GRAPH AND HYPERGRAPH PACKING

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ABSTRACT

Packing problems in combinatorics concern the edge disjoint embedding of a family of guest (hyper)graphs into a given host (hyper)graph. Questions of this type are intimately connected to the field of design theory, and have a variety of significant applications. The area has seen important progress in the last two decades, with a number of powerful new methods developed. Here, I will survey some major results contributing to this progress, alongside background, and some ideas concerning the methods involved.

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1. INTRODUCTION AND BACKGROUND

Assume that we want to test the efficacy of n drugs. One challenge when setting up an experiment for this is that the efficacy may vary with different characteristics of the individuals a drug is used by, such as age, ethnicity, sex or existing medical conditions; and we want to control for such variances. One method applied to address this in statistical experiments is *blocking*: Individuals are grouped into blocks of similar characteristics, so that within one block we can directly compare outcomes. For simplicity, let us assume that each of these blocks has size q, which we shall think of as relatively small, allowing for fine-grained control. Let us also assume that the number n of drugs tested is large compared to q. In this scenario, if we want to gain an overall picture of the (pairwise) relative efficacy of the drugs, and each individual is given one of the drugs, we also want the property that each pair of drugs is tested on two individuals from the same block. The most efficient way of guaranteeing this is to require additionally that each pair of drugs appears only in one block. Mathematically, what this is asking for is a certain type of combinatorial (block) design (for a definition see Section 3).

Various generalizations are natural: We could ask for r-wise comparisons instead of pairwise comparisons; we could ask that each set of r drugs is contained in exactly λ instead of only one block; we could allow blocks of different sizes; or we could ask that for each block there is always a collection of other blocks, disjoint among themselves and the chosen block, that partition the set of administered drugs (this is called a resolvable design; a different example of this is given below). All these and many others have been considered in what is known as design theory (for a comprehensive overview of the area, see [16]). These designs are also special types of so-called (hyper)graph packings (we formally introduce packings in Section 2). Two of the most fundamental questions concerning these mathematical objects are: For which parameters do these packings exist? And how can they be constructed? Combinatorics has recently seen particularly rich progress concerning these questions, alongside the development of powerful new methods. In this survey I will outline some of the most important of these results and methods.

The connection of designs to statistical experiments was formalized by Fisher in the first half of the 20th century (see, e.g., [26]). Historically, however, questions related to designs appeared already earlier; the following description is based on [66]. In the 1830s, motivated by the study of certain plane cubic curves, Plücker discovered a particular design, the so-called 9-point affine plane. Later extensions of his work by Fano were important for the development of projective geometry. Soon after, designs featured in recreational mathematics. The 1844 edition of the *Lady's and Gentleman's Diary*, according to its title page a journal "designed principally for the amusement and instruction of students in mathematics: comprising many useful and entertaining particulars, interesting to all persons engaged in that delightful pursuit", presented the following prize problem, posed by the editor, Revd. Woolhouse:

- Determine the number of combinations that can be made out of n symbols, p symbols in each; with this limitation, that no combination of q symbols, which may appear in any one of them shall be repeated in any other.

This asks for a partial design maximizing the number of blocks. In the 1850 edition of the *Lady's and Gentleman's Diary*, Revd. Kirkman posed what is nowadays known as "Kirkman's schoolgirls problem":

- Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.

This asks for a resolvable design with $n = 15$, block size $q = 3$, and with $r = 2$. A quest for solutions, generalizations, along with rediscoveries, and discussions over priority ensued. See [66] for a detailed historical account and the mathematicians involved in these early developments. I will return to designs in Section 3.

Another recreationally motivated packing problem was posed by Ringel in 1967 (see [32]) and had the well-being of the mathematical community in mind: At Oberwolfach meetings, each meal the participants are assigned a seat at one of the possibly differently sized tables. Is a succession of assignments possible for these meals, such that no participant sits next to another participant twice? This asks for a packing of so-called cycle factors in the complete graph on n vertices, where n is the number of meeting participants, and each cycle represents one of the tables. I will return to this in Section 4.

These are just some examples. There is a wealth of other prominent problems that can be formulated as graph or hypergraph packing problems, including the search for Latin squares, or orthogonal Latin squares. Applications also arise in the construction of certain codes, or in information security. (See, for example, [40] for more details.) In this survey though I will not explore these further, but concentrate on (mainly recent) mathematical progress concerning packings instead.

Organization. It is not my goal in this contribution to exhaustively survey the vast amount of results that have so far been obtained concerning designs and packings. Instead, I aim to highlight some important recent progress and to discuss some newly developed techniques that made this progress possible. The remainder is organized as follows. In Section 2 some notation and basic definitions are introduced. In Section 3 we discuss recent breakthrough results concerning designs. Sections 4 and 5 concentrate on packing results for cycles, and trees, respectively. Section 6 considers packing problems for more general classes of graphs, while Section 7 briefly mentions related results for hypergraphs. In Section 8, finally, some important open problems are collected.

2. NOTATION AND BASIC DEFINITIONS

We denote the set $\{1, \dots, n\}$ by $[n]$. A *graph* $G = (V, E)$ consists of a set of vertices V (which is always finite here) and a set of edges $E \subseteq \binom{V}{2}$, where each edge contains two different elements of V. An r*-uniform hypergraph* $H = (V, E)$ generalizes this notion in that it allows r vertices in each edge, that is, it requires $E \subseteq \binom{V}{r}$. We write $V(H)$ and $E(H)$ for the vertices and edges of H, respectively, and $e(H)$ for $|E(H)|$. For a vertex u and a set of vertices S in a graph G, the *neighborhood* $N_G(u)$ of u is the set of all vertices v such that uv is an edge, the *degree* of v is $|N_G(v)|$, and $N_G(S) = \bigcap_{u \in S} N_G(u)$ is the *common neighborhood* of S. Similarly, for a set U of $r-1$ vertices in an r-uniform hypergraph H, the neighborhood $N_H(U)$ of U is the set of all vertices v such that Uv is an edge. For a (hyper)graph H and a vertex set S, we write $H \setminus S$ for the sub(hyper)graph of H induced on vertex set $V(H) \setminus S$. A sub(hyper)graph of H is called *spanning* if it uses all vertices of H.

A *complete graph* K_n on n vertices is a graph in which all pairs of vertices form an edge. Analogously, the complete r-uniform hypergraph $K_n^{(r)}$ contains all r-sets as edges. A *path* in a graph is a sequence of different vertices $v_1, \ldots, v_\ell$ such that $v_i v_{i+1}$ is an edge for every $i \in [\ell - 1]$; a *cycle* in a graph is a path $v_1, \ldots, v_\ell$ plus the edge $v_\ell v_1$. A *tree* on n vertices is a graph with $n-1$ edges without cycles. A graph is *r-regular* if each vertex has degree r. A *cycle factor* of a graph H on n vertices is a 2-regular subgraph of H on n vertices. For a (hyper)graph F, an *F-factor* in a (hyper)graph H on n vertices is a spanning sub(hyper)graph of H consisting of vertex disjoint copies of F. A *Hamilton cycle* in a graph is a cycle using all the vertices. In an r-uniform hypergraph H, a *tight Hamilton cycle* is given by an ordering $v_1, \ldots, v_n$ of the vertices of H such that $v_i, \ldots, v_{i+r-1}$ forms an edge for each $i \in [n]$, where indices are taken modulo n. (The Hamilton cycle is then formed by the involved edges.)

