maps g and π_M. Secondly, given an $f \in C^\infty(M \times N)$ and a $q \in N$, the 'slice map' f_q, given by $M \ni p \mapsto f(p,q) \in \mathbb{R}$ is smooth too, since $f_q = f \circ i_q$. We will use these observations later on to show that the 'tangent space of $M \times N$ at (p,q)' can be identified with $T_pM \times T_qN$ in Exercise 2.8.

Exercise 1.18. (Smoothness is a local property.) Let M, N be smooth manifolds. Show that $f : M \to N$ is smooth if and only if for every U open in M, $f|_U : U \to N$ is smooth. Here U has the induced smooth structure from that of M.

The operations $+, \cdot : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ are defined pointwise:

$$\left.\begin{array}{r}(f+g)(p) = f(p)+g(p)\\ (f\cdot g)(p) = f(p)\cdot g(p)\end{array}\right\}\text{ for all } p \in M.$$

It can be checked that $f+g,\ f\cdot g \in C^\infty(M)$, and that with these operations, $(C^\infty(M), +, \cdot)$ is a ring, with the additive identity being the zero function $\mathbf{0} \in C^\infty(M)$ (given by $M \ni p \mapsto \mathbf{0}(p) := 0 \in \mathbb{R}$), and the multiplicative identity $\mathbf{1} \in C^\infty(M)$ (given by $M \ni p \mapsto \mathbf{1}(p) := 1 \in \mathbb{R}$). However, $C^\infty(M)$ is not a field, since not every[3] $f \in C^\infty(M)\backslash\{\mathbf{0}\}$ will have a multiplicative inverse. We will see later that the set of 'smooth vector fields' on a manifold has the natural structure of a module over the ring $C^\infty(M)$.

We will meet geodesics later on, which will be the 'straightest' possible curves in the Lorentzian manifold, describing paths of 'freely falling' particles. The straight lines in Euclidean space and great circles on the sphere S^2 are geodesics. In any case, they are 'smooth' curves.

Definition 1.9. (Smooth curve.)
A smooth map $\gamma : I \to M$, where I is an open interval in $\mathbb{R}$, is called a *smooth curve*. If $I \subset \mathbb{R}$ is any interval, not necessarily open, then a curve $\gamma : I \to M$ is a *smooth curve* if there exists an open interval $\widetilde{I} \supset I$, and a smooth curve $\widetilde{\gamma} : \widetilde{I} \to M$ such that $\widetilde{\gamma}|_I = \gamma$.

Just like in linear algebra, where one aim is to classify vector spaces up to isomorphisms, in differential geometry, the notion analogous to an isomorphism is that of a diffeomorphism.

Definition 1.10. (Diffeomorphism.)
Let M, N be smooth manifolds. A bijection $f : M \to N$ such that f and $f^{-1} : N \to M$ are both smooth, is called a *diffeomorphism*, and M and N are then said to be *diffeomorphic*.

[3] Consider an $f \in C^\infty(M)\backslash\{\mathbf{0}\}$ that has a zero at some point. In fact, in Chapter 2, we will construct nonzero functions that vanish outside a neighbourhood of a point.

Example 1.14. (Chart maps are diffeomorphisms.) Let (U, φ) be a chart from an atlas defining the smooth manifold M, and consider U as a smooth manifold with the atlas $\{(U, \varphi)\}$. Recall from Example 1.13 that the chart map $\varphi : U \to \varphi(U)$ is smooth. Also, it is a bijection onto the open set $\varphi(U)$. We show that its inverse $\varphi^{-1} : \varphi(U) \to U$ is smooth too. For all $\varphi(p) \in \varphi(U)$, with $p \in U$, we take the admissible chart $(\varphi(U), \mathrm{id}_{\varphi(U)})$ for $\varphi(U)$ containing $\varphi(p)$, and take the admissible chart (U, φ) for U. Then $\varphi^{-1}(\varphi(U)) = U$. Moreover, $\varphi \circ \varphi^{-1} \circ (\mathrm{id}_{\varphi(U)})^{-1} = \mathrm{id}|_{\varphi(U)} : \varphi(U) \to \varphi(U)$, which is clearly C^∞. As this happens with each point in $\varphi(U)$, we conclude that $\varphi^{-1} : \varphi(U) \to U$ is smooth. ◇

Exercise 1.19. Let M be an affine space over V, considered as a smooth manifold in the usual way. For a $\mathbf{v} \in V$, define $\gamma_{\mathbf{v}} : \mathbb{R} \to M$ by $\gamma_{\mathbf{v}}(t) = p + t\mathbf{v}$, $t \in \mathbb{R}$. Show that $\gamma_{\mathbf{v}}$ is a smooth curve.