Definition 2.1 (Packing, decomposition). For a collection $G_1, \ldots, G_t$ of (hyper)graphs, and another (hyper)graph H, the family $(G_1, \ldots, G_t)$ is said to *pack into H* if there are edge-disjoint copies of $G_1, \ldots, G_t$ in H. The packing is called *perfect* if it uses exactly all the edges of H once, that is, if $\sum_{i \in [t]} e(G_i) = e(H)$. It is called *almost-perfect* if $\sum_{i \in [t]} e(G_i) = (1 - o(1))e(H)$. A perfect packing of $(G_1, \ldots, G_t)$ is also called a *decomposition* of H into $(G_1, \ldots, G_t)$. We occasionally refer to H as the *host (hyper)graph* and to the G_i as the *guest (hyper)graphs* of the packing.

More generally, given a natural number λ, a *λ-fold packing* of $(G_1, \ldots, G_t)$ into H allows edges of H to be used more than once but not more than λ times, where we require, however, that the embeddings use distinct subgraphs of H. A packing (in the sense above) thus is a 1-fold packing. A λ-fold packing is *perfect* if it uses each edge of H exactly λ times. A perfect λ-fold packing of $(G_1, \ldots, G_t)$ into H is also called a *λ-fold decomposition* of H into $(G_1, \ldots, G_t)$.

3. DESIGNS

Given a set X of size n and a family S of distinct subsets of X, each of size q, we say that S is a *design* with parameters (n, q, r, λ) if every subset Y of X of size r is contained in exactly λ members of S. For example, it is easy to see that for $X = [7]$ the family $S = \{123, 145, 167, 246, 257, 356, 347\}$, where we denote the subset $\{s_1, s_2, s_3\}$ of S

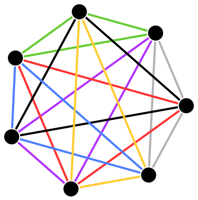

FIGURE 1
A design with parameters $(7, 3, 2, 1)$ as a packing of triangles in K_7.

by $s_1s_2s_3$, is a design with parameters $(7,3,2,1)$. In the language of hypergraphs, the problem of finding a design with parameters (n,q,r,λ) translates to the problem of finding a perfect λ-fold packing of complete r-uniform hypergraphs $K_q^{(r)}$ on q vertices in a complete r-uniform hypergraph $K_n^{(r)}$ on n vertices. A design is *resolvable* if (in the language of hypergraphs) it is also a λ-fold packing of $K_q^{(r)}$-factors. For example, the design with parameters $(7, 3, 2, 1)$ corresponds to a packing of triangles in the complete graph K_7 which uses each edge exactly once (see Figure 1). This cannot be a resolvable design since 7 is not divisible by 3.

What are necessary conditions for a design with certain parameters to exist? Firstly, it is clear that, for example, no design with parameters $(5, 3, 2, 1)$ can exist because the complete graph K_5 on 5 vertices has $\binom{5}{2} = 10$ edges, which is not divisible by 3. On the other hand, for parameters $(6, 3, 2, 1)$ the number of edges in K_6 is $\binom{6}{2} = 15$, hence divisible by 3; but each vertex of K_6 is contained in 5 edges, which cannot all edge-disjointly be covered by triangles because each triangle would use 2 edges at the vertex. The conditions resulting from simple obstacles like this are called divisibility conditions. In full generality they are as follows.

Definition 3.1 (Divisibility conditions). The parameters (n, q, r, λ) satisfy the *divisibility conditions*, if $\binom{q-i}{r-i}$ divides $\lambda\binom{n-i}{r-i}$ for every $0 \le i \le r-1$.

It is clear that these conditions are necessary because any set I of i vertices in the complete r-uniform hypergraph on n vertices is contained in $\binom{n-i}{r-i}$ edges, each of which we need to cover λ times; and for this we can use, from any copy of the complete r-uniform hypergraph using the vertices of I, the $\binom{q-i}{r-i}$ edges touching I. The existence conjecture for designs states that these conditions are also sufficient, apart from some small counterexamples.

Conjecture 3.2 (existence conjecture for designs). *Given q, r, and λ, there is n_0 so that for each $n \ge n_0$, if (n, q, r, λ) satisfy the divisibility conditions, then there is a design with these parameters.*

This conjecture and related problems inspired much work. The case of packing triangles in a complete graph, that is, $r = 2$, $q = 3$, and $\lambda = 1$ was already solved by Kirkman (see [66]). It took much longer to solve the graph case for all q: In a celebrated series of papers, Wilson settled the problem for $r = 2$ in the 1970s [67, 68, 70]. Ray-Chaudhuri and

Wilson [59] established the existence of resolvable designs in the graph case. In the 1980s Teirlinck [65] proved that nontrivial designs exist for all r (and some q and λ). Already before that, natural variations of the problem were considered. Graver and Jurkat [31] and independently Wilson [69] proved that the divisibility conditions are sufficient for so-called integral designs, where we allow the assignment of arbitrary integer weights to q-sets (instead of just weights 0 and 1) and these have to add up to λ on each r-set. Rödl [61], on the other hand, established the existence of almost-designs: families of subsets of size q that cover all but a small fraction of the r-sets of the ground set. The following theorem makes this precise for the case $\lambda = 1$.

Theorem 3.3 (Rödl [61]). *Given $r, q \in \mathbb{N}$ with $1 \le r \le q$ and given $\gamma > 0$, there is n_0 such that for each $n \ge n_0$, there is a partition of the edges of $K_n^{(r)}$ into edge-disjoint copies of $K_q^{(r)}$ and a leftover set of size at most γn^r.*

This result did not only represent important progress, but fundamentally influenced Combinatorics through the novel technique its proof introduced: the so-called *Rödl nibble*, which has been used to resolve a multitude of other important problems.

Let us briefly sketch the basic idea for the proof of Theorem 3.3 in the special case $r = 2$ and $q = 3$. In this case we want to pack roughly $\frac{1}{6}(1-\gamma')n^2$ triangles in the complete graph H on n vertices. We approach this by embedding triangles *randomly*. Now, it is clear that we cannot simply randomly throw in all the triangles at once since this would lead to lots of overlaps of triangles on edges. However, if instead we randomly throw in only a small constant proportion, say $\alpha_1 n^2$ triangles, then the following will be true. Among the $\alpha_1 n^2$ randomly embedded triangles in expectation only a small proportion, of order $\alpha_1^2 n^2$, will overlap. So, assuming we have a typical outcome of random choices, we can simply discard all overlapping triangles, and the remainder will still be a packing of more than $\frac{1}{2}\alpha_1 n^2$ triangles. Rödl's idea now was to iterate this procedure in the following way. We remove from H all edges that have been used in the packing just obtained. One can then show that what remains of H has good quasirandomness properties. This is why another random embedding round of triangles will be successful: We embed $\alpha_2 n^2$ triangles randomly into H, with high probability not many of these will overlap, which we can again discard. We can then again delete all used edges from H and proceed to the next round, and so on, until almost all edges are used (at which point the error hidden in the quasirandomness condition gets out of control). The choice of the constants α_i here is somewhat delicate but we shall not discuss this here further (see, e.g., [6, CHAPTER 4] for more details).

Returning to perfect packings, Kuperberg, Lovett, and Peled [51] proved the existence of nontrivial designs for a large range of parameters (but with λ comparatively large) before, in a celebrated breakthrough, Keevash [36] resolved the existence conjecture. The result Keevash obtains is stronger in that it allows more generally for packings in all hypergraphs with certain quasirandomness properties.

An n-vertex r-uniform hypergraph H is called (ε, a)-*typical*, if every set $A \subseteq \binom{V(H)}{r-1}$, that is, of subsets of $V(H)$ of size $r-1$, such that $|A| \le a$ satisfies $|\bigcap_{U \in A} N_H(U)| = (1 \pm \varepsilon) d^{|A|} n$, where $d = |E(H)|/\binom{n}{r}$ is the *density* of H. The mandated sizes of common neighborhoods in this definition is what we would expect to see in a random graph of density d. The divisibility conditions are adapted to this setting of an incomplete host hypergraph in the obvious way: We say that H is $K_q^{(r)}$*-divisible* if for each $0 \leq i \leq r$ and every set I of i vertices in H we have that $\binom{q-i}{r-i}$ divides $|\{e \in E(H) : I \subseteq e\}|$. Keevash's result then reads as follows.

Theorem 3.4 (Keevash [36]). *Given $q > r \geq 1$ and $\lambda \geq 1$, there exist $\varepsilon_0, \alpha > 0$ and $s, n_0 \in \mathbb{N}$ so that the following holds. Let H be a $K_q^{(r)}$-divisible (ε, s)-typical r-uniform hypergraph on $n \geq n_0$ vertices with $e(H) = d\binom{n}{r}$ edges such that $d \geq n^{-\alpha}$ and $\varepsilon \leq \varepsilon_0 d^{s^2}$. Then H has a λ-fold decomposition into $K_q^{(r)}$-copies.*

Keevash's proof of this result combines the nibble method with a powerful new approach, which he calls *randomized algebraic construction*. In its underlying philosophy, this in turn can be seen as inspired by the so-called *absorbing method*, an important contemporary technique in combinatorics pioneered by Krivelevich [48] and Rödl, Ruciński, and Szemerédi [62]. Roughly, the idea of the absorbing method is as follows: We set aside at the start a clever structure, which we call the absorber. We then obtain an almost-solution to our problem (often with the help of certain random processes, or greedy methods), leaving a small leftover. We then use the absorber to incorporate the leftover into the almost-solution, obtaining a full solution. In Keevash's proof the absorber is constructed by using edge-disjoint $K_q^{(r)}$-copies satisfying certain algebraic relations (whose definition uses some randomness); showing that this construction can absorb any leftover is complicated and requires a sequence of intricate steps. See [39] for a more detailed outline of this approach, and [37] for a detailed discussion in the special case of triangle decompositions.

Using a different influential variation of the absorbing method called *iterative absorption*, Glock, Kühn, Lo, and Osthus [30] provided a different proof of this result and more: They establish the existence of hypergraph F-designs, that is, perfect packings of copies of an arbitrary fixed r-uniform hypergraph F into the complete r-uniform hypergraph on n vertices (where n is large compared to the number of vertices in F) when suitable divisibility conditions are satisfied. For graphs, this result is due to Wilson [71].

Given F and $0 \leq i \leq r-1$, we define d_i to be the greatest common divisor of all $|\{e \in E(F) : I \subseteq e\}|$ such that I is a set of i vertices in F. We say that an r-uniform hypergraph H is F*-divisible* if for each $0 \leq i \leq r$ and every set I of i vertices in H we have that d_i divides $|\{e \in E(H) : I \subseteq e\}|$.

Theorem 3.5 (Glock, Kühn, Lo, and Osthus [30]). *Given $q > r \geq 1$, and $\varepsilon, d > 0$ such that $\varepsilon \leq 0.9(d/2)^s/(\tilde{q}^r 4^{\tilde{q}})$, where $\tilde{q} := 2q \cdot q!$ and $s := 2^r\binom{\tilde{q}+r}{r}$, there are $n_0 \in \mathbb{N}$ and $\gamma > 0$ such that the following holds. Let F be any r-uniform hypergraph on q vertices and H be an F-divisible (ε, s)-typical r-uniform hypergraph on $n \geq n_0$ vertices with $e(H) = d\binom{n}{r}$ edges, and let $\lambda \leq \gamma n$. Then H has a λ-fold decomposition into F-copies.*

The basic idea of iterative absorption is to repeatedly apply an absorbing-type technique, making the leftover more and more structured in every round, until it is structured enough so that it can be absorbed entirely. In the case of [30] "structured enough" means that before the last absorption step, the leftover will be contained in a constant-sized part of H, so that only a constant number of different possibilities for the leftover remain. For these constant number of possibilities, one can prepare at the very start by setting aside for each of them a suitable absorber. The details are involved and require lots of complex ideas, also building on previous work in [12, 13, 29, 45, 50]. An excellent exposition of the application of iterative absorption for obtaining triangle decompositions can be found in [11].

Keevash [38] greatly extended Theorem 3.4 to a decomposition result allowing a partite setting. This means that the host hypergraph as well as the guest hypergraphs come with a partition and we require that vertices of part i in a guest hypergraph gets embedded into part i of the host hypergraph. This immediately allows the construction of resolvable designs via the following observation: Assume we want to obtain a resolvable $K_q^{(r)}$-decomposition of H, a hypergraph on n vertices. Then we construct an auxiliary hypergraph $\tilde{H}$ whose vertices come in two parts, one containing the vertices of H, and the other containing $(r-1)n/q$ new vertices $(u_{i,j})_{i\in[n/q],j\in[r-1]}$. The edges of $\tilde{H}$ are the edges of H and additionally for each vertex $v \in V(H)$ and each $i \in [n/q]$ we add the edge $\{v, u_{i,1}, \dots, u_{i,r-1}\}$. Also, let G be the graph obtained from $K_q^{(r)}$, which forms one part of G, by adding $r-1$ new vertices $y_1, \dots, y_{r-1}$, which form the second part, and adding all edges $\{x, y_1, \dots, y_{r-1}\}$ with $x \in V(K_q^{(r)})$. Then, a decomposition of $\tilde{H}$ into copies of G which respects the partition automatically gives a resolvable $K_q^{(r)}$-decomposition of H.

In fact, the main result in [38] is even more general in that it allows scenarios where the edges of the hypergraphs are colored, where multihypergraphs and ordered edges are allowed. This result is powerful and general, and stating it is complex. To convey an idea of what can be handled, let us look at the special case of graphs, which is handled in Theorem 3.10 and taken from [2]. A similar formulation of a special case is given in [41, THEOREM 3.4]. See also [40] for more general special cases.

Theorem 3.10 handles *partially directed multigraphs*, that is, $(V, E \dot\cup D)$, where V is the vertex set, E is a set of undirected edges, and D is a set of directed edges, with multiedges in E and D allowed, antiparallel directed edges allowed, but no loops. For an integer D, a partially directed multigraph is $[D]$-*edge-colored* if each edge (directed or undirected) is assigned a color from $[D]$. We are interested in obtaining decompositions in this colored setting: Let $\mathcal{G}$ be a family of $[D]$-edge-colored partially directed multigraphs on vertex set $[q]$, and let H be a $[D]$-edge-colored partially directed multigraph on vertex set $[n]$. We say that H has a $\mathcal{G}$-*decomposition* if the edges of G can be partitioned into copies of partially directed multigraphs from $\mathcal{H}$ that preserve the coloring. We further operate in a partite setting, where we allow edges inside the parts. The restrictions on the host graph H and the guest graphs G are as follows.

Definition 3.6 (Compatible partite partially directed multigraphs). Let $D, q, n \in \mathbb{N}$, let $\mathcal{P} = \{P_1, \dots, P_t\}$ be a partition of $[q]$ and $\mathcal{P}' = \{P_1', \dots, P_t'\}$ be a partition of $[n]$. Let $\mathcal{G}$ be a family of partially directed multigraphs on $[q]$, and H be a partially directed multigraph on $[n]$, with $\mathcal{G}$ and H all $[D]$-edge-colored. Further, for each color $d \in [D]$, assume we are given a pair $(i, j) \in [t]^2$, which call the *color location* of d, and each color is specified as being either a *directed color* or an *undirected color*.

In this case, we say that H and $\mathcal{G}$ are (n, q)*-compatible partite partially directed multigraphs* with partitions $\mathcal{P}$ and $\mathcal{P}'$ if the following hold for each color d and its color location (i, j):

(i) If d is a directed (undirected) color, then all edges in H and in $\mathcal{G}$ of color d are directed (undirected).