Exercise 1.20. Let U, V be open subsets of $\mathbb{R}^m, \mathbb{R}^n$, respectively. We consider U, V as smooth manifolds with the smooth structures $[\{(U, \mathrm{id}_U)\}], [\{(V, \mathrm{id}_V)\}]$, respectively. Show that $f : U \to V$ is smooth if and only if f is C^∞.

Exercise 1.21. Let $\mathbb{R}$ be equipped with the two incompatible atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ given in Example 1.6. Prove that $(\mathbb{R}, [\mathcal{A}_1])$ is diffeomorphic to $(\mathbb{R}, [\mathcal{A}_2])$. (From our earlier considerations, the incompatibility of $\mathcal{A}_1$ with $\mathcal{A}_2$ can be expressed by saying that the *identity map* fails to be a diffeomorphism between the smooth manifolds $(\mathbb{R}, [\mathcal{A}_1])$ and $(\mathbb{R}, [\mathcal{A}_2])$. However, this exercise shows that there may nevertheless be other maps which serve as a diffeomorphism.)

Exercise 1.22. Let M, N be smooth manifolds, and $f : M \to N$ be a diffeomorphism. If (U, φ) is an admissible chart for M, then it is easy to see that $(f(U), \varphi \circ f^{-1})$ is a chart for N. Show that $(f(U), \varphi \circ f^{-1})$ is an admissible chart for N.

Exercise 1.23. Let M be a smooth manifold. Show that the set

$$\mathrm{Diff}(M) := \{f : M \to M \,|\, f \text{ is a diffeomorphism}\},$$

together with the operation $\circ$ of composition of maps, forms a group.

Exercise 1.24. (Lie group and left-translation diffeomorphisms.) A *Lie group* is a group $(G, \cdot)$ equipped with a smooth structure, such that the multiplication map $G \times G \ni (p, q) \mapsto p \cdot q \in G$, and the inverse map $G \ni q \mapsto q^{-1} \in G$, are smooth. Given $p \in G$, the *left-translation by* p is the map $L_p : G \to G$ defined by $G \ni q \mapsto p \cdot q$. Show that L_p is a diffeomorphism for each $p \in G$.

Exercise 1.25. (Submanifolds.) Suppose that M is an m-dimensional smooth manifold. A subset $N \subset M$ is said to be a *submanifold* of dimension $n \leqslant m$ if for each $p \in N$, there exists an admissible chart (U, φ) of M such that $p \in U$, and $\varphi(U \cap N) = \widetilde{U} \times \{\mathbf{0}\} \subset \mathbb{R}^n \times \mathbb{R}^{m-n} = \mathbb{R}^m$, where $\widetilde{U}$ is an open subset of $\mathbb{R}^n$. Then (U, ϕ) is called an *allowed chart for* N. Let $\pi : \mathbb{R}^m \to \mathbb{R}^n$ be the projection map onto the first n components.

Prove that

$$\mathcal{A}_N := \{(U \cap N, \pi \circ \varphi|_{U \cap N}) : (U, \varphi) \text{ is an allowed chart for } N\}$$

is an atlas for N. So N is a smooth manifold with the smooth structure $[\mathcal{A}_N]$. Show that the inclusion map $i : N \hookrightarrow M$ is smooth.

Exercise 1.26. Let M_1, M_2 be smooth manifolds and N_1, N_2 be submanifolds of M_1, M_2, respectively. If $f : M_1 \to M_2$ is a smooth map such that $f(N_1) \subset N_2$, then show that $f|_{N_1} : N_1 \to N_2$ is also smooth.

Before beginning with the second chapter, we make a remark on some notation which will be used from now on. For a smooth manifold M, we will often take for granted that its dimension is denoted by m. Charts will often be denoted by (U, φ), but also by $(U, \mathbf{x})$, where the understanding is that the component functions of the map $\mathbf{x} : U \to \mathbb{R}^m$ are denoted by $x^i : U \to \mathbb{R}$, $1 \leqslant i \leqslant m$. Moreover, given a function $f : M \to \mathbb{R}$, a point $p \in M$, and an admissible chart (U, φ), we will denote the partial derivative of $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ with respect to the i^{th} variable at the point $\varphi(p)$ by

$$\frac{\partial(f \circ \varphi^{-1})}{\partial u^i}(\varphi(p)).$$