(ii) In H all edges of color d start in P'_i and end in P'_j, and in all $G \in \mathcal{G}$ all edges of color d start in P_i and end in P_j.

(iii) For each $G \in \mathcal{G}$, there are no parallel (directed or undirected) or antiparallel edges in G. (In H parallel and antiparallel edges are allowed.)

Keevash's result requires a number of very general conditions under which the desired decompositions exist, which we define next in our specific setup. We start with the divisibility conditions. For a color d and an edge-colored partially directed multigraph F, we let $e_d(F)$ denote the number of edges in F colored d, and for a vertex v in F we let $\deg_{F,d}(x)$, $\deg^{\text{out}}_{F,d}(x)$, and $\deg^{\text{out}}_{F,d}(x)$, respectively, denote the number of undirected edges of color d incident to v in D, the number of directed edges of color d leaving v in F, and the number of directed edges of color d entering v in F, respectively.

Definition 3.7 (Partite divisibility conditions). Let $\mathcal{G}$, H, and $\mathcal{P}$, $\mathcal{P}'$ be as in Definition 3.6. We say that $(H, \mathcal{P}')$ is $(\mathcal{G}, \mathcal{P})$*-divisible* if the following hold:

(i) There are integers $(m_G)_{G\in\mathcal{G}}$ such that for each $d \in [D]$ we have

$$e_d(H) = \sum_{G\in\mathcal{G}} m_G \cdot e_d(G).$$

(ii) For each $i \in [t]$ and every vertex $v \in P'_i$, there are integers $(m_{G,x})_{G\in\mathcal{G},x\in P_i}$ such that for each undirected color $d \in [D]$ we have

$$\deg_{H,d}(v) = \sum_{G\in\mathcal{G},x\in P_i} m_{G,x} \cdot \deg_{G,d}(x),$$

and for each directed color $d \in [D]$ we have

$$\deg^{\text{out}}_{H,d}(v) = \sum_{G\in\mathcal{G},x\in P_i} m_{G,x} \cdot \deg^{\text{out}}_{G,d}(x) \quad \text{and}$$

$$\deg^{\text{in}}_{H,d}(v) = \sum_{G\in\mathcal{G},x\in P_i} m_{G,x} \cdot \deg^{\text{in}}_{G,d}(x).$$

Further, the following regularity condition is required, which can be seen as a robust fractional decomposition requirement mandating that suitably weighted copies of the guest graphs G in the host graph H are distributed regularly on edges of H. We denote the fact that a colored subgraph G' of H is a copy of some $G \in \mathcal{G}$ (with colors preserved) by writing $G' \sim_H \mathcal{G}$.

Definition 3.8 (Regularity condition). Let $\mathcal{G}$, H, $\mathcal{P}$, $\mathcal{P}'$, q, and n be as in Definition 3.6 and let $c, \omega > 0$ be reals. We say that $(H, \mathcal{P}')$ is $(\mathcal{G}, \mathcal{P}, c, \omega)$*-regular*, if there are weights $(w_{G'})_{G' \sim_H \mathcal{G}}$ with $w_{G'} \in [\omega \cdot n^{2-q}, \frac{1}{\omega} \cdot n^{2-q}]$ such that for each edge $e \in E(H)$ we have

$$\sum_{G' \sim_H \mathcal{G} : e \in E(G')} w_{G'} = (1 \pm c).$$

Finally, Keevash's result uses the following condition that considers any guest graph vertex $x \in V(G_i)$ and requires that for every choice of linear-sized sets A_y for each other vertex y in G_i, where A_y is chosen in the part of $\mathcal{P}'$ where y should be embedded, the common neighborhood of these sets A_y in the part of $\mathcal{P}'$ where x should be embedded is of linear size. We first need to make precise what we mean by common neighborhood in this setting. For any two vertices y, x in some guest graph G with $x \in P_i$ and a set A of vertices in the host graph H, we define the neighborhood of A in H mandated by yx as

$$N_H^{(y,x,G)}(A) = \begin{cases} N_{H,d}(A) \cap P_i' & \text{if } yx \in E \text{ is undirected,} \\ N_{H,d}^{\text{out}}(A) \cap P_i' & \text{if } yx \in E \text{ is directed towards } x, \\ N_{H,d}^{\text{in}}(A) \cap P_i' & \text{if } yx \in E \text{ is directed towards } y, \\ P_i' & \text{if there is no edge with endpoints } x \text{ and } y, \end{cases}$$

where d is the unique color of the (directed or undirected) edge with endpoints x and y (if it exists), $N_{H,d}(A)$ is the set of common neighbors of A in H of color d, and $N_{H,d}^{\text{out}}(A)$, $N_{H,d}^{\text{in}}(A)$ are defined analogously.

Definition 3.9 (Vertex extendibility). Let $\mathcal{G}$, H, $\mathcal{P}$, $\mathcal{P}'$, q, and n be as in Definition 3.6, let h be an integer, and $\omega > 0$ be a real. We say that $(H, \mathcal{P}')$ is $(\mathcal{G}, P, \omega, h)$*-vertex extendable*, if the following holds for every guest graph $G \in \mathcal{G}$ and each of its vertices $x \in [q]$. For every choice of pairwise disjoint vertex sets $(A_y)_{y \in [q] \setminus \{x\}}$ in H, one for each vertex in G other than x, of size $|A_y| \leq h$ and with $A_y \subseteq P_j'$ whenever $y \in P_j$, we have

$$\left| \bigcap_{y \in [q] \setminus \{x\}} N_H^{(y,x,H)}(A_y) \right| \geq \omega n.$$

The following is a special case of Keevash's result [**40, Theorem 19**] (see also [**2**] for more detailed explanations).

Theorem 3.10. *Given $q, D \in \mathbb{N}$, and $\sigma > 0$, there exist $\omega_0 > 0$, $n_0 \in \mathbb{N}$, such that for $q' = \max\{q, 8 + \log_2(1/\sigma)\}$, $h' = 2^{50q'^3}$, $\delta = 2^{-10^3 q'^5}$ the following holds for every $n > n_0$ and $\omega \in (n^{-\delta}, \omega_0)$.*

Let H and $\mathcal{G} = (G_i)_{i \in [m]}$ be $[D]$-edge-colored and (n, q)-compatible partite partially directed multigraphs with partitions $\mathcal{P} = (P_i)_{i \in [t]}$ and $\mathcal{P}' = (P_i')_{i \in [t]}$, respectively, such that $|P_i'| \geq \sigma n$ for each $i \in [t]$. If $(H, \mathcal{P}')$ is $(\mathcal{G}, \mathcal{P})$-divisible, $(\mathcal{G}, \mathcal{P}, \omega^{h^{20}}, \omega)$-regular, and $(\mathcal{G}, \mathcal{P}, \sqrt[q'h]{\omega}, h)$-vertex-extendable, then H has a $\mathcal{G}$-decomposition.

Let us illustrate the power of this result with a simple example (that will be useful for packing spanning trees in Section 5). Assume we are given a partially directed multigraph H

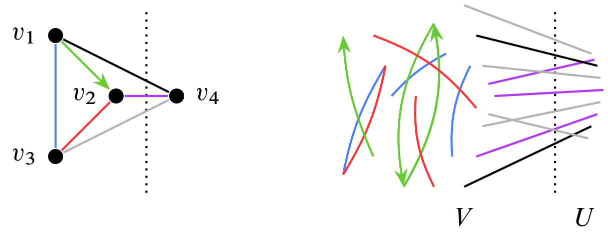

FIGURE 2
A diamond and an illustration of a colored partially directed multigraph with partition $U\dot{\cup}V$ that can be decomposed into diamond-copies with the help of Theorem 3.10.

with a partition $V(H) = V\dot{\cup}U$ such that U is much smaller than V. There are edges of 6 different colors in H; edges of the first color (let us call it green) are directed, the others are undirected; edges of the first three colors (green and, say, blue and red) run within V and edges of the other three colors (say, grey, black, and purple) run between V and U. Our goal is to pack colored *diamonds* in H such that all green, blue, red, and grey edges are used in H. Here, our diamonds D have four vertices $v_1, \ldots, v_4$, a directed green edge v_1v_2, and undirected edges v_1v_3 of color blue, v_2v_3 of color red, v_1v_4 of color black, v_2v_4 of color purple, and v_3v_4 of color grey. See Figure 2 for an illustration. In such a packing we do not need to use all black and purple edges of H (which makes the conditions we shall need in order to obtain such a packing easier). In order to apply Theorem 3.10 in this setting, we thus let $\mathcal{G}$ have three elements: D, a graph containing a single black edge, and a graph containing a single purple edge. We can then apply Theorem 3.10, if appropriate divisibility conditions for the edges of the different colors are satisfied, and the colored edges in H are suitably quasirandomly distributed: The quasirandomness conditions will imply the more general regularity and extendibility conditions. For more details, see [2, SECTION 4].

Another direction of generalization that received considerable attention concerns packings of small graphs into graphs with sufficiently high minimum degrees. Here, the most famous (and still open) problem is a conjecture of Nash-Williams concerning triangle packings, to which I will return in Section 8. Refer to the survey [49] for more details on progress concerning problems of this type.

4. PACKING CYCLES

Let us now turn to a famous problem concerning the packing of cycle factors into the complete graph that we already mentioned in the introduction.

Problem 4.1 (Oberwolfach problem [32]). Let C be any 2-regular graph on an odd number n of vertices. Can we perfectly pack copies of C into K_n?

Here, n has to be odd so that each vertex in K_n has an even number of neighbors. If such a perfect packing can be obtained, this means that n workshop participants can be placed around the tables represented by the cycles in C for $\frac{n-1}{2}$ meals so that no two sit next to each other twice. Several variations of this problem were considered, for example, the so-called Hamilton–Waterloo problem, where two different 2-regular graphs C_1 and C_2 have to be used a prescribed number of times in the packing, representing a meeting that takes place at the two close venues Hamilton and Waterloo. The Oberwolfach problem and its variants have inspired a vast number of research papers, but only recently was the problem solved for large n, independently by Glock, Joos, Kim, Kühn, Osthus [28] and Keevash, Staden [41]. (These papers are also good references for further background and previous results.) The results obtained by both groups are more general (in particular they also both solve the Hamilton–Waterloo problem), but in different ways.

The result in [28] allows for more general classes of graph to be packed into complete graphs. A graph on n vertices is called *μ-separable* if it contains a set S of at most μn vertices such that in $H \setminus S$ each component has at most μn vertices.

Theorem 4.2 (Glock, Joos, Kim, Kühn, Osthus [28]). *For all $\Delta \in \mathbb{N}$ and $\alpha > 0$, there are $\mu > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. If $\mathcal{C}$ and $\mathcal{G}$ are families of n vertex graphs containing together at most $\binom{n}{2}$ edges such that*

(i) *$\mathcal{C}$ consists of at least αn copies of a 2-regular n-vertex graph,*

(ii) *each graph in $\mathcal{G}$ is a μ-separable r-regular n-vertex graph with $r \leq \Delta$,*

then $\mathcal{C} \cup \mathcal{G}$ pack into K_n.

In their proof, Glock, Joos, Kim, Kühn, and Osthus use results on hypergraph matchings by Alon and Yuster [7], as well as an approximate decomposition result by Condon, Kim, Kühn, and Osthus [17] (see also Section 6), and a special case of the colored partite designs results of [38], alongside many new ideas. Here I will not describe more in detail how a reduction to these design results can be obtained for this packing problem; but in the next section I provide more details of a reduction of this type in a different setting.

Theorem 4.2 allows for the packing of a linear fraction of copies of the same cycle factor and any collection of separable regular graphs. In particular, it allows the perfect packing of any family of cycle factors in which one cycle factor appears linearly often. The result in [41], on the other hand, allows the perfect packing of any family of cycle factors (but not more generally separable graphs), also in host graphs which are not necessarily complete. Recall that an n-vertex graph H is called (ε, s)-typical, if every set $S \subseteq V(H)$ of at most s vertices has $(1 \pm \varepsilon)d^{|S|}n$ common neighbors in H, where $d = |E(H)|/\binom{n}{2}$.

Theorem 4.3 (Keevash, Staden [41]). *For all $\delta > 0$ there are $\varepsilon > 0$, s, and n_0 such that the following holds for any $n \geq n_0$ and $r \geq \delta n$. If $\mathcal{C}$ is a family of 2-regular n-vertex graphs with $|\mathcal{C}| = r$, and H is (ε, s)-typical, then $\mathcal{C}$ packs into H.*

For proving this result, Keevash and Staden also use the colored partite designs results of [38] as a tool. Here, a direct reduction is only possible if all cycles in $\mathcal{C}$ have lengths bounded by some constant. For nonconstant cycle lengths a more intricate reduction to colored partite designs is needed, embedding constant-length paths and connecting them into larger cycles; this requires a lot of extra work and ideas.

Packing, more generally, cycles in regular host graphs was also considered; see [18] for results concerning the packing of Hamilton cycles, and the survey [49] for more background and related results.

5. PACKING TREES

In the area of tree packings, there are two influential conjectures, which are remarkable for their elegant statements. The first of these was formulated by Ringel [60] in 1968.

Conjecture 5.1 (Ringel's conjecture [60]). *For each $n \in \mathbb{N}$ and for each tree T on $n+1$ vertices, we have that $2n+1$ copies of T pack into the complete graph K_{2n+1}.*

The second of these conjectures is attributed to Gyárfás (see [33]) and from 1978.

Conjecture 5.2 (Gyárfás's tree packing conjecture). *For each $n \in \mathbb{N}$ and for each family of trees $(T_s)_{s\in[n]}$ such that T_s has s vertices for each $s \in [n]$, we have that $(T_s)_{s\in[n]}$ packs into the complete graph K_n.*

These conjectures have in common that they ask for perfect packings, because trees on s vertices have $s-1$ edges and we have $(2n+1)(n+1) = \binom{2n+1}{2}$ and $\sum_{s=1}^{n}(s-1) = \binom{n}{2}$. However, they differ in that Ringel's conjecture concerns the packing of copies of the same tree, which has roughly half the number of vertices of the host graph, and the tree packing conjecture concerns the packing of different trees, some of which have essentially the same number of vertices as the host graph. Thus, it is not surprising that Gyárfás's tree packing conjecture turned out more challenging than Ringel's conjecture.

Both these conjectures inspired much work, and we shall turn to recent progress shortly, concentrating on the highlights. But first I will mention connections to some other well-studied combinatorial objects. Indeed, when it turned out that Ringel's conjecture was difficult, even for special classes of trees, then it was suggested that some symmetry might help in attacking the problem. To this end, Rosa [63] introduced a notion which by now we call graceful labelings. A *graceful labeling* of a graph H is an injective mapping $f : V(H) \to \{1, \dots, e(H)+1\}$ such that the induced edge labels $|f(x) - f(y)|$ for $xy \in E(H)$ are distinct. It is easy to see that if H has a graceful labeling, then there is a packing of k copies of H into the complete graph on k vertices as long as $k \geq 2e(H)+1$: Given a graceful labeling f of H consider the embeddings $f_i : V(H) \to V(K_k)$ with $V(K_k) = [k]$ and $f_i(x) = f(x) + i$ for $0 \leq i < k$, where $k \geq 2e(H)+1$ guarantees that these embeddings are edge disjoint. This means that f_0 embeds H as mandated by the labeling, and then we take translates (or "rotations") of this embedding to pack all other copies of H. Consequently, the following conjecture, which is attributed to Kotzig (see [63]), implies Ringel's conjecture (Conjecture 5.1).

Conjecture 5.3 (Graceful labeling conjecture). *Every tree has a graceful labelling.*

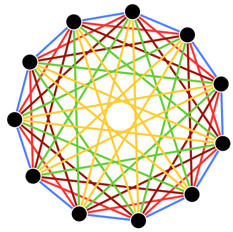

FIGURE 3
The near distance coloring of K_{11}.

This conjecture is still open, but Adamaszek, Allen, Grosu, and Hladký [1] proved an approximate version for almost all trees: They showed that every n-vertex tree with maximum degree at most $cn/\log n$ has a labeling satisfying the gracefulness property which uses labels from $[(1+\varepsilon)n]$, where $\varepsilon > 0$, $c > 0$ is small compared to ε, and n is sufficiently large.

What proved more important for the resolution of Ringel's conjecture (for large n) is the following connection to rainbow copies in edge-colored graphs. A *rainbow* copy of a graph G in an edge-colored graph H is a copy of G in H whose edges have pairwise distinct colors. The *near distance coloring* of the complete graph K_k on vertex set $[k]$ assigns to each edge $uv \in E(K_k)$ the smallest number d as color such that $u + d = v$ or $v + d = u$ modulo k (see Figure 3 for an example). Again, it is easy to see that a rainbow copy of H in K_k under the near distance coloring gives a packing of k (uncolored) copies of H in K_k (uncolored) by taking the rainbow copy and considering translates as before. Using this formulation, Montgomery, Pokrovskiy, and Sudakov [57] proved that Ringel's conjecture holds for large n. (Preliminary results were obtained in [15, 34], but these results work more generally also for Gyárfás's conjecture, so they are listed below.)

Theorem 5.4 (Montgomery, Pokrovskiy, Sudakov [57]). *For every sufficiently large n and every tree T on $n+1$ vertices, there is a rainbow copy of T in the near-distance coloring of K_{2n+1}.*

For proving this result, Montgomery, Pokrovskiy, and Sudakov distinguish different cases, depending on whether the tree under consideration has many nonneighboring leaves, many bare paths, or most vertices in the tree are leaves with many neighboring leaves. Here, a bare path is a path in the tree whose internal vertices have no neighbors outside the path. These cases are exhaustive because trees have average degree smaller than 2. In the first two cases, completing the rainbow copy relies on a version of absorption introduced in [54]. Techniques from Montgomery, Pokrovskiy, and Sudakov's earlier papers [55, 56] are also used.

Keevash and Staden [42] prove the following generalization of Ringel's conjecture to quasirandom graphs.

Theorem 5.5 (Keevash, Staden [42]). *There exists $s \in \mathbb{N}$ such that for all $p > 0$ there are $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ with $pn \in \mathbb{N}$ the following holds. Let T be a tree on $pn + 1$ vertices. Let H be a graph on $2n + 1$ vertices with $pn(2n + 1)$ edges which is (ε, s)-typical. Then H can be decomposed into $2n + 1$ copies of T.*

Their proof distinguishes similar cases as the previous result: almost all vertices belong to large stars; many leaves are in small stars; and many vertices are in disjoint bare paths. They also apply a recent result on pseudorandom hypergraph matchings by Ehard, Glock, and Joos [21], their methods developed for proving Theorem 4.3, a result of Barát, Gyarfás, and Sárközy [10] on rainbow matchings in bipartite multigraphs, alongside many new ideas.

As indicated before, for Gyárfás's conjecture (Conjecture 5.2) less is known. An almost perfect packing version for trees with constant maximum degree was obtained in [15], applying a version of the Rödl nibble. Ferber and Samotij [25] considered trees with maximum degree up to $cn/\log n$, proving these can be almost-perfectly packed into random host graphs (in fact, they have more general results for sparse random host graphs). Getting perfect packing results turned out much harder. Joos, Kim, Kühn, and Osthus [34] proved that Gyárfás's conjecture holds for families of trees with constant maximum degree if n is sufficiently large. Their proof uses an array of important tools developed previously: Szemerédi's regularity lemma, robust expanders, random walks, iterative absorption (we refer to [34] for more details) and a blow-up lemma for approximate decompositions (that we shall return to in Section 6).

In [2,3] it is shown that Gyárfás's conjecture holds when n is sufficiently large for all families of trees with maximum degree $cn/\log n$ for some universal constant $c > 0$. In fact, the following more general result is obtained, which applies to quasirandom host graphs, and also implies a version of Gyárfás's conjecture for families of different trees (of the same size) under the same maximum degree restriction.

Theorem 5.6. *For each $\delta, d > 0$ there exist $c, \varepsilon > 0$ and $n_0, s \in \mathbb{N}$ such that for each $n \geq n_0$ in any (ε, s)-typical graph H on n vertices with at least dn^2 edges we can pack any family $(T_t)_{t\in[N]}$ of trees satisfying*

(i) $\sum_{t\in[N]} e(T_t) \leq e(H)$ *and* $\Delta(T_t) \leq \frac{cn}{\log n}$ *for all* $t \in [N]$,

(ii) $\delta n \leq v(T_t) \leq (1-\delta)n$ *for all* $1 \leq t \leq (\frac{1}{2}+\delta)n$ *and* $v(T_t) \leq n$ *for all* $(\frac{1}{2}+\delta)n < t \leq N$.

Let me briefly sketch some proof ideas used for obtaining this result. Similarly to the approaches above, we distinguish two cases: Either a linear number of the nonspanning trees we are given contain a linear number of leaves, or a linear number of them contain a linear number of disjoint bare paths of length 11. In both cases we first remove these leaves/paths and obtain an almost perfect packing of what remains using a random packing process which we outline in the subsequent section, with a leftover graph $\tilde{H}$. We show that this random process preserves many nice properties, which we shall need to complete the packing, for example, that $\tilde{H}$ is quasirandom. In the first case (which is treated in [3]), it then remains to pack the omitted leaves. We obtain this by another random process, in one round randomly mapping leaves "dangling" at one host graph vertex to remaining host graph edges, moving on to the next host graph vertex in the next round, an so on. Here, a leaf "dangles" at a host graph vertex v if its neighbor has been embedded to v. For being successful in this process, we further need to use a random orientation of the host graph and exploit superregularity properties of certain auxiliary graphs representing all choices we have for embedding leaves "dangling" at v. More precisely, our random process will select a random perfect matching in this auxiliary graph, and then we need to show that this does not too negatively affect the auxiliary graphs for the remaining host graph vertices v'. Some of these ideas are inspired by methods from [34].

In the second case (which is treated in [2]), it remains to pack the remaining bare paths, and also a small number of leaves that we also omitted when obtaining the almost perfect packing. These leaves were omitted for the following reason: All vertices in bare paths have degree 2, which creates some obvious parity restrictions when we want to pack the bare paths. We now first embed the omitted leaves in such a way that we obtain the necessary parity of edges remaining at each host graph vertex. Each of the bare paths that now remain has 11 vertices. We will then, in a sequence of carefully tailored intermediate stages, embed some of these paths completely and some of these paths partly, until we arrive at the following scenario, where only a set of bare paths of length 3 remain to be packed which are paths in a subset $(T_t)_{t\in\mathcal{S}}$ of our trees. Our goal is to apply Keevash's partite and colored designs result (in the form of Theorem 3.10) to pack these. In the scenario we obtain, the remainder H^* of the host graph has an even number of vertices which are partitioned into two sets $V_{\boxminus} = \{\boxminus_i : i \in [\ell]\}$ and $V_{\boxplus} = \{\boxplus_i : i \in [\ell]\}$. We call each pair $\{\boxminus_i, \boxplus_i\}$ a *terminal pair*, and each of the remaining paths x, x', y', y is *anchored* at some terminal pair, that is x is embedded to some $\boxminus_i$ and y is embedded to the corresponding $\boxplus_i$, but the edges xy, yz, zw still need to be embedded. When these will be embedded, then we insist that x' is also embedded in $V_{\boxminus}$ and y' is also embedded in $V_{\boxplus}$, which implies that in our intermediate stages mentioned above we will need to guarantee that $H^*[V_{\boxplus}]$ and $H^*[V_{\boxplus}]$ have the same number of edges.

We shall apply the designs result, Theorem 3.10, on the following auxiliary partially directed colored multigraph, which we call chest and which has vertex parts $V = \{1, \dots, |V_{\boxminus}|\}$ and $U = \mathcal{S}$. The chest has the following colored edges running within V: A blue undirected edge ij for each edge $\boxminus_i\boxminus_j \in E(H^*)$, a red undirected edge ij for each edge $\boxplus_i\boxplus_j \in E(H^*)$, and a green directed edge from i to j for each edge $\boxminus_i\boxplus_j \in E(H^*)$. This means that the embedding of a remaining path x, x', y', y to vertices $\boxminus_i, \boxminus_j, \boxplus_k, \boxplus_i$ in H^* corresponds to a triangle i, j, k in the chest, in which ij is blue, jk is green and directed to k, and ki is red. The chest further has the following edges running from V to U: a grey undirected edge ti for each remaining path in T_t anchored at $\{\boxminus_i, \boxplus_i\}$, a black undirected edge ti for each for each terminal $\boxminus_i$ that does not host any vertex of T_t yet (embedded in previous stages), and a purple undirected edge ti for each terminal $\boxplus_i$ that does not host any vertex of T_t, yet. The reason for inserting these edges is that we need to

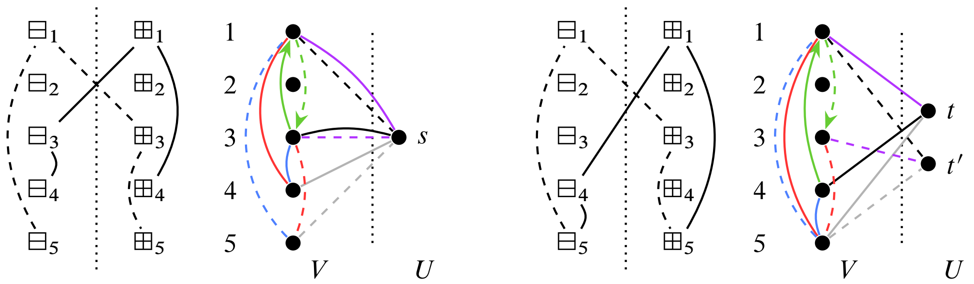

FIGURE 4

The two left-hand pictures show an example of two remaining paths from the same tree T_t embedded in H^* to the paths $\boxminus_4 \boxminus_3 \boxplus_1 \boxplus_4$ and $\boxminus_5 \boxminus_1 \boxplus_3 \boxplus_5$ and the corresponding diamonds in the chest. The two right-hand pictures show an example of two remaining paths from different trees T_t and $T_{t'}$ embedded in H^* to the paths $\boxminus_5 \boxminus_4 \boxplus_1 \boxplus_5$ and $\boxminus_5 \boxminus_1 \boxplus_3 \boxplus_5$, respectively, and the corresponding diamonds in the chest. The solid and dashed lines in this picture are only used to distinguish the two paths/diamonds.

guarantee that the remaining paths are embedded vertex disjointly if they come from the same tree and that they also do not use vertices previously used for this tree; in addition we need that for a given terminal pair exactly the trees T_t for which some remaining bare path is anchored at this terminal pair receive exactly one path with these endpoints in the packing (this is guaranteed by the grey edges). A packing of the remaining paths in H^* which satisfies these properties corresponds precisely to a packing of diamonds (recall the definition from Section 3) in the chest using all red, green, blue, and grey edges. See Figure 4 for an illustration. Consequently, our strategy is to use the intermediate stages for ensuring the necessary divisibility, regularity, and extendibility properties of the chest so that we can then apply Theorem 3.10 to complete our perfect packing. (For more details, see [2, SECTION 3].)

6. PACKING MORE GENERAL GRAPH CLASSES

The results on designs in Section 3 concern packings of graphs that are small in comparison to the host graph. The results on cycle and tree packings discussed in the previous two sections concern the packing of graphs from very special classes of graphs. It is natural to ask for which other types of guest graphs one can hope for analogous packing results.

Early progress was made by Messuti, Rödl, and Schacht [52], who used the results of Ray-Chaudhury and Wilson [59] on resolvable graph decompositions to obtain almost perfect packings of almost spanning graphs from minor-closed families of graphs with bounded maximum degree. I omit the definition of minors here; suffice it to say that graphs embeddable on a fixed surface are a special case. Ferber, Lee, and Mousset [24] generalized this result, moving from almost spanning graphs to spanning graphs.

In [5] more generally families of D-degenerate graphs are considered. A graph is called *D-degenerate* if there is an ordering of its vertices such that each vertex has at most D

neighbors preceding it. Many important classes of graphs are degenerate, such as trees, which are 1-degenerate, or planar graphs, which are 5-degenerate.

Theorem 6.1 ([5]). *For every $D \in \mathbb{N}$ and $\eta > 0$, there are $n_0 \in \mathbb{N}$ and $c, \varepsilon > 0$ such that for each $n \geq n_0$ the following holds. Suppose that $(G_s)_{s \in \mathcal{S}}$ is a family of D-degenerate graphs, each on at most n vertices and of maximum degree $\frac{cn}{\log n}$, whose total number of edges is at most $(p - \eta)\binom{n}{2}$, and suppose that H is an $(\varepsilon, 2D+3)$-typical n-vertex graph with $p\binom{n}{2}$ edges. Then $(G_s)_{s \in \mathcal{S}}$ packs into H.*

The techniques developed for proving this result form the starting point for the results on tree packings obtained in [2, 3] discussed in the previous section. Accordingly, the results of [2, 3] also more generally apply to certain classes of D-degenerate guest graphs with many leaves and many bare paths, respectively. The details are more complex, and we omit them here.

In the proof of this result we use the following natural random packing process. We embed the guest graphs G_s one after the other. When constructing an embedding ϕ_s of G_s, we proceed vertex by vertex, following a degeneracy order $x_1, x_2, \ldots$. When embedding x_i, we consider all previously embedded neighbors $y_1, \ldots, y_\ell$ of x_i, of which there are at most D. It is clear that we need to embed x_i into the set X_i that is given by the common neighborhood in the host graph of the $\phi_s(y_j)$ with $j \in [\ell]$ minus the set U_i of vertices used already for earlier vertices of G_s, that is, $U_i = \{\phi(x_j) : j < i\}$. We choose a random vertex in X_i as $\phi(x_i)$. Then we delete from the host graph all edges used for embedding x_i, that is, all edges $\phi_s(y_j)\phi_s(x_i)$ with $j \in [\ell]$. This process will of course only have a chance of succeeding if all of our guest graphs are a bit smaller than the host graph H. To obtain this setting, we omit a small (linear) number of vertices from each guest graph G_s that is too large before running the random process. We also set aside a small random proportion of the edges of H, which we use after running the random packing process to pack the omitted vertices greedily. While the described random packing process is easy, analyzing it is not: In order to prove Theorem 6.1, we show that the sets X_i always stay as large as expected because the random process preserves pseudorandomness of the (changing) host graph, as well as a suitably random distribution of the sets U_i, among other nice properties. This then allows us to complete the packing.

Kim, Kühn, Osthus, and Tyomkyn [44] provide an important general purpose tool for obtaining almost-perfect decompositions, namely a blow-up lemma for decompositions. The blow-up lemma [46] is an integral part of the powerful *regularity method*, complementing Szemerédi's celebrated regularity lemma [64]. For simplicity, we only state the bipartite version here, which displays the essence of the setup; the generalization of this setup to more general partite graphs is standard. For a bipartite graph H with partition (V_1, V_2) and sets $V_1' \subseteq V_1$, $V_2' \subseteq V_2$, we let $d_H(V_1', V_2') = e_H(V_1', V_2')/(|V_1'||V_2'|)$ be the *density* of (V_1', V_2'). We say that H is (ε, d)-*regular* if $d_H(V_1', V_2') = d \pm \varepsilon$ for each $V_1' \subseteq V_1$, $V_2' \subseteq V_2$ with $|V_1'| \geq \varepsilon|V_1|$, $|V_2'| \geq \varepsilon|V_2|$. Further, H is (ε, d)-superregular, if it is (ε, d)-regular and each vertex in V_1 (respectively V_2) has $(d \pm \varepsilon)|V_2|$ (respectively $(d \pm \varepsilon)|V_1|$) neighbors in V_2 (respectively V_1).

Theorem 6.2. *For every $\alpha > 0$, there are $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$ and $d \geq \alpha$. Suppose H is a bipartite graph with partition classes of size n, which is (ε, d)-superregular. Suppose $(G_s)_{s\in\mathcal{S}}$ with $|\mathcal{S}| \leq \alpha^{-1}n$ is a family of bipartite graphs with partition classes of site n, with maximum degree $\Delta(G_s) \leq \alpha^{-1}$ for each $s \in \mathcal{S}$, whose total number of edges is at most $(1-\alpha)dn^2$. Then there is a packing of $(G_s)_{s\in\mathcal{S}}$ into H.*

Ehard and Joos [23] provide a simplified proof of this result, and are also able to obtain a generalization, yielding packings with stronger quasirandomness properties. Applications of this result are manifold. It has been applied as a tool (among other techniques) in [34] and [43] for obtaining packings of trees, in [17] for obtaining a packing version of the so-called bandwidth theorem, and in [47] for decompositions in more general graphs.

7. PACKING LARGE HYPERGRAPHS

Analogues for hypergraph packings have been considered for various results discussed in the preceding sections. In particular, Keevash's result on decompositions in the coloured and partite setting discussed in Section 3 does more generally allow k-partite hypergraphs (under suitable conditions). In the same section we also already mentioned the results on general hypergraph F-designs by Glock, Kühn, Lo, and Osthus [30]. Here, I will just briefly mention some important further developments.

Almost-perfect decompositions of regular hypergraphs satisfying certain quasirandomness properties into Hamilton cycles (of different types) were obtained by Bal and Frieze [9]. Packings of more general tight cycle factors in hypergraphs with large co-degrees were considered by Joos, Kühn, and Schülke [35]. Results on almost-perfect decompositions quasirandom hypergraphs into arbitrary families of hypergraphs of bounded maximum degree are proved by Ehard and Joos [22]. Turning to hypergraphs with larger degrees, in [4] the almost-perfect packing result for D-degenerate graphs of [5] and the perfect packing result for D-degenerate graphs with many leaves of [3] are generalized to hypergraphs.

I close this section by remarking that graph and hypergraph packings are intimately related to the problem of finding perfect matchings in certain hypergraphs, highlighting the importance of new results in this direction, such as those by Ehard, Glock, and Joos [21], for the area. For example, consider the problem of finding an F-factor of $K_n^{(r)}$ for some r-regular hypergraph F. This is equivalent to finding a perfect matching in the $e(F)$-uniform hypergraph with vertex set $E(K_n^{(r)})$ which has an edge $\tilde{F} = \{e_1, \ldots, e_{e(F)}\}$ whenever $\tilde{F}$ is a copy of F (not necessarily induced) in $K_n^{(r)}$. More details on connections between packings, hypergraph matchings, and also rainbow subgraphs can be found in [21].

8. SOME OPEN PROBLEMS

I close with a small collection of what I consider some important open problems in the area. Firstly, it remains to resolve Gyárfás's tree packing conjecture in full

Problem 8.1. Solve the tree packing conjecture (Conjecture 5.2) in full.

Similarly, the graceful labeling conjecture also remains open.

Problem 8.2. Solve the graceful labeling conjecture (Conjecture 5.3) in full.

Another famous conjecture that I only mentioned in passing so far concerns triangle packings.

Conjecture 8.3 (Nash-Williams [58]). *For large n every K_3-divisible graph H on n vertices with $\delta(H) \geq \frac{3}{n}/4$ has a K_3-decomposition.*

By a result of Barber, Kühn, Lo, and Osthus [13], the approximate version of this conjecture follows from a fractional version. Recent progress on fractional triangle decompositions was made by Dross [20] and Delcourt and Postle [19], but it remains open to show that a fractional triangle decomposition exists in n-vertex graphs of minimum degree $3n/4$. Decomposition problems for other graphs F than the triangle were considered in [12, 29, 53].

For perfectly packing cycle factors in a graph H, we clearly cannot impose any nontrivial minimum degree condition; instead we need the host graph H to be regular. The following conjecture from [28] concerns packings of cycle factors in sufficiently dense regular graphs.

Conjecture 8.4 (Glock, Joos, Kim, Kühn, Osthus [28]). *Any large n-vertex r-regular graph H with even $r \geq \frac{3}{4}n + o(n)$ has a decomposition into G-copies for any 2-regular graph G on n vertices.*

For more general classes of graphs than cycles, trees and F-factors, little is known so far when it comes to perfect packings. The following conjecture from [28] concerns packings of arbitrary regular graphs into the complete graph.

Conjecture 8.5 (Glock, Joos, Kim, Kühn, Osthus [28]). *For all $\Delta \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ so that for $n \geq n_0$ any family $(G_i)_{i \in [t]}$ of n-vertex graphs such that G_i is r_i-regular with $r_i \leq \Delta$ and $\sum_{i \in [t]} r_i = n - 1$ packs into K_n.*

Similarly, for hypergraphs many problems remain. In particular, showing that $K_n^{(r)}$ has a decomposition into tight Hamilton cycles (under appropriate divisibility conditions) is still open. This was conjectured by Bailey and Stevens [8]. Glock, Kühn, and Osthus propose the following more general conjecture.

Conjecture 8.6 (Glock, Kühn, and Osthus [49]). *For fixed k and large n, every vertex disjoint union G of tight k-uniform cycles, each of length at least $2k - 1$, with in total n vertices, decomposes $K_n^{(k)}$ if k divides $\binom{n-1}{k-1}$.*

Another direction that has not yet seen much progress is that of packing problems in sparse graphs. As mentioned earlier, tree packing problems in this context were considered in [25]. Packings of Hamilton cycles in sparse random graphs were considered in [14, 27, 45]. It would be interesting to obtain similar results for other families of guest graphs.

Problem 8.7. For which families of graphs and probabilities p is the following true? Given a family $(G_i)_{i\in[t]}$ of graphs on at most $(1-\varepsilon)n$ vertices with in total at most $(1-\varepsilon)p\binom{n}{2}$ edges, we can pack $(G_i)_{i\in[t]}$ into the random graph $G(n,p)$.

